Journal of Global Optimization **15:** 41–54, 1999.

Existence Theorems of Hartman–Stampacchia Type for Hemivariational Inequalities and Applications

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(Received 27 February 1998; accepted in revised form 15 October 1998)

Abstract. We give some versions of theorems of Hartman-Stampacchia's type for the case of Hemivariational Inequalities on compact or on closed and convex subsets in infinite and finite dimensional Banach spaces. Several problems from Nonsmooth Mechanics are solved with these abstract results.

Key words: Hemivariational inequalities, Clarke subdifferential, Monotone operator, Set valued mappings

1. Introduction and the main results

The well-known theorem of Hartman-Stampacchia (see [3], Lemma 3.1, or [5], Theorem I.3.1) asserts that if V is a finite dimensional Banach space, $K \subset V$ is compact and convex, $A : K \to V^\star$ is continuous, then there exists $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u\rangle \geq 0\,. \tag{1}$$

If we weaken the hypotheses and consider the case where K is a closed and convex subset of the finite dimensional space V, Hartman and Stampacchia proved (see [5], Theorem I.4.2) that a necessary and sufficient condition which ensures the existence of a solution to Problem (1) is that there is some $R > 0$ such that a solution u of (1) with $\|u\| \leq R$ satisfies $\|u\| < R$.

The purpose of this paper is to extend these classical results in the framework of hemivariational inequalities. These inequalities appear as a generalization of variational inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but give rise to substationarity problems. The mathematical theory of hemivariational inequalities, as well as their applications in Mechanics, Engineering or Economics, has been developed by P.D. Panagiotopoulos (see monographs [6, 8, 9] and the references cited therein for a treatment of this theory and further comments).

Let V be a real Banach space and let $T : V \to L^p(\Omega, \mathbf{R}^k)$ be a linear and continuous operator, where $1 \leq p < \infty$, $k \geq 1$, and Ω is a bounded open set in $\mathbf{R}^N$. Throughout this paper, K is a subset of V, $A : K \to V^*$ an operator and $j = j(x, y) : \Omega \times \mathbf{R}^k \to \mathbf{R}$ is a Carathéodory function which is locally Lipschitz with respect to the second variable $y \in \mathbf{R}^k$ and satisfies the following assumption

(j) there exist $h_1 \in L^{\frac{p}{p-1}}(\Omega, \mathbf{R})$ and $h_2 \in L^\infty(\Omega, \mathbf{R})$ such that

$$|z| \leq h_1(x) + h_2(x)|y|^{p-1},$$

for a.e. $x \in \Omega$, every $y \in \mathbf{R}^k$ and $z \in \partial j(x, y)$. Denoting by $Tu = \hat{u}$, $u \in V$, our aim is to study the problem

(P) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0.$$

We have denoted by $j^0(x, y; h)$ the (partial) Clarke derivative of the locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbf{R}^k$ with respect to the direction $h \in \mathbf{R}^k$, where $x \in \Omega$, and by $\partial j(x, y)$ the Clarke generalized gradient of this mapping at $y \in \mathbf{R}^k$, that is

$$j^0(x, y; h) = \limsup_{\substack{y' \to y \\ t \downarrow 0}} \frac{j(x, y' + th) - j(x, y')}{t};$$

$$\partial j(x, y) = \{z \in \mathbf{R}^k : \langle z, h\rangle \leq j^0(x, y; h), \quad \text{for all } h \in \mathbf{R}^k\}$$

The euclidean norm in $\mathbf{R}^k$, $k \geq 1$, and the duality pairing between a Banach space and its dual will be denoted by $|\cdot|$, resp. $\langle \cdot\,, \cdot\rangle$. We also denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega, \mathbf{R}^k)$ defined by

$$\|\hat{u}\|_p = \left(\int_\Omega |\, \hat{u}(x)\,|^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

In order to state our existence results for the problem (P), we need the following definitions.

DEFINITION 1. *The operator $A : K \to V^*$ is w^*-demicontinuous if for any sequence $\{u_n\} \subset K$ converging to u, the sequence $\{Au_n\}$ converges to Au for the w^*-topology in V^*.*

DEFINITION 2. *The operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K if for any finite dimensional space $F \subset V$, which intersects K, the operator $A|_{K\cap F}$ is demicontinuous, that is $\{Au_n\}$ converges weakly to Au in V^* for each sequence $\{u_n\} \subset K \cap F$ which converges*[⋆] *to u.*

⋆ By 'converges' we always mean 'strongly (or norm) converges'.

REMARK 1. In reflexive Banach spaces the following hold: (a) the w^*-demicontinuity and demicontinuity are the same; (b) a demicontinuous operator $A : K \to V^*$ is continuous on finite dimensional subspaces of K.

The following result is a generalized form of the Hartman–Stampacchia theorem for the case of hemivariational inequalities in infinite dimensional real Banach spaces; namely it generalizes Theorem 6 in [13] and Theorem 2.1 in [14] for the framework of such inequalities.

THEOREM 1. *Let K be a compact and convex subset of the infinite dimensional Banach space V and let j satisfy the condition (j). If the operator $A : K \to V^*$ is w^*-demicontinuous, then the problem (P) admits a solution.*

In finite dimensional Banach spaces the above theorem has the following equivalent form.

COROLLARY 1. *Let V be a finite dimensional Banach space and let K be a compact and convex subset of V. If the assumption (j) is fulfilled and if $A : K \to V^*$ is a continuous operator, then the problem (P) has at least a solution.*

In Section 2 the proof of Theorem 1 will be based on Corollary 1; for this reason Corollary 1 will be proved before this theorem.

REMARK 2. The condition of w^*-demicontinuity on the operator $A : K \to V^*$ in Theorem 1 may be replaced equivalently by the assumption:

(A_1) the mapping $K \ni u \to \langle Au, v\rangle$ is weakly upper semi-continuous, for each $v \in V$.

REMARK 3. If A is w^*-demicontinuous, $\{u_n\} \subset K$ and $u_n \to u$, then

$$\lim_{n\to\infty} \langle Au_n, u_n\rangle = \langle Au, u\rangle.$$

Weaking more the hypotheses on K by assuming that K is a closed, bounded and convex subset of the Banach space V, we need some more about the operators A and T (see Theorem 2). We first recall that an operator $A : K \to V^*$ is said to be monotone if, for every $u, v \in K$,

$$\langle Au - Av, u - v\rangle \geq 0.$$

Thus we can formulate the following result, which is the corresponding variant for hemivariational inequalities of Theorem 1.1 in [3].

THEOREM 2. *Let V be a reflexive infinite dimensional Banach space and let $T : V \to L^p(\Omega, \mathbf{R}^k)$ be a linear and compact operator. Assume K is a closed, bounded and convex subset of V and $A : K \to V^*$ is monotone and continuous on finite dimensional subspaces of K. If j satisfies the condition (j) then the problem (P) has at least one solution.*

We also give a generalization of Theorem III.1.7. in [5] by

THEOREM 3. *Assume that the same hypotheses as in Theorem 2 hold without the assumption of boundedness of K. Then a necessary and sufficient condition for the hemivariational inequality (P) to have a solution is that there exists $R > 0$ with the property that at least one solution of the problem*

$$\begin{cases} u_R \in K \cap \{u \in V;\ \|u\| \leq R\}; \\ \langle Au_R, v - u_R\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \hat{v}(x) - \hat{u}_R(x))\, dx \geq 0, \\ \text{for every } v \in K \text{ with } \|v\| \leq R, \end{cases} \tag{2}$$

satisfies the inequality $\|u_R\| < R$.

A basic tool in our proofs will be the following auxiliary result.

LEMMA 1. *(a) If it is satisfied the assumption (j) and V_1, V_2 are nonempty subsets of V, then the mapping $V_1 \times V_2 \to \mathbf{R}$ defined by:*

$$(u, v) \to \int_\Omega j^0(x, \hat{u}(x), \hat{v}(x))\, dx \tag{3}$$

is upper semi-continuous.

(b) Moreover, if $T : V \to L^p(\Omega, \mathbf{R}^k)$ is a linear compact operator, then the above mapping is weakly upper semi-continuous.

Proof. (a) Let $\{(u_m, v_m)\}_{m\in\mathbf{N}} \subset V_1 \times V_2$ be a sequence converging to $(u, v) \in V_1 \times V_2$, as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbf{R}^k)$ is continuous, it follows that

$$\hat{u}_m \to \hat{u}, \quad \hat{v}_m \to \hat{v} \text{ in } L^p(\Omega, \mathbf{R}^k), \quad \text{as } m \to \infty$$

There exists a subsequence $\{(\hat{u}_n, \hat{v}_n)\}$ of the sequence $\{(\hat{u}_m, \hat{v}_m)\}$ such that

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))\, dx = \lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))\, dx.$$

By Proposition 4.11 in [4], one may suppose the existence of two functions $\hat{u}_0, \hat{v}_0 \in L^p(\Omega, \mathbf{R}^+)$, and of two subsequences of $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ denoted again by the same symbols and such that:

$$|\hat{u}_n(x)| \leq \hat{u}_0(x), \quad |\hat{v}_n(x)| \leq \hat{v}_0(x),$$

$$\hat{u}_n(x) \to \hat{u}(x), \quad \hat{v}_n(x) \to \hat{v}(x), \quad \text{as } n \to \infty$$

for a.e. $x \in \Omega$. On the other hand, for each x where holds true the condition (j) and for each $y, h \in \mathbf{R}^k$, there exists $z \in \partial j(x, y)$ such that

$$j^0(x, y; h) = \langle z, h\rangle = \max\{\langle w, h\rangle :\ w \in \partial j(x, y)\},$$

(see [1], Prop 2.1.2). Now, by (j),

$$|j^0(x, y; h)| \leq |z|\,|h| \leq (h_1(x) + h_2(x)|y|^{p-1}) \cdot |h|\,.$$

Consequently, denoting $F(x) = (h_1(x) + h_2(x)|\hat{u}_0(x)|^{p-1})|\hat{v}_0(x)|$, we find that

$$|j^0(x, \hat{u}_n(x); \hat{v}_n(x))| \leq F(x),$$

for all $n \in \mathbf{N}$ and for a.e. $x \in \Omega$.

From Holder's Inequality and from the condition (j) for the functions h_1 and h_2 it follows that $F \in L^1(\Omega, \mathbf{R})$. Fatou's Lemma yields

$$\lim_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}_n(x))\, dx \leq \int_\Omega \limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x))\, dx.$$

Next, by the upper-semicontinuity of the mapping $j^0(x, .; .)$ (see [1], Prop. 2.1.1) we get that

$$\limsup_{n\to\infty} j^0(x, \hat{u}_n(x); \hat{v}_n(x)) \leq j^0(x, \hat{u}(x); \hat{v}(x))\,,$$

for a.e. $x \in \Omega$, because

$$\hat{u}_n(x) \to \hat{u}(x) \ \text{ and } \quad \hat{v}_n(x) \to \hat{v}(x), \quad \text{as } n \to \infty$$

for a.e. $x \in \Omega$. Hence

$$\limsup_{m\to\infty} \int_\Omega j^0(x, \hat{u}_m(x); \hat{v}_m(x))\, dx \leq \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x))\, dx,$$

which proves the upper-semicontinuity of the mapping defined by (3).

(b) Let $\{(u_m, v_m)\}_{m\in\mathbf{N}} \subset V_1 \times V_2$ be now a sequence weakly-converging to $\{u, v\} \in V_1 \times V_2$, as $m \to \infty$. Thus $u_m \rightharpoonup u,\ v_m \rightharpoonup v$ weakly as $m \to \infty$. Since $T : V \to L^p(\Omega, \mathbf{R}^k)$ is a linear compact operator, it follows that

$$\hat{u}_m \to \hat{u}, \quad \hat{v}_m \to \hat{v} \ \text{ in } \ L^p(\Omega, \mathbf{R}^k).$$

From now on the proof follows the same proof as in the case (a). □

2. Proof of the theorems

2.1. PROOF OF COROLLARY 1

Arguing by contradiction, for every $u \in K$, there is some $v = v_u \in K$ such that

$$\langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < 0.$$

For every $v \in K$, set

$$N(v) = \{u \in K;\, \langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx < 0\}.$$

For any fixed $v \in K$ the mapping $K \to \mathbf{R}$ defined by

$$u \longmapsto \langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx$$

is upper semi-continuous, by Lemma 1 and the continuity of A. Thus, by the definition of the upper semi-continuity, $N(v)$ is an open set. Our initial assumption implies that $\{N(v);\, v \in K\}$ is a covering of K. Hence, by the compactness of K, there exist $v_1, \ldots, v_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} N(v_j).$$

Let $\rho_j(u)$ be the distance from u to $K \setminus N(v_j)$. Then ρ_j is a Lipschitz map which vanishes outside $N(v_j)$ and the functionals

$$\psi_j(u) = \frac{\rho_j(u)}{\sum_{i=1}^{n} \rho_i(u)}$$

define a partition of the unity subordinated to the covering $\{\rho_1, \cdots, \rho_n\}$. Moreover, the mapping

$$p(u) = \sum_{j=1}^{n} \psi_j(u) v_j$$

is continuous and maps K into itself, because of the convexity of K. Thus, by Brouwer's fixed point Theorem, there exists u_0 in the convex closed hull of $\{v_1, \ldots, v_n\}$ such that $p(u_0) = u_0$. Define

$$q(u) = \langle Au, p(u) - u\rangle + \int_\Omega j^0(x, \hat{u}(x); p(\hat{u})(x) - \hat{u}(x))\, dx.$$

The convexity of the map $j^0(\hat{u}; \cdot)$ (see [1], Lemma 1) and the fact that $\sum_{j=1}^{n} \psi_j(u) = 1$ imply

$$q(u) \le \sum_{j=1}^{n} \psi_j(u)\, \langle Au, v_j - u\rangle + \sum_{j=1}^{n} \psi_j(u) \int_\Omega j^0(x, \hat{u}(x); \hat{v}_j(x) - \hat{u}(x))\, dx.$$

For arbitrary $u \in K$, there are only two possibilities: if $u \notin N(v_i)$, then $\psi_i(u) = 0$. On the other hand, for all $1 \le j \le n$ (there exists at least such an indice) such that $u \in N(v_j)$, we have $\psi_j(u) > 0$. Thus, by the definition of $N(v_j)$,

$$q(u) < 0, \quad \text{for every } u \in K.$$

But $q(u_0) = 0$, which gives a contradiction. □

2.2. PROOF OF THEOREM 1

For this proof we need Lemma 2 below. Let F be an arbitrary finite dimensional subspace of V which intersects K. Let $i_{K\cap F}$ be the canonical injection of $K \cap F$ into K and i_F^* be the adjoint of the canonical injection i_F of F into V. Then:

LEMMA 2. *The operator*

$$B : K \cap F \to F^*, \quad B = i_F^* A i_{K\cap F}$$

is continuous.

REMARK 4. The above lemma also holds true if the operator A is continuous on finite dimensional subspaces of K.

Proof. For any $v \in K$, set

$$S(v) = \left\{ u \in K;\, \langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0 \right\}.$$

Step 1. $S(v)$ is a closed set.

We first observe that $S(v) \neq \emptyset$, since $v \in S(v)$. Let $\{u_n\} \subset S(v)$ be an arbitrary sequence which converges to u as $n \to \infty$. We have to prove that $u \in S(v)$. By the part (a) of Lemma 1 we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} [\langle Au_n, v - u_n\rangle + \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))]\ dx \\ &= \lim_{n\to\infty} \langle Au_n, v - u_n\rangle + \limsup_{n\to\infty} \int_\Omega j^0(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))\ dx \\ &\leq \langle Au, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\ dx. \end{aligned}$$

This is equivalent to $u \in S(v)$.

Step 2. The family $\{S(v);\, v \in K\}$ has the finite intersection property.

Let $\{v_1, \ldots, v_n\}$ be an arbitrary finite subset of K and let F be the linear space spanned by this family. Applying Corollary 1 to the operator B defined in Lemma 2, we find $u \in K \cap F$ such that $u \in \cap_{j=1}^n S(v_j)$, which means that the family of closed sets $\{S(v);\, v \in K\}$ has the finite intersection property. But the set K is compact. Hence

$$\bigcap_{v\in K} S(v) \neq \emptyset,$$

which means that the problem (P) has at least one solution. □

2.3. PROOF OF THEOREM 2

Let F be an arbitrary finite dimensional subspace of V, which intersects K. Consider the canonical injections $i_{K\cap F} : K \cap F \to K$ and $i_F : F \to V$ and let $i_F^* : V^* \to F^*$ be the adjoint of i_F. Applying Corollary 1 to the continuous operator $B = i_F^\star A i_{K\cap F}$ (see Remark 4) we find some u_F in the compact set $K \cap F$ such that, for every $v \in K \cap F$,

$$\langle i_F^\star A i_{K\cap F} u_F, v - u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\, dx \geq 0. \tag{4}$$

But

$$0 \leq \langle Av - Au_F, v - u_F\rangle = \langle Av, v - u_F\rangle - \langle Au_F, v - u_F\rangle. \tag{5}$$

Hence, by (4), (5) and the observation that $\langle i_F^\star A i_{K\cap F} u_F, v - u_F\rangle =$ $\langle Au_F, v - u_F\rangle$, we have

$$\langle Av, v - u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\, dx \geq 0, \tag{6}$$

for any $v \in K \cap F$. The set K is weakly closed as a closed convex set; thus it is weakly compact because it is bounded and V is a reflexive Banach-space.

Now, for every $v \in K$ define

$$M(v) = \left\{ u \in K;\, \langle Av, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0 \right\}.$$

The set $M(v)$ is weakly closed by the part (b) of Lemma 1 and by the fact that this set is weakly sequentially dense (see, e.g., [2], pp. 145-149 or [10], p. 3). We now show that the set $M = \cap_{v\in K} M(v) \subset K$ is non-empty. To prove this, it suffices to prove that

$$\bigcap_{j=1}^{n} M(v_j) \neq \emptyset, \tag{7}$$

for any $v_1, \ldots, v_n \in K$. Let F be the finite dimensional linear space spanned by $\{v_1, \ldots, v_n\}$. Hence, by (6), there exists $u_F \in F$ such that, for every $v \in K \cap F$,

$$\langle Av, v - u_F\rangle + \int_\Omega j^0(x, \hat{u}_F(x); \hat{v}(x) - \hat{u}_F(x))\, dx \geq 0.$$

This means that $u_F \in M(v_j)$, for every $1 \leq j \leq n$, which implies (7). Consequently, it follows that $M \neq \emptyset$. Therefore there is some $u \in K$ such that, for every $v \in K$,

$$\langle Av, v - u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0. \tag{8}$$

We shall prove that from (8) we can conclude that u is a solution of Problem (P). Fix $w \in K$ and $\lambda \in (0, 1)$. Putting $v = (1 - \lambda)u + \lambda w \in K$ in (8) we find

$$\langle A((1-\lambda)u + \lambda w), \lambda(w - u)\rangle + \int_\Omega j^0(x, \hat{u}(x); \lambda(\hat{w} - \hat{u})(x))\, dx \geq 0. \quad (9)$$

But $j^0(x, \hat{u}; \lambda\hat{v}) = \lambda\, j^0(x, \hat{u}; \hat{v})$, for any $\lambda > 0$. Therefore (9) may be written, equivalently,

$$\langle A((1-\lambda)u + \lambda w), w - u\rangle + \int_\Omega j^0(x, \hat{u}(x); (\hat{w} - \hat{u})(x))\, dx \geq 0. \quad (10)$$

Let F be the vector space spanned by u and w. Taking into account the demicontinuity of the operator $A_{|K\cap F}$ and passing to the limit in (10) as $\lambda \longrightarrow 0$, we obtain that u is a solution of Problem (P). □

REMARK 5. As the set $K \cap \{u \in V;\ \|u\| \leq R\}$ is a closed bounded and convex set in V, it follows from Theorem 2 that the problem (2) in the formulation of our Theorem 3 has at least one solution for any fixed $R > 0$.

2.4. PROOF OF THEOREM 3

The necessity is evident.

Let us now suppose that there exists a solution u_R of (2) with $\|u_R\| < R$. We prove that u_R is solution of (P). For any fixed $v \in K$, we choose $\varepsilon > 0$ small enough so that $w = u_R + \varepsilon(v - u_R)$ satisfies $\|w\| < R$. Hence, by (2),

$$\langle Au_R, \varepsilon(v - u_R)\rangle + \int_\Omega j^0(x, \hat{u}_R(x); \varepsilon(\hat{v} - \hat{u}_R)(x))\, dx \geq 0$$

and, using again the positive homogeneity of the map $v \longmapsto j^0(u; v)$, the conclusion follows. □

3. Applications

3.1. NONCOERCIVE HEMIVARIATIONAL INEQUALITIES

We consider noncoercive forms of the coercive and semicoercive hemivariational problems treated in [6] (pp. 65-77). The results are more general from the point of view of the absence of the coercivity or the semicoercivity assumption, but less general from the point of view of the boundedness of the set K. For this purpose, let us assume that V is a real Hilbert space and that the continuous injections

$$V \subset [L^2(\Omega, \mathbf{R}^k)]^N \subset V^\star$$

hold, where $V^\star$ denotes the dual space of V. Moreover let $T : V \to L^2(\Omega, \mathbf{R}^k)$, $T(u) = \hat{u}$, $\hat{u}(x) \in \mathbf{R}^k$ be a linear and continuous mapping. Consider the operator

A appearing in our abstract framework as $Au = A_1 u + f$, where $f \in V^\star$ is a prescribed element, while A_1 satisfies, respectively, the assumptions of Theorems 1, 2 or 3. Then the theorem 1 holds for the problem

(P_1) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v-u\rangle + \int_\Omega j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0.$$

Moreover, if T is a linear compact operator, then Theorems 2 and 3 also hold for the above problem.

Suppose further that Γ is the Lipschitz boundary of Ω and that the linear mapping $T : V \to L^2(\Gamma, \mathbf{R}^k)$ is continuous. Then the theorem 1 holds for the problem

(P_2) Find $u \in K$ such that, for every $v \in K$,

$$\langle Au, v-u\rangle + \int_\Gamma j^0(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))\, dx \geq 0.$$

Furthermore, if T is compact, then Theorems 2 and 3 remain valid for (P_2).

3.2. NONMONOTONE LAWS IN NETWORKS WITH CONVEX CONSTRAINTS

We shall give now an application in Economics concerning a network flow problem. We follow the basic ideas of Prager [7, 11], and for the consideration of the nonlinearities we combine them with the notion of nonconvex superpotential. We refer to [6] (p. 191) for the derivation of the formulas.

The generally non-monotone nonlinearity is caused by the law relating the two branch variables of the network, the 'flow intensity' and the 'price differential' which here can also be vectors. The problem is formulated as a hemivariational inequality and the existence of its solution is discussed further. We consider networks with directed branches. The nodes are denoted by Latin letters and the branches by Greek letters. We suppose that we have m nodes and ν branches. We take as branch variables the 'flow intensity' s_γ and the 'price differential' e_γ. As node variables the 'amount of flow' p_k and the 'shadow price' u_k are considered. The terminology has been taken from [11]. Moreover each branch may have an 'initial price differential' vector e^0_γ. The above given quantities are assembled in vectors e, e^0, u, s, p. The node-branch incidence matrix G is denoted by G, where the lines of G are linearly independent. Upper index T denotes the transpose of a matrix or a vector. The network law is a relation between the 'flow intensity' s_γ and the 'price differential' e_γ. We accept that s_γ is a nonmonotone function of the e_γ expressed by the relation

$$e_\gamma - e^0_\gamma \in \partial j_\gamma(s_\gamma) + \frac{1}{2}\partial s^T_\gamma C_\gamma s_\gamma\,, \tag{11}$$

where k_γ is a positive definite symmetric matrix and ∂ is the generalized gradient. The graph of the $s_\gamma - e_\gamma$ law is called γ-characteristic.

The problem to be solved consists in the determination for the whole network of the vectors s, e, u, for given vectors p and e_0.

Further let $C = \operatorname{diag}[C_1, \ldots, C_\gamma, \cdots]$ and let the summation $\sum_\gamma$ be extended over all branches. Now we consider the graph which corresponds to the network and a corresponding tree. The tree results from the initial graph by cuting all the branches creating the closed loops. Let us denote by s_T (resp. s_M) the part of the vector s corresponding to the tree branches (resp. to the cut branches giving rise to closed loops). Then we may write instead of $Gs = p$ the relation

$$G_T s_T + G_M s_M = p\,.$$

Here G_T is nonsingular and thus we may write that

$$s = \begin{bmatrix} s_T \\ s_M \end{bmatrix} = \begin{bmatrix} G_T^{-1} \\ 0 \end{bmatrix} p + \begin{bmatrix} -G_T^{-1} G_M \\ I \end{bmatrix} s_M = s_0 + B s_M\,, \tag{12}$$

where I denotes the unit matrix. Using (11) and (12) we obtain (cf. [6]) a hemivariational inequality with respect to s_M which reads: find $s_M \in \mathbf{R}^{n_1}$ (n_1 is the dimension of s_M) such that

$$\begin{aligned} &\sum_\gamma j_\gamma^0((s_0 + Bs_M)_\gamma, (Bs_M^\star - Bs_M)_\gamma) + s_M^T B^T C B(s_M^\star - s_M) \\ &\quad + s_0^T C B(s_M^\star - s_M) + e^{0T} B(s_M^\star - s_M) \geq 0 \quad \forall s_M^\star \in \mathbf{R}^{n_1}\,. \end{aligned} \tag{13}$$

Let us now assume that the flow intensities s_M are constrained to belong to a bounded and closed convex subset $K \subset \mathbf{R}^{n_1}$ (box constraints are very common). Thus the problem takes the form: find $s_M \in K$ which satisfies (13), for every $s_M^\star \in K$.

Since the rank of B is equal to the number of its columns and C is symmetric and positive definite the same happens for $B^T C B$. In the finite dimensional case treated here, one can easily verify that Corollary 1 holds, if $j_\gamma(\cdot,\cdot)$ satisfies the condition (j). Thus (13) has at least one solution.

3.3. ON THE NONCONVEX SEMIPERMEABILITY PROBLEM

Let us put ourselves within the framework of [6] (p. 185), where we have studied nonconvex semipermeability problems. We consider an open, bounded, connected subset Ω of $\mathbf{R}^3$ referred to a fixed Cartesian coordinate system $0x_1x_2x_3$ and we formulate the equation

$$-\Delta u = f \qquad \text{in } \Omega \tag{14}$$

for stationary problems.

Here u represents the temperature in the case of heat conduction problems, whereas in problems of hydraulics and electrostatics the pressure and the electric potential are represented, respectively. We denote further by Γ the boundary of Ω and we assume that Γ is sufficiently smooth ($C^{1,1}$-boundary is sufficient). If $n = \{n_i\}$ denotes the outward unit normal to Γ then $\partial u/\partial n$ is the flux of heat, fluid or electricity through Γ for the aforementioned classes of problems.

We may consider the interior and the boundary semipermeability problems.

In the first class of problems the classical boundary conditions

$$u = 0 \quad \text{on } \Gamma \tag{15}$$

are assumed to hold, whereas in the second class the boundary conditions are defined as a relation between $\partial u/\partial n$ and u. In the first class the semipermeability conditions are obtained by assuming that $f = \bar{f} + \bar{\bar{f}}$ where $\bar{\bar{f}}$ is given and $\bar{f}$ is a known function of u. Here, we consider (15) for the sake of simplicity. All these problems may be put in the following general framework. For the first class we seek a function u such as to satisfy (14), (15) with

$$f = \bar{f} + \bar{\bar{f}}, \qquad -\bar{f} \in \partial j_1(x, u) \text{ in } \Omega\,. \tag{16}$$

For the second class we seek a function u such that (14) is satisfied together with the boundary condition

$$-\frac{\partial u}{\partial n} \in \partial j_2(x, u) \quad \text{on } \Gamma_1 \subset \Gamma \qquad \text{and} \qquad u = 0 \text{ on } \Gamma \setminus \Gamma_1\,. \tag{17}$$

Both $j_1(x, \cdot)$ and $j_2(x, \cdot)$ are locally Lipschitz functions and ∂ denotes the generalized gradient. Note, that if $q = \{q_i\}$ denotes the heat flux vector and $k > 0$ is the coefficient of thermal conductivity of the material we may write by Fourier's law that $q_i n_i = -k\partial u/\partial n$.

Let us introduce the notations

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and

$$(f, u) = \int_\Omega f u \, dx\,.$$

We may ask in addition that u is constrained to belong to a convex bounded closed set $K \subset V$ due to some technical reasons, e.g., constraints for the temperature or the pressure of the fluid, etc.

The hemivariational inequalities correspond to the two classes of problems. Let for the first class $V = H_0^1(\Omega)$ and $\bar{\bar{f}} \in L^2(\Omega)$; for the second class $V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma \setminus \Gamma_1\}$ and $f \in L^2(\Omega)$. Then from the Green–Gauss theorem applied to (14), with the definition of (16) and (17) we are led to the following two hemivariational inequalities for the first and for the second class of semipermeability problems respectively

(i) Find $u \in K$ such that

$$a(u, v-u) + \int_{\Omega} j_1^0(x, u(x); v(x) - u(x))\, dx \geq (\bar{\bar{f}}, v-u) \qquad \forall v \in K\,. \tag{18}$$

(ii) Find $u \in K$ such that

$$a(u, v-u) + \int_{\Gamma_1} j_2^0(x, u(x); v(x) - u(x))\, d\Gamma \geq (f, v-u) \qquad \forall v \in K\,. \tag{19}$$

Since $a(\cdot,\cdot)$ is (strongly) monotone on V both in (i) and (ii) and the embeddings $V \subset L^2(\Omega)$ and $V \subset L^2(\Gamma_1)$ are compact we can prove the existence of solutions of (i) and of (ii) by applying Theorem 2 if j_1 and j_2 satisfy the condition (j).

3.4. ADHESIVELY SUPPORTED ELASTIC PLATE BETWEEN TWO RIGID SUPPORTS

Let us consider a Kirchoff plate. The elastic plate is referred to a right-handed orthogonal Cartesian coordinate system $Ox_1x_2x_3$. The plate has constant thickness h_1, and the middle surface of the plate coincides with the Ox_1x_2-plane. Let Ω be an open, bounded and connected subset of $\mathbf{R}^2$ and suppose that the boundary Γ is Lipschitzian ($C^{0,1}$-boundary). The domain Ω is occupied by the plate in its undeformed state. On $\Omega' \subset \Omega$ (Ω' is such that $\overline{\Omega}' \cap \Gamma = \emptyset$) the plate is bonded to a support through an adhesive material. We denote by $\zeta(x)$ the deflection of the point $x = (x_1, x_2, x_3)$ and by $f = (0, 0, f_3)$, $f_3 = f_3(x)$ (hereafter called f for simplicity) the distributed load of the considered plate per unit area of the middle surface. Concerning the laws for adhesive forces and the formulation of the problems we refer to [9]. Here we make the additional assumption that the displacements of the plate are prevented by some rigid supports. Thus we may put as an additional assumption the following one:

$$z \in K, \tag{20}$$

where K is a convex closed bounded subset of the displacement space. One could have, e.g., that $a_0 \leq z \leq b_0$ etc.

We assume that any type of boundary conditions may hold on Γ. Here we assume that the plate boundary is free. Indeed there is no need to guarantee that the strain energy of the plate is coercive. Thus the whole space $H^2(\Omega)$ is the kinematically admissible set of the plate. If one takes now into account the relation (20), then $z \in K \subset H^2(\Omega)$, where K is a closed convex bounded subset of $H^2(\Omega)$ and the problem has the following form:

Find $\zeta \in K$ such as to satisfy

$$a(\zeta, z-\zeta) + \int_{\Omega'} j^0(\zeta, z-\zeta)\, d\Omega \geq (f, z-\zeta) \qquad \forall z \in K. \tag{21}$$

Here $a(\cdot, \cdot)$ is the elastic energy of the Kirchoff plate, i.e.

$$a(\zeta, z) = k \int_{\Omega} [(1 - \nu)\, \zeta_{,\alpha\beta}\, z_{,\alpha\beta} + \nu\, \Delta\zeta\, \Delta z]\, d\Omega \quad \alpha, \beta = 1, 2, \tag{22}$$

where $k = Eh^3/12(1 - \nu^2)$ is the bending rigidity of the plate with E and ν the modulus of elasticity and the Poisson ratio respectively, and h is its thickness. Moreover j is the binding energy of the adhesive which is a locally Lipschitz function on $H^2(\Omega)$ and $f \in L^2(\Omega)$ denotes the external forces. Furthermore, if j fulfills the growth condition (j) then, taking into consideration that $a(\cdot, \cdot)$ appearing in (22) is continuous monotone, we can deduce, by applying Theorem 2, the existence of a solution of the problem (21).

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