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Sublinear eigenvalue problems on compact Riemannian manifolds with applications in Emden–Fowler equations

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Abstract. Let (M,g) be a compact Riemannian manifold without boundary, with $\dim M \geq 3$, and $f : \mathbb{R} \to \mathbb{R}$ a continuous function which is sublinear at infinity. By various variational approaches, existence of multiple solutions of the eigenvalue problem

$$-\Delta_g \omega + \alpha(\sigma)\omega = \tilde{K}(\lambda,\sigma) f(\omega), \qquad \sigma \in M,\ \omega \in H_1^2(M),$$

is established for certain eigenvalues $\lambda > 0$, depending on further properties of f and on explicit forms of the function $\tilde{K}$. Here, Δ_g stands for the Laplace–Beltrami operator on (M,g), and α, $\tilde{K}$ are smooth positive functions. These multiplicity results are then applied to solve Emden–Fowler equations which involve sublinear terms at infinity.

1. Introduction and statement of main results. Let us consider the following parametrized Emden–Fowler (or Lane–Emden) equation:

$$(EF)_\lambda \qquad -\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, K is smooth on the d-dimensional unit sphere S^d, $d \geq 3$, $s \in \mathbb{R}$, and $\lambda > 0$ is a parameter. The equation $(EF)_\lambda$ has been extensively studied in the *pure superlinear* case, i.e., when f has the form $f(t) = |t|^{p-1}t$, $p > 1$ (see Cotsiolis–Iliopoulos [3], Vázquez–Véron [9]). In these papers, the authors obtained existence and multiplicity of solutions for $(EF)_\lambda$, applying either minimization or minimax methods. Note that in the pure superlinear case the presence of the parameter $\lambda > 0$ is not relevant due to the rescaling technique. One of the purposes of the present paper is to guarantee *multiple* solutions of $(EF)_\lambda$ for certain $\lambda > 0$ when the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is

(a) *not* necessarily of *pure power* type, and
(b) *sublinear at infinity* (see (1.2) below).

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The solutions of $(EF)_\lambda$ are being sought in the particular form

$$u(x) = u(|x|, x/|x|) = u(r,\sigma) = r^s\omega(\sigma). \tag{1.1}$$

Here, $(r,\sigma) \in (0,\infty) \times S^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$. This type of transformation is also used by Bidaut-Véron and Véron [2], where the asymptotics of a special form of $(EF)_\lambda$ has been studied. By means of (1.1), the equation $(EF)_\lambda$ reduces to

$$-\Delta_{g_0}\omega + s(1-s-d)\omega = \lambda K(\sigma)f(\omega), \quad \sigma \in S^d, \tag*{$(1.1)_\lambda$}$$

where Δ_{g_0} denotes the Laplace–Beltrami operator on (S^d, g_0) and g_0 is the canonical metric induced from $\mathbb{R}^{d+1}$.

Note in particular that when $s = -d/2$ or $s = -d/2 + 1$, and $f(t) = |t|^{4/(d-2)}t$, the existence of a smooth solution $\omega > 0$ of $(1.1)_\lambda$ can be viewed as an affirmative answer to the famous Yamabe problem on S^d (see also the Nirenberg problem); for these topics we refer the reader to Aubin [1], Cotsiolis–Iliopoulos [4], Hebey [5], and references therein. In these cases the right hand side of $(1.1)_\lambda$ involves the critical Sobolev exponent.

As we pointed out before, our aim is to study $(EF)_\lambda$ (specially, problem $(1.1)_\lambda$) in the *sublinear* case. Since $1 - d < s < 0$ implies the coercivity of the operator $\omega \mapsto -\Delta_{g_0}\omega + s(1-s-d)\omega$, the form of $(1.1)_\lambda$ motivates the study of the following general *eigenvalue problem,* denoted by $(P)_\lambda$, which constitutes the main objective of our paper:

Find $\lambda \in (0,\infty)$ and $\omega \in H_1^2(M)$ such that

$$-\Delta_g\omega + \alpha(\sigma)\omega = \tilde{K}(\lambda,\sigma)f(\omega), \quad \sigma \in M, \tag*{$(\tilde{1.1})_\lambda$}$$

where we assume

- (A_1) *(M, g) is a smooth compact d-dimensional Riemannian manifold without boundary, $d \geq 3$;*
- (A_2) *$\alpha \in C^\infty(M)$ and $\tilde{K} \in C^\infty((0,\infty) \times M)$ are positive functions;*
- (f_1) *$f : \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous and sublinear at infinity, i.e.,*

$$\lim_{|t|\to\infty} \frac{f(t)}{t} = 0. \tag{1.2}$$

A typical case when (1.2) holds is

- $(f_1^{q,c})$ *There exist $q \in (0,1)$ and $c > 0$ such that $|f(t)| \leq c|t|^q$ for every $t \in \mathbb{R}$.*

We simply say that $\omega_\lambda \in H_1^2(M)$ is a solution of $(P)_\lambda$ if $\omega = \omega_\lambda$ satisfies $(\tilde{1.1})_\lambda$. Above, Δ_g is the Laplace–Beltrami operator on (M, g); its expression in local coordinates is $\Delta_g\omega = g^{ij}(\partial_{ij}\omega - \Gamma_{ij}^k\partial_k\omega)$. $H_1^2(M)$ is the usual Sobolev space on M, endowed with its natural norm $\|\cdot\|_{H_1^2}$ (see [5] or Section 2 for details).

The presence of the parameter $\lambda > 0$ in $(P)_\lambda$ is indispensable. Indeed, if we consider a sublinear function at infinity which is, in addition, uniformly Lipschitz (with Lipschitz constant $L > 0$), and $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma)$, with $K \in C^\infty(M)$ positive, one can prove that for $0 < \lambda < (1/L)(\min_M \alpha / \max_M K) =: \lambda_L$ we have only the $\omega = \omega_\lambda = 0$ solution of $(P)_\lambda$, as the standard contraction principle on the Hilbert space $H_1^2(M)$ shows. For a concrete example, let us consider the function $f(t) = \ln(1 + t^2)$ and assume that $K(\sigma)/\alpha(\sigma) = \mathrm{const} = \mu_0 \in (0, \infty)$. Then, for every $0 < \lambda < \min_M \alpha / \mu_0 \max_M \alpha$, problem $(P)_\lambda$ has only the trivial solution; however, when $\lambda > 5/4\mu_0$, problem $(P)_\lambda$ has *three* distinct constant solutions which are precisely the fixed points of the function $t \mapsto \lambda\mu_0 \ln(1 + t^2)$. Note that one of the solutions is the trivial one.

In the generic case, for a fixed $\lambda > 0$, the function $\omega_\lambda(\sigma) = c \in \mathbb{R}$ is a solution of $(P)_\lambda$ if and only if $\alpha(\sigma)c = \tilde{K}(\lambda, \sigma) f(c)$ for a.e. $\sigma \in M$. In particular, when $\omega_\lambda(\sigma) = c \neq 0$, the function $\sigma \mapsto \tilde{K}(\lambda, \sigma)/\alpha(\sigma)$ is constant; let us denote this value by $\mu_\lambda > 0$. Thus, nonzero constant solutions of $(P)_\lambda$ appear as fixed points of the function $t \mapsto \mu_\lambda f(t)$.

In order to obtain multiple solutions of $(P)_\lambda$ not only in the case when $\sigma \mapsto \tilde{K}(\lambda, \sigma)/\alpha(\sigma)$ is constant for certain $\lambda > 0$, we will use variational arguments; weak solutions of $(P)_\lambda$ will be found as critical points of the energy functional associated with $(P)_\lambda$ (see Section 2). Due to (f_1), these elements are actually classical solutions of $(P)_\lambda$.

Our first result concerns the case when $f : \mathbb{R} \to \mathbb{R}$ satisfies

(f_2) $\lim_{t \to 0} f(t)/t = 0$.

Before stating this result, let us note that the usual norm on the space $L^p(M)$ will be denoted by $\|\cdot\|_p$, $p \in [1, \infty]$.

Theorem 1.1. *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* (f_1), (f_2) *and* $\sup_{t \in \mathbb{R}} F(t) > 0$, *where* $F(t) = \int_0^t f(\tau)\, d\tau$. *Assume also that* (A_1) *and* (A_2) *are satisfied with* $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma)$, *and* $K \in C^\infty(M)$ *is positive. Then there exists* $\tilde{\lambda} > 0$ *such that for every* $\lambda > \tilde{\lambda}$ *problem* $(P)_\lambda$ *has at least two distinct, nontrivial solutions.*

Remark 1.1. As we mentioned above, when the nonlinearity f is a uniformly Lipschitz function (with Lipschitz constant $L > 0$), we have extra information on the eigenvalues:

(a) problem $(P)_\lambda$ has only the trivial solution whenever $\lambda \in (0, \lambda_L)$;
(b) problem $(P)_\lambda$ has at least two nontrivial solutions whenever $\lambda > \tilde{\lambda}$.

Moreover, the proof of Theorem 1.1 shows that the number $\tilde{\lambda}$ is less than or equal to the value

$$\lambda^* := \frac{1}{2}\frac{\|\alpha\|_1}{\|K\|_1}\left(\max_{t\neq 0}\frac{F(t)}{t^2}\right)^{-1}.$$

Consequently, Theorem 1.1 is valid for any $\lambda > \lambda^*$ which can be easily computed.

A direct consequence of Theorem 1.1 applied for $(1.1)_\lambda$ is the following

THEOREM 1.2. *Assume that* $1-d<s<0$. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a function as in Theorem* 1.1 *and* $K\in C^\infty(S^d)$ *positive. Then there exists* $\tilde\lambda>0$ *such that for every* $\lambda>\tilde\lambda$ *problem* $(EF)_\lambda$ *has at least two distinct, nontrivial solutions.*

In order to obtain a new kind of multiplicity result concerning $(P)_\lambda$ (specially, $(1.1)_\lambda$ and $(EF)_\lambda$), we require:

- (f_3) *There exists* $\mu_0\in(0,\infty)$ *such that the set of all global minima of the function* $t\mapsto \tilde F_{\mu_0}(t):=\frac{1}{2}t^2-\mu_0F(t)$ *has at least* $m\geq 2$ *connected components.*

Note that (f_3) implies that the function $t\mapsto\tilde F_{\mu_0}(t)$ has at least $m-1$ local maxima. Thus, the function $t\mapsto\mu_0 f(t)$ has at least $2m-1$ fixed points. In particular, if for some $\lambda>0$ one has $\tilde K(\lambda,\sigma)/\alpha(\sigma)=\mu_0$ for every $\sigma\in M$, then problem $(P)_\lambda$ has at least $2m-1\geq 3$ constant solutions. On the other hand, the following general result can be shown.

THEOREM 1.3. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a function which satisfies* (f_1) *and* (f_3). *Assume that* (A_1) *and* (A_2) *are satisfied with* $\tilde K(\lambda,\sigma)=\lambda K(\sigma)+\mu_0\alpha(\sigma)$, *and* $K\in C^\infty(M)$ *is positive. Then*

- (a) *for every* $\eta>\max\{0,\|\alpha\|_1\min_{t\in\mathbb{R}}\tilde F_{\mu_0}(t)\}$ *there exists* $\tilde\lambda_\eta>0$ *such that for every* $\lambda\in(0,\tilde\lambda_\eta)$ *problem* $(P)_\lambda$ *has at least* $m+1$ *solutions* $\omega_\lambda^{1,\eta},\dots,\omega_\lambda^{m+1,\eta}\in H_1^2(M)$;
- (b) *if* $(f_1^{q,c})$ *holds then for each* $\lambda\in(0,\tilde\lambda_\eta)$ *there is a set* $I_\lambda\subset\{1,\dots,m+1\}$ *with* $\mathrm{card}(I_\lambda)=m$ *such that*

$$\|\omega_\lambda^{i,\eta}\|_{H_1^2}<\frac{t_{\eta,q,c}}{\min\{1,\min_M\alpha^{1/2}\}},\qquad i\in I_\lambda,$$

where $t_{\eta,q,c}>0$ *is the greatest solution of the equation*

$$\frac{1}{2}t^2-\frac{\mu_0c\|\alpha\|_1^{(1-q)/2}}{q+1}t^{q+1}-\eta=0,\qquad t>0.$$

A consequence of Theorem 1.3 in the context of $(EF)_\lambda$ reads as follows.

THEOREM 1.4. *Assume that* $1-d<s<0$. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a function as in Theorem* 1.3 *and* $K\in C^\infty(S^d)$ *be a positive function. Then there exists* $\lambda_0>0$ *such that for every* $\lambda\in(0,\lambda_0)$ *the problem*

$$-\Delta u = |x|^{s-2}[\lambda K(x/|x|) + \mu_0 s(1-s-d)]f(|x|^{-s}u), \qquad x \in \mathbb{R}^{d+1} \setminus \{0\},$$

has at least $m+1$ *solutions.*

The proof of Theorem 1.1 is based on the paper of Pucci–Serrin [6], while that of Theorem 1.3 relies on a recent abstract critical point theorem of Ricceri (see [7] and [8]).

EXAMPLES. (a) Let $f(t) = \ln(1+t^2)$. Then Theorems 1.1–1.2 apply.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = \min\{t_+ - \sin(\pi t_+), 2(m-1)\}$ where $m \in \mathbb{N}\setminus\{1\}$ is fixed and $t_+ = \max\{t, 0\}$. Clearly, (f_1) is satisfied, while for $\mu_0 = 1$, the assumption (f_3) also holds. Indeed, the function $t \mapsto \tilde{F}_1(t)$ has precisely m global minima: $0, 2, \ldots, 2(m-1)$. Moreover, $\min_{t\in\mathbb{R}} \tilde{F}_1(t) = 0$. Therefore, one can apply Theorems 1.3 and 1.4.

In the next section we recall some basic facts on Sobolev spaces defined on compact Riemannian manifolds. In Section 3 we prove Theorems 1.1 and 1.2 while in the last section we deal with the proofs of Theorems 1.3 and 1.4.

2. Preliminaries. Let (M, g) be a smooth compact d-dimensional Riemannian manifold without boundary, $d \geq 3$, and let $\alpha \in C^\infty(M)$ be a positive function. For every $\omega \in C^\infty(M)$, set

$$\|\omega\|^2_{H^2_\alpha} = \int_M \langle \nabla\omega, \nabla\omega\rangle \, d\sigma_g + \int_M \alpha(\sigma)\langle \omega, \omega\rangle \, d\sigma_g,$$

where $\langle\cdot,\cdot\rangle$ is the inner product on covariant tensor fields associated to g, $\nabla\omega$ is the covariant derivative of ω, and $d\sigma_g$ is the Riemannian measure. The Sobolev space $H^2_\alpha(M)$ is defined as the completion of $C^\infty(M)$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Clearly, $H^2_\alpha(M)$ is a Hilbert space endowed with the inner product

$$\langle \omega_1, \omega_2\rangle_{H^2_\alpha} = \int_M \langle \nabla\omega_1, \nabla\omega_2\rangle \, d\sigma_g + \int_M \alpha(\sigma)\langle \omega_1, \omega_2\rangle \, d\sigma_g, \qquad \omega_1, \omega_2 \in H^2_\alpha(M).$$

Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent to the standard norm $\|\cdot\|_{H^2_1}$; actually, the latter is just $\|\cdot\|_{H^2_\alpha}$ with $\alpha = 1$. Moreover, we have

$$\min\{1, \min_M \alpha^{1/2}\}\|\omega\|_{H^2_1} \leq \|\omega\|_{H^2_\alpha} \leq \max\{1, \|\alpha\|^{1/2}_\infty\}\|\omega\|_{H^2_1} \tag{2.1}$$

for $\omega \in H^2_\alpha(M)$. Note that $H^2_\alpha(M)$ is compactly embedded in $L^p(M)$ for every $p \in [1, 2d/(d-2))$; the Sobolev embedding constant will be denoted by $S_p > 0$.

Let $\lambda > 0$. The energy functional $\mathcal{E}_\lambda : H^2_1(M) \to \mathbb{R}$ associated with problem $(P)_\lambda$ is

$$\mathcal{E}_\lambda(\omega) = \frac{1}{2}\|\omega\|^2_{H^2_\alpha} - \int_M \tilde{K}(\lambda, \sigma)F(\omega(\sigma))\, d\sigma_g, \tag{2.2}$$

where $F(t) = \int_0^t f(\tau)\,d\tau$. Due to our initial assumptions (A_1), (A_2) and (f_1), the functional $\mathcal{E}_\lambda$ is well-defined, it belongs to $C^1(H_1^2(M),\mathbb{R})$, and its critical points are precisely the weak (so classical) solutions of problem $(P)_\lambda$. By (f_1), for every $\varepsilon > 0$ sufficiently small there is $c(\varepsilon) > 0$ such that $|f(t)| \le \varepsilon|t| + c(\varepsilon)$ for every $t \in \mathbb{R}$. Consequently, for every $\omega \in H_1^2(M)$, we have

$$\mathcal{E}_\lambda(\omega) \ge \frac{1}{2}\,(1 - \varepsilon\|\tilde{K}(\lambda,\cdot)\|_\infty S_2^2)\|\omega\|_{H_\alpha^2}^2 - c(\varepsilon)\|\tilde{K}(\lambda,\cdot)\|_\infty S_1\|\omega\|_{H_\alpha^2}.$$

Therefore, the functional $\mathcal{E}_\lambda$ is coercive and bounded from below on $H_1^2(M)$. Moreover, it satisfies the standard Palais–Smale condition (see Zeidler [10, Example 38.25]).

3. Proof of Theorems 1.1 and 1.2. In this section we assume the hypotheses of Theorem 1.1 hold. We define $\mathcal{N}, \mathcal{F} : H_1^2(M) \to \mathbb{R}$ by

$$\mathcal{N}(\omega) = \frac{1}{2}\,\|\omega\|_{H_\alpha^2}^2 \quad \text{and} \quad \mathcal{F}(\omega) = \int_M K(\sigma)F(\omega(\sigma))\,d\sigma_g, \quad \omega \in H_1^2(M). \tag{3.1}$$

PROPOSITION 3.1. $\lim_{\varrho\to 0^+} \sup\{\mathcal{F}(\omega)/\mathcal{N}(\omega) : 0 < \mathcal{N}(\omega) < \varrho\} = 0$.

Proof. Due to (f_2), for small $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(t)| < \varepsilon(2\|K\|_\infty S_2^2)^{-1}|t|$ for every $|t| < \delta(\varepsilon)$. On account of (f_1), one may fix $1 < \nu < (d+2)/(d-2)$ and $c(\varepsilon) > 0$ such that $|f(t)| < c(\varepsilon)|t|^\nu$ for every $|t| \ge \delta(\varepsilon)$. Combining these two facts, after an integration, we obtain

$$|F(t)| \le \varepsilon(4\|K\|_\infty S_2^2)^{-1}t^2 + c(\varepsilon)(\nu+1)^{-1}|t|^{\nu+1} \quad \text{for every } t \in \mathbb{R}.$$

Fix a $\varrho > 0$ and any $\omega \in H_\alpha^2(M)$ with $\mathcal{N}(\omega) < \varrho$. By the above estimate,

$$\begin{aligned}
\mathcal{F}(\omega) &\le \frac{\varepsilon}{4}\,\|\omega\|_{H_\alpha^2}^2 + \frac{c(\varepsilon)}{\nu+1}\,\|K\|_\infty S_{\nu+1}^{\nu+1}\|\omega\|_{H_\alpha^2}^{\nu+1} \\
&< \frac{\varepsilon}{4}\,\|\omega\|_{H_\alpha^2}^2 + \frac{c(\varepsilon)}{\nu+1}\,\|K\|_\infty S_{\nu+1}^{\nu+1}(2\varrho)^{(\nu-1)/2}\|\omega\|_{H_\alpha^2}^2 \\
&= (\varepsilon/4 + c'(\varepsilon)\varrho^{(\nu-1)/2})\|\omega\|_{H_\alpha^2}^2.
\end{aligned}$$

Thus there exists $\varrho(\varepsilon) > 0$ such that for every $0 < \varrho < \varrho(\varepsilon)$, we have

$$0 \le \sup\{\mathcal{F}(\omega)/\mathcal{N}(\omega) : 0 < \mathcal{N}(\omega) < \varrho\} \le \varepsilon/2 + 2c'(\varepsilon)\varrho^{(\nu-1)/2} < \varepsilon,$$

which completes the proof.

Proof of Theorem 1.1. Let us define

$$\tilde{\lambda} = \inf\{\mathcal{N}(\omega)/\mathcal{F}(\omega) : \mathcal{F}(\omega) > 0\}.$$

Fix $\lambda > \tilde{\lambda}$. On one hand, there exists $\omega_\lambda^0 \in H_1^2(M)$ such that $\mathcal{F}(\omega_\lambda^0) > 0$ and $\lambda > \mathcal{N}(\omega_\lambda^0)/\mathcal{F}(\omega_\lambda^0)$. On account of (2.2) and (3.1), we have

$$\mathcal{E}_\lambda(\omega_\lambda^0) = \mathcal{N}(\omega_\lambda^0) - \lambda\mathcal{F}(\omega_\lambda^0) < 0. \tag{3.2}$$

On the other hand, due to Proposition 3.1, there exists $\delta > 0$ such that for all $\varrho \in (0, \delta)$ one has

$$\sup\{\mathcal{F}(\omega)/\mathcal{N}(\omega) : 0 < \mathcal{N}(\omega) < \varrho\} < 1/\lambda.$$

In other words, for every $\omega \in H_1^2(M) \setminus \{0\}$ satisfying $\mathcal{N}(\omega) < \varrho$ we have $\mathcal{F}(\omega)/\mathcal{N}(\omega) < 1/\lambda$ and hence

$$\mathcal{E}_\lambda(\omega) = \mathcal{N}(\omega) - \lambda\mathcal{F}(\omega) > 0 = \mathcal{E}_\lambda(0).$$

Consequently, $0 \in H_1^2(M)$ is a local minimum point for $\mathcal{E}_\lambda$, but not a global one, in view of (3.2). Since the functional $\mathcal{E}_\lambda$ is bounded from below on $H_1^2(M)$ and it satisfies the standard Palais–Smale condition, the global minimum of $\mathcal{E}_\lambda$ is achieved. Applying [6, Theorems 1 and 4], we obtain a third critical point of $\mathcal{E}_\lambda$ which is not 0. This concludes our proof.

Proof of Theorem 1.2. Let us choose $(M, g) = (S^d, g_0)$, and $\alpha(\sigma) = s(1-s-d)$ for every $\sigma \in S^d$ in Theorem 1.1. Thus, for every $\lambda > \tilde{\lambda}$, problem $(1.1)_\lambda$ has at least two distinct, nontrivial solutions $\omega_\lambda^1, \omega_\lambda^2 \in H_1^2(S^d)$. On account of (1.1), the elements $u_\lambda^i(x) = |x|^s \omega_\lambda^i(x/|x|)$, $i \in \{1, 2\}$, are solutions of $(EF)_\lambda$.

Remark 3.1. For every $k \in \{1, \ldots, d-2\}$, let $G_k = O(k+1) \times O(d-k) \subset O(d+1)$. The action of the group G_k on $H_1^2(S^d)$ is defined by $g\omega(\sigma) = \omega(g^{-1}\sigma)$ for every $g \in G_k$, $\omega \in H_1^2(S^d)$, $\sigma \in S^d$. Assume that K in $(1.1)_\lambda$ is constant. One can prove that the energy functional $\mathcal{E}_\lambda : H_1^2(S^d) \to \mathbb{R}$ is G_k-invariant for every $k \in \{1, \ldots, d-2\}$, i.e., $\mathcal{E}_\lambda(g\omega) = \mathcal{E}_\lambda(\omega)$ for every $g \in G_k$ and $\omega \in H_1^2(S^d)$. Now, let

$$H_{G_k}(S^d) = \{\omega \in H_1^2(S^d) : g\omega = \omega \text{ for every } g \in G_k\}$$

be the fixed point space of $H_1^2(S^d)$ under the action of G_k. Let $\mathcal{E}_\lambda^{G_k}$ be the restriction of the functional $\mathcal{E}_\lambda$ to the space $H_{G_k}(S^d)$. Now, we follow the same arguments as in Theorem 1.1 with $\mathcal{E}_\lambda^{G_k}$ and $H_{G_k}(S^d)$ instead of $\mathcal{E}_\lambda$ and $H_1^2(S^d)$, respectively. Therefore, for every $\lambda > \tilde{\lambda}$ and $k \in \{1, \ldots, d-2\}$ we can guarantee the existence of at least two different, nontrivial critical points of $\mathcal{E}_\lambda^{G_k}$ belonging to $H_{G_k}(S^d)$. On the other hand, the principle of symmetric criticality of Palais implies that every critical point of $\mathcal{E}_\lambda^{G_k}$ will also be a critical point of $\mathcal{E}_\lambda$, thus a solution of $(1.1)_\lambda$. If such a solution is not constant, say $\omega_{\lambda,k} \in H_{G_k}(S^d)$, it cannot belong to $H_{G_l}(S^d)$ whenever $l \neq k$. Indeed, when $k \neq l$, the group generated topologically by G_k and G_l acts transitively on S^d, that is, the orbit of every element from S^d under that group is the whole sphere S^d. Consequently, $H_{G_k}(S^d) \cap H_{G_l}(S^d)$ contains only the a.e. constant functions defined on S^d. Therefore, a nonconstant solution of $(1.1)_\lambda$ which belongs to $H_{G_k}(S^d)$ will not appear in $H_{G_l}(S^d)$, $l \neq k$. In this way, the number of solutions of $(1.1)_\lambda$ and $(EF)_\lambda$ can increase.

4. Proof of Theorems 1.3 and 1.4. We assume the hypotheses of Theorem 1.3 hold. Using the notation from the previous section (see (3.1)), we define the functional $\mathcal{N}_{\mu_0} : H_1^2(M) \to \mathbb{R}$ by

$$\mathcal{N}_{\mu_0}(\omega) = \mathcal{N}(\omega) - \mu_0 \int_M \alpha(\sigma) F(\omega(\sigma))\, d\sigma_g, \qquad \omega \in H_1^2(M).$$

PROPOSITION 4.1. *The set of all global minima of the functional $\mathcal{N}_{\mu_0}$ has at least m connected components in the weak topology on $H_1^2(M)$.*

Proof. First, for every $\omega \in H_1^2(M)$ we have

$$\begin{aligned}\mathcal{N}_{\mu_0}(\omega) &= \frac{1}{2}\|\omega\|^2_{H^2_\alpha} - \mu_0 \int_M \alpha(\sigma) F(\omega(\sigma))\, d\sigma_g \\ &= \frac{1}{2}\int_M |\nabla \omega|^2\, d\sigma_g + \int_M \alpha(\sigma)\tilde{F}_{\mu_0}(\omega(\sigma))\, d\sigma_g \\ &\geq \|\alpha\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_{\mu_0}(t).\end{aligned}$$

Moreover, if we consider $\omega(\sigma) = \omega_{\tilde{t}}(\sigma) = \tilde{t}$ for a.e. $\sigma \in M$, where $\tilde{t} \in \mathbb{R}$ is a minimum point of the function $t \mapsto \tilde{F}_{\mu_0}(t)$, then we have equality in the previous estimate. Thus,

$$\inf_{\omega \in H_1^2(M)} \mathcal{N}_{\mu_0}(\omega) = \|\alpha\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_{\mu_0}(t).$$

Moreover, if $\omega \in H_1^2(M)$ is not a constant function, then $|\nabla\omega|^2 = g^{ij}\partial_i\omega\partial_j\omega > 0$ on a positive measure set in M. In this case,

$$\mathcal{N}_{\mu_0}(\omega) = \frac{1}{2}\int_M |\nabla \omega|^2\, d\sigma_g + \int_M \alpha(\sigma)\tilde{F}_{\mu_0}(\omega(\sigma))\, d\sigma_g > \|\alpha\|_1 \inf_{t\in\mathbb{R}} \tilde{F}_{\mu_0}(t).$$

Consequently, there is a one-to-one correspondence between the sets

$$\mathrm{Min}(\mathcal{N}_{\mu_0}) = \{\omega \in H_1^2(M) : \mathcal{N}_{\mu_0}(\omega) = \inf_{\omega \in H_1^2(M)} \mathcal{N}_{\mu_0}(\omega)\}$$

and

$$\mathrm{Min}(\tilde{F}_{\mu_0}) = \{t \in \mathbb{R} : \tilde{F}_{\mu_0}(t) = \inf_{t\in\mathbb{R}} \tilde{F}_{\mu_0}(t)\}.$$

Indeed, let θ be the function that associates to every $t \in \mathbb{R}$ the equivalence class of those functions which are a.e. equal to t on the whole M. Then $\theta : \mathrm{Min}(\tilde{F}_{\mu_0}) \to \mathrm{Min}(\mathcal{N}_{\mu_0})$ is actually a homeomorphism, where $\mathrm{Min}(\mathcal{N}_{\mu_0})$ is considered with the relativization of the weak topology on $H_1^2(M)$. On account of the hypothesis (f_3), the set $\mathrm{Min}(\tilde{F}_{\mu_0})$ has at least $m \geq 2$ connected components. Therefore, the same is true for the set $\mathrm{Min}(\mathcal{N}_{\mu_0})$, which completes the proof.

Since $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma) + \mu_0\alpha(\sigma)$, the energy functional associated to $(P)_\lambda$ has the form $\mathcal{E}_\lambda = \mathcal{N}_{\mu_0} - \lambda\mathcal{F}$, where $\mathcal{F}$ comes from (3.1). In order to prove Theorem 1.3, we recall a recent critical point result of Ricceri [7, Theorem 8].

THEOREM 4.1. *Let H be a separable and reflexive real Banach space, and let $\mathcal{N}, \mathcal{G} : H \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functionals, with $\mathcal{N}$ coercive. Assume that the functional $\mathcal{N} + \lambda\mathcal{G}$ satisfies the Palais–Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of $\mathcal{N}$ has at least m connected components in the weak topology, with $m \geq 2$. Then, for every $\eta > \inf_H \mathcal{N}$, there exists $\overline{\lambda} > 0$ such that for every $\lambda \in (0, \overline{\lambda})$, the functional $\mathcal{N} + \lambda\mathcal{G}$ has at least $m + 1$ critical points, m of which are in $\mathcal{N}^{-1}((-\infty, \eta))$.*

Proof of Theorem 1.3. Let us choose $H = H_1^2(M)$, $\mathcal{N} = \mathcal{N}_{\mu_0}$ and $\mathcal{G} = -\mathcal{F}$ in Theorem 4.1. Due to Proposition 4.1 and to basic properties of the functions $\mathcal{N}_{\mu_0}$, $\mathcal{F}$, all the hypotheses of Theorem 4.1 are satisfied.

Then, for every $\eta > \max\{0, \|\alpha\|_1 \min_{t\in\mathbb{R}} \tilde{F}_{\mu_0}(t)\}(\geq \inf_{\omega\in H_1^2(M)} \mathcal{N}_{\mu_0}(\omega)\})$ there is a $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ the function $\mathcal{N}_{\mu_0} - \lambda\mathcal{F}$ has at least $m+1$ critical points; let us denote them by $\omega_\lambda^{1,\eta}, \ldots, \omega_\lambda^{m+1,\eta} \in H_1^2(M)$. Clearly, they are solutions of problem $(P)_\lambda$, which proves (a).

We know in addition that m elements among $\omega_\lambda^{1,\eta}, \ldots, \omega_\lambda^{m+1,\eta}$ belong to the set $\mathcal{N}_{\mu_0}^{-1}((-\infty, \eta))$. Let $\tilde{\omega}$ be such an element, i.e.,

$$\mathcal{N}_{\mu_0}(\tilde{\omega}) = \frac{1}{2}\,\|\tilde{\omega}\|_{H_\alpha^2}^2 - \mu_0 \int_M \alpha(\sigma)F(\tilde{\omega}(\sigma))\,d\sigma_g < \eta. \tag{4.1}$$

Assume that $(f_1^{q,c})$ holds. Then $|F(t)| \leq \frac{c}{q+1}|t|^{q+1}$ for every $t \in \mathbb{R}$. By the Hölder inequality, one has

$$\int_M \alpha(\sigma)|\tilde{\omega}(\sigma)|^{q+1}\,d\sigma_g \leq \|\alpha\|_1^{(1-q)/2}\|\tilde{\omega}\|_{H_\alpha^2}^{q+1}. \tag{4.2}$$

Since $\eta > 0$, the equation

$$\frac{1}{2}\,t^2 - \frac{\mu_0 c\|\alpha\|_1^{(1-q)/2}}{q+1}\,|t|^{q+1} - \eta = 0 \tag{4.3}$$

always has a positive solution. On account of (4.1) and (4.2), the number $\|\tilde{\omega}\|_{H_\alpha^2}$ is less than the greatest solution $t_{\eta,q,c} > 0$ of (4.3). It remains to apply (2.1) to prove (b).

Proof of Theorem 1.4. It follows directly by Theorem 1.3.

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