ELSEVIER

J. Math. Pures Appl. 81 (2002) 827–846

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

Entire solutions blowing up at infinity for semilinear elliptic systems

Florica-Corina Şt. Cîrstea [a,1], Vicenţiu D. Rădulescu [b,*]

[a] *School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia*

[b] *Department of Mathematics, University of Craiova, 1100 Craiova, Romania*

Accepted 15 April 2002

Abstract

We consider the system $\Delta u = p(x)g(v)$, $\Delta v = q(x)f(u)$ in $\mathbb{R}^N$, where f, g are positive and non-decreasing functions on $(0,\infty)$ satisfying the Keller–Osserman condition and we establish the existence of positive solutions that blow-up at infinity.

Résumé

On considère le système $\Delta u = p(x)g(v)$, $\Delta v = q(x)f(u)$ sur $\mathbb{R}^N$, où f, g sont fonctions positives et croissantes sur $(0,\infty)$, qui satisfont la condition de Keller–Osserman et on établit l'existence des solutions positives qui explosent à l'infini.

MSC: 35B40; 35B50; 35J45

Keywords: Nonlinear elliptic system; Entire explosive solution; Keller–Osserman condition; Maximum principle; Semilinear elliptic systems; Blow-up

1. Introduction and the main results

Consider the following semilinear elliptic system:

* Correspoding author.

E-mail addresses: florica@sci.vu.edu.au (F.-C. Şt. Cîrstea), vicrad@yahoo.com (V.D. Rădulescu).

1 The research was supported by the Australian Government through DETYA and Victoria University under the IPRS Programme.

PII: S0021-7824(02)01265-5

$$\begin{cases} \Delta u = p(x)g(v) & \text{in } \mathbb{R}^N, \\ \Delta v = q(x)f(u) & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geqslant 3$ and $p, q \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ $(0 < \alpha < 1)$ are non-negative and radially symmetric functions. Throughout this paper we assume that $f, g \in C^{0,\beta}_{\mathrm{loc}}[0,\infty)$ $(0 < \beta < 1)$ are positive and non-decreasing on $(0,\infty)$.

We are concerned here with the existence of positive *entire large solutions* of (1), that is positive classical solutions which satisfy $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$. Set $\mathbb{R}^+ = (0,\infty)$ and define:

$$\mathcal{G} = \big\{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ \bigm| (\exists) \text{ an entire radial solution of (1)} \\ \text{so that } \big(u(0), v(0)\big) = (a,b)\big\}.$$

The case of pure powers in the non-linearities was treated by Lair and Shaker in [4]. They proved that $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$ if $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \geqslant 0$ with $0 < \gamma, \theta \leqslant 1$. Moreover, they established that all positive entire radial solutions of (1) are *large* provided that

$$\int_0^\infty t p(t)\,\mathrm{d}t = \infty, \qquad \int_0^\infty t q(t)\,\mathrm{d}t = \infty. \tag{2}$$

If, in turn

$$\int_0^\infty t p(t)\,\mathrm{d}t < \infty, \qquad \int_0^\infty t q(t)\,\mathrm{d}t < \infty, \tag{3}$$

then all positive entire radial solutions of (1) are *bounded*.

Our purpose is to generalize the above results to a larger class of systems. More precisely, we prove:

Theorem 1. *Assume that*

$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \quad \textit{for all } c > 0. \tag{4}$$

Then $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$. Moreover, the following hold:

(i) *If p and q satisfy* (2)*, then all positive entire radial solutions of* (1) *are large.*
(ii) *If p and q satisfy* (3)*, then all positive entire radial solutions of* (1) *are bounded.*

Furthermore, if f, g are locally Lipschitz continuous on $(0,\infty)$ and (u,v), $(\tilde{u},\tilde{v})$ denote two positive entire radial solutions of (1)*, then there exists a positive constant C such that for all $r \in [0,\infty)$, we have*

$$\max\left\{\left|u(r)-\tilde{u}(r)\right|, \left|v(r)-\tilde{v}(r)\right|\right\} \leqslant C \max\left\{\left|u(0)-\tilde{u}(0)\right|, \left|v(0)-\tilde{v}(0)\right|\right\}.$$

If f and g satisfy the stronger regularity $f, g \in C^1[0,\infty)$, then we drop the assumption (4) and require, in turn,

$$(\mathrm{H}_1) \quad f(0)=g(0)=0, \qquad \liminf_{u\to\infty} \frac{f(u)}{g(u)} =: \sigma > 0$$

and the Keller–Osserman condition (see [3,9]),

$$(\mathrm{H}_2) \quad \int_1^\infty \frac{\mathrm{d}t}{\sqrt{G(t)}} < \infty, \quad \text{where } G(t) = \int_0^t g(s)\,\mathrm{d}s.$$

Observe that assumptions (H_1) and (H_2) imply that f satisfies condition (H_2), too.

The significance of the growth condition (H_2) in the scalar case will be stated in the next section.

Set $\eta = \min\{p,q\}$. If η is not identically zero at infinity and assumption (3) holds, then we prove:

Property 1. $\mathcal{G} \neq \emptyset$ (see Lemma 4).

Property 2. $\mathcal{G}$ is *bounded* (see Lemma 5).

Property 3. $F(\mathcal{G}) \subset \mathcal{G}$ (see Lemma 6), where

$$F(\mathcal{G}) = \left\{(a,b) \in \partial\mathcal{G} \;\middle|\; a>0 \text{ and } b>0\right\}.$$

For $(c,d) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$, define:

$$R_{c,d} = \sup\Big\{r>0 \;\Big|\; (\exists) \text{ a radial solution of (1) in } B(0,r) \text{ so that } \big(u(0),v(0)\big) = (c,d)\Big\}. \tag{5}$$

Property 4. $0 < R_{c,d} < \infty$ provided that $\nu = \max\{p(0), q(0)\} > 0$ (see Lemma 7).

Our main result in this case is:

Theorem 2. *Let $f, g \in C^1[0,\infty)$ satisfy (H_1) and (H_2). Assume (3) holds, η is not identically zero at infinity and $\nu > 0$. Then any entire radial solution (u,v) of (1) with $(u(0),v(0)) \in F(\mathcal{G})$ is large.*

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 3$, denote a smooth bounded domain or the whole space $\mathbb{R}^N$. Assume $\rho \not\equiv 0$ is non-negative such that $\rho \in C^{0,\alpha}(\overline{\Omega})$, if Ω is bounded and $\rho \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ otherwise. Consider the problem:

$$\Delta u = \rho(x)h(u) \quad \text{in } \Omega, \tag{6}$$

where the non-linearity $h \in C^1[0,\infty)$ satisfies

$$(\mathrm{A}_1) \quad h(0) = 0,\ h' \geqslant 0,\ h > 0 \quad \text{on } (0,\infty).$$

Proposition 1. *Let $\Omega = B(0,R)$ for some $R > 0$ and let ρ be radially symmetric in Ω. Then Eq.* (6) *subject to the Dirichlet boundary condition*

$$u = c\ (\text{const}) > 0 \quad \textit{on } \partial\Omega, \tag{7}$$

has a unique non-negative solution u_c, which, moreover, is positive and radially symmetric.

Proof. By Proposition 2.1 in [7] (see also [1, Theorem 5]), problem (6) + (7) has a unique non-negative solution u_c which, moreover, is positive. If u_c were not radially symmetric, then a different solution could be obtained by rotating it, which would contradict the uniqueness of the solution. □

By a *large solution* of Eq. (6) we mean a solution $u \geqslant 0$ in Ω satisfying $u(x) \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbb{R}^N$) or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbb{R}^N$). In the latter case, the solution is called an *entire large solution*. We point out that, if there exists a large solution of Eq. (6), then it is *positive*. Indeed, assume that $u(x_0) = 0$ for some $x_0 \in \Omega$. Since u is a large solution we can find a smooth domain $\omega \Subset \Omega$ such that $x_0 \in \omega$ and $u > 0$ on $\partial\omega$. Thus, by Theorem 5 in [1], the problem:

$$\begin{cases} \Delta\zeta = \rho(x)h(\zeta) & \text{in } \omega, \\ \zeta = u & \text{on } \partial\omega, \\ \zeta \geqslant 0 & \text{in } \omega, \end{cases}$$

has a unique solution, which is positive. By uniqueness, $\zeta = u$ in ω, which is a contradiction. This shows that any large solution of Eq. (6) cannot vanish in Ω.

Cf. Keller [3] and Osserman [9], if Ω is bounded and $\rho \equiv 1$, then Eq. (6) has a large solution if and only if h satisfies

$$(\mathrm{A}_2) \quad \int_1^\infty \frac{\mathrm{d}t}{\sqrt{H(t)}} < \infty, \quad \text{where } H(t) = \int_0^t h(s)\,\mathrm{d}s.$$

This fact leads to:

Lemma 1. *Eq.* (6), *considered in bounded domains, can have large solutions only if h satisfies the Keller–Osserman condition* (A_2).

Proof. Suppose, a priori, that Eq. (6) has a large solution u_∞. For any $n \geqslant 1$, consider the problem:

$$\begin{cases} \Delta u = \|\rho\|_\infty h(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \\ u \geqslant 0 & \text{in } \Omega. \end{cases}$$

By Proposition 2.1 in [7], this problem has a unique solution, say u_n, which, moreover, is positive in $\overline{\Omega}$. By the maximum principle,

$$0 < u_n \leqslant u_{n+1} \leqslant u_\infty \quad \text{in } \Omega,\ \forall n \geqslant 1.$$

Thus, for every $x \in \Omega$, it makes sense to define $\bar{u}(x) = \lim_{n\to\infty} u_n(x)$. Since (u_n) is uniformly bounded on every compact set $\omega \Subset \Omega$, standard elliptic regularity implies that $\bar{u}$ is a large solution of the problem $\Delta u = \|\rho\|_\infty h(u)$ in Ω. □

Therefore, in the rest of this section, we consider Eq. (6) assuming always that (A_1) and (A_2) hold. In this situation, by Lemma 1 in [1],

$$\int_1^\infty \frac{\mathrm{d}t}{h(t)} < \infty. \tag{8}$$

Typical examples of non-linearities satisfying (A_1) and (A_2) are:

(i) $h(u) = e^u - 1$;
(ii) $h(u) = u^p$, $p > 1$;
(iii) $h(u) = u[\ln(u+1)]^p$, $p > 2$.

For the proofs of the propositions that will be stated below, we refer the reader to [1].

Proposition 2 [1, Theorem 1]. *Let Ω be a bounded domain. Assume that ρ satisfies*:

(ρ_1) *for every $x_0 \in \Omega$ with $\rho(x_0) = 0$, there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega}_0 \subset \Omega$ and $\rho|_{\partial\Omega_0} > 0$.*

Then Eq. (6) *possesses a large solution.*

Corollary 1. *Let $\Omega = B(0, R)$ for some $R > 0$. If ρ is radially symmetric in Ω and $\rho|_{\partial\Omega} > 0$, then there exists a radial large solution of Eq.* (6).

Proof. By Proposition 1, the large solution constructed in the same way as in the proof of [1, Theorem 1] will be radially symmetric. □

Proposition 3 [1, Theorem 2]. *Consider Eq.* (6) *with* $\Omega = \mathbb{R}^N$ *assuming that* ρ *satisfies*

(ρ_1') *there exists a sequence of smooth bounded domains* $(\Omega_n)_{n\geqslant 1}$ *such that* $\overline{\Omega}_n \subset \Omega_{n+1}$,

$$\mathbb{R}^N = \bigcup_{n=1}^{\infty} \Omega_n \text{ and } (\rho_1) \text{ holds in } \Omega_n, \text{ for any } n \geqslant 1.$$

$$(\rho_2) \quad \int_0^\infty r\phi(r)\,\mathrm{d}r < \infty, \quad \text{where } \phi(r) = \max\{\rho(x)\colon |x| = r\}.$$

Then Eq. (6) *has an entire large solution.*

Remark 1. Theorem 4 in [1] asserts that (8) is a necessary condition for the existence of entire large solutions to Eq. (6) if ρ satisfies (ρ_2) and for which h is not assumed to fulfill (A_2).

Remark 2. If ρ is radially symmetric in $\mathbb{R}^N$ and not identically zero at infinity, then (ρ_1') is fulfilled.

Indeed, we can find an increasing sequence of positive numbers $(R_n)_{n\geqslant 1}$ such that $R_n \to \infty$ and $\rho > 0$ on $\partial B(0, R_n)$, for any $n \geqslant 1$. Therefore, (ρ_1') is satisfied on $\Omega_n = B(0, R_n)$.

Corollary 2. *Let* $\Omega \equiv \mathbb{R}^N$. *Assume that* ρ *is radially symmetric in* $\mathbb{R}^N$, *not identically zero at infinity such that* (ρ_2) *is fulfilled. Then Eq.* (6) *has a radial entire large solution.*

Proof. By Remark 2 and Corollary 1, the entire large solution constructed as in the proof of Theorem 2 in [1] will be radially symmetric. $\square$

We supplied in [1] an example of function ρ with properties stated in Corollary 2. More precisely,

$$\begin{cases} \rho(r) = 0 & \text{for } r = |x| \in [n - 1/3, n + 1/3],\ n \geqslant 1; \\ \rho(r) > 0 & \text{in } \mathbb{R}_+ \setminus \bigcup_{n=1}^{\infty} [n - 1/3, n + 1/3]; \\ \rho \in C^1[0,\infty) & \text{and} \quad \max_{r\in[n,n+1]} \rho(r) = \dfrac{1}{n^3}. \end{cases}$$

3. Auxiliary results

We refer to [5–8,10] for various results related to blow-up boundary solutions for elliptic equations.

Lemma 2. *Condition* (2) *holds if and only if* $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$, *where*

$$A(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\,\mathrm{d}s\,\mathrm{d}t,$$

$$B(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r > 0.$$

Proof. Indeed, for any $r > 0$,

$$\begin{aligned} A(r) &= \frac{1}{N-2}\left[\int_0^r tp(t)\,\mathrm{d}t - \frac{1}{r^{N-2}}\int_0^r t^{N-1} p(t)\,\mathrm{d}t\right] \\ &\leqslant \frac{1}{N-2}\int_0^r tp(t)\,\mathrm{d}t. \end{aligned} \tag{9}$$

On the other hand,

$$\begin{aligned} &\int_0^r tp(t)\,\mathrm{d}t - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\,\mathrm{d}t \\ &\quad = \frac{1}{r^{N-2}}\int_0^r \left(r^{N-2} - t^{N-2}\right) tp(t)\,\mathrm{d}t \\ &\quad \geqslant \frac{1}{r^{N-2}}\left[r^{N-2} - \left(\frac{r}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\,\mathrm{d}t. \end{aligned}$$

This combined with (9) yields

$$\frac{1}{N-2}\int_0^r tp(t)\,\mathrm{d}t \geqslant A(r) \geqslant \frac{1}{N-2}\left[1 - \left(\frac{1}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\,\mathrm{d}t.$$

Our conclusion follows now by letting $r \to \infty$. □

Lemma 3. *Assume that condition* (3) *holds. Let* f *and* g *be locally Lipschitz continuous functions on* $(0, \infty)$. *If* (u, v) *and* $(\tilde{u}, \tilde{v})$ *denote two bounded positive entire radial solutions of* (1), *then there exists a positive constant* C *such that for all* $r \in [0, \infty)$, *we have*

$$\max\left\{\left|u(r) - \tilde{u}(r)\right|, \left|v(r) - \tilde{v}(r)\right|\right\} \leqslant C \max\left\{\left|u(0) - \tilde{u}(0)\right|, \left|v(0) - \tilde{v}(0)\right|\right\}.$$

Proof. We first see that radial solutions of (1) are solutions of the ordinary differential equations system:

$$\begin{cases} u''(r) + \dfrac{N-1}{r} u'(r) = p(r) g\big(v(r)\big), & r > 0, \\ v''(r) + \dfrac{N-1}{r} v'(r) = q(r) f\big(u(r)\big), & r > 0. \end{cases} \tag{10}$$

Define $K = \max\{|u(0) - \tilde{u}(0)|, |v(0) - \tilde{v}(0)|\}$. Integrating the first equation of (10), we get:

$$u'(r) - \tilde{u}'(r) = r^{1-N} \int_0^r s^{N-1} p(s) \big(g\big(v(s)\big) - g\big(\tilde{v}(s)\big)\big) \, \mathrm{d}s.$$

Hence

$$\big|u(r) - \tilde{u}(r)\big| \leqslant K + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|g\big(v(s)\big) - g\big(\tilde{v}(s)\big)\big| \, \mathrm{d}s \, \mathrm{d}t. \tag{11}$$

Since (u, v) and $(\tilde{u}, \tilde{v})$ are bounded entire radial solutions of (1) we have:

$$\begin{aligned} \big|g\big(v(r)\big) - g\big(\tilde{v}(r)\big)\big| &\leqslant m \big|v(r) - \tilde{v}(r)\big| \quad \text{for any } r \in [0, \infty), \\ \big|f\big(u(r)\big) - f\big(\tilde{u}(r)\big)\big| &\leqslant m \big|u(r) - \tilde{u}(r)\big| \quad \text{for any } r \in [0, \infty), \end{aligned}$$

where m denotes a Lipschitz constant for both functions f and g. Therefore, using (11) we find:

$$\big|u(r) - \tilde{u}(r)\big| \leqslant K + m \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|v(s) - \tilde{v}(s)\big| \, \mathrm{d}s \, \mathrm{d}t. \tag{12}$$

Arguing as above, but now with the second equation of (10), we obtain:

$$\big|v(r) - \tilde{v}(r)\big| \leqslant K + m \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) \big|u(s) - \tilde{u}(s)\big| \, \mathrm{d}s \, \mathrm{d}t. \tag{13}$$

Define:

$$X(r) = K + m \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|v(s) - \tilde{v}(s)\big| \, \mathrm{d}s \, \mathrm{d}t,$$

$$Y(r) = K + m \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) \big|u(s) - \tilde{u}(s)\big| \, \mathrm{d}s \, \mathrm{d}t.$$

It is clear that X and Y are non-decreasing functions with $X(0)=Y(0)=K$. By a simple calculation together with (12) and (13) we obtain:

$$\begin{aligned}\left(r^{N-1}X'\right)'(r)&=mr^{N-1}p(r)\left|v(r)-\tilde{v}(r)\right|\leqslant mr^{N-1}p(r)Y(r),\\ \left(r^{N-1}Y'\right)'(r)&=mr^{N-1}q(r)\left|u(r)-\tilde{u}(r)\right|\leqslant mr^{N-1}q(r)X(r).\end{aligned}\tag{14}$$

Since Y is non-decreasing, we have:

$$X(r)\leqslant K+mY(r)A(r)\leqslant K+\frac{m}{N-2}Y(r)\int_0^r tp(t)\,\mathrm{d}t\leqslant K+mC_pY(r),\tag{15}$$

where $C_p=(1/(N-2))\int_0^\infty tp(t)\,\mathrm{d}t$. Using (15) in the second inequality of (14) we find:

$$\left(r^{N-1}Y'\right)'(r)\leqslant mr^{N-1}q(r)\left(K+mC_pY(r)\right).$$

Integrating twice this inequality from 0 to r, we obtain:

$$Y(r)\leqslant K(1+mC_q)+\frac{m^2}{N-2}C_p\int_0^r tq(t)Y(t)\,\mathrm{d}t,$$

where $C_q=(1/(N-2))\int_0^\infty tq(t)\,\mathrm{d}t$. From Gronwall's inequality, we deduce:

$$Y(r)\leqslant K(1+mC_q)\mathrm{e}^{\frac{m^2}{N-2}C_p\int_0^r tq(t)\,\mathrm{d}t}\leqslant K(1+mC_q)\mathrm{e}^{m^2C_pC_q}$$

and similarly for X. The conclusion follows now from the above inequality, (12) and (13). □

4. Proof of Theorem 1

Since the radial solutions of (1) are solutions of the ordinary differential equations system (10) it follows that the radial solutions of (1) with $u(0)=a>0$, $v(0)=b>0$ satisfy:

$$u(r)=a+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g\left(v(s)\right)\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0,\tag{16}$$

$$v(r)=b+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f\left(u(s)\right)\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0.\tag{17}$$

Define $v_0(r)=b$ for all $r \geqslant 0$. Let $(u_k)_{k\geqslant 1}$ and $(v_k)_{k\geqslant 1}$ be two sequences of functions given by:

$$u_k(r)=a+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g\big(v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0,$$

$$v_k(r)=b+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f\big(u_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0.$$

Since $v_1(r)\geqslant b$, we find $u_2(r)\geqslant u_1(r)$ for all $r\geqslant 0$. This implies $v_2(r)\geqslant v_1(r)$ which further produces $u_3(r)\geqslant u_2(r)$ for all $r\geqslant 0$. Proceeding at the same manner we conclude that

$$u_k(r)\leqslant u_{k+1}(r)\quad\text{and}\quad v_k(r)\leqslant v_{k+1}(r),\quad \forall r\geqslant 0\text{ and }k\geqslant 1.$$

We now prove that the non-decreasing sequences $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ are bounded from above on bounded sets. Indeed, we have:

$$u_k(r)\leqslant u_{k+1}(r)\leqslant a+g\big(v_k(r)\big)A(r),\quad \forall r\geqslant 0 \tag{18}$$

and

$$v_k(r)\leqslant b+f\big(u_k(r)\big)B(r),\quad \forall r\geqslant 0. \tag{19}$$

Let $R>0$ be arbitrary. By (18) and (19) we find:

$$u_k(R)\leqslant a+g\Big(b+f\big(u_k(R)\big)B(R)\Big)A(R),\quad \forall k\geqslant 1$$

or, equivalently,

$$1\leqslant \frac{a}{u_k(R)}+\frac{g(b+f(u_k(R))B(R))}{u_k(R)}A(R),\quad \forall k\geqslant 1. \tag{20}$$

By the monotonicity of $(u_k(R))_{k\geqslant 1}$, there exists $\lim_{k\to\infty}u_k(R):=L(R)$. We claim that $L(R)$ is finite. Assume the contrary. Then, by taking $k\to\infty$ in (20) and using (4) we obtain a contradiction. Since $u_k'(r)$, $v_k'(r)\geqslant 0$ we get that the map $(0,\infty)\ni R\to L(R)$ is non-decreasing on $(0,\infty)$ and

$$u_k(r)\leqslant u_k(R)\leqslant L(R),\quad \forall r\in[0,R],\ \forall k\geqslant 1, \tag{21}$$

$$v_k(r)\leqslant b+f\big(L(R)\big)B(R),\quad \forall r\in[0,R],\ \forall k\geqslant 1. \tag{22}$$

It follows that there exists $\lim_{R\to\infty}L(R)=\overline{L}\in(0,\infty]$ and the sequences $(u_k(r))_{k\geqslant 1}$, $(v_k(r))_{k\geqslant 1}$ are bounded above on bounded sets. Thus, we can define $u(r):=\lim_{k\to\infty}u_k(r)$

and $v(r) := \lim_{k\to\infty} v_k(r)$ for all $r \geqslant 0$. By standard elliptic regularity theory we obtain that (u, v) is a positive entire solution of (1) with $u(0) = a$ and $v(0) = b$.

We now assume that, in addition, condition (3) is fulfilled. According to Lemma 2 we have that $\lim_{r\to\infty} A(r) = \overline{A} < \infty$ and $\lim_{r\to\infty} B(r) = \overline{B} < \infty$. Passing to the limit as $k \to \infty$ in (20) we find:

$$1 \leqslant \frac{a}{L(R)} + \frac{g(b + f(L(R))B(R))}{L(R)} A(R) \leqslant \frac{a}{L(R)} + \frac{g(b + f(L(R))\overline{B}\,)}{L(R)}\,\overline{A}.$$

Letting $R \to \infty$ and using (4) we deduce $\overline{L} < \infty$. Thus, taking into account (21) and (22), we obtain:

$$u_k(r) \leqslant \overline{L} \quad \text{and} \quad v_k(r) \leqslant b + f\bigl(\overline{L}\bigr)\overline{B}, \quad \forall r \geqslant 0,\ \forall k \geqslant 1.$$

So, we have found upper bounds for $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ which are independent of r. Thus, the solution (u, v) is bounded from above. This shows that any solution of (16) and (17) will be bounded from above provided (3) holds. Thus, we can apply Lemma 3 to achieve the second assertion of (ii).

Let us now drop the condition (3) and assume that (2) is fulfilled. In this case, Lemma 2 tells us that $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$. Let (u, v) be an entire positive radial solution of (1). Using (16) and (17) we obtain:

$$u(r) \geqslant a + g(b)A(r), \quad \forall r \geqslant 0,$$
$$v(r) \geqslant b + f(a)\,B(r), \quad \forall r \geqslant 0.$$

Taking $r \to \infty$ we get that (u, v) is an entire large solution. This concludes the proof of Theorem 1. □

We now give some examples of non-linearities f and g which satisfy the assumptions of Theorem 1 (see [2]).

(1) Let

$$f(t) = \sum_{j=1}^{l} a_j t^{\gamma_j}, \qquad g(t) = \sum_{k=1}^{m} b_k t^{\theta_j} \qquad \text{for } t > 0$$

with $a_j, b_k, \gamma_j, \theta_k > 0$ and $f(t) = g(t) = 0$ for $t \leqslant 0$. Assume that $\gamma\theta < 1$, where

$$\gamma = \max_{1\leqslant j\leqslant l} \gamma_j, \qquad \theta = \max_{1\leqslant k\leqslant m} \theta_k.$$

(2) Let

$$f(t) = \bigl(1 + t^2\bigr)^{\gamma/2} \quad \text{and} \quad g(t) = \bigl(1 + t^2\bigr)^{\theta/2} \quad \text{for } t \in \mathbb{R}$$

with $\gamma, \theta > 0$ and $\gamma\theta < 1$.

(3) Let

$$f(t)=\begin{cases} t^{\gamma} & \text{if } 0\leqslant t\leqslant 1,\\ t^{\theta} & \text{if } t\geqslant 1,\end{cases}$$

and

$$g(t)=\begin{cases} t^{\theta} & \text{if } 0\leqslant t\leqslant 1,\\ t^{\gamma} & \text{if } t\geqslant 1,\end{cases}$$

with $\gamma,\theta>0$, $\gamma\theta<1$ and $f(t)=g(t)=0$ for $t\leqslant 0$.

(4) Let $g(t)=t$ for $t\in\mathbb{R}$, $f(t)=0$ for $t\leqslant 0$ and

$$f(t)=t\left(-\ln\left(\left(\frac{2}{\pi}\right)\arctan t\right)\right)^{\gamma}\quad \text{for } t>0$$

where $\gamma\in(0,1/2)$.

5. Proof of Theorem 2

Let $f,g\in C^1[0,\infty)$ satisfy (H_1) and (H_2). Suppose that η is not identically zero at infinity and (3) holds. We first give the proofs of Properties 1–4 which are the main tools used to deduce Theorem 2.

Lemma 4. $\mathcal{G}\neq\emptyset$.

Proof. By Corollary 2, the problem:

$$\Delta\psi=(p+q)(x)(f+g)(\psi)\quad \text{in } \mathbb{R}^N,$$

has a positive radial entire large solution. Since ψ is radial, we have:

$$\psi(r)=\psi(0)+\int_0^r t^{1-N}\int_0^t s^{N-1}(p+q)(s)(f+g)\big(\psi(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad \forall r\geqslant 0.$$

We claim that $(0,\psi(0)]\times(0,\psi(0)]\subseteq\mathcal{G}$. To prove this, fix $0<a,b\leqslant\psi(0)$ and let $v_0(r)\equiv b$ for all $r\geqslant 0$. Define the sequences $(u_k)_{k\geqslant 1}$ and $(v_k)_{k\geqslant 1}$ by:

$$u_k(r)=a+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g\big(v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad \forall r\in[0,\infty),\ \forall k\geqslant 1,\tag{23}$$

$$v_k(r)=b+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f\big(u_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad \forall r\in[0,\infty),\ \forall k\geqslant 1.\tag{24}$$

We first see that $v_0 \leqslant v_1$ which produces $u_1 \leqslant u_2$. Consequently, $v_1 \leqslant v_2$ which further yields $u_2 \leqslant u_3$. With the same arguments, we obtain that (u_k) and (v_k) are non-decreasing sequences. Since $\psi'(r) \geqslant 0$ and $b = v_0 \leqslant \psi(0) \leqslant \psi(r)$ for all $r \geqslant 0$ we find:

$$\begin{aligned} u_1(r) &\leqslant a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\bigl(\psi(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)\bigl(\psi(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t = \psi(r). \end{aligned}$$

Thus $u_1 \leqslant \psi$. It follows that

$$\begin{aligned} v_1(r) &\leqslant b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\bigl(\psi(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)\bigl(\psi(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t = \psi(r). \end{aligned}$$

Similar arguments show that

$$u_k(r) \leqslant \psi(r) \quad \text{and} \quad v_k(r) \leqslant \psi(r), \quad \forall r \in [0,\infty),\ \forall k \geqslant 1.$$

Thus, (u_k) and (v_k) converge and $(u,v) = \lim_{k\to\infty}(u_k, v_k)$ is an entire radial solution of (1) such that $(u(0), v(0)) = (a,b)$. This completes the proof. □

An easy consequence of the above result is:

Corollary 3. *If* $(a,b) \in \mathcal{G}$, *then* $(0,a] \times (0,b] \subseteq \mathcal{G}$.

Proof. Indeed, the process used before can be repeated by taking:

$$u_k(r) = a_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\bigl(v_{k-1}(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1,$$

$$v_k(r) = b_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\bigl(u_k(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1,$$

where $0 < a_0 \leqslant a$, $0 < b_0 \leqslant b$ and $v_0(r) \equiv b_0$ for all $r \geqslant 0$.

Letting (U,V) be the entire radial solution of (1) with central values (a,b) we obtain as in Lemma 4,

$$u_k(r) \leqslant u_{k+1}(r) \leqslant U(r), \quad \forall r \in [0, \infty),\ \forall k \geqslant 1,$$
$$v_k(r) \leqslant v_{k+1}(r) \leqslant V(r), \quad \forall r \in [0, \infty),\ \forall k \geqslant 1.$$

Set $(u, v) = \lim_{k\to\infty}(u_k, v_k)$. We see that $u \leqslant U$, $v \leqslant V$ on $[0, \infty)$ and (u, v) is an entire radial solution of (1) with central values (a_0, b_0). This shows that $(a_0, b_0) \in \mathcal{G}$, so that our assertion is proved. □

Lemma 5. *$\mathcal{G}$ is bounded.*

Proof. Set $0 < \lambda < \min\{\sigma, 1\}$ and let $\delta = \delta(\lambda)$ be large enough so that

$$f(t) \geqslant \lambda g(t), \quad \forall t \geqslant \delta. \tag{25}$$

Since η is radially symmetric and not identically zero at infinity, we can assume $\eta > 0$ on $\partial B(0, R)$ for some $R > 0$. Corollary 1 ensures the existence of a positive large solution ζ of the problem

$$\Delta\zeta = \lambda\eta(x)g\left(\frac{\zeta}{2}\right) \quad \text{in } B(0, R).$$

Arguing by contradiction: let us assume that $\mathcal{G}$ is not bounded. Then, there exists $(a, b) \in \mathcal{G}$ such that $a + b > \max\{2\delta, \zeta(0)\}$. Let (u, v) be the entire radial solution of (1) such that $(u(0), v(0)) = (a, b)$. Since $u(x) + v(x) \geqslant a + b > 2\delta$ for all $x \in \mathbb{R}^N$, by (25), we find:

$$f\big(u(x)\big) \geqslant f\left(\frac{u(x)+v(x)}{2}\right) \geqslant \lambda g\left(\frac{u(x)+v(x)}{2}\right) \quad \text{if } u(x) \geqslant v(x)$$

and

$$g\big(v(x)\big) \geqslant g\left(\frac{u(x)+v(x)}{2}\right) \geqslant \lambda g\left(\frac{u(x)+v(x)}{2}\right) \quad \text{if } v(x) \geqslant u(x).$$

It follows that

$$\begin{aligned} \Delta(u+v) &= p(x)g(v) + q(x)f(u) \geqslant \eta(x)\big(g(v) + f(u)\big) \\ &\geqslant \lambda\eta(x)g\left(\frac{u+v}{2}\right) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

On the other hand, $\zeta(x) \to \infty$ as $|x| \to R$ and $u, v \in C^2(\overline{B(0, R)})$. Thus, by the maximum principle, we conclude that $u + v \leqslant \zeta$ in $B(0, R)$. But this is impossible since $u(0) + v(0) = a + b > \zeta(0)$. □

Lemma 6. *$F(\mathcal{G}) \subset \mathcal{G}$.*

Proof. Let $(a,b)\in F(\mathcal{G})$. We claim that $(a-1/n_0, b-1/n_0)\in\mathcal{G}$ provided $n_0\geqslant 1$ is large enough so that $\min\{a,b\}>1/n_0$. Indeed, if this is not true, by Corollary 3,

$$D := \left[a-\frac{1}{n_0},\infty\right)\times\left[b-\frac{1}{n_0},\infty\right)\subseteq(\mathbb{R}^+\times\mathbb{R}^+)\setminus\mathcal{G}.$$

So, we can find a small ball B centered in (a,b) such that $B\Subset D$, i.e., $B\cap\mathcal{G}=\emptyset$. But this will contradict the choice of (a,b). Consequently, there exists (u_{n_0}, v_{n_0}) an entire radial solution of (1) such that $(u_{n_0}(0), v_{n_0}(0)) = (a-1/n_0, b-1/n_0)$. Thus, for any $n\geqslant n_0$, we can define:

$$u_n(r) = a-\frac{1}{n}+\int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g\big(v_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0,$$

$$v_n(r) = b-\frac{1}{n}+\int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f\big(u_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t,\quad r\geqslant 0.$$

Using Corollary 3 once more, we conclude that $(u_n)_{n\geqslant n_0}$ and $(v_n)_{n\geqslant n_0}$ are non-decreasing sequences. We now prove that (u_n) and (v_n) converge on $\mathbb{R}^N$. To this aim, let $x_0\in\mathbb{R}^N$ be arbitrary. But η is not identically zero at infinity so that, for some $R_0>0$, we have $\eta>0$ on $\partial B(0,R_0)$ and $x_0\in B(0,R_0)$.

Since $\sigma=\liminf_{u\to\infty} f(u)/g(u)>0$, we find $\tau\in(0,1)$ such that

$$f(t)\geqslant\tau g(t),\quad \forall t\geqslant\frac{a+b}{2}-\frac{1}{n_0}.$$

Therefore, on the set where $u_n\geqslant v_n$, we have:

$$f(u_n)\geqslant f\left(\frac{u_n+v_n}{2}\right)\geqslant\tau g\left(\frac{u_n+v_n}{2}\right).$$

Similarly, on the set where $u_n\leqslant v_n$, we have:

$$g(v_n)\geqslant g\left(\frac{u_n+v_n}{2}\right)\geqslant\tau g\left(\frac{u_n+v_n}{2}\right).$$

It follows that, for any $x\in\mathbb{R}^N$,

$$\begin{aligned}\Delta(u_n+v_n) &= p(x)g(v_n)+q(x)f(u_n)\geqslant\eta(x)\big[g(v_n)+f(u_n)\big]\\ &\geqslant \tau\eta(x)g\left(\frac{u_n+v_n}{2}\right).\end{aligned}$$

On the other hand, by Corollary 1, there exists a positive large solution of

$$\Delta\zeta=\tau\eta(x)g\left(\frac{\zeta}{2}\right)\quad\text{in } B(0,R_0).$$

The maximum principle yields $u_n + v_n \leqslant \zeta$ in $B(0, R_0)$. So, it makes sense to define $(u(x_0), v(x_0)) = \lim_{n\to\infty}(u_n(x_0), v_n(x_0))$. Since x_0 is arbitrary, the functions u, v exist on $\mathbb{R}^N$. Hence (u, v) is an entire radial solution of (1) with central values (a, b), i.e., $(a, b) \in \mathcal{G}$. $\square$

Lemma 7. *If, in addition,* $\nu = \max\{p(0), q(0)\} > 0$, *then* $0 < R_{c,d} < \infty$ *where* $R_{c,d}$ *is defined by* (5).

Proof. Since $\nu > 0$ and $p, q \in C[0,\infty)$, there exists $\varepsilon > 0$ such that $(p+q)(r) > 0$ for all $0 \leqslant r < \varepsilon$. Let $0 < R < \varepsilon$ be arbitrary. By Corollary 1, there exists a positive radial large solution of the problem

$$\Delta\psi_R = (p+q)(x)(f+g)(\psi_R) \quad \text{in } B(0, R).$$

Moreover, for any $0 \leqslant r < R$,

$$\psi_R(r) = \psi_R(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)\big(\psi_R(s)\big)\,\mathrm{d}s\,\mathrm{d}t.$$

It is clear that $\psi_R'(r) \geqslant 0$. Thus, we find:

$$\psi_R'(r) = r^{1-N} \int_0^r s^{N-1}(p+q)(s)(f+g)\big(\psi_R(s)\big)\,\mathrm{d}s \leqslant C(f+g)\big(\psi_R(r)\big),$$

where $C > 0$ is a positive constant such that $\int_0^\varepsilon (p+q)(s)\,\mathrm{d}s \leqslant C$.

Since $f + g$ satisfies (A_1) and (A_2), we may then invoke Lemma 1 in [1] to conclude

$$\int_1^\infty \frac{\mathrm{d}t}{(f+g)(t)} < \infty.$$

Therefore, we get:

$$-\frac{\mathrm{d}}{\mathrm{d}r} \int_{\psi_R(r)}^\infty \frac{\mathrm{d}s}{(f+g)(s)} = \frac{\psi_R'(r)}{(f+g)(\psi_R(r))} \leqslant C \quad \text{for any } 0 < r < R.$$

Integrating from 0 to R and recalling that $\psi_R(r) \to \infty$ as $r \nearrow R$, we obtain:

$$\int_{\psi_R(0)}^\infty \frac{\mathrm{d}s}{(f+g)(s)} \leqslant CR.$$

Letting $R \searrow 0$ we conclude that

$$\lim_{R\searrow 0} \int_{\psi_R(0)}^{\infty} \frac{\mathrm{d}s}{(f+g)(s)} = 0.$$

This implies that $\psi_R(0) \to \infty$ as $R \searrow 0$. So, there exists $0 < \widetilde{R} < \varepsilon$ such that $0 < c, d \leqslant \psi_{\widetilde{R}}(0)$. Set

$$u_k(r) = c + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1, \tag{26}$$

$$v_k(r) = d + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(u_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1, \tag{27}$$

where $v_0(r) = d$ for all $r \in [0,\infty)$. As in Lemma 4, we find that (u_k) respectively, (v_k) are non-decreasing and

$$u_k(r) \leqslant \psi_{\widetilde{R}}(r) \quad \text{and} \quad v_k(r) \leqslant \psi_{\widetilde{R}}(r), \quad \forall r \in \big[0, \widetilde{R}\big),\ \forall k \geqslant 1.$$

Thus, for any $r \in [0, \widetilde{R})$, there exists $(u(r), v(r)) = \lim_{k\to\infty}(u_k(r), v_k(r))$ which is, moreover, a radial solution of (1) in $B(0, \widetilde{R})$ such that $(u(0), v(0)) = (c, d)$. This shows that $R_{c,d} \geqslant \widetilde{R} > 0$. By the definition of $R_{c,d}$ we also derive

$$\lim_{r\nearrow R_{c,d}} u(r) = \infty \quad \text{and} \quad \lim_{r\nearrow R_{c,d}} v(r) = \infty. \tag{28}$$

On the other hand, since $(c, d) \notin \mathcal{G}$, we conclude that $R_{c,d}$ is finite. □

Proof of Theorem 2 completed.

Let $(a, b) \in F(\mathcal{G})$ be arbitrary. By Lemma 6, $(a, b) \in \mathcal{G}$ so that we can define (U, V) an entire radial solution of (1) with $(U(0), V(0)) = (a, b)$. Obviously, for any $n \geqslant 1$, $(a + 1/n, b + 1/n) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$. By Lemma 7, $R_{a+1/n,b+1/n}$ (in short, R_n) defined by (5) is a positive number. Let (U_n, V_n) be the radial solution of (1) in $B(0, R_n)$ with the central values $(a + 1/n, b + 1/n)$. Thus,

$$U_n(r) = a + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(V_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0, R_n), \tag{29}$$

$$V_n(r) = b + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(U_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0, R_n). \tag{30}$$

In view of (28) we have:

$$\lim_{r\nearrow R_n} U_n(r)=\infty \quad \text{and} \quad \lim_{r\nearrow R_n} V_n(r)=\infty, \quad \forall n\geqslant 1.$$

We claim that $(R_n)_{n\geqslant 1}$ is a non-decreasing sequence. Indeed, if (u_k), (v_k) denote the sequences of functions defined by (26) and (27) with $c=a+1/(n+1)$ and $d=b+1/(n+1)$, then

$$\begin{aligned} u_k(r)\leqslant u_{k+1}(r)\leqslant U_n(r),\\ v_k(r)\leqslant v_{k+1}(r)\leqslant V_n(r),\end{aligned} \quad \forall r\in[0,R_n),\ \forall k\geqslant 1. \tag{31}$$

This implies that $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ converge for any $r\in[0,R_n)$. Moreover, $(U_{n+1},V_{n+1})=\lim_{k\to\infty}(u_k,v_k)$ is a radial solution of (1) in $B(0,R_n)$ with central values $(a+1/(n+1), b+1/(n+1))$. By the definition of R_{n+1}, it follows that $R_{n+1}\geqslant R_n$ for any $n\geqslant 1$.

Set $R:=\lim_{n\to\infty} R_n$ and let $0\leqslant r<R$ be arbitrary. Then, there exists $n_1=n_1(r)$ such that $r<R_n$ for all $n\geqslant n_1$. From (31) we see that $U_{n+1}\leqslant U_n$ (respectively, $V_{n+1}\leqslant V_n$) on $[0,R_n)$ for all $n\geqslant 1$. So, there exists $\lim_{n\to\infty}(U_n(r),V_n(r))$ which, by (29) and (30), is a radial solution of (1) in $B(0,R)$ with central values (a,b). Consequently,

$$\lim_{n\to\infty} U_n(r)=U(r) \quad \text{and} \quad \lim_{n\to\infty} V_n(r)=V(r) \quad \text{for any } r\in[0,R). \tag{32}$$

Since $U_n'(r)\geqslant 0$, from (30) we find:

$$V_n(r)\leqslant b+\frac{1}{n}+f\big(U_n(r)\big)\int_0^\infty t^{1-N}\int_0^t s^{N-1}q(s)\,\mathrm{d}s\,\mathrm{d}t.$$

This yields

$$V_n(r)\leqslant C_1U_n(r)+C_2 f\big(U_n(r)\big), \tag{33}$$

where C_1 is an upper bound of $(V(0)+1/n)/(U(0)+1/n)$ and

$$C_2=\int_0^\infty t^{1-N}\int_0^t s^{N-1}q(s)\,\mathrm{d}s\,\mathrm{d}t\leqslant \frac{1}{N-2}\int_0^\infty sq(s)\,\mathrm{d}s<\infty.$$

Define $h(t)=g(C_1t+C_2f(t))$ for $t\geqslant 0$. It is easy to check that h satisfies (A_1) and (A_2). So, by Lemma 1 in [1] we can define:

$$\Gamma(s)=\int_s^\infty \frac{\mathrm{d}t}{h(t)}, \quad \text{for all } s>0.$$

But U_n verifies

$$\Delta U_n = p(x)g(V_n)$$

which, combined with (33), implies

$$\Delta U_n \leqslant p(x)h(U_n).$$

A simple calculation shows that

$$\begin{aligned}\Delta\Gamma(U_n) &= \Gamma'(U_n)\Delta U_n + \Gamma''(U_n)|\nabla U_n|^2\\ &= \frac{-1}{h(U_n)}\Delta U_n + \frac{h'(U_n)}{[h(U_n)]^2}|\nabla U_n|^2\\ &\geqslant \frac{-1}{h(U_n)}p(r)hU_n) = -p(r),\end{aligned}$$

which we rewrite as

$$\left(r^{N-1}\frac{\mathrm{d}}{\mathrm{d}r}\Gamma(U_n)\right)' \geqslant -r^{N-1}p(r) \quad \text{for any } 0 < r < R_n.$$

Fix $0 < r < R$. Then $r < R_n$ for all $n \geqslant n_1$ provided n_1 is large enough. Integrating the above inequality over $[0, r]$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}r}\Gamma(U_n) \geqslant -r^{1-N}\int_0^r s^{N-1}p(s)\,\mathrm{d}s.$$

Integrating this new inequality over $[r, R_n]$ we obtain:

$$-\Gamma\big(U_n(r)\big) \geqslant -\int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall n \geqslant n_1,$$

since $U_n(r) \to \infty$ as $r \nearrow R_n$ implies $\Gamma(U_n(r)) \to 0$ as $r \nearrow R_n$. Therefore,

$$\Gamma\big(U_n(r)\big) \leqslant \int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall n \geqslant n_1.$$

Letting $n \to \infty$ and using (32) we find:

$$\Gamma\big(U(r)\big) \leqslant \int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t,$$

or, equivalently

$$U(r) \geqslant \Gamma^{-1}\left(\int_r^R t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t\right).$$

Passing to the limit as $r \nearrow R$ and using the fact that $\lim_{s\searrow 0}\Gamma^{-1}(s)=\infty$, we deduce:

$$\lim_{r\nearrow R} U(r) \geqslant \lim_{r\nearrow R}\Gamma^{-1}\left(\int_r^R t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t\right)=\infty.$$

But (U,V) is an entire solution so that we conclude $R=\infty$ and $\lim_{r\to\infty}U(r)=\infty$. Since (3) holds and $V'(r)\geqslant 0$ we find:

$$\begin{aligned} U(r) &\leqslant a+g\bigl(V(r)\bigr)\int_0^\infty t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant a+g\bigl(V(r)\bigr)\frac{1}{N-2}\int_0^\infty tp(t)\,\mathrm{d}t, \quad \forall r\geqslant 0. \end{aligned}$$

We deduce $\lim_{r\to\infty}V(r)=\infty$, otherwise we obtain that $\lim_{r\to\infty}U(r)$ is finite, a contradiction. Consequently, (U,V) is an entire large solution of (1). This concludes our proof. □

References

[1] F. Cîrstea, V. Rădulescu, Blow-up boundary solutions of semilinear elliptic problems, Nonlinear Anal. 48 (2002) 521–534.

[2] R. Dalmasso, Existence and uniqueness of positive solutions of semilinear elliptic systems, Nonlinear Anal. 39 (2000) 559–568.

[3] J.B. Keller, On solution of $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957) 503–510.

[4] A.V. Lair, A.W. Shaker, Existence of entire large positive solutions of semilinear elliptic systems, J. Differential Equations 164 (2000) 380–394.

[5] A.C. Lazer, P.J. McKenna, On a problem of Bieberbach and Rademacher, Nonlinear Anal. 21 (1993) 327–335.

[6] A.C. Lazer, P.J. McKenna, Asymptotic behavior of solutions of boundary blow-up problems, Differential Integral Equations 7 (1994) 1001–1020.

[7] M. Marcus, On solutions with blow-up at the boundary for a class of semilinear elliptic equations, in: G. Buttazzo, et al. (Eds.), Developments in Partial Differential Equations and Applications to Mathematical Physics, Plenum, New York, 1992, pp. 65–77.

[8] M. Marcus, L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 237–274.

[9] R. Osserman, On the inequality $\Delta u \geqslant f(u)$, Pacific J. Math. 7 (1957) 1641–1647.

[10] C. Yarur, Existence of continuous and singular ground states for semilinear elliptic systems, Electron. J. Differential Equations 1 (1998) 1–27.