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Existence and Uniqueness of Positive Solutions to a Semilinear Elliptic Problem in $\mathbb{R}^N$

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Let $p \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ with $p > 0$ and let $f \in C^1((0,\infty),(0,\infty))$ be such that $\lim_{u\searrow 0} f(u)/u = +\infty$, f is bounded at infinity, and the mapping $u \longmapsto f(u)/(u+\beta)$ is decreasing on $(0,\infty)$, for some $\beta > 0$. We prove that the problem $-\Delta u = p(x)f(u)$ in $\mathbb{R}^N$, $N > 2$, has a unique positive $C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ solution that vanishes at infinity provided $\int_0^\infty r\Phi(r)dr < \infty$, where $\Phi(r) = \max\{p(x); |x| = r\}$. Furthermore, it is showed that this condition is nearly optimal. Our results extend previous works by Lair–Shaker and Zhang, while the proofs are based on two theorems on bounded domains, due to Brezis–Oswald and Crandall–Rabinowitz–Tartar.

1. INTRODUCTION

Consider the problem

$$\begin{aligned} -\Delta u &= p(x)f(u) \qquad \text{in } \mathbb{R}^N \\ u &> 0 \qquad \text{in } \mathbb{R}^N \\ u(x) &\to 0 \qquad \text{as } |x| \to \infty, \end{aligned} \tag{1}$$

where $N > 2$ and the function p satisfies the following hypotheses:

(p1) $p \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.
(p2) $p > 0$ in $\mathbb{R}^N$.

This problem has been intensively studied in the case where $f(u) = u^{-\gamma}$, with $\gamma > 0$. For instance, in the case of a bounded domain $\Omega \subset \mathbb{R}^N$, Lazer and McKenna proved in [7] that the problem

$$-\Delta u = p(x)u^{-\gamma}, \qquad \text{in } \Omega,$$

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has a unique classical solution if p is a sufficiently smooth function that is positive on $\overline{\Omega}$. The existence of entire positive solutions on $\mathbb{R}^N$ for $\gamma \in (0, 1)$ and under certain additional hypotheses has been established by Edelson [4] and Kusano–Swanson [5]. For instance, Edelson proved the existence of a solution provided that

$$\int_1^\infty r^{N-1+\lambda(N-2)} \Phi(r) dr < \infty,$$

for some $\lambda \in (0, 1)$, where $\Phi(r) = \max_{|x|=r} p(x)$. This result is generalized for any $\gamma > 0$ via the sub- and super solutions method in Shaker [8] or by other methods by Dalmasso [3]. Lair and Shaker continued in [6] the study of (1) for $f(u) = u^{-\gamma}$, $\gamma > 0$. They proved the existence of a solution under the hypothesis

(p3) $\int_0^\infty r \cdot \Phi(r) dr < \infty$, where $\Phi(r) = \max_{|x|=r} p(x)$.

Zhang studied in [9] the case of a nonlinearity $f \in C^1((0, \infty), (0, \infty))$ that decreases on $(0, \infty)$ and satisfies $\lim_{u \searrow 0} f(u) = +\infty$.

Our aim is to extend the results of Lair, Shaker and Zhang for the case of a nonlinearity that is not necessarily decreasing on $(0, \infty)$. More exactly, let f: $(0, \infty) \to (0, \infty)$ be a C^1 function that satisfies the following assumptions:

(f1) There exists $\beta > 0$ such that the mapping $u \longmapsto f(u)/(u + \beta)$ is decreasing on $(0, \infty)$.

(f2) $\lim_{u \searrow 0} f(u)/u = +\infty$ and f is bounded in a neighborhood of $+\infty$.

Our main result is the following:

THEOREM 1. *Under hypotheses* (f1), (f2), (p1)–(p3), *problem* (1) *has a unique positive global solution* $u \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$.

Theorem 1 shows that (p3) is sufficient for the existence of the unique solution to problem (1). The following result shows that condition (p3) is nearly necessary.

THEOREM 2. *Suppose p is a positive radial function that is continuous on $\mathbb{R}^N$ and satisfies*

$$\int_0^\infty r p(r) dr = \infty.$$

Then problem (1) *has no positive radial solution*.

2. UNIQUENESS

Suppose u and v are arbitrary solutions of problem (1). Let us show that $u \leq v$ or, equivalently, $\ln(u(x)+\beta) \leq \ln(v(x)+\beta)$, for any $x \in \mathbb{R}^N$. Assume the contrary. Since we have

$$\lim_{|x|\to\infty} (\ln(u(x)+\beta) - \ln(v(x)+\beta)) = 0,$$

we deduce that $\max_{\mathbb{R}^N}(\ln(u(x)+\beta) - \ln(v(x)+\beta))$ exists and is positive. At that point, say x_0, we have

$$\nabla(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) = 0,$$

so

$$\frac{1}{u(x_0)+\beta} \cdot \nabla u(x_0) = \frac{1}{v(x_0)+\beta} \cdot \nabla v(x_0). \tag{2}$$

By (f1) we obtain

$$\frac{f(u(x_0))}{u(x_0)+\beta} < \frac{f(v(x_0))}{v(x_0)+\beta}. \tag{3}$$

So, by (2) and (3),

$$\begin{aligned}
0 &\geq \Delta(\ln(u(x_0)+\beta) - \ln(v(x_0)+\beta)) \\
&= \frac{1}{u(x_0)+\beta} \cdot \Delta u(x_0) - \frac{1}{v(x_0)+\beta} \cdot \Delta v(x_0) \\
&\quad - \frac{1}{(u(x_0)+\beta)^2} \cdot |\nabla u(x_0)|^2 + \frac{1}{(v(x_0)+\beta)^2} \cdot |\nabla v(x_0)|^2 \\
&= \frac{1}{u(x_0)+\beta}\,\Delta u(x_0) - \frac{1}{v(x_0)+\beta}\,\Delta v(x_0) \\
&= -p(x_0)\left(\frac{f(u(x_0))}{u(x_0)+\beta} - \frac{f(v(x_0))}{v(x_0)+\beta}\right) > 0,
\end{aligned}$$

which is a contradiction. Hence $u \leq v$. A similar argument can be made to produce $v \leq u$, forcing $u = v$.

3. EXISTENCE

We first show that our hypothesis (f1) implies that $\lim_{u\searrow 0} f(u)$ exists, finite or $+\infty$. Indeed, since $f(u)/(u+\beta)$ is decreasing, there exists $L := \lim_{u\searrow 0} f(u)/(u+\beta) \in (0, +\infty]$. It follows that $\lim_{u\searrow 0} f(u) = L\beta$.

To prove the existence of a solution to (1), we need to employ a corresponding result by Brezis and Oswald (see [1]) for bounded domains. They considered the problem

$$\begin{aligned} -\Delta u &= g(x,u) \qquad \text{in } \Omega \\ u \geq 0, \quad u &\not\equiv 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $g(x,u)$: $\Omega \times [0,\infty) \to \mathbb{R}$.

Assume that

$$\begin{aligned} &\text{for a.e. } x \in \Omega \text{ the function } u \to g(x,u) \text{ is continuous on } [0,\infty) \\ &\text{and the function } u \to g(x,u)/u \text{ is decreasing on } (0,\infty); \end{aligned} \tag{5}$$

$$\text{for each } u \geq 0 \text{ the function } x \to g(x,u) \text{ belongs to } L^\infty(\Omega); \tag{6}$$

$$\exists C > 0 \text{ such that } g(x,u) \leq C(u+1) \text{ a.e. } x \in \Omega, \qquad \forall u \geq 0. \tag{7}$$

Set

$$a_0(x) = \lim_{u \searrow 0} g(x,u)/u \quad \text{and} \quad a_\infty(x) = \lim_{u \to \infty} g(x,u)/u,$$

so that $-\infty < a_0(x) \leq +\infty$ and $-\infty \leq a_\infty(x) < +\infty$.

Under these hypotheses on g, Brezis and Oswald proved in [1] that there is at most one solution of (4). Moreover, a solution of (4) exists if and only if

$$\lambda_1(-\Delta - a_0(x)) < 0 \tag{8}$$

and

$$\lambda_1(-\Delta - a_\infty(x)) > 0, \tag{9}$$

where $\lambda_1(-\Delta - a(x))$ denotes the first eigenvalue of the operator $-\Delta - a(x)$ with zero Dirichlet condition. The precise meaning of $\lambda_1(-\Delta - a(x))$ is

$$\lambda_1(-\Delta - a(x)) = \inf_{\varphi \in H_0^1,\ \|\varphi\|_2 = 1} \left(\int |\nabla\varphi|^2 - \int_{[\varphi \neq 0]} a\varphi^2 \right).$$

Note that $\int_{[\varphi\neq 0]} a\varphi^2$ makes sense if $a(x)$ is any measurable function such that either $a(x) \leq C$ or $a(x) \geq -C$ a.e. on Ω.

Let us consider the problem

$$\begin{aligned} -\Delta u_k &= p(x)f(u_k), \qquad \text{if } \mid x \mid < k, \\ u_k(x) &= 0, \qquad \text{if } \mid x \mid = k. \end{aligned} \tag{10}$$

The following two distinct situations may occur:

Case 1. f is bounded on $(0, \infty)$. In this case, as we have initially observed, $\lim_{u \searrow 0} f(u)$ exists and it is finite, so f can be extended by continuity at the origin.

To obtain a solution to problem (10), it is enough to verify that the hypotheses of the Brezis–Oswald theorem are fulfilled. Obviously, (5) and (6) hold. Now, using (p1), (p2), and the fact that f is bounded, we easily deduce that (7) is satisfied. We observe that $a_0(x) = \lim_{u \searrow 0} p(x)f(u)/u = +\infty$ and $a_\infty(x) = \lim_{u \to +\infty} p(x)f(u)/u = 0$. Then (8) and (9) are also fulfilled. Thus by Theorem 1 in [1] problem (10) has a unique solution u_k, which, by the maximum principle, is positive in $|x| < k$.

Case 2. $\lim_{u \searrow 0} f(u) = +\infty$. We will apply the method of sub- and supersolutions to find a solution to the problem (10). We first observe that 0 is a subsolution for this problem.

We construct in what follows a positive supersolution. By the boundedness of f in a neighborhood of $+\infty$, there exists $A > 0$ such that $f(u) \leq A$, for any $u \in (1, +\infty)$. Let f_0: $(0, 1] \to (0, +\infty)$ be a continuous nonincreasing function such that $f_0 \geq f$ on $(0, 1]$. We can assume without loss of generality that $f_0(1) = A$. Set

$$g(u) = \begin{cases} f_0(u), & \text{if } 0 < u \leq 1, \\ A, & \text{if } u > 1. \end{cases}$$

Then g is a continuous nonincreasing function on $(0, +\infty)$. Let h: $(0, \infty) \to (0, \infty)$ be a C^1 nonincreasing function such that $h \geq g$. Thus, by Theorem 1.1 in [2], the problem

$$\begin{cases} -\Delta U = p(x)h(U) & \text{if } \mid x \mid < k, \\ U = 0, & \text{if } \mid x \mid = k, \end{cases}$$

has a positive solution. Now, since $h \geq f$ on $(0, +\infty)$, it follows that U is a supersolution for problem (10).

In both cases studied above we define $u_k = 0$ for $|x| > k$. Using a maximum principle argument as already done above to prove the uniqueness, we can show that $u_k \leq u_{k+1}$ on $\mathbb{R}^N$.

We now prove the existence of a positive function $v \in C^2(\mathbb{R}^N)$ for which $u_k \leq v$ on $\mathbb{R}^N$. As in [6] we construct first a positive radially symmetric function w such that $-\Delta w = \Phi(r)$ $(r = \mid x \mid)$ on $\mathbb{R}^N$ and $\lim_{r \to \infty} w(r) = 0$. We obtain

$$w(r) = K - \int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1} \Phi(\sigma)\, d\sigma\, d\zeta,$$

where

$$K = \int_0^\infty \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta, \tag{11}$$

provided the integral is finite. Integration by parts gives

$$\begin{aligned}
&\int_0^r \zeta^{1-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta \\
&\qquad = -(n-2)^{-1} \int_0^r \frac{d}{d\zeta}\zeta^{2-n} \int_0^\zeta \sigma^{n-1}\Phi(\sigma)\, d\sigma\, d\zeta \\
&\qquad = (n-2)^{-1}\left(-r^{2-n} \int_0^r \sigma^{n-1}\Phi(\sigma)\, d\sigma + \int_0^r \zeta\Phi(\zeta)\, d\zeta\right).
\end{aligned} \tag{12}$$

Now, by l'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{r\to\infty}\left(-r^{2-n} \int_0^r \sigma^{n-1}\Phi(\sigma)\, d\sigma + \int_0^r \zeta\Phi(\zeta)\, d\zeta\right) \\
&\qquad = \lim_{r\to\infty} \frac{-\int_0^r \sigma^{n-1}\Phi(\sigma)\, d\sigma + r^{n-2}\int_0^r \zeta\Phi(\zeta)\, d\zeta}{r^{n-2}} \\
&\qquad = \lim_{r\to\infty} \int_0^r \zeta\Phi(\zeta)\, d\zeta = \int_0^\infty \zeta\Phi(\zeta)\, d\zeta < \infty.
\end{aligned}$$

It follows that $K = 1/(n-2) \cdot \int_0^\infty \zeta\Phi(\zeta)\, d\zeta < \infty$.

Clearly, we have

$$w(r) < \frac{1}{n-2} \cdot \int_0^\infty \zeta\Phi(\zeta) d\zeta \qquad \forall r > 0.$$

Let v be a positive function such that $w(r) = (1/c) \cdot \int_0^{v(r)} t/f(t)\, dt$, where $c > 0$ will be chosen such that $Kc \le \int_0^c t/f(t)\, dt$.

We prove that we can find $c > 0$ with this property.

By our hypothesis (f2) we obtain that $\lim_{x\to\infty} \int_0^x t/f(t)\, dt = +\infty$. Now using l'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{\int_0^x t/f(t)\, dt}{x} = \lim_{x\to\infty} \frac{x}{f(x)} = +\infty.$$

From this we deduce that there exists $x_1 > 0$ such that $\int_0^x t/f(t)\, dt \ge Kx$ for all $x \ge x_1$. It follows that for any $c \ge x_1$ we have $Kc \le \int_0^c t/f(t)\, dt$.

But w is a decreasing function, and this implies that v is a decreasing function, too. Then

$$\int_0^{v(r)} \frac{t}{f(t)} dt \le \int_0^{v(0)} \frac{t}{f(t)} dt = c \cdot w(0) = c \cdot K \le \int_0^c \frac{t}{f(t)} dt.$$

It follows that $v(r) \le c$ for all $r > 0$.

From $w(r) \to 0$ as $r \to \infty$ we deduce that $v(r) \to 0$ as $r \to \infty$.

By the choice of v we have

$$\nabla w = \frac{1}{c} \cdot \frac{v}{f(v)} \nabla v \quad \text{and} \quad \Delta w = \frac{1}{c} \frac{v}{f(v)} \Delta v + \frac{1}{c} \left(\frac{v}{f(v)} \right)' |\nabla v|^2. \tag{13}$$

The hypothesis $u \longmapsto f(u)/(u+\beta)$ is a decreasing function on $(0, \infty)$ implies that $u \longmapsto f(u)/u$ is a decreasing function on $(0, \infty)$. From (13) we deduce that

$$\Delta v < c \frac{f(v)}{v} \Delta w = -c \frac{f(v)}{v} \Phi(r) \leq -f(v)\Phi(r). \tag{14}$$

By (10) and (14) and using in an essential manner the hypothesis (f1), as already done for proving the uniqueness, we obtain that $u_k \leq v$ for $|x| \leq k$ and, hence, for all $\mathbb{R}^N$.

Now we have a bounded increasing sequence,

$$u_1 \leq u_2 \leq \cdots \leq u_k \leq u_{k+1} \leq \cdots \leq v,$$

with v vanishing at infinity. Thus there exists a function, say $u \leq v$, such that $u_k \to u$ pointwise in $\mathbb{R}^N$.

Now, using the same argument as in [6], it is easy to prove that $u \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$, and thus u is a classical solution of problem (1).

4. PROOF OF THEOREM 2

Suppose (1) has such a solution, $u(r)$. Then

$$u''(r) + \frac{n-1}{r} u'(r) = -f(u(r))p(r).$$

We set $\ln(u(r)+1) = \tilde{u}(r) > 0$ for all $r > 0$:

$$\Delta \tilde{u}(r) = \frac{1}{u(r)+1} \Delta u(r) - \frac{1}{(u(r)+1)^2} |\nabla u|^2.$$

Then $\tilde{u}(r)$ satisfies

$$\tilde{u}'' + \frac{n-1}{r} \tilde{u}' + \frac{1}{(u(r)+1)^2} |\nabla u|^2 = -\frac{f(u(r))}{u(r)+1} p(r). \tag{15}$$

Multiplying Eq. (15) by r^{n-1} and integrating on $(0, \zeta)$ yields

$$\tilde{u}'(\zeta)\zeta^{n-1} + \int_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2} |\nabla u|^2 d\sigma = -\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1} p(\sigma)\sigma^{n-1} d\sigma. \tag{16}$$

Now we multiply (16) by ζ^{1-n} and integrate over $(0,r)$. Hence

$$\tilde{u}(r)-\tilde{u}(0)+\int_0^r \zeta^{1-n}\int_0^\zeta \frac{\sigma^{n-1}}{(u(\sigma)+1)^2}|\nabla u|^2 d\sigma\, d\zeta$$
$$=-\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta.$$

We observe that $\tilde{u}(r)<\tilde{u}(0)\ \forall r>0$ implies $u(r)<u(0)\ \forall r>0$.

If $\beta\geq 1$ then the function $u\longmapsto f(u)/(u+1)$ is decreasing on $(0,\infty)$. This implies

$$\frac{f(u(\sigma))}{u(\sigma)+1}>\frac{f(u(0))}{u(0)+1}. \tag{17}$$

Since $\tilde{u}$ is positive, we have

$$\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta\leq \tilde{u}(0)\qquad \text{for all } r>0.$$

Substituting (17) into this expression, we obtain

$$\int_0^r \zeta^{1-n}\int_0^\zeta p(\sigma)\sigma^{n-1}d\sigma\, d\zeta\leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty.$$

We can use integration by parts and l'Hôpital's rule (as we did in proving that the integral in (11) is finite) to rewrite this as

$$\frac{1}{n-2}\lim_{r\to\infty}\int_0^r tp(t)dt\leq \frac{u(0)+1}{f(u(0))}\tilde{u}(0)<\infty,$$

contradicting the hypothesis.

If $\beta<1$ then the function $u\longmapsto (u+\beta)/(u+1)$ is increasing on $(0,\infty)$. In this case we have

$$\tilde{u}(0)>\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta$$
$$=\int_0^r \zeta^{1-n}\int_0^\zeta \frac{f(u(\sigma))}{u(\sigma)+\beta}\cdot\frac{u(\sigma)+\beta}{u(\sigma)+1}p(\sigma)\sigma^{n-1}d\sigma\, d\zeta$$
$$\geq \frac{f(u(0))}{u(0)+\beta}\beta\int_0^r \zeta^{1-n}\int_0^\zeta p(\sigma)\sigma^{n-1}d\sigma\, d\zeta,$$

which implies

$$\int_0^r \zeta^{1-n}\int_0^\zeta p(\sigma)\sigma^{n-1}d\sigma\, d\zeta<\frac{\tilde{u}(0)(u(0)+\beta)}{\beta\cdot f(u(0))}<\infty\qquad \text{for all } r>0.$$

We obtain again that

$$\frac{1}{n-2}\lim_{r\to\infty}\int_0^r tp(t)dt \le \frac{u(0)+\beta}{\beta\cdot f(u(0))}\tilde{u}(0) < \infty,$$

contradicting the hypothesis.

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