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Solutions and Approximate Solutions of Quasi-Equilibrium Problems in Banach Spaces

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Abstract This paper deals with quasi-equilibrium problems in the setting of real Banach spaces. By a fixed point theory approach, we obtain existence results under mild conditions of continuity, improving some previous results in this area. By a selection theory approach, we make use of the Michael selection theorem to overcome the separability of the Banach spaces and generalize some results obtained recently in the literature. Finally, we deal with the existence of approximate solutions for quasi-equilibrium problems, and by arguments combining selection theory and fixed point theory, we obtain some qualitative results for quasi-equilibrium problems involving sub-lower semicontinuous set-valued mappings.

Keywords Quasi-equilibrium problem · Set-valued mapping · Semicontinuity · Fixed point · Continuous selection

Mathematics Subject Classification 47J20 · 49J35 · 49J53 · 46N10

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1 Introduction

The so-called equilibrium problem in the sense of Blum, Muu and Oettli or inequality of Ky Fan-type, as known today, has been considered in [1,2] as an important and general framework for describing, in a common formulation, various problems arising in different areas of mathematics, including optimization problems, mathematical economic problems and Nash equilibrium problems. Historically, this formulation has been first used as a pure mathematical object in the work by Fan [3] on minimax inequality problems, which has been followed for a long time by several studies on equilibrium problems considered under different headings, for instance, in [4,5]. It is worth mentioning that one of the interests of this common formulation, called simply the equilibrium problem, is that many techniques developed for a particular case may be extended, with suitable adaptations, to the equilibrium problem, and then they can be applied to other particular cases.

Although the equilibrium problem subsumes several kinds of problems, there are many models described by variational inequalities involving constraints that depend on the solution itself. In this direction, there are the quasi-variational inequalities considered early in the literature in connection with stochastic impulse control problems, where the constraint set is subject to modifications, see, for instance, [4,6] and the references therein. For more recent existence results for quasi-variational inequalities with applications to Nash equilibria of generalized games, we also refer, for instance, to [7,8].

In the spirit to describe in a more again general framework most of problems arising in nonlinear analysis, it has been considered and adopted recently in the literature the notion of quasi-equilibrium problem, which appears as an equilibrium problem in which the constraint set is subject to modifications. The quasi-equilibrium problem is a unified formulation which encompasses many relevant problems such as quasi-variational inequalities, mixed quasi-variational like inequalities and all the special cases of the equilibrium problem, see [9,10]. See also the seminal paper [4], where this formulation has been first used as a pure mathematical object. The first existence results have been established and applied to different optimization problems including Nash equilibrium problems under constraints and quasi-variational inequalities for monotone operators.

We would like to precise here that many problems related to the term equilibrium and arising from different areas of sciences can be mathematically modeled as special cases of the unified formulation called the equilibrium problem. Also, the equilibrium problem subsumes many mathematical special cases which are in relation to the term equilibrium such as Nash equilibrium problems and economic equilibrium problems. Then, one can naturally guess that this is the reason for which the term equilibrium problem has been chosen to name this unified formulation. It is now well known that the last decades have witnessed an exceptional growth in theoretical advances on the equilibrium problem and its applications in concrete cases. Maybe, the simplicity of this formulation is the principal reason which has allowed all these advancements. We point out that the equilibrium problem has never been introduced in order to deal directly with other problems which are not described by the old existing concepts. As we will see later, the equilibrium problem is an inequality involving a bifunction, which is not necessarily a variational inequality, see, for example, [11] to find various bifunctions defined for some important models of real-life problems. Also, if we assume that a problem is directly modeled as an equilibrium problem by using an inequality involving a bifunction, then nothing can impose that this inequality is a variational inequality. Unfortunately, this is not the task for which the concept of the equilibrium problem has been introduced, but to describe various existing concepts in a common way in order to deeply study them altogether.

Let us point out again that there are some other unified mathematical formulations encompassing different special cases of the equilibrium problem which have been also considered in the literature. We especially think about the general conditions considered in [12] as a structure giving rise to what is called there "equilibrium problems of a certain type." These general conditions express what is called "the common laws," and it is shown that many equilibrium problems arising from different areas of sciences fulfill the common laws. Similar to the equilibrium problem considered in this paper, the equilibrium problems of this above type subsumes many problems of nonlinear analysis as particular cases. This is an important different point of view, which has also allowed to obtain different results on concrete cases, and especially on the traffic equilibrium problem.

In this paper, we deal with existence of solutions and approximate solutions of the quasi-equilibrium problem. After presenting the necessary background, we follow in Sect. 3 an approach based on fixed point theory to solve the quasi-equilibrium problem under mild conditions of semicontinuity and hemicontinuity introduced and used recently in [13–17]. In this approach, we are interested in solving the equilibrium problem defined on the images of a given set-valued mapping. Then, we seek a fixed point to a related set-valued mapping defined in the sequel and called "the selection set-valued mapping." An example of a bifunction and an application to variational inequalities have been also given in order to highlight our techniques developed in this section. In Sect. 4, we follow a selection theory approach and make use of the Michael selection theorem for paracompact Hausdorff topological spaces. We obtain existence results in the settings of real Banach spaces instead of separable real Banach spaces considered recently in the literature with the Michael selection theorem version for perfectly normal spaces, which is more restrictive in our purpose. Section 5 is devoted to the existence of approximate solutions of the quasi-equilibrium problem. In this section, we impose connection between the involved set-valued mapping and the bifunction and make use of the notion of sub-lower semicontinuous set-valued mappings, introduced in relationship with the notion of approximate continuous selection, to carry out existence of approximate solutions of the quasi-equilibrium problem. This approach combines arguments and techniques from fixed point theory and selection theory and has been already considered for lower semicontinuous set-valued mappings, see, for instance, [7,8,10].

2 Notations and Preliminaries

We first recall the notion of the equilibrium problem we are going to consider in this paper. Let C be a nonempty subset of a Hausdorff topological space E, and

$\Phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction, called equilibrium bifunction iff $\Phi(x, x) = 0$, for every $x \in C$. The *equilibrium problem* is a problem of the form

$$\text{find } x^* \in C \text{ such that } \Phi\left(x^*, y\right) \geq 0 \quad \forall y \in C, \tag{EP}$$

where the set C is called the constraint set.

Although we will give later an example of a bifunction and an application of our techniques to the special case of a variational inequality, we recall here the special case of *Nash equilibrium problems*. A noncooperative game is described through a number N of players, and each player i has a strategy set $K_i \subset \mathbb{R}^n$, also called the feasible set of player i, and aims at minimizing a loss function $f_i : K \to \mathbb{R}$ with $K = \prod_{i=1}^{N} K_i$. For $x = (x_1, \ldots, x_n) \in K$ and $y_i \in K_i$, let $x(y_i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$ denote the vector obtained from x by replacing x_i by y_i. A *Nash equilibrium* is any $x^* \in K$ such that

$$f_i\left(x^*\right) \leq f_i\left(x^*\left(y_i\right)\right) \quad \forall y_i \in K_i, \quad \forall i = 1, \ldots, N.$$

Literally speaking, at a Nash equilibrium point x^*, no player can reduce its loss by unilaterally changing its strategy. The problem of finding a Nash equilibrium amounts to solving the equilibrium problem (EP) corresponding to the so-called Nikaido–Isoda bifunction Φ defined on $K \times K$ by

$$\Phi(x, y) = \sum_{i=1}^{N} \left(f_i\left(x\left(y_i\right)\right) - f_i(x)\right).$$

Now, we turn to the quasi-equilibrium problem, which will be especially considered and studied here in the paper. A *quasi-equilibrium problem* is a problem of the form:

$$\text{find } x^* \in A\left(x^*\right) \text{ such that } \Phi\left(x^*, y\right) \geq 0 \quad \forall y \in A\left(x^*\right), \tag{QEP}$$

where $A : C \rightrightarrows C$ is a set-valued mapping on C. In order words, a quasi-equilibrium problem is an equilibrium problem in which the constraint set is subject to modifications depending on the considered point.

Before going further on existence of solutions and approximate solutions of the quasi-equilibrium problem, we give the necessary background on continuity and convexity and develop some preliminary results we need in the sequel.

Let X be a Hausdorff topological space, $x \in X$ and $f : X \longrightarrow \mathbb{R}$ be a function. Recall that f is said to be *lower semicontinuous* at x iff for every $\varepsilon > 0$, there exists an open neighborhood U of x such that

$$f(y) \geq f(x) - \varepsilon \quad \forall y \in U.$$

The function f is said to be *upper semicontinuous* at x iff $(-f)$ is lower semicontinuous at x. If X is a metric space, then f is lower (*resp.* upper) semicontinuous at $x \in X$ if and only if for every sequence $(x_n)_n$ in X converging to x, we have

$$f(x) \leq \liminf_{n\to+\infty} f(x_n) \quad (resp.\ f(x) \geq \limsup_{n\to+\infty} f(x_n)),$$

where $\liminf_{n\to+\infty} f(x_n) = \sup_n \inf_{k\geq n} f(x_k)$ and $\limsup_{n\to+\infty} f(x_n) = \inf_n \sup_{k\geq n} f(x_k)$.

Following [14–17], f is said to be lower (*resp*. upper) semicontinuous on a subset S of X iff it is lower (*resp*. upper) semicontinuous at every point of S. Note that for $f : X \to \mathbb{R}$, the property for f of being lower (*resp*. upper) semicontinuous on a subset S of X is stronger than the lower (*resp*. upper) semicontinuity of the restriction $f_{|S} : S \to \mathbb{R}$ of f on S.

If X is a real topological Hausdorff vector space, then there is the notion of hemicontinuity for real-valued functions defined on X, which is, in turn, the semicontinuity on line segments. A function $f : X \longrightarrow \mathbb{R}$ is said to be *lower hemicontinuous* at x iff for every $\varepsilon > 0$ and every $z \in X$, there exists $t_z \in [0, 1]$ such that

$$f(tz + (1-t)x) \geq f(x) - \varepsilon \quad \forall t \in \left[0, t_z\right].$$

The function f is said to be *upper hemicontinuous* at x iff $(-f)$ is lower hemicontinuous at x. It is said to be *hemicontinuous* at x iff it is lower and upper hemicontinuous at x. Following [13], f is said to be lower (*resp*. upper) hemicontinuous on a subset S of X iff it is lower (*resp*. upper) hemicontinuous at every point of S. It is said to be hemicontinuous on S iff it is lower and upper hemicontinuous on S.

In the sequel, we need the following notions of convexity of real-valued functions defined on a real topological Hausdorff vector space X. A function $f : X \longrightarrow \mathbb{R}$ is said to be

1. *Quasi-convex* on X iff, for every $x_1, x_2 \in X$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\} \quad \forall \lambda \in [0, 1];$$

2. *Semistrictly quasi-convex* on X iff, for every $x_1, x_2 \in C$ such that $f(x_1) \neq f(x_2)$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad \forall \lambda \in]0, 1[;$$

3. *Explicitly quasi-convex* on X iff it is quasi-convex and semistrictly quasi-convex.

Note that there is not any inclusion relationship between the class of semistrictly quasi-convex functions and that of quasi-convex functions. However, if f is a lower semicontinuous and semistrictly quasi-convex function, then f is explicitly quasi-convex, see, for example, [18,19].

There are several notions related to monotonicity of bifunctions, which play an important role in the results on existence of solutions of the equilibrium problem. Recall that a bifunction $\Phi : X \times X \longrightarrow \mathbb{R}$ is said to be *pseudo-monotone* on X iff

$$\Phi(x, y) \geq 0 \Longrightarrow \Phi(y, x) \leq 0, \quad \forall x, y \in X.$$

Clearly, if Φ is pseudo-monotone on C, then for every $x \in C$, $\Phi(x, x) = 0$ if and only if $\Phi(x, x) \geq 0$.

For two Hausdorff topological spaces X and Y, a set-valued mapping F from X to Y will be denoted by $F : X \rightrightarrows Y$. The graph of F is the set

$$\operatorname{grph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}.$$

If $X = Y$, we denote by $\operatorname{fix}(F)$ the fixed points set of F. That is,

$$\operatorname{fix}(F) := \{x \in X : x \in F(x)\}.$$

For a subset B of Y, we denote, respectively, by

$$F^{-1}(B) := \{x \in X : F(x) \cap B \neq \emptyset\} \quad \text{and} \quad F^{+}(B) := \{x \in X : F(x) \subset B\},$$

the lower and upper inverse set of B by F. For $y \in Y$, we denote the fiber of F at y by $F^{-1}(y)$ instead of $F^{-1}(\{y\})$.

Recall that a set-valued mapping $F : X \rightrightarrows Y$ is said to be lower semicontinuous at a point $x_0 \in X$ iff whenever V is an open subset of Y such that $F(x_0) \cap V \neq \emptyset$, $F^{-1}(V)$ is a neighborhood of x_0. It turns out that F is lower semicontinuous at $x \in X$ if and only if F is continuous at $x \in X$ as a function from X to the set of subsets of Y endowed with the lower Vietoris topology.

By analogy, a set-valued mapping $F : X \rightrightarrows Y$ is said to be upper semicontinuous at a point $x_0 \in X$ iff it is continuous at $x \in X$ as a function from X to the set of subsets of Y endowed with the upper Vietoris topology. That is, F is upper semicontinuous at $x \in X$ iff whenever V is an open subset of Y such that $F(x_0) \subset V$, the upper inverse set of V by F is a neighborhood of x_0.

The set-valued mapping F is said to be lower (*resp*. upper) semicontinuous on X iff it is lower (*resp*. upper) semicontinuous at every point of X. Clearly, F is lower (*resp*. upper) semicontinuous on X if and only if the lower (*resp*. upper) inverse set of any open subset V of Y by F is open. Also, if F has open fibers at every point y of a subset B of Y, then $F^{-1}(B)$ is open. Thus, a set-valued mapping $F : X \rightrightarrows Y$ with open fibers (at every point of Y) is lower semicontinuous on X. The converse is not true in general.

For a subset S of X, we denote by $\operatorname{cl}(S)$ the closure of S with respect to X, except if it is explicitly specified. If $(X, \|.\|)$ is a Banach space, $\varepsilon > 0$ and $x_0 \in X$, we denote by $B(x_0, \varepsilon) = \{x \in X : \|x - x_0\| < \varepsilon\}$ the open ball around x_0 with radius ε. For a subset S of X, we write

$$B(S, \varepsilon) := \bigcup_{x \in S} B(x, \varepsilon).$$

For a set-valued mapping $F : X \rightrightarrows Y$ and $B \subset Y$, the set-valued mapping $F \cap B$ is defined by $(F \cap B)(x) = F(x) \cap B$, for every $x \in X$.

Recall that a single-valued mapping $f : X \to Y$ is said to be a selection of a set-valued mapping $F : X \rightrightarrows Y$ iff $f(x) \in F(x)$, for every $x \in X$.

In the sequel, we will also make use of the following notations. If $\Phi : X \times X \longrightarrow \mathbb{R}$ is a bifunction and K is a subset of X, then the restriction of Φ on $K \times K$ will be denoted by $\Phi_{|K}$. Also, for a set-valued mapping $F : X \rightrightarrows Y$, $F_{|K}$ denotes the restriction of F on K.

3 Existence of Solutions of the Quasi-Equilibrium Problem: A Fixed Point Theory Approach

In this section, we deal with the existence of solutions of the quasi-equilibrium problem by following a fixed point theory approach. This approach has already been considered in [4], and it seems to be the more natural technique to handle such a problem.

In [13–17], various results concerning existence of solutions of the equilibrium problem have been carried out under weakened conditions of semicontinuity and hemicontinuity. The following result is the real topological Hausdorff vector space version of the generalizations of the Ky Fan minimax inequality theorem recently obtained in [14,15,17]. It is based on some techniques such as those used in [17, Proposition 2.2], which remain valid in the general setting of Hausdorff topological spaces.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real topological Hausdorff vector space E. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction and suppose that the following assumptions hold:*

1. *$\Phi(x, x) \geq 0$, for every $x \in C$;*
2. *Φ is quasi-convex in its second variable on C;*
3. *there exist a compact subset K of C and $y_0 \in K$ such that*

$$\Phi(x, y_0) < 0 \quad \forall x \in C \setminus K;$$

4. *Φ is upper semicontinuous in its first variable on K.*

Then, the equilibrium problem (EP) *has a solution.*

In the presence of pseudomonotonocity and explicit quasi-convexity, the upper semicontinuity of the bifunction Φ in its first variable can be weakened to upper hemicontinuity.

Theorem 3.2 *Let C be a nonempty, closed and convex subset of a real topological Hausdorff vector space E. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose that the following assumptions hold:*

1. *Φ is pseudo-monotone on C;*
2. *Φ is explicitly quasi-convex in its second variable on C;*
3. *there exists a compact subset K of C and $y_0 \in K$ such that*

$$\Phi(x, y_0) < 0 \quad \forall x \in C \setminus K;$$

4. *Φ is upper hemicontinuous in its first variable on K;*
5. *Φ is lower semicontinuous in its second variable on K.*

Then, the equilibrium problem (EP) *has a solution.*

Although the fundamental role of the equilibrium problem is to unify different abstract and practice problems in a common way in order to study them, we provide here the following example of an equilibrium bifunction defined on a real Banach space. This example is constructed to emphasize the importance of Theorem 3.1, where the involved bifunction is not upper semicontinuous in its first variable on the whole space. Note that the compact set K used here, and in Theorem 3.1 and Theorem 3.2, is called in the literature the set of coerciveness.

Example 3.1 Let $(E, \|.\|)$ be a real Banach space and take K a compact subset of X such that $0 \in K \subset B(0, 1)$. Define $\Phi : E \times E \rightrightarrows \mathbb{R}$ by

$$\Phi(x, y) = \begin{cases} \dfrac{\|y\|^2 - \|x\|^2}{2}, & \text{if } \|x\| = 2, \\ \|y\|^2 - \|x\|^2, & \text{otherwise.} \end{cases}$$

Clearly, all the conditions of Theorem 3.1 are satisfied with $y_0 = 0$. To show that Φ is not upper semicontinuous in its first variable on the whole space E, let $y \in E$ be such that $\|y\| > 2$. Let $(x_n)_n$ be a sequence in E converging to $x \in E$ such that $\|x\| = 2$ and $\|x_n\| \neq 2$, for every n. Clearly,

$$\limsup_{n \to +\infty} \Phi(x_n, y) = \|y\|^2 - 4 > \frac{\|y\|^2 - 4}{2} = \Phi(2, y).$$

Then, ϕ is not upper semicontinuous in its variable at any $x \in E$ such that $\|x\| = 2$.

Now, we give an application of our techniques on the equilibrium problem developed above, and especially Theorem 3.2, to the special case of nonlinear variational inequalities. In this example, the operator L is not necessarily hemicontinuous on the whole space.

Consider the special case of *a nonlinear variational inequality* of the form

$$\text{find } x^* \in C \text{ such that } \langle Lx^*, y - x^* \rangle \geq 0 \quad \forall y \in C, \tag{VI}$$

where C is a nonempty convex subset of a real Banach space $(E, \|.\|)$, E^* the dual of E, $L : C \to E^*$ is an operator and $\langle , \rangle$ denotes the duality pairing between E^* and E.

Clearly, $x^* \in C$ is a solution of the variational inequality (VI) if and only if x^* is a solution of the equilibrium problem (EP) with the bifunction $\Phi_L : C \times C \to \mathbb{R}$ defined by

$$\Phi_L(x, y) = \langle Lx, y - x \rangle.$$

The bifunction Φ_L is linear and continue in its second variable on C endowed with the weak topology. However, the upper semicontinuity of Φ_L in its first variable is too strong in many applications since L can be chosen only hemicontinuous. Clearly, Φ_L is hemicontinuous in its first variable on a subset S contained in C whenever L is hemicontinuous on S.

Also, the operator L is said to be pseudo-monotone on C iff whenever $x, y \in C$, we have

$$\langle Lx, y - x\rangle \geq 0 \Longrightarrow \langle Ly, y - x\rangle \leq 0,$$

which is equivalent to the pseudo-monotonicity of the bifunction Φ_L.

Finally, a notion of coerciveness for operators exists in the literature, which generalizes that for bilinear forms on Hilbert spaces. The operator L is said to be coercive on C iff there exists $y_0 \in C$ such that

$$\lim_{\substack{\|x\| \to +\infty \\ x \in C}} \frac{\langle Lx, x - y_0\rangle}{\|x\|} = +\infty.$$

It is not hard to see that if L is coercive on C, then there exists $R > 0$ such that $y_0 \in \overline{B}(0, R)$ and

$$\langle Lx, y_0 - x\rangle < 0 \quad \forall x \in C \setminus \overline{B}(0, R),$$

where $\overline{B}(0, R) = \{x \in E : \|x\| \leq R\}$ is the closed ball around 0 with radius R. We put $K_R = \overline{B}(0, R)$ and call (y_0, K_R) an adapted couple of coerciveness of L (which may not be unique).

Proposition 3.1 *Let E be a real reflexive Banach space, C be a nonempty, closed and convex subset of E and $L : C \to E^*$ be an operator. Assume that*

1. *L is pseudo-monotone on C;*
2. *L is coercive on C and let (y_0, K_R) be an adapted couple of coerciveness of L;*
3. *L is hemicontinuous on K_R.*

Then, the variational inequality (VI) *has a solution.*

Proof Consider the space E endowed with the weak topology and take Φ_L the bifunction defined above. Since K_R is weakly compact, the result holds by applying Theorem 3.2. □

Now, we continue developing our techniques on the equilibrium problem. We will be interested in the Minty lemma for the equilibrium problem, which deals in particular with properties such as compactness and convexity of the sets of solutions. We will see in particular that the set of solutions in Proposition 3.1 is nonempty, weakly compact and convex, since K_R is weakly compact and convex.

In the sequel, for $y \in C$, we define the following sets:

$$\Phi^+(y) = \{x \in C : \Phi(x, y) \geq 0\} \quad \text{and} \quad \Phi^-(y) = \{x \in C : \Phi(y, x) \leq 0\}.$$

Clearly, $x^* \in C$ is a solution of the equilibrium problem (EP) if and only if $x^* \in \bigcap_{y \in C} \Phi^+(y)$.

Under assumptions of Theorem 3.2, we obtain that the set of solutions Sol (Φ, C) of the equilibrium problem (EP) is nonempty and

$$\mathrm{Sol}\,(\Phi, C) = \bigcap_{y\in C} \Phi^{+}(y) \subset \bigcap_{y\in C} \mathrm{cl}\left(\Phi^{+}(y)\right) \subset K,$$

and there exists equality under assumptions of Theorem 3.1. We remark that the set $\bigcap_{y\in C} \mathrm{cl}\left(\Phi^{+}(y)\right)$ is compact. Also, by the pseudo-monotonicity of Φ on C, we have

$$\Phi^{+}(y) \subset \Phi^{-}(y) \quad \forall y \in C,$$

and by the explicit quasi-convexity of Φ in its second variable on C and the hemicontinuity in the first variable on K, we prove that

$$\left(\bigcap_{y\in C} \Phi^{-}(y)\right) \cap K \subset \bigcap_{y\in C} \Phi^{+}(y).$$

The quasi-convexity of Φ in its second variable on C yields that the set $\Phi^{-}(y)$ is convex, for every $y \in C$.

Theorem 3.3 *Assume that the hypotheses of Theorem 3.1 or Theorem 3.2 hold. Then, the set of solutions Sol* (Φ, C) *of the equilibrium problem* (EP) *is a nonempty set. If in addition,*

1. *K is convex;*
2. *Φ is pseudo-monotone on C;*
3. *Φ is semistrictly quasi-convex in its second variable on C,*

then Sol (Φ, C) *is nonempty, compact and convex.*

Proof It results from the explanation above that

$$\mathrm{Sol}\,(\Phi, C) = \left(\bigcap_{y\in C} \Phi^{-}(y)\right) \cap K.$$

This completes the proof. □

Now, we are in position to formulate our existence results for the quasi-equilibrium problem. As mentioned above, we are following here the approach which has already been considered and intensively investigated in [4].

Clearly, a point $x^* \in C$ is a solution of the quasi-equilibrium problem (QEP) if and only if x^* is a fixed point of the set-valued mapping $S : C \rightrightarrows C$ defined by

$$S(x) = \{z \in A(x) : \Phi(z, y) \geq 0 \quad \forall y \in A(z)\},$$

and called in the literature, the selection set-valued mapping.

Theorem 3.4 *Let C be a nonempty subset of a real Banach space E and $A : C \rightrightarrows C$ be a set-valued mapping with nonempty, closed and convex values. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction and suppose that for every $x \in C$, there exists a nonempty compact and convex subset K_x of $A(x)$ and $y_x \in K_x$ such that $\Phi_{|A(x)}$ satisfies the hypotheses in Theorem 3.3 with K_x the set of coerciveness. Then, the selection set-valued mapping S has nonempty, compact and convex values.*

Proof It suffices to apply Theorem 3.3 to $\Phi_{|A(x)}$, for every $x \in C$. □

Now, we formulate an existence result for the quasi-equilibrium problem by applying the Kakutani fixed point theorem.

Theorem 3.5 *Under the assumptions of Theorem 3.4, we suppose further that there exists a nonempty, closed and convex subset C_0 of C such that*

1. *$S(C_0)$ is a relatively compact subset of C_0;*
2. *grph $\left(S_{|C_0}\right)$ is closed in $C_0 \times C_0$.*

Then, the quasi-equilibrium problem (QEP) *has a solution.*

Proof Put $K = \text{clconv}\,(\text{cl}\,(S(C_0)))$, the closed convex hull of $S(C_0)$. Clearly, K is a nonempty, compact and convex subset of C_0, $S(K) \subset K$, grph $\left(S_{|K}\right)$ is closed in $K \times K$ and $S_{|K}$ has nonempty, closed and convex values. Then, $S_{|K}$ is a Kakutani mapping. That is, $S_{|K}$ is upper semicontinuous and has nonempty, compact and convex values. Then, by applying the Kakutani fixed point theorem (see [20,21]), $S_{|K}$ has a fixed point $x^* \in K$, which is a solution of the quasi-equilibrium problem (QEP). □

We note that the conditions in Theorem 3.5 involve the selection set-valued mapping itself, which is not in the initial data of the quasi-equilibrium problem (QEP). Now, we provide assumptions only on the involved data of the quasi-equilibrium problem (QEP) such that the conditions in Theorem 3.5 will be satisfied.

Theorem 3.6 *Under the hypotheses of Theorem 3.4, we assume further that for $C_0 := \text{clconv}\left(\bigcup_{x \in C} K_x\right)$, the following conditions hold:*

1. *C_0 is a compact subset of C;*
2. *$(A \cap C_0)_{|C_0}$ is upper semicontinuous;*
3. *$\Phi_{|C_0}$ is upper semicontinuous on $C_0 \times C_0$;*
4. *for every converging sequence $(x_n)_n$ in C_0 to x and for every $y \in A(x)$, there exists a sequence $(y_n)_n$ converging to y such that $y_n \in A(x_n) \cap C_0$, for every n.*

Then, the equilibrium problem (QEP) *has a solution.*

Proof The set C_0 is a nonempty compact and convex subset of C. Since for every $x \in C$, $S(x) \subset K_x$, $S(C_0)$ is contained in C_0. In order to apply Theorem 3.5, it remains to prove that grph $\left(S_{|C_0}\right)$ is closed in $C_0 \times C_0$. To do this, take a sequence $(x_n, z_n)_n$ in $C_0 \times C_0$ converging in $C_0 \times C_0$ to (x, z) such that $z_n \in S(x_n)$, for every n. We will prove that $z \in S(x)$. We have $z_n \in A(x_n)$ for every n, and

$$\Phi(z_n, y) \geq 0 \quad \forall y \in A(x_n).$$

Since A has closed values, then by the upper semicontinuity of $(A \cap C_0)_{|C_0}$, we have $z \in A(x)$. Now, let $y \in A(x)$ and let $(y_n)_n$ be a converging sequence in C_0 to y such that $y_n \in A(x_n) \cap C_0$, for every n. Then, by the upper semicontinuity of $\Phi_{|C_0}$, we have

$$\Phi(z, y) \geq \limsup_{n \to +\infty} \Phi(z_n, y_n) \geq 0.$$

Since y is arbitrary in $A(x)$, we conclude that $z \in S(x)$, which completes the proof. □

Remark 3.1 If we take into account the recent advancement in the rich area of fixed point theory, we point out that all our existence results for the quasi-equilibrium problem, obtained above, remain true in the case of complete Hausdorff locally convex vector topological spaces instead of real Banach spaces, see [20,21].

4 Existence of Solutions of the Quasi-Equilibrium Problem: A Selection Theory Approach

In this section, we deal with the existence of solutions of the quasi-equilibrium problem by following a selection theory approach. This direction has already been considered in [7] in the setting of finite dimensional spaces and developed in [10] for separable Banach spaces.

One of the most known and important results in the selection theory area is the Michael selection theorem, which states that every lower semicontinuous set-valued mapping from a paracompact Hausdorff topological space X with nonempty, closed and convex values in a Banach space has a continuous selection. Motivated by the problem of extending continuous functions defined on closed subsets, E. Michael obtained in his paper [22] characterizations of various kinds of topological properties such as paracompactness, normality, collectionwise normality and perfect normality by means of existence of continuous selections of lower semicontinuous set-valued mappings with values in Banach spaces. Every metric space is both paracompact and perfectly normal, and both these two properties are stronger than collectionwise normality.

In our study, the quasi-equilibrium problem (QEP) will be considered in a real Banach space, and instead of the Michael selection theorem for perfectly normal spaces considered in the above-mentioned papers, we use here the Michael selection theorem for paracompact Hausdorff topological spaces, which is also the more suitable theorem in many analysis studies. The perfectly normal version is more restrictive since it requires separable Banach spaces and imposes that the involved set-valued mapping must have values in the family of convex subsets containing the inside points of their closures, see [7,10,22] for more details.

Beside the existence of continuous selections of lower semicontinuous set-valued mappings, there is the notion of selectionable set-valued mappings, which will be important in our purpose. This notion will be also interesting since it will prevent us to repeat the proofs of some known facts of the selection theory.

Let X and Y be two Hausdorff topological spaces. Following [23], a set-valued mapping $F : X \rightrightarrows Y$ is said to be *locally selectionable* at a point $x_0 \in X$ iff for every $y_0 \in F(x_0)$, there exist an open neighborhood U_{x_0} of x_0 and a continuous function $f_{x_0} : U_{x_0} \to Y$ such that $f_{x_0}(x_0) = y_0$ and

$$f_{x_0}(x) \in F(x) \quad \forall x \in U_{x_0}.$$

The set-valued mapping F is said to be locally selectionable on X iff it is locally selectionable at every point of X.

We can consult [23] to see that every locally selectionable set-valued mapping on X is lower semicontinuous on X. Also, every locally selectionable set-valued mapping on a paracompact Hausdorff topological space with nonempty convex values in a topological Hausdorff vector space has a continuous selection; see [23, Proposition 10.2].

Now, we formulate the following result on the existence of solutions of the quasi-equilibrium problem.

Theorem 4.1 *Let C be a nonempty subset of a real Banach space E, $\Phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $A : C \rightrightarrows C$ be a set-valued mapping. Suppose further that there exist a nonempty, closed and convex subset C_0 of C and a compact subset K of C_0 such that the following conditions hold:*

1. *$A_{|C_0}$ is lower semicontinuous on C_0 and has nonempty, closed and convex values in K;*
2. *fix $\left(A_{|C_0}\right)$ is nonempty closed subset, and $\Phi(x, x) = 0$, for every $x \in$ fix $\left(A_{|C_0}\right)$;*
3. *the restriction of Φ on fix $\left(A_{|C_0}\right) \times C$ is quasi-convex in its second variable;*
4. *the restriction of Φ on fix $\left(A_{|C_0}\right) \times C$ is upper semicontinuous.*

Then, the equilibrium problem (QEP) *has a solution.*

Proof Define the set-valued mapping $F : \text{fix}\left(A_{|C_0}\right) \rightrightarrows C$ by

$$F(x) := \{y \in C : \Phi(x, y) < 0\}.$$

Clearly, F has convex values, and by the upper semicontinuity of the restriction of Φ on fix $\left(A_{|C_0}\right) \times C$, the graph of F is open in fix $\left(A_{|C_0}\right) \times C$.

Now, consider the set-valued mapping $G = K \cap F : \text{fix}\left(A_{|C_0}\right) \rightrightarrows C$ defined by

$$G(x) := A(x) \cap F(x).$$

The restriction of A on fix $\left(A_{|C_0}\right)$ being a lower semicontinuous set-valued mapping from the paracompact Hausdorff topological space fix $\left(A_{|C_0}\right)$ to the real Banach space E with nonempty, closed and convex values, then by the Michael selection theorem, for every $x_0 \in$ fix $\left(A_{|C_0}\right)$ and for every $y_0 \in A(x_0)$, there exists a continuous selection f_{x_0} of $A_{|\text{fix}(A_{|C_0})}$ such that $f_{x_0}(x_0) = y_0$, see [23, Corollary 11.1]. That is, $A_{|\text{fix}(A_{|C_0})}$ is locally selectionable set-valued mapping at every point of fix $\left(A_{|C_0}\right)$. Since F has an open graph in fix $\left(A_{|C_0}\right) \times C$, it results by [23, Proposition 10.4] that if $x_0 \in$ fix $\left(A_{|C_0}\right)$ such that $G(x_0) \neq \emptyset$, then G is locally selectionable set-valued mapping at x_0.

We claim that there exists $x_0 \in \operatorname{fix}\left(A_{|C_0}\right)$ such that $G(x_0) = \emptyset$, which proves that x_0 is a solution of the quasi-equilibrium problem (QEP). Assume by contradiction that $G(x_0) \neq \emptyset$, for every $x \in \operatorname{fix}\left(A_{|C_0}\right)$. It results that G is locally selectionable set-valued mapping with nonempty convex values from the paracompact Hausdorff topological space $\operatorname{fix}\left(A_{|C_0}\right)$ to the real Banach space E. Then, by [23, Proposition 10.2], G has a continuous selection g. Define the set-valued mapping $H : C_0 \rightrightarrows E$ by

$$H(x) := \begin{cases} \{g(x)\}, & \text{if } x \in \operatorname{fix}\left(A_{|C_0}\right), \\ A(x), & \text{if } x \notin \operatorname{fix}\left(A_{|C_0}\right). \end{cases}$$

The set-valued mapping H is lower semicontinuous on C_0. Indeed, let $x_0 \in C_0$ and V be an open subset of E such that $H(x_0) \cap V \neq \emptyset$. If $x_0 \notin \operatorname{fix}\left(A_{|C_0}\right)$, by the lower semicontinuity of A, let U be an open neighborhood of x_0 such that

$$U \cap \operatorname{fix}\left(A_{|C_0}\right) = \emptyset \quad \text{and} \quad A(x) \cap V \neq \emptyset \quad \forall x \in U.$$

Then, $H(x) \cap V \neq \emptyset$, for every $x \in U$. Otherwise, suppose that $x_0 \in \operatorname{fix}\left(A_{|C_0}\right)$. Then, by continuity of g on $\operatorname{fix}\left(A_{|C_0}\right)$, let U_1 be an open neighborhood of x_0 in C_0 such that

$$g(x) \in V \quad \forall x \in U_1 \cap \operatorname{fix}\left(A_{|C_0}\right).$$

On the other hand, by lower semicontinuity of A on C_0, let U_2 be an open neighborhood of x_0 in C_0 such that

$$A(x) \cap V \neq \emptyset \quad \forall x \in U_2.$$

Clearly, $H(x) \cap V \neq \emptyset$, for every $x \in U_1 \cap U_2$. Hence, H is lower semicontinuous at x_0. Now, by applying the Michael selection theorem, the set-valued mapping H has a continuous selection f. Since $H(C_0) \subset A(C_0) \subset K$, $f : C_0 \to C_0$ is a compact mapping, and it follows from the Schauder fixed point theorem (see [20, Theorem 4.14]) that f has a fixed point. This means that there exists $x \in C_0$ such that $x = f(x) \in A(x)$. Therefore, $x \in \operatorname{fix}\left(A_{|C_0}\right)$, and then $x \in G(x) \subset F(x)$. It follows that $\Phi(x, x) < 0$, which yields a contradiction and completes the proof. □

Remark 4.1 As stated in [10], in the proof of Theorem 4.1, we remark that the condition of the restriction of Φ on $\operatorname{fix}\left(A_{|C_0}\right) \times C$ is quasi-convex in its second variable that can be replaced by the following weaker condition: the set

$$F(x) := \{y \in C : \Phi(x, y) < 0\}$$

is convex, for every $x \in \operatorname{fix}\left(A_{|C_0}\right)$.

As in the previous section, let us now give some conditions on the initial data for which the conditions in Theorem 4.1 are satisfied. We remark that all the conditions involve only the initial data of the quasi-equilibrium problem (QEP), the unique point which maybe need to be discussed is that about the fixed points set of the set-valued mapping $A_{|C_0}$. By a similar statement involving the Michael selection theorem and the Schauder fixed point theorem, as in the proof of the theorem above, we give here the following sufficient conditions under which the fixed points set of the set-valued mapping A is nonempty and closed. Note that every set-valued mapping with closed graph has closed values. The converse is true under additional conditions such as the upper semicontinuity.

Proposition 4.1 *Let C be a nonempty subset of a real Banach space E and $A : C \rightrightarrows C$ be a set-valued mapping. Suppose further that there exist a nonempty, closed and convex subset C_0 of C and a compact subset K of C_0 such that the following conditions hold:*

1. *$A_{|C_0} : C_0 \rightrightarrows C$ is lower semicontinuous;*
2. *$A_{|C_0} : C_0 \rightrightarrows C$ has nonempty, closed and convex values in K;*
3. *the graph of $A_{|C_0}$ is closed in $C_0 \times C$.*

Then, fix $\left(A_{|C_0}\right)$ is a nonempty, closed and compact set.

5 Existence of Approximate Solutions of the Quasi-Equilibrium Problem

Like approximate selections, approximate solutions are well-known and important tools, which have already been used in quasi-variational inequality studies and in many other areas of nonlinear analysis. Such notions have been also used recently for the quasi-equilibrium problem, see, for instance, [7–9] and the references therein.

In the sequel, for $\varepsilon > 0$ and a set-valued mapping $F : X \rightrightarrows Y$, we denote by $F_\varepsilon : X \rightrightarrows Y$ the set-valued mapping defined by

$$F_\varepsilon (x) := B (F (x), \varepsilon).$$

For $\varepsilon > 0$, we call in what follows an ε-solution of the quasi-equilibrium problem (QEP), any $x_\varepsilon \in \mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$ such that

$$\Phi (x_\varepsilon, y) \geq 0 \quad \forall y \in A_\varepsilon (x_\varepsilon) \cap C,$$

where the closure is taken with respect to the subset C. An approximate solution of the quasi-equilibrium problem (QEP) is any ε-solution of the quasi-equilibrium problem (QEP), for any $\varepsilon > 0$.

We remark that the set-valued mapping A_ε has open values. Then, the techniques developed in the previous sections fail to be applied to A_ε.

Now, we present the notion of sub-lower semicontinuity considered in [24] in the realm of topological vector spaces, see also [25,26] for other notions such as almost lower semicontinuity and quasi lower semicontinuity. All these notions are weaker than that of lower semicontinuity and fit very well with the notion of approximate continuous selections. However, the notion of sub-lower semicontinuity seems to be more adapted to our purpose.

Let X be a Hausdorff topological space and Y be a normed vector space. A set-valued mapping $F : X \rightrightarrows Y$ is said to be sub-lower semicontinuous at $x \in X$ iff for every $\varepsilon > 0$, there exist $z_x \in F(x)$ and a neighborhood U_x of x such that

$$z_x \in F_\varepsilon\left(x'\right) \quad \forall x' \in U_x.$$

The set-valued mapping F is said to be sub-lower semicontinuous on X iff it is sub-lower semicontinuous at every point of X.

In the sequel, we will make use of the following result where the proof holds from classical arguments, see [8,22–25]. It provides us with a localization of the continuous selections of sub-lower semicontinuous set-valued mappings.

Lemma 5.1 *Let X be a paracompact Hausdorff topological space, Y be a normed vector space, S be a convex subset of Y, $F : X \rightrightarrows Y$ be a set-valued mapping and $\varepsilon > 0$. Suppose that for every $x \in X$, there exist $z_x \in F(x)$ and an open neighborhood U_x of x such that*

$$z_x \in F_\varepsilon\left(x'\right) \cap S \quad \forall x' \in U_x.$$

Then, there exists a continuous selection $f : X \to S$ of F_ε.

Proof For every $x \in X$, let U_x be an open neighborhood of x and $z_x \in F(x)$ such that $z_x \in F_\varepsilon\left(x'\right) \cap S$, for every $x' \in U_x$. Let $(O_i)_{i\in I}$ be an open refinement of the open cover $(U_x)_{x\in X}$ of the paracompact Hausdorff topological space X, and let $(p_i)_{i\in I}$ be a partition of unity subordinated to $(O_i)_{i\in I}$. For every $i \in I$, take $x_i \in X$ such that $O_i \subset U_{x_i}$, and define the function $f : X \to Y$ by

$$f(x) = \sum_{i\in I} p_i(x)\, z_{x_i}.$$

Then f is continuous since it is locally a finite sum of continuous functions. For every $i \in I$ such that $p_i(x) \neq 0$, we have $x \in U_{x_i}$, and then $z_{x_i} \in F_\varepsilon(x)$. By the convexity of $F(x)$, $F_\varepsilon(x)$ is also convex, and then $f(x) \in F_\varepsilon(x)$. Also, since S is convex and $z_{x_i} \in S$ for every $i \in I$, $f(x) \in S$, for every $x \in X$. □

An adaptation of the proof of the above lemma to our purpose yields the following important tool for the existence of approximate solutions of the quasi-equilibrium problem. This result is presented for sub-lower semicontinuous and can be compared to [8, Lemma 2.1] and [9, Theorem 2.3].

Lemma 5.2 *Let X be a paracompact Hausdorff topological space, Y be a normed vector space, S be a convex subset of Y, $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty convex values in S, $\Psi : X \times S \to \mathbb{R}$ a bifunction, $\varepsilon > 0$ and $\alpha \in \mathbb{R}$. We define*

$$B_{\Psi,\alpha}(x) = \{y \in S : \Psi(x, y) < \alpha\},$$

and suppose that for every $x \in X$, the following conditions hold:

1. *the set* $F_\varepsilon(x) \cap B_{\Psi,\alpha}(x)$ *is nonempty and convex;*
2. *there exist* $z_x \in F(x)$ *and an open neighborhood* U_x *of* x *such that*

$$z_x \in F_\varepsilon\left(x'\right) \cap B_{\Psi,\alpha}\left(x'\right) \quad \forall x' \in U_x.$$

Then, there exists a continuous selection $f_\varepsilon : X \to S$ *of* F_ε *such that* $\Psi(x, f_\varepsilon(x)) < \alpha$*, for every* $x \in X$.

Proof For every $x \in X$, let $z_x \in F(x)$ and take U_x defined by condition (2). By proceeding as in Lemma 5.1, the convexity of $F_\varepsilon(x) \cap B_{\Psi,\alpha}(x)$ for every $x \in X$, yields a continuous selection $f_\varepsilon : X \to S$ of the set-valued mapping $H_\varepsilon : X \rightrightarrows Y$ defined by

$$H_\varepsilon(x) = F_\varepsilon(x) \cap B_{\Psi,\alpha}(x).$$

Thus, $f_\varepsilon(x) \in F_\varepsilon(x)$ and $\Psi(x, f_\varepsilon(x)) < \alpha$, for every $x \in X$. □

Remark 5.1 While the convexity of $B_{\Psi,\alpha}(x)$ in the above lemma requires conditions only on Ψ and it is satisfied if Ψ is quasi-convex in its second variable on X, the other conditions seem to be more complicated and require connections between Ψ and F.

Here, we give the following result which provides sufficient conditions involving Ψ and F in order to satisfy hypotheses (1) and (2) of the above lemma.

Proposition 5.1 *Let* X *be a paracompact Hausdorff topological space,* Y *be a normed vector space,* S *be a convex subset of* Y*,* $F : X \rightrightarrows Y$ *be a set-valued mapping with nonempty convex values in* S*,* $\Psi : X \times S \to \mathbb{R}$ *be a bifunction,* $\varepsilon > 0$ *and* $\alpha \in \mathbb{R}$.

1. *If* Ψ *is quasi-convex in its second variable on* X*, then for every* $x \in X$*,* $B_{\Psi,\alpha}(x)$ *is convex.*
2. *If* $\inf_{y \in F_\varepsilon(x)} \Psi(x, y) < \alpha$ *for some* $x \in X$*, then* $F_\varepsilon(x) \cap B_{\Psi,\alpha}(x) \neq \emptyset$.
3. *If one of the following two conditions holds:*
   (a) F *is lower semicontinuous on* X *and* Ψ *is upper semicontinuous in its first variable on* X *and* $F(x) \cap B_{\Psi,\alpha}(x) \neq \emptyset$*, for every* $x \in X$*;*
   (b) F *is sub-lower semicontinuous on* X *and* Ψ *is upper semicontinuous in its first variable on* X *and* $F(x) \subset B_{\Psi,\alpha}(x)$*, for every* $x \in X$*,*

   then condition (2) of Lemma 5.2 is satisfied.

Proof We verify only the last condition, the other conditions being obvious or already discussed. Let $x \in X$.

In the case where condition (3a) is satisfied, F is lower semicontinuous. Let $z_x \in F(x) \cap B_{\Psi,\alpha}(x)$, and by lower semicontinuity of F, let U_x^1 be an open neighborhood of x such that $F\left(x'\right) \cap B(z_x, \varepsilon) \neq \emptyset$, for every $x' \in U_x^1$. By upper semicontinuity of Ψ in its first variable, let U_x^2 be an open neighborhood of x such that $z_x \in B_{\Psi,\alpha}\left(x'\right)$, for every $x' \in U_x^2$. Clearly, for every $x' \in U_x = U_x^1 \cap U_x^2$, $z_x \in F_\varepsilon\left(x'\right) \cap B_{\Psi,\alpha}\left(x'\right)$.

In the case where condition (3b) is satisfied, F is sub-lower semicontinuous. Let $z_x \in F(x)$ and U_1^x be as in the definition of sub-lower semicontinuity. Since

$z_x \in B_{\Psi,\alpha}(x)$, we choose by upper semicontinuity of Ψ in its first variable, an open neighborhood U_x^2 of x such that $z_x \in B_{\Psi,\alpha}\left(x'\right)$, for every $x' \in U_x^2$. As above, the result comes by taking $U_x = U_x^1 \cap U_x^2$. □

In [9], the nonemptiness of the fixed points set of A_ε, which is crucial for the existence of approximate solutions of the quasi-equilibrium problem (QEP), has been obtained by applying the Fan–Browder fixed point theorem since the lower semicontinuity of A implies that the set-valued mapping A_ε has open fibers. It can be showed as follows. For $y \in C$, we have

$$A_\varepsilon^{-1}(y) = \{x \in C : y \in B(A(x), \varepsilon)\} = \{x \in C : A(x) \cap B(y, \varepsilon) \neq \emptyset\} \\ = A^{-1}(B(y, \varepsilon)).$$

We remark that this fact has been used only to prove the existence of fixed points of A_ε. The existence of a fixed point of any selection of A_ε will suffice to overcome the strong condition of the openness of the fibers of A_ε.

Now, we present an existence result of approximate solutions of the quasi-equilibrium problem (QEP) in the case of sub-lower semicontinuous set-valued mappings.

Theorem 5.1 *Let C be a nonempty, closed and convex subset of a real Banach space E, $A : C \rightrightarrows C$ be a set-valued mapping, and $\Phi : C \times C \to \mathbb{R}$ be a bifunction. Suppose further that the following conditions hold:*

1. *A is sub-lower semicontinuous on C;*
2. *there exists a compact subset K of C such that A has nonempty convex values in K.*

Then, for every $\varepsilon > 0$, the set-valued mapping $A_\varepsilon : C \rightrightarrows C$ has a nonempty fixed points set.

In addition, we assume that the following hypotheses are fulfilled:

1. *Φ is quasi-convex in its second variable;*
2. *Φ is upper semicontinuous in its first variable on C;*
3. *there exists $\varepsilon_0 > 0$ such that $\Phi(x, x) \geq 0$, for every $x \in B(A(x), \varepsilon_0) \cap C$;*
4. *for every $0 < \varepsilon < \varepsilon_0$,*
   (a) the function defined on cl (fix $(A_\varepsilon \cap C)$) by

$$x \mapsto \inf_{y \in A_\varepsilon(x) \cap C} \Phi(x, y)$$

   attains its supremum γ_ε on cl (fix $(A_\varepsilon \cap C)$) and this supremum is finite;
   (b) $A(x) \subset B_{\Phi, \gamma_\varepsilon + \frac{1}{n}}(x)$, for every $x \in C$ and $n \in \mathbb{N}^*$.

Then, for every $0 < \varepsilon < \varepsilon_0$, the quasi-equilibrium problem (QEP) *has an ε-solution.*

Proof Let $\varepsilon > 0$. The set-valued mapping A_ε has a nonempty fixed points set. Indeed, put $K_0 = \text{clconv}(K)$, which is a nonempty, compact and convex subset of C, and $A(C) \subset K_0$. The set-valued mapping $A : C \rightrightarrows E$ is sub-lower semicontinuous and has nonempty convex values in the convex subset K_0. Thus, by Lemma 5.1, we consider a continuous selection $f_\varepsilon : C \to K_0$ of A_ε. By the Schauder fixed point theorem, f_ε has a fixed point x_ε^*, which is necessarily in K_0. Thus, $\mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$ is nonempty. Note that the set $\mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$ is contained in K_0, then it is compact.

Now, for $0 < \varepsilon < \varepsilon_0$, let $x_\varepsilon \in \mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$ be such that

$$\sup_{x \in \mathrm{cl}(\mathrm{fix}(A_\varepsilon \cap C))} \inf_{y \in A_\varepsilon(x) \cap C} \Phi\,(x, y) = \inf_{y \in A_\varepsilon(x_\varepsilon) \cap C} \Phi\,(x_\varepsilon, y) = \gamma_\varepsilon.$$

Put $\alpha_{\varepsilon,n} = \gamma_\varepsilon + \frac{1}{n}$, for $n \in \mathbb{N}^*$.

By taking $X = S = K_0$ and $Y = E$, it follows by Lemma 5.2 applied to A and Φ that there exists a continuous selection $f_\varepsilon : K_0 \to K_0$ of A_ε such that

$$\Phi\,(x, f_\varepsilon\,(x)) < \alpha_{\varepsilon,n} \quad \forall x \in K_0.$$

Again by the Schauder fixed point theorem, let $\overline{x}_\varepsilon \in K_0$ be a fixed point of f_ε. That is,

$$\overline{x}_\varepsilon = f_\varepsilon\,(\overline{x}_\varepsilon) \in A_\varepsilon\,(\overline{x}_\varepsilon) \cap K_0 \subset B\,(A\,(x_\varepsilon)\,, \varepsilon_0) \cap C.$$

Therefore

$$0 \le \Phi\,(\overline{x}_\varepsilon, \overline{x}_\varepsilon) = \Phi\,(\overline{x}_\varepsilon, f_\varepsilon\,(\overline{x}_\varepsilon)) < \alpha_{\varepsilon,n} = \gamma_\varepsilon + \frac{1}{n}.$$

By letting $n \to +\infty$, we obtain that $\inf\limits_{y \in A_\varepsilon(x_\varepsilon) \cap C} \Phi\,(x_\varepsilon, y) = \gamma_\varepsilon \ge 0$. It follows that we have $x_\varepsilon \in \mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$ and

$$\Phi\,(x_\varepsilon, y) \ge 0 \quad \forall y \in A_\varepsilon\,(x_\varepsilon) \cap C,$$

which states that the quasi-equilibrium problem (QEP) has an ε-solution and completes the proof. □

Remark 5.2 Let us point out that the function $x \mapsto \inf_{y \in A_\varepsilon(x) \cap C} \Phi\,(x, y)$ defined on the set $\mathrm{cl}\,(\mathrm{fix}\,(A_\varepsilon \cap C))$, in the above theorem, is supposed to have a finite supremum. It is well known that the Berge maximum theorem is an important tool usually used to deal with such properties when the set-valued A is lower semicontinuous. Unfortunately, and even if A is lower semicontinuous and the above function is proper, nothing can guarantee that its supremum is finite if no additional conditions on Φ and on the values of A are assumed.

6 Conclusions

The equilibrium problem and, by consequent, the quasi-equilibrium problem studied in the paper have been introduced mainly to describe in a unified way various problems arising in nonlinear analysis and in mathematics in general. The family of problems that can be expressed as an equilibrium problem is growing as far as the other related areas are being developed. Recently, it has been proved that quasi-hemivariational inequalities, which constitute an important variational formulation for several classes of mechanical problems, can be also expressed as an equilibrium problem. On the other hand, and as already mentioned, one of the interests of such a unified formulation is that many techniques and methods developed for solving a special case may be adapted, with suitable modification, to the other special cases. Motivated by these facts, it has been proved in some recent works that the techniques on weakening semicontinuity and hemicontinuity to the set of coerciveness developed to the equilibrium problem can be applied to various special cases such as quasi-hemivariational inequalities and can be used with other techniques such as the Ekeland variational principle. These techniques have been also highlighted here by an example and an application to nonlinear variational inequalities.

In this direction, we have been concerned here in the paper with the quasi-equilibrium problem, which constitutes a relevant mathematical formulation including the equilibrium problem and other concepts such as quasi-variational inequalities. We remark that in the approach based on fixed point theory developed in the paper, our techniques on weakening semicontinuity and hemicontinuity are applied easily and directly to the quasi-equilibrium problem. And because of our conviction of always looking for optimal conditions when dealing with such problems, we have also considered the approach based on selection theory. In such a way, we have been able to obtain results improving some recent properties in the literature. We have been also interested in approximate solutions of the quasi-equilibrium problem and highlighted the necessary background for their existence by using the notion of sub-lower semicontinuous set-valued mappings. This study is motivated by the importance of approximate solutions in general in many areas of mathematics, but also by some recent works on approximate solutions of the quasi-equilibrium problem and its special cases.

The techniques developed in the two approaches based on fixed point theory and on selection theory as well as those developed for approximate solutions are given under general settings. In such a way, they can be easily applied to several particular cases.

Finally, we point out that this subject is under perpetual advancement, and it may be also interesting to look for weakened conditions on convexity when dealing with existence of solutions and approximate solutions of the quasi-equilibrium problem. The convergence of the sequence of approximate solutions of the quasi-equilibrium problem is also a challenge which has to be considered in the future.

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