

Normalized solutions for the upper critical Choquard equation with nonlocal perturbations

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Abstract

This paper investigates the qualitative properties of normalized solutions to the upper critical Choquard equation with nonlocal perturbations:

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}-2}u + \mu(I_\beta * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases}$$

where $N \geq 3$, $\alpha, \beta \in (0, N)$, $p \in \left(\frac{N+\beta}{N}, \frac{N+\beta}{N-2}\right)$, $\mu \in \mathbb{R}$, $c > 0$, $\lambda \in \mathbb{R}$ is an unknown Lagrange multiplier, and I_α, I_β denote the Riesz potentials. For $\mu > 0$, we establish the existence of normalized solutions in several regimes, that is, when $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$ (mass-subcritical), $p = \frac{N+\beta+2}{N}$ (mass-critical), and $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$ (mass-supercritical). For $\mu \leq 0$, we derive a non-existence result. Particularly, to obtain sharp energy estimates crucial for restoring compactness, we classify analyses by ranges of α, β across different dimensions, developing tailored scaling techniques within each range to control energy levels below the corresponding compactness thresholds. This enables us to resolve open problems in sharp energy estimation for the mass-subcritical regime: we cover full parameter ranges for $N = 3, 4$ and extend admissible parameter ranges for $N \geq 5$, while providing a more comprehensive characterization of α, β to advance related research. Moreover, the framework applies directly to special cases including $\alpha = \beta$ and van der Waals-type potentials ($p = \frac{N+\alpha}{N-2}$ with $\alpha < \beta$), improving upon existing literature in these settings. We anticipate that the energy estimation techniques introduced in this paper will be extended to wider classes of nonlocal critical elliptic equations with mass constraint.

Keywords: Nonlinear Schrödinger equation; Normalized solution; Ground state; Upper critical Choquard equation; Mixed nonlinearities.

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1 Introduction

This paper investigates the normalized solutions to the following upper critical Choquard problem

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with nonlocal perturbation:

$$\begin{cases} -\Delta u + \lambda u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}} \right) |u|^{\frac{N+\alpha}{N-2}-2} u + \mu (I_\beta * |u|^p) |u|^{p-2} u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \end{cases} \quad (1.1)$$

where $N \geq 3$, $\alpha, \beta \in (0, N)$, $\mu > 0$, $p \in \left(\frac{N+\beta}{N}, \frac{N+\beta}{N-2} \right)$, $c > 0$, $\lambda \in \mathbb{R}$ is an unknown Lagrange multiplier, and I_ξ ($\xi = \alpha, \beta$) is the Riesz potential defined for $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\xi(x) := \frac{\mathcal{A}_{N,\xi}}{|x|^{N-\xi}} \quad \text{with} \quad \mathcal{A}_{N,\xi} := \frac{\Gamma\left(\frac{N-\xi}{2}\right)}{2^\xi \pi^{N/2} \Gamma\left(\frac{\xi}{2}\right)}, \quad (1.2)$$

with Γ denoting the Gamma function. From a variational perspective, normalized solutions (solutions with prescribed mass $c > 0$) to problem (1.1) correspond to critical points of the energy functional $\Phi_\mu : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined as:

$$\Phi_\mu(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}} \right) |u|^{\frac{N+\alpha}{N-2}} dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx, \quad (1.3)$$

constrained on the mass manifold

$$\mathcal{S}_c := \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\}. \quad (1.4)$$

We say that u is a ground state solution of (1.1) if it minimizes the energy functional Φ_μ among all solutions on $\mathcal{S}_c$, i.e.,

$$\Phi_\mu|'_{\mathcal{S}_c}(u) = 0 \text{ and } \Phi_\mu(u) = \inf \left\{ \Phi_\mu(u) : u \in \mathcal{S}_c, \Phi_\mu|'_{\mathcal{S}_c}(u) = 0 \right\}.$$

Problem (1.1) arises when seeking standing wave solutions $\psi(t, x) = e^{i\lambda t} u(x)$ with prescribed mass (L^2 -norm) for the time-dependent nonlinear Choquard equation:

$$i \frac{\partial \psi}{\partial t} = \Delta \psi + (I_\alpha * F(\psi)) f(\psi), \quad \text{in } \mathbb{R}^N \times \mathbb{R} \quad (N \geq 1), \quad (1.5)$$

where t denotes time, $\psi : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{C}$ is complex-valued, i is the imaginary unit, and $f(\psi) = F'(\psi)$. The solitary wave ansatz yields the Choquard-type equation:

$$-\Delta u + \lambda u = (I_\alpha * F(u)) f(u), \quad \text{in } \mathbb{R}^N. \quad (1.6)$$

In the classical case $N = 3$, $\alpha = 2$, and $f(u) = u$, (1.6) reduces to the Choquard-Pekar equation, originating in Pekar's quantum-theoretic model of the stationary polaron [1]. Subsequent works [2, 3, 4, 5, 6] extended its applications to plasma physics and self-gravitating matter. Mathematical analysis began with Lieb [6] and Lions [7], which established existence of radial positive solutions for prescribed frequency $\lambda > 0$ and prescribed mass $c > 0$, respectively. This foundation stimulated extensive study of (1.6) under both prescribed frameworks.

For the prescribed frequency approach ($\lambda > 0$), Moroz-Van Schaftingen [8] considered the special case $f(u) = |u|^{p-2}u$ in (1.6), deriving the optimal range $\frac{N+\alpha}{N} < p < 2_\alpha^*$ for existence of ground states, where $2_\alpha^* := \frac{N+\alpha}{N-2}$ for $N \geq 3$ and $2_\alpha^* := +\infty$ for $N = 1, 2$. The endpoints of this interval are extremal values for the Hardy-Littlewood-Sobolev inequality and sometimes called lower and upper critical exponents. The Pohozaev identity precludes nontrivial solutions at critical exponents, but solvability can be restored via suitable subcritical perturbations (e.g., $\mu|u|^{q-2}u$ for $2 < q < 2_\alpha^*$ or $\mu(I_\alpha * |u|^q)|u|^{q-2}u$ for $\frac{N+\alpha}{N} < q < 2_\alpha^*$ under suitable conditions on $\mu > 0$); see [9, 10, 11, 12, 13] for upper and lower critical cases. Here, $2^* := \frac{2N}{N-2}$ ($N \geq 3$) or $+\infty$ ($N = 1, 2$).

Particularly, as $\alpha \rightarrow 0$, $(I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u \rightarrow |u|^{2^*-2}u$ for $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $N \geq 3$, since $2_\alpha^* \rightarrow 2^*/2$ and $(I_\alpha * F(u))f(u) \rightarrow F(u)f(u)$ formally. The upper critical exponent $\frac{N+\alpha}{N-2}$ plays an

analogous role to the Sobolev critical exponent $\frac{2N}{N-2}$ in local semilinear elliptic equations for $N \geq 3$. Consequently, upper critical exponent problems for Choquard equations can be viewed as nonlocal counterparts of the Sobolev critical Schrödinger equation (Brezis-Nirenberg type problem [14]):

$$-\Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{2^*-2} u, \quad x \in \mathbb{R}^N \quad (N \geq 3). \quad (1.7)$$

This paper focuses on the upper critical Choquard problem under the prescribed mass framework.

The prescribed mass approach has garnered increasing interest in recent years due to its physical significance, for example, the mass (L^2 -norm) is a preserved quantity of the evolution and variational characterization of such solutions can help to analyze their orbital stability/instability. Mathematically, this approach presents greater complexity than prescribed frequency problems. Specifically, while nontrivial weak limits typically solve prescribed frequency problems, the non-compact embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ complicates verification of the constraint condition for prescribed mass problems. The subsequent subsection reviews relevant literature motivating our work and highlights novel aspects of our study of (1.1).

1.1 Previous developments and perspectives

Within the framework of normalized solutions, in addition to the lower and upper critical exponents, a new mass-critical exponent $\frac{N+\alpha+2}{N}$ arises. This critical threshold determines the boundedness below of the constrained functional on $\mathcal{S}_c$, thereby fundamentally influencing our choice of approaches for the search of constrained critical points. If $F(u)$ in (1.6) grows faster than $|u|^{\frac{N+\alpha+2}{N}}$ at infinity, the functional is unbounded below on $\mathcal{S}_c$ (mass-supercritical); otherwise, it is bounded below (mass-subcritical). The mass-critical case occurs precisely when the boundedness depends explicitly on the parameter $c > 0$. For the prototypical case $f(u) = |u|^{p-2}u$, the ranges $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, $p = \frac{N+\alpha+2}{N}$, and $\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$ correspond to mass-subcritical, mass-critical, and mass-supercritical scenarios, respectively.

For the mass-subcritical power nonlinearity $f(u) = |u|^{p-2}u$ with $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$ in (1.6), Ye [15] established the existence of normalized ground states via global minimization on $\mathcal{S}_c$. Regarding mass-supercritical nonlinearities of the form $f(u) \sim \sum_{i=1}^k |u|^{p_i-2}u$ with $\frac{N+\alpha+2}{N} < p_i < 2_\alpha^*$ ($1 \leq i \leq k$, $N \geq 3$), Li-Ye [16] obtained the existence of normalized solutions of mountain-pass type using minimax procedures from [17]. This result was subsequently extended by Bartsch-Liu-Liu [18] to broader classes of mass-supercritical nonlinearities ($N \geq 1$). Upper critical Choquard equations present significantly greater complexity under the prescribed mass constraint compared to their subcritical counterparts, primarily due to inherent non-compactness issues. In [19, 20], Li studied the upper critical Choquard equation with local homogeneous perturbation:

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^{2_\alpha^*}) |u|^{2_\alpha^*-2} u + \mu |u|^{q-2} u, & x \in \mathbb{R}^N \quad (N \geq 3), \\ \int_{\mathbb{R}^N} |u|^2 dx = c. \end{cases} \quad (1.8)$$

Adapting strategies from [21, 22, 23, 24] originally developed for normalized solutions of the Sobolev critical Schrödinger equation (1.7), she proved the existence of two radial solutions for $2 < q < 2 + \frac{4}{N}$ and one radial solution for $2 + \frac{4}{N} \leq q < 2^*$, while simultaneously analyzing qualitative properties and stability. Here, $2 + \frac{4}{N}$ is the mass-critical exponent for Schrödinger equations. Further developments concerning mixed Choquard-type and local power nonlinearities can be found in Yao-Chen-Rădulescu-Sun [25].

In contrast to local perturbations, Ye-Shen-Yang [26] and Shang-Ma [27] considered the nonlocal perturbation case, and investigated the existence and multiplicity of normalized solutions for the upper critical Choquard problem (1.1) with $\alpha = \beta$ using methods from [23] and [21], respectively. Their

main results are summarized in Table 1 and Table 2, respectively. Recently, Ding-Wang [28] partially unified and improved their results, with key findings systematically summarized in Table 3.

N	α	p	μ, c	Type of solution
$N \geq 3$	$(0, N)$	$\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A ground state solution with negative energy
$N \geq 4$		$p = \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A ground state solution
$N = 4$		$\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$	$0 < \mu c^k < \infty$	A ground state solution
$N \geq 5$			$0 < \mu c^k < \theta < \infty$	

Table 1: Existence results on (1.1) with $\alpha = \beta$ in [26]

N	α	p	μ, c	Type of solution
$N \geq 3$	$(0, N)$	$\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A ground state solution with negative energy
$N = 3$	$(0, 1)$	$p = \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A mountain pass solution
$N = 4$	$(0, 4)$			
$N \geq 5$	$[N - 4, N)$			
$N = 3$	$(0, 1)$	$\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$	$0 < \mu c^k < \infty$	A mountain pass solution
$N = 4$	$(0, 4)$		$0 < \mu c^k < \theta < \infty$	
$N \geq 5$	$[N - 4, N)$			

Table 2: Existence results on (1.1) with $\alpha = \beta$ in [27]

N	α	p	μ, c	Type of solution
$N \geq 3$	$(0, N)$	$\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A ground state solution with negative energy
$3 \leq N \leq 5$	$[\max\{N-2, 2N-6\}, N)$	$2 \leq p < \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A mountain pass solution with positive energy
$N = 5$	$(0, \frac{7}{4})$	$\max\{\frac{8+\alpha}{6}, \frac{7+2\alpha}{6}\} < p < \frac{7+\alpha}{5}$		
$N \geq 6$	$(0, N-2)$	$\max\{\frac{2N+\alpha-2}{2N-4}, \frac{N+\alpha-2}{N-2}\} < p < \frac{N+\alpha+2}{N}$		
$N = 3$	$(0, 1)$	$p = \frac{N+\alpha+2}{N}$	$0 < \mu c^k < \theta < \infty$	A ground state solution
$N \geq 4$	$(0, N)$			
$N = 3$	$(0, \min\{p-1, 3\})$	$\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$	$0 < \mu c^k < \infty$	A ground state solution
$N \geq 4$	$(0, N)$			

Table 3: Existence results on (1.1) with $\alpha = \beta$ in [28]

Here $k = k(N, \alpha, p) > 0, \theta = \theta(N, \alpha, p) > 0$, which may take different values in different works and under different situations.

Compared to the extensively studied case $\alpha = \beta$, the solution set for problems with differing potential ranges ($\alpha \neq \beta$) exhibits richer structure. Jia-Luo [29] pioneered the investigation of the critical problem:

$$\begin{cases} -\Delta u + \lambda u = (|x|^{-4} * |u|^2) u + \mu (|x|^{-\hat{\beta}} * |u|^2) u, & x \in \mathbb{R}^N \quad (N \geq 5), \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases} \quad (1.9)$$

which corresponds to the special case of our problem (1.1) with parameters $\alpha = N-4, \frac{N+\alpha}{N-2} = 2, p = 2$, and $\beta = N - \hat{\beta}$. Here, for $\hat{\beta} \in (0, 2), \hat{\beta} = 2$, and $\hat{\beta} \in (2, 4)$, it follows that, respectively, $\frac{N+\beta}{N} < 2 < \frac{N+\beta+2}{N}, 2 = \frac{N+\beta+2}{N}, \frac{N+\beta+2}{N} < 2 < \frac{N+\beta}{N-2}$. Accordingly, the regimes $\hat{\beta} \in (0, 2), \hat{\beta} = 2$, and $\hat{\beta} \in (2, 4)$ correspond to mass-subcritical, mass-critical, and mass-supercritical nonlocal perturbations. Building on techniques from [23, 24] for normalized solutions of the Sobolev-critical Schrödinger equation (1.7) via Pohozaev manifold decomposition, Jia-Luo established the following existence results:

- when $0 < \hat{\beta} < 2$, for sufficiently small $\mu c^{(4-\hat{\beta})/2}$, (1.9) admits two solutions: one local minimizer and one saddle point;
- for $\hat{\beta} = 2$ (with μc below a threshold) and $2 < \hat{\beta} < 4$, (1.9) possesses a normalized ground state solution.

However, for the mass-subcritical case ($0 < \hat{\beta} < 2$), the existence proof for the second solution requires careful re-examination. *The energy level estimate (3.21) on page 22 appears to be insufficient* for recovering the compactness of (PS) sequences, which is a crucial step for the second solution.

Very recently, Chen-Chen [30] investigated the general critical problem (1.1). Adopting techniques from [21], the authors proved that for any fixed $\mu > 0$, there exist

$$\bar{c} = \left(\frac{1}{2D} \right)^{\frac{2_{\alpha}^* - p\gamma_p}{p(1-\gamma_p)(2_{\alpha}^* - 1)}} > 0, \quad W_c := \{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \bar{\rho}\}$$

where

$$D := \frac{\mu}{2p} C_{N,p} \left(\frac{2_{\alpha}^*(1-\gamma_p)}{2p(2_{\alpha}^* - 1)} \mu C_{N,p} \mathcal{S}_{\alpha}^{2_{\alpha}^*} \right)^{\frac{p\gamma_p - 1}{2_{\alpha}^* - p\gamma_p}} + \frac{1}{22_{\alpha}^*} \mathcal{S}_{\alpha}^{-2_{\alpha}^*} \left(\frac{2_{\alpha}^*(1-\gamma_p)}{2p(2_{\alpha}^* - 1)} C_{N,p} \mathcal{S}_{\alpha}^{2_{\alpha}^*} \right)^{\frac{2_{\alpha}^* - 1}{2_{\alpha}^* - p\gamma_p}},$$

$$\bar{\rho} := \left(\frac{\mu 2_{\alpha}^*(1-\gamma_p)}{2p(2_{\alpha}^* - 1)} C_{N,p} \bar{c}^{p(1-\gamma_p)} \mathcal{S}_{\alpha}^{2_{\alpha}^*} \right)^{\frac{1}{2_{\alpha}^* - p\gamma_p}},$$

$$\gamma_p := \frac{Np - (N + \beta)}{2p}, \quad (1.10)$$

and $C_{N,p} = \frac{p}{\|Q_p\|_2^{2p-2}}$ (Q_p is defined in (2.5)), such that the following existence results hold:

Theorem C. [30, Theorems 1.1 and 1.2] *Let $N \geq 3$, $0 < \alpha < \beta < N$, and $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$. For any $\mu > 0$ and any $c \in (0, \bar{c})$:*

- (i) (1.1) admits a ground state solution u_c that is a local minimizer of Φ_{μ} in W_c satisfying $\Phi_{\mu}(u_c) = \inf_{W_c} \Phi_{\mu} < 0 < \inf_{u \in \partial W_c} \Phi_{\mu}(u)$;
- (ii) (1.1) has another solution with positive energy if $N > \frac{2\beta+4p-1}{2(p-1)}$.

A detailed re-examination, however, suggests that the explicit expressions for $\bar{c}$ and $\bar{\rho}$ in Theorem C may require further clarification. We will subsequently provide revised expressions to refine the statement of Theorem C. It is also worth noting that the dimensional constraint in (ii) implicitly requires

$$N \geq 9, \quad 0 < \beta < \frac{N}{4} - 2, \quad \text{and} \quad \frac{2N + 2\beta - 1}{2(N - 2)} < p < \frac{N + \beta + 2}{N}.$$

Now a natural question arises: Does (1.1) admit normalized solutions with positive energy for $3 \leq N < 9$ in the mass-subcritical perturbation? To the best of our knowledge, no relevant results exist.

Motivated by the aforementioned work, we systematically investigate the existence and multiplicity of normalized solutions to (1.1) for all $N \geq 3$ and the full range $p \in \left(\frac{N+\beta}{N}, \frac{N+\beta}{N-2} \right)$, addressing the open problem. The intricate interplay between nonlocal terms induces distinct geometric structures of Φ across different p -regimes, necessitating separate analyses of three cases: $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$ (mass-subcritical), $p = \frac{N+\beta+2}{N}$ (mass-critical), and $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$ (mass-supercritical).

1.2 Statement of the main results

Before stating our main results, we introduce some definitions. Define the Pohozaev type function

$$\mathcal{P}_\mu(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu \frac{Np - (N + \beta)}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx. \quad (1.11)$$

It is straightforward to verify that

$$\begin{aligned} \Phi_\mu(t^{N/2}u_t) &= \frac{t^2}{2} \|\nabla u\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx \\ &\quad - \frac{\mu}{2p} t^{Np-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \end{aligned} \quad (1.12)$$

and

$$\mathcal{P}_\mu(u) = \frac{d}{dt} \Phi_\mu(t^{N/2}u_t) \Big|_{t=1}.$$

We recall that any solution of (1.1) lies in the L^2 -Pohozaev manifold given by

$$\mathcal{M}_c := \{u \in \mathcal{S}_c : \mathcal{P}_\mu(u) = 0\}. \quad (1.13)$$

For any given $c > 0$, we define

$$\hat{m}_\mu(c) := \inf_{u \in \mathcal{M}_c} \Phi_\mu(u). \quad (1.14)$$

Clearly, if a solution u_c of (1.1) satisfies $\Phi_\mu(u_c) = \inf_{u \in \mathcal{M}_c} \Phi_\mu(u)$, then it is a ground state solution.

To derive sharp energy estimates, following [13, Lemma 1.2] (which addresses the Choquard-type Brezis-Nirenberg problem), we define the best constant $\mathcal{S}_\alpha$ for the Hardy-Littlewood-Sobolev inequality:

$$\mathcal{S}_\alpha := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left[\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx \right]^{\frac{N-2}{N+\alpha}}}. \quad (1.15)$$

Setting

$$U(x) := \frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}, \quad (1.16)$$

we have the identity

$$(\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(\alpha+2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{\alpha}{2}} \int_{\mathbb{R}^N} |\nabla U|^2 dx = \int_{\mathbb{R}^N} (I_\alpha * |U|^{\frac{N+\alpha}{N-2}}) |U|^{\frac{N+\alpha}{N-2}} dx = (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}}, \quad (1.17)$$

where the constants $\mathcal{A}_{N,\alpha}$ and $\mathcal{C}_{N,\alpha}$ are defined by equations (1.2) and (2.3), respectively. Combining (1.15) and (1.17), we see that $U(x)$ and the extremal function of $\mathcal{S}_\alpha$ differ only by a constant coefficient.

Following the argument in [30, Lemmas 3.1 and 3.4], for any fixed $\mu > 0$, we obtain the explicit value

$$c_0 = \left(\frac{(2_\alpha^* - 1) \|Q_p\|_2^{2p-2}}{2^* - \gamma_p p} \right)^{\frac{2_\alpha^* - \gamma_p p}{(2_\alpha^* - 1)(1 - \gamma_p)p}} \left(\frac{(1 - \gamma_p p) 2_\alpha^* \mathcal{S}_\alpha^{2_\alpha^*}}{(2_\alpha^* - 1) \|Q_p\|_2^{2p-2}} \right)^{\frac{1 - \gamma_p p}{(2_\alpha^* - 1)(1 - \gamma_p)p}} \mu^{-\frac{1}{(1 - \gamma_p)p}}, \quad (1.18)$$

such that, for each $c \in (0, c_0)$, the set $V_c \subset \mathcal{S}_c$ satisfies the property

$$m_\mu(c) := \inf_{u \in V_c} \Phi_\mu(u) < 0 < \inf_{u \in \partial V_c} \Phi_\mu(u). \quad (1.19)$$

Here the sets are defined by:

$$V_c := \{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \rho_0\}, \quad \partial V_c := \{u \in \mathcal{S}_c : \|\nabla u\|_2^2 = \rho_0\}, \quad (1.20)$$

where $\rho_0 := \rho(c_0)$ and the radius function $\rho(c)$ is given by

$$\rho(c) := \left(\frac{2_\alpha^*(1 - p\gamma_p)\mu\mathcal{S}_\alpha^{2_\alpha^*}}{(2_\alpha^* - 1)\|Q_p\|_2^{2p-2}} \right)^{\frac{1}{2_\alpha^* - p\gamma_p}} c^{\frac{p(1-\gamma_p)}{2_\alpha^* - p\gamma_p}}, \quad \forall c > 0. \quad (1.21)$$

A minor modification of the proof of [30, Theorem 1.1] now yields the existence of a first solution, as stated below.

Theorem 1.1. *Let $N \geq 3$, $\alpha, \beta \in (0, N)$, and $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$. For any $\mu > 0$, there exists a $c_0 = c_0(\mu) > 0$ (defined in (1.18)) such that, for any $c \in (0, c_0)$, (1.1) admits a ground state solution u_c , which is a local minimizer of Φ_μ in V_c satisfying $\Phi_\mu(u_c) = m_\mu(c) < 0$.*

For $\mu > 0$, by developing new compactness arguments and refined energy estimates, we establish three existence results for the cases $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$, $p = \frac{N+\beta+2}{N}$, and $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$:

Theorem 1.2. *Let $N \geq 3$, $\alpha, \beta \in (0, N)$, $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$, and $\mu > 0$. For any $c \in (0, c_0)$, problem (1.1) admits a second solution $(u_c, \lambda_c) \in H_{\text{rad}}^1(\mathbb{R}^N) \times (0, +\infty)$ satisfying*

$$0 < \Phi_\mu(u_c) < m_\mu(c) + \frac{2 + \alpha}{2(N + \alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}},$$

under the following dimensional regimes:

(i) $N = 3, 4$: No additional constraints on α , β , and p (beyond the general assumptions).

(ii) $N = 5$:

- $\alpha \in [1, 2) \cup [4, 5)$ (no additional constraints);
- $\alpha \in (0, 1) \cup [2, 4)$, $\beta \in (0, \frac{7}{4})$, $4\beta - 5\alpha < 2$, and

$$\max \left\{ \frac{7+2\beta}{6}, \frac{8+\beta}{6}, \frac{8+2\beta-\alpha}{6} \right\} < p < \frac{7+\beta}{5}.$$

(iii) $N \geq 6$: $\beta \in (0, N-2)$, $4\beta - N\alpha < 2N - 8$, and

$$\max \left\{ \frac{N+\beta-2}{N-2}, \frac{2N-2+\beta}{2N-4}, \frac{2N-2+2\beta-\alpha}{2N-4} \right\} < p < \frac{N+\beta+2}{N}.$$

Theorem 1.3. *Let $N \geq 3$, $\alpha, \beta \in (0, N)$, $p = \frac{N+\beta+2}{N}$, and $c > 0$. Then problem (1.1) has a solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ satisfying*

$$\Phi_\mu(u_c) = \inf_{\mathcal{M}_c} \Phi_\mu,$$

under one of the following conditions:

(i) $N = 3$: $\beta \in (0, 1)$ and $0 < \mu < \frac{\|Q_p\|_2^{2p-2}}{2c^{(\beta+2)/3}}$.

(ii) $N \geq 4$: $0 < \mu < \frac{\|Q_p\|_2^{2p-2}}{2c^{(\beta+2)/N}}$.

Theorem 1.4. *Let $N \geq 3$, $\alpha, \beta \in (0, N)$, $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$, $c > 0$, and $\mu > 0$. Then problem (1.1) has a solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ satisfying*

$$\Phi_\mu(u_c) = \inf_{\mathcal{M}_c} \Phi_\mu,$$

under one of the following conditions:

(i) $N = 3$:

- $\beta \in (0, 1)$, $\frac{5+\beta}{3} < p < 3 + \beta$, and $\mu > 0$;

- $\beta \in (1, 3)$, $\frac{5+\beta}{3} < p \leq 1 + \beta$, and $\mu > \mu^* > 0$, with μ^* sufficiently large;
- $\beta \in [1, 3)$, $1 + \beta < p < 3 + \beta$, and $\mu > 0$.

(ii) $N \geq 4$: No additional constraints on α , β , p , and μ (beyond the general assumptions).

For $\mu \leq 0$, we establish the following non-existence result:

Theorem 1.5. *Let $N \geq 3$, $\alpha, \beta \in (0, N)$, $\frac{N+\beta}{N} < p < \frac{N+\beta}{N-2}$, $\mu \leq 0$, and $c > 0$. Then problem (1.1) has no solutions in $H^1(\mathbb{R}^N) \times (0, +\infty)$.*

Remark 1.6. (i) Theorems 1.2-1.4 extend and cover the relevant results in [29, 30].

(ii) Our analysis can directly apply to the special case $\alpha = \beta$, with principal findings summarized in Table 4 below. Crucially, the new compactness techniques developed in this work enable us to generalize and extend the existence results established in [26, 27, 28] (see Tables 1-3) by relaxing dimensional and parametric restrictions on (N, α, p) .

N	α	p	μ	c	Conclusion
$N = 3, 4$	$(0, N)$	$\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$	$\mu > 0$	$(0, c_0)$	(1.1) has a mountain pass solution with positive energy
$N = 5$	$[1, 2) \cup [4, 5)$				
	$(0, 1)$				
$N \geq 6$	$(0, N - 2)$	$\max\{\frac{2N+\alpha-2}{2N-4}, \frac{N+\alpha-2}{N-2}\} < p < \frac{N+\alpha+2}{N}$	$0 < \mu < \mu_0$	$c > 0$	(1.1) has a ground state solution
$N = 3$	$(0, 1)$	$p = \frac{N+\alpha+2}{N}$			
$N \geq 4$	$(0, N)$				
$N = 3$	$(1, 3)$	$\frac{5+\alpha}{3} < p \leq 1 + \alpha$	$\mu > \mu^* > 0$	$c > 0$	(1.1) has a ground state solution
	$[1, 3)$	$1 + \alpha < p < 3 + \alpha$			
	$(0, 1)$	$\frac{5+\alpha}{3} < p < 3 + \alpha$			
$N \geq 4$	$(0, N)$	$\frac{N+\alpha+2}{N} < p < \frac{N+\alpha}{N-2}$			

Table 4: Our results when $\alpha = \beta$

Remark 1.7. (i) In the mass-subcritical regime $\left(\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}\right)$, establishing the existence of a second solution presents significant analytical challenges, primarily due to compactness issues. The core difficulty lies in obtaining rigorous upper bounds for the minimax level $M_\mu(c)$:

$$M_\mu(c) < m_\mu(c) + \frac{2+\alpha}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}. \quad (1.22)$$

To the best of our knowledge, the existing literature establishes this inequality only in certain spatial dimensions N and within restricted ranges of the parameters α , β , and p . Even in the simpler case $\alpha = \beta$, the result is known only for limited ranges of α and p . **The validity of (1.22) across the full parameter space has remained an open problem.** This work resolves this fundamental issue for $N = 3, 4$, proving (1.22) holds unconditionally for all $0 < \alpha, \beta < N$ and $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$, thereby guaranteeing a second solution. For $N \geq 5$, we substantially extend the known admissible ranges beyond all previous results, though these may not be optimal.

(ii) In the L^2 -supercritical case, our existence results provide a more comprehensive analysis regarding α and β , significantly improving and extending the related research.

As a by-product of this work, our approach can be adapted to address the following Schrödinger equation with van der Waals type potentials:

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}-2}u + \mu(I_\beta * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases} \quad (1.23)$$

which corresponds to the special case of problem (1.1) with $p = \frac{N+\alpha}{N-2}$ and $\alpha < \beta$. In this direction, we obtain the following results:

Corollary 1.8. *For problem (1.23), the conclusions of Theorems 1.2-1.4 hold under the following corresponding conditions, respectively:*

- (A) $N = 4$ with $\alpha \in (0, 1)$ and $2\alpha + 2 < \beta < 4$;
 $N = 5$ with $\alpha \in [1, 2)$ and $\frac{5\alpha+4}{3} < \beta < 5$;
 $N \geq 7$ with $\alpha \in (0, N-4)$ and $\alpha + \frac{2\alpha+4}{N-2} < \beta < \min\{2 + \alpha, 1 + \frac{3}{2}\alpha\}$.
- (B) $N \geq 4$ with $0 < \alpha < N - \frac{4}{N} - 2$, $\beta = \alpha + \frac{2\alpha+4}{N-2}$, and $0 < \mu < \frac{\|Q_{(N+\alpha)/(N-2)}\|_2^{(2\alpha+4)/(N-2)}}{2c^{(\alpha+2)/(N-2)}}$.
- (C) $N = 3$ with $0 < \alpha < \beta < 1$ and $\mu > 0$;
 $N = 3$ with $\alpha \in (0, 1)$, $2 + \alpha \leq \beta < 3$, and $\mu > \mu^* > 0$, with μ^* sufficiently large;
 $N = 3$ with $\alpha \in (0, 1)$, $1 \leq \beta < 2 + \alpha$, and $\mu > 0$;
 $N = 3$ with $\alpha \in [1, 3)$, $\alpha < \beta < 3$, and $\mu > 0$;
 $N \geq 4$ with $0 < \alpha < \beta < \min\{\alpha + \frac{2\alpha+4}{N-2}, N\}$ and $\mu > 0$.

Remark 1.9. As $\alpha, \beta \rightarrow 0$, (1.1) formally reduces to

$$\begin{cases} -\Delta u + \lambda u = |u|^{2^*-2}u + \mu|u|^{2p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases} \quad (1.24)$$

where $p \in (1, \frac{N}{N-2})$. It is proved in [21, 23, 31] that (1.24) admits prescribed mass solutions under the L^2 -subcritical ($2 < 2p < 2 + \frac{4}{N}$), L^2 -critical ($2p = 2 + \frac{4}{N}$), and L^2 -supercritical ($2 + \frac{4}{N} < 2p < \frac{2N}{N-2}$) perturbation. Thus, Theorems 1.2-1.4 may be viewed as an generalization of the results in [21, 23, 31] to Brezis-Nirenberg-type Choquard equation. In particular, motivated by Ding-Ji-Pucci [32], it would be interesting to extend the present approach to the $(2, q)$ -Laplacian equation with an additional $-\Delta_q u$ term, where the results are expected to depend on the value of q .

Remark 1.10. As $\beta \rightarrow 0$, (1.1) formally reduces to

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}-2}u + \mu|u|^{2p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases} \quad (1.25)$$

where $p \in (1, \frac{N}{N-2})$. It is proved in [19, 20] that (1.25) admits nontrivial solutions. Thus, the main results in the present paper may be viewed as an generalization of the results in [19, 20] to Choquard equations with nonlocal perturbation.

Remark 1.11. As $\alpha \rightarrow 0$, (1.1) formally reduces to

$$\begin{cases} -\Delta u + \lambda u = |u|^{2^*-2}u + \mu(I_\beta * |u|^p)|u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c. \end{cases} \quad (1.26)$$

When $N = 3$, the existence and multiplicity of normalized solutions for problem (1.26) were investigated in [33, 34]. Therefore, the main results of this paper can be regarded as an extension of those in [33, 34] to Choquard equations for all $N \geq 3$.

The paper is organized as follows. In Section 2, we give the variational setting and preliminaries. In section 3, we study the second normalized solution of problem (1.1) under the basis of Theorem 1.1 and then complete the proof of Theorem 1.2. In section 4, we complete the proof of Theorems 1.3 and 1.4. In Section 5, we establish the non-existence result, and prove Theorem 1.5.

Throughout the paper, we make use of the following notations:

- $L^s(\mathbb{R}^N)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^N)$ and $t > 0$, we set $u_t(x) := u(tx)$;
- For any $x \in \Omega$ and $r > 0$, $B_r(x) := \{y \in \Omega : |y - x| < r\}$ and $B_r = B_r(0)$;
- $\theta(N) := \frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}$ is the measure of the unit ball in $\mathbb{R}^N$;
- $C_1, C_2, \dots$ denote positive constants possibly different in different places.

2 Variational framework and preliminaries

In this section, we give some preliminary results which will be often used throughout the rest of the paper.

Lemma 2.1. (Hardy-Littlewood-Sobolev inequality,[35]) Let $N \geq 3, s, r > 1, 0 < \alpha < N$ with $\frac{N-\alpha}{N} + \frac{1}{s} + \frac{1}{r} = 2$, $g \in L^s(\mathbb{R}^N)$, and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $\mathcal{C}_{N,\alpha,s,r}$, independent of g, h , such that

$$\int_{\mathbb{R}^N} (I_\alpha * g) h dx \leq \mathcal{C}_{N,\alpha,s,r} \|g\|_s \|h\|_r. \quad (2.1)$$

In particular,

$$\int_{\mathbb{R}^N} (I_\alpha * g) h dx \leq \mathcal{C}_{N,\alpha} \|g\|_{2N/(N+\alpha)} \|h\|_{2N/(N+\alpha)}, \quad (2.2)$$

where

$$\mathcal{C}_{N,\alpha} := \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{-\frac{\alpha}{N}}. \quad (2.3)$$

Lemma 2.2. ([15, Lemma 2.3]) Let $N \geq 1, \beta \in (0, N)$, and $\frac{N+\beta}{N} < p < \frac{N+\beta}{N-2}$, then

$$\int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \leq \frac{p}{\|Q_p\|_2^{2p-2}} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{(N+\beta)-(N-2)p}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{Np-(N+\beta)}{2}}, \quad (2.4)$$

where equality holds for $u = Q_p$, and Q_p is a nontrivial solution of

$$-\frac{Np-(N+\beta)}{2} \Delta Q_p + \frac{N+\beta-(N-2)p}{2} Q_p = (I_\beta * |Q_p|^p) |Q_p|^{p-2} Q_p. \quad (2.5)$$

Lemma 2.3. ([8, 36]) If there exist $u \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that

$$-\Delta u + \lambda u = (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}-2} u + \mu (I_\beta * |u|^p) |u|^{p-2} u, \quad x \in \mathbb{R}^N,$$

then

$$\mathcal{P}_\mu(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu \frac{Np-(N+\beta)}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx = 0.$$

Lemma 2.4. ([35], Page 107:(6) and (9)) Let $N \geq 3$. If $s \in (1, \frac{N}{\alpha})$, then for every $v \in L^s(\mathbb{R}^N)$, $I_\alpha * v \in L^{\frac{Ns}{N-\alpha s}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * v|^{\frac{Ns}{N-\alpha s}} dx \leq C(N, \alpha, s) \left(\int_{\mathbb{R}^N} |v|^s dx \right)^{\frac{N}{N-\alpha s}},$$

where $C(N, \alpha, s) > 0$ depends only on N, α , and s .

Lemma 2.5. Let $N \geq 3$ and $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$. Assume that $u_n \rightarrow u$ a.e. $x \in \mathbb{R}^N$ and $\sup_{n \in \mathbb{N}} \|u_n\|_{\frac{2Np}{N+\alpha}} < +\infty$. Then for any $v \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^{p-2} u_n v dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^{p-2} u v dx. \quad (2.6)$$

Proof. Note that $|u_n|^p \in L^s(\mathbb{R}^N)$ with $s = \frac{2N}{N+\alpha} \in (1, \frac{N}{\alpha})$ and $\frac{Ns}{N-\alpha} = \frac{2N}{N-\alpha}$. Then, by Lemma 2.4, we have $I_\alpha * |u_n|^p \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |I_\alpha * |u_n|^p|^{\frac{2N}{N-\alpha}} dx \leq C(N, \alpha, s) \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N-\alpha}}. \quad (2.7)$$

By Hölder inequality and (2.7), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} |(I_\alpha * |u_n|^p) |u_n|^{p-2} u_n|^{\frac{2Np}{2Np-(N+\alpha)}} dx \\ & \leq \left(\int_{\mathbb{R}^N} |I_\alpha * |u_n|^p|^{\frac{2N}{N-\alpha}} dx \right)^{\frac{(N-\alpha)p}{2Np-(N+\alpha)}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{(N+\alpha)(p-1)}{2Np-(N+\alpha)}} \\ & \leq C_1 \left[\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N-\alpha}} \right]^{\frac{(N-\alpha)p}{2Np-(N+\alpha)}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{(N+\alpha)(p-1)}{2Np-(N+\alpha)}} \\ & = C_1 \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\alpha}} dx \right)^{\frac{(N+\alpha)(2p-1)}{2Np-(N+\alpha)}}. \end{aligned}$$

This, together with [8, Lemma 2.6.], shows that

$$(I_\alpha * |u_n|^p) |u_n|^{p-2} u_n \rightharpoonup (I_\alpha * |u|^p) |u|^{p-2} u \text{ in } L^{\frac{2Np}{2Np-(N+\alpha)}}(\mathbb{R}^N). \quad (2.8)$$

It follows (2.8) that (2.6) holds. $\square$

Let H be a real Hilbert space whose norm and scalar product will be denoted respectively by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$. Let E be a real Banach space with norm $\|\cdot\|_E$. We assume throughout this section that

$$E \hookrightarrow H \hookrightarrow E^* \quad (2.9)$$

with continuous injections, where E^* is the dual space of E . Thus H is identified with its dual space. We will always assume in the sequel that E and H are infinite dimensional spaces. We consider the manifold

$$M := \{u \in E : \|u\|_H = 1\}. \quad (2.10)$$

M is the trace of the unit sphere of H in E and is, in general, unbounded. Throughout the paper, M will be endowed with the topology inherited from E . Moreover M is a submanifold of E of codimension 1 and its tangent space at a given point $u \in M$ can be considered as a closed subspace of E of codimension 1, namely

$$T_u M := \{v \in E : (u, v)_H = 0\}. \quad (2.11)$$

We consider a functional $\varphi : E \rightarrow \mathbb{R}$ which is of class $\mathcal{C}^1$ on E . We denote by $\varphi|_M$ the trace of φ on M . Then $\varphi|_M$ is a $\mathcal{C}^1$ functional on M , and for any $u \in M$,

$$\langle \varphi|_M'(u), v \rangle = \langle \varphi'(u), v \rangle, \forall v \in T_u M. \quad (2.12)$$

In the sequel, for any $u \in M$, we define the norm $\|\varphi|_M'(u)\|$ by

$$\|\varphi|_M'(u)\| = \sup_{v \in T_u M, \|v\|_E=1} |\langle \varphi'(u), v \rangle|. \quad (2.13)$$

Let $E \times \mathbb{R}$ be equipped with the scalar product

$$((u, \tau), (v, \sigma))_{E \times \mathbb{R}} := (u, v)_E + \tau\sigma, \quad \forall (u, \tau), (v, \sigma) \in E \times \mathbb{R},$$

and corresponding norm

$$\|(u, \tau)\|_{E \times \mathbb{R}} := \sqrt{\|u\|_E^2 + \tau^2}, \quad \forall (u, \tau) \in E \times \mathbb{R}.$$

Next, we consider a functional $\tilde{\varphi} : E \times \mathbb{R} \rightarrow \mathbb{R}$ which is of class $\mathcal{C}^1$ on $E \times \mathbb{R}$. We denote by $\varphi|_{M \times \mathbb{R}}$ the trace of $\tilde{\varphi}$ on $M \times \mathbb{R}$. Then $\varphi|_{M \times \mathbb{R}}$ is a $\mathcal{C}^1$ functional on $M \times \mathbb{R}$, and for any $(u, \tau) \in M \times \mathbb{R}$,

$$\langle \tilde{\varphi}'|_{M \times \mathbb{R}}(u, \tau), (v, \sigma) \rangle := \langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle, \quad \forall (v, \sigma) \in \tilde{T}_{(u, \tau)}(M \times \mathbb{R}), \quad (2.14)$$

where

$$\tilde{T}_{(u, \tau)}(M \times \mathbb{R}) := \{(v, \sigma) \in E \times \mathbb{R} : (u, v)_H = 0\}. \quad (2.15)$$

In the sequel, for any $(u, \tau) \in M \times \mathbb{R}$, we define the norm $\|\tilde{\varphi}'|_{M \times \mathbb{R}}(u)\|$ by

$$\|\tilde{\varphi}'|_{M \times \mathbb{R}}(u, \tau)\| = \sup_{(v, \sigma) \in \tilde{T}_{(u, \tau)}(M \times \mathbb{R}), \|(v, \sigma)\|_{E \times \mathbb{R}}=1} |\langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle|. \quad (2.16)$$

Lemma 2.6. [37, Lemma 3] *Let $\varphi \in \mathcal{C}^1(E, \mathbb{R})$ and $\{u_n\} \subset M$ be a bounded sequence in E . Then the following are equivalent:*

- (i) $\|\varphi'_M(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\varphi'(u_n) - \langle \varphi'(u_n), u_n \rangle u_n$ in E^* as $n \rightarrow \infty$.

Lemma 2.7. [38, Corollary 2.15] *Assume that $\tilde{\theta} \in \mathbb{R}$, $\tilde{\varphi} \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R})$ and $\tilde{\Upsilon} \subset M \times \mathbb{R}$ is closed set. Let*

$$\tilde{\Gamma} := \left\{ \tilde{\gamma} \in \mathcal{C}([0, 1], M \times \mathbb{R}) : \tilde{\gamma}(0) \in \tilde{\Upsilon}, \tilde{\varphi}(\tilde{\gamma}(1)) < \tilde{\theta} \right\}. \quad (2.17)$$

Assume that $\tilde{\varphi}$ satisfies

$$\tilde{a} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0, 1]} \tilde{\varphi}(\tilde{\gamma}(t)) > \tilde{b} := \sup_{\gamma \in \Gamma} \max\{\tilde{\varphi}(\tilde{\gamma}(0)), \tilde{\varphi}(\tilde{\gamma}(1))\}. \quad (2.18)$$

Let $\{\tilde{\gamma}_n\} \subset \tilde{\Gamma}$ be such that

$$\sup_{t \in [0, 1]} \tilde{\varphi}(\tilde{\gamma}_n(t)) \leq \tilde{a} + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (2.19)$$

Then there exists a sequence $(v_n, \tau_n) \subset M \times \mathbb{R}$ satisfying

- (i) $\tilde{a} - \frac{2}{n} \leq \tilde{\varphi}(v_n, \tau_n) \leq \tilde{a} + \frac{2}{n}$;
- (ii) $\min_{t \in [0, 1]} \|(v_n, \tau_n) - \tilde{\gamma}_n(t)\|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
- (iii) $\|\tilde{\varphi}'|_{M \times \mathbb{R}}(v_n, \tau_n)\| \leq \frac{8}{\sqrt{n}}$.

3 L^2 -subcritical perturbation case

In this section, we will study the second solution of equation (1.1) under the basis of Theorem 1.1. Assume that $\mu > 0$ and let $c_0, \rho_0 := \rho(c_0)$, and V_c be given in (1.18), (1.21), and (1.20). For any $c \in (0, c_0)$, the set $V_c \subset \mathcal{S}_c$ have the property:

$$m_\mu(c) := \inf_{u \in V_c} \Phi_\mu(u) < 0 < \inf_{u \in \partial V_c} \Phi_\mu(u), \quad (3.1)$$

where $m_\mu(c)$ is reached by a positive, radially symmetric non-increasing function, denoted $u_c \in V_c$ that satisfies, for a $\lambda_c > 0$,

$$-\Delta u_c + \lambda_c u_c - (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}})|u_c|^{\frac{N+\alpha}{N-2}-2}u_c - \mu(I_\beta * |u_c|^p)|u_c|^{p-2}u_c = 0, \quad x \in \mathbb{R}^N. \quad (3.2)$$

To apply Lemma 2.7, we let $E = H_{\text{rad}}^1(\mathbb{R}^N)$ and $H = L^2(\mathbb{R}^N)$. Define the inner product on E and H as

$$(u, v)_E := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) dx, \quad (u, v)_H := \frac{1}{c} \int_{\mathbb{R}^N} uv dx, \quad \forall u, v \in E,$$

and the corresponding norms as

$$\|u\|_E := \sqrt{(u, u)_E}, \quad \|u\|_H := \sqrt{(u, u)_H}, \quad \forall u \in E.$$

After identifying H with its dual, we have $E \hookrightarrow H \hookrightarrow E^*$ with continuous injections. From the definition of M given in (2.10), one has

$$M := \{u \in E : \|u\|_H = 1\} = \left\{ u \in E : \int_{\mathbb{R}^N} u^2 dx = c \right\} = \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N).$$

Obviously, $\Phi_\mu \in \mathcal{C}^1(E, \mathbb{R})$, and

$$\langle \Phi'_\mu(u), u \rangle = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}})|u|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |u|^p)|u|^p dx.$$

Let us define a continuous map $\eta : H_{\text{rad}}^1(\mathbb{R}^N) \times \mathbb{R} \rightarrow H_{\text{rad}}^1(\mathbb{R}^N)$ by

$$\eta(v, t)(x) := e^{Nt/2} v(e^t x) \text{ for } v \in H_{\text{rad}}^1(\mathbb{R}^N), \quad \forall t \in \mathbb{R}, \quad x \in \mathbb{R}^N,$$

and consider the following auxiliary functional:

$$\begin{aligned} \tilde{\Phi}_\mu(v, t) := \Phi_\mu(\eta(v, t)(x)) &= \frac{1}{2} e^{2t} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{N-2}{2(N+\alpha)} e^{\frac{2(N+\alpha)}{N-2}t} \int_{\mathbb{R}^N} (I_\alpha * |v|^{\frac{N+\alpha}{N-2}})|v|^{\frac{N+\alpha}{N-2}} dx \\ &\quad - \frac{\mu}{2p} e^{Npt-(N+\beta)t} \int_{\mathbb{R}^N} (I_\beta * |v|^p)|v|^p dx. \end{aligned} \quad (3.3)$$

We see that $\tilde{\Phi}_\mu$ is of class $\mathcal{C}^1$, and for any $(w, s) \in H_{\text{rad}}^1(\mathbb{R}^N) \times \mathbb{R}$,

$$\begin{aligned} \langle \tilde{\Phi}'_\mu(v, t), (w, s) \rangle &= \langle \tilde{\Phi}'_\mu(v, t), (w, 0) \rangle + \langle \tilde{\Phi}'_\mu(v, t), (0, s) \rangle \\ &= e^{2t} \int_{\mathbb{R}^N} \nabla v \cdot \nabla w dx - e^{\frac{2(N+\alpha)}{N-2}t} \int_{\mathbb{R}^N} (I_\alpha * |v|^{\frac{N+\alpha}{N-2}})|v|^{\frac{N+\alpha}{N-2}-2} v w dx \\ &\quad - \mu e^{Npt-(N+\beta)t} \int_{\mathbb{R}^N} (I_\beta * |v|^p)|v|^{p-2} v w dx + s \left(e^{2t} \int_{\mathbb{R}^N} |\nabla v|^2 dx \right. \\ &\quad \left. - e^{\frac{2(N+\alpha)}{N-2}t} \int_{\mathbb{R}^N} (I_\alpha * |v|^{\frac{N+\alpha}{N-2}})|v|^{\frac{N+\alpha}{N-2}} dx \right. \\ &\quad \left. - \mu \frac{Np - (N+\beta)}{2p} e^{Npt-(N+\beta)t} \int_{\mathbb{R}^N} (I_\beta * |v|^p)|v|^p dx \right) \\ &= \langle \Phi'_\mu(\eta(v, t)), \eta(w, t) \rangle + s \mathcal{P}_\mu(\eta(v, t)). \end{aligned} \quad (3.4)$$

Let

$$u(x) := \eta(v, t) = e^{Nt/2} v(e^t x), \quad \phi(x) := \eta(w, t) = e^{Nt/2} w(e^t x). \quad (3.5)$$

Then

$$(u, \phi)_H = \frac{1}{c} \int_{\mathbb{R}^N} u(x) \phi(x) dx = \frac{1}{c} \int_{\mathbb{R}^N} v(y) w(y) dy = (v, w)_H.$$

This shows that

$$\phi \in T_u(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \iff (w, s) \in \tilde{T}_{(v, \tau)}((\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}), \quad \forall t, s \in \mathbb{R}. \quad (3.6)$$

It follows from (3.4), (3.5), and (3.6) that

$$|\mathcal{P}_\mu(u)| = \left| \langle \tilde{\Phi}'_\mu(v, t), (0, 1) \rangle \right| \leq \left\| \tilde{\Phi}'_\mu|_{(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}}(v, t) \right\| \quad (3.7)$$

and

$$\begin{aligned} \left\| \Phi'_\mu|_{\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)}(u) \right\| &= \sup_{\phi \in T_u(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N))} \frac{1}{\|\phi\|_E} |\langle \Phi'_\mu(u), \phi \rangle| \\ &= \sup_{\phi \in T_u(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N))} \frac{1}{\sqrt{\|\nabla \phi\|_2^2 + \|\phi\|_2^2}} |\langle \Phi'_\mu(\eta(v, t)), \eta(w, t) \rangle| \\ &= \sup_{\phi \in T_u(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N))} \frac{1}{\sqrt{\|\nabla \phi\|_2^2 + \|\phi\|_2^2}} |\langle \tilde{\Phi}'_\mu(v, t), (w, 0) \rangle| \\ &\leq \sup_{(w, 0) \in \tilde{T}_{(v, t)}((\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R})} \frac{e^{|t|}}{\|(w, 0)\|_{E \times \mathbb{R}}} |\langle \tilde{\Phi}'_\mu(v, t), (w, 0) \rangle| \\ &\leq e^{|t|} \left\| \tilde{\Phi}'_\mu|_{(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}}(v, t) \right\|. \end{aligned} \quad (3.8)$$

Lemma 3.1. *For any $\mu > 0$ and any $c \in (0, c_0)$, there exists $\kappa > 0$ such that*

$$M_\mu(c) := \inf_{\gamma \in \Gamma_c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \kappa > \sup_{\gamma \in \Gamma_c} \max\{\Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1))\}, \quad (3.9)$$

where

$$\Gamma_c := \{\gamma \in \mathcal{C}([0, 1], \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) : \gamma(0) = u_c, \Phi_\mu(\gamma(1)) < 2m_\mu(c)\}. \quad (3.10)$$

Proof. Set $\kappa := \inf_{u \in \partial V_c} \Phi_\mu(u)$. By (3.1), $\kappa > 0$. Let $\gamma \in \Gamma_c$ be arbitrary. Since $\gamma(0) = u_c \in V_c$, we have $\|\nabla \gamma(0)\|_2^2 < \rho_0$. If $\gamma(1) \in V_c$, from $\Phi_\mu(\gamma(1)) < 2m_\mu(c) < 0$, we can deduce

$$m_\mu(c) = \inf_{u \in V_c} \Phi_\mu(u) \leq \Phi_\mu(\gamma(1)) < 2m_\mu(c) < 0,$$

which is a contradiction. Therefore $\gamma(1) \notin V_c$. According to $\Phi_\mu(\gamma(1)) < 2m_\mu(c) < 0 < \inf_{u \in \partial V_c} \Phi_\mu(u)$, we have $\gamma(1) \notin \partial V_c$. By continuity of $\gamma(t)$ on $[0, 1]$ and $\|\nabla \gamma(0)\|_2^2 < \rho_0$, $\|\nabla \gamma(1)\|_2^2 > \rho_0$, there exists $t_0 \in (0, 1)$ such that $\|\nabla \gamma(t_0)\|_2^2 = \rho_0$ and so

$$\max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \Phi_\mu(\gamma(t_0)) \geq \inf_{u \in \partial V_c} \Phi_\mu(u) = \kappa.$$

Due to the arbitrariness of γ , it follows that

$$M_\mu(c) = \inf_{\gamma \in \Gamma_c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \kappa > 0.$$

By (3.1) we have

$$\Phi_\mu(\gamma(0)) = \Phi_\mu(u_c) = m_\mu(c) < 0. \quad (3.11)$$

From (3.10) and (3.11), one has

$$\sup_{\gamma \in \Gamma_c} \max\{\Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1))\} \leq 0 < \kappa.$$

Thus (3.9) holds. □

Lemma 3.2. *For any $\mu > 0$ and any $c \in (0, c_0)$, there exists a sequence $\{u_n\} \subset \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ such that*

$$\Phi_\mu(u_n) \rightarrow M_\mu(c) > 0, \quad \Phi'_\mu|_{\mathcal{S}_c}(u_n) \rightarrow 0, \quad \text{and} \quad \mathcal{P}_\mu(u_n) \rightarrow 0. \quad (3.12)$$

Proof. By Theorem 1.1, $u_c \in \mathcal{S}_c \cap H_{\text{rad}}(\mathbb{R}^N)$. Let $\tilde{\Phi}_\mu$ be defined by (3.3),

$$\tilde{\Gamma}_c := \left\{ \tilde{\gamma} \in \mathcal{C}([0, 1], (\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}) : \tilde{\gamma}(0) = (u_c, 0), \tilde{\Phi}_\mu(\tilde{\gamma}(1)) < 2m_\mu(c) \right\}, \quad (3.13)$$

and

$$\tilde{M}_\mu(c) := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0, 1]} \tilde{\Phi}_\mu(\tilde{\gamma}(t)). \quad (3.14)$$

It is easy to verify that $\Gamma_c = \{\eta \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}_c\}$. By (3.9), there exists $\kappa > 0$ such that

$$\begin{aligned} \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0, 1]} \tilde{\Phi}_\mu(\tilde{\gamma}(t)) &= \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0, 1]} \Phi_\mu(\eta(\tilde{\gamma}(t))) \\ &= \inf_{\gamma \in \Gamma_c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \\ &\geq \kappa > \sup_{\gamma \in \Gamma_c} \max\{\Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1))\} \\ &= \sup_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max\{\Phi_\mu(\eta(\tilde{\gamma}(0))), \Phi_\mu(\eta(\tilde{\gamma}(1)))\} \\ &= \sup_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max\{\tilde{\Phi}_\mu(\tilde{\gamma}(0)), \tilde{\Phi}_\mu(\tilde{\gamma}(1))\}. \end{aligned}$$

This shows that $\tilde{M}_\mu(c) = M_\mu(c)$ and (2.18) holds with $\tilde{\varphi} = \tilde{\Phi}_\mu$.

For any $n \in \mathbb{N}$, (3.9) implies that there exists $\gamma_n \in \Gamma_c$ such that

$$\max_{t \in [0, 1]} \Phi_\mu(\gamma_n(t)) \leq M_\mu(c) + \frac{1}{n}.$$

Set $\tilde{\gamma}_n = (\gamma_n, 0)$. Then $\tilde{\gamma}_n \in \tilde{\Gamma}_c$. It is easy to see

$$\begin{aligned} \max_{t \in [0, 1]} \tilde{\Phi}_\mu(\tilde{\gamma}_n(t)) &= \max_{t \in [0, 1]} \tilde{\Phi}_\mu(\gamma_n(t), 0) = \max_{t \in [0, 1]} \Phi_\mu(\eta(\gamma_n(t), 0)) = \max_{t \in [0, 1]} \Phi_\mu(\gamma_n(t)) \\ &\leq M_\mu(c) + \frac{1}{n} = \tilde{M}_\mu(c) + \frac{1}{n}. \end{aligned}$$

Then applying Lemma 2.7 to $\tilde{\Phi}$, there exists a sequence $\{(v_n, t_n)\} \subset (\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}$ satisfying

- (i) $M_\mu(c) - \frac{2}{n} = \tilde{M}_\mu(c) - \frac{2}{n} \leq \tilde{\Phi}_\mu(v_n, t_n) \leq \tilde{M}_\mu(c) + \frac{2}{n} = M_\mu(c) + \frac{2}{n};$
- (ii) $\min_{t \in [0, 1]} \|(v_n, t_n) - (\gamma_n, 0)\|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}};$
- (iii) $\left\| \tilde{\Phi}_\mu|_{(\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)) \times \mathbb{R}}(v_n, t_n) \right\| \leq \frac{8}{\sqrt{n}}.$

Let $u_n = \eta(v_n, t_n)$. It follows from (i)-(iii), (3.7), and (3.8) that $\{u_n\} \subset \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ is a (PS) sequence for Φ_μ restricted to $\mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ at level $M_\mu(c)$, with $\mathcal{P}_\mu(u_n) \rightarrow 0$. Since the problem is invariant under rotations, $\{u_n\} \subset \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ is also the (PS) sequence for Φ_μ restricted to $\mathcal{S}_c$ at level $M_\mu(c)$, with $\mathcal{P}_\mu(u_n) \rightarrow 0$. □

Let $K_N := [N(N-2)]^{(N-2)/4}$. Now we define functions $U_n(x) := \Theta_n(|x|)$, where

$$\Theta_n(r) = K_N \begin{cases} \left(\frac{n}{1+n^2 r^2} \right)^{\frac{N-2}{2}}, & 0 \leq r < 1; \\ \left(\frac{n}{1+n^2} \right)^{\frac{N-2}{2}} (2-r), & 1 \leq r < 2; \\ 0, & r \geq 2. \end{cases} \quad (3.15)$$

Computing directly, we have

$$\|U_n\|_2^2 = O\left(\frac{\xi(n)}{n^2}\right), \quad n \rightarrow \infty, \quad (3.16)$$

$$\xi(n) := \int_0^n \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \begin{cases} O(n), & N = 3; \\ O(\log(1+n^2)), & N = 4; \\ O(1), & N \geq 5, \end{cases} \quad (3.17)$$

$$\|\nabla U_n\|_2^2 = (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} + O\left(\frac{1}{n^{N-2}}\right), \quad n \rightarrow \infty, \quad (3.18)$$

$$\int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx = (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} - O\left(\frac{1}{n^{(N+\alpha)/2}}\right), \quad (3.19)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\beta * U_n^t) U_n^s dx \\ & \geq \mathcal{A}_{N,\beta} K_N^{t+s} \int_{B_1(0)} \int_{B_1(0)} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{(N-2)t}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{(N-2)s}{2}}}{|x-y|^{N-\beta}} dx dy \\ & = \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \int_{B_n(0)} \int_{B_n(0)} \frac{\left(\frac{1}{1+|z|^2}\right)^{\frac{(N-2)t}{2}} \left(\frac{1}{1+|w|^2}\right)^{\frac{(N-2)s}{2}}}{|z-w|^{N-\beta}} dz dw \\ & \geq \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \int_{B_1(0)} \int_{B_1(0)} \frac{\left(\frac{1}{1+|z|^2}\right)^{\frac{(N-2)t}{2}} \left(\frac{1}{1+|w|^2}\right)^{\frac{(N-2)s}{2}}}{|z-w|^{N-\beta}} dz dw \\ & \geq \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \frac{1}{2^{(N-\beta)+(N-2)(t+s)/2}} \left(\int_{B_1(0)} 1 dz\right)^2 \\ & = O\left(\frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}}\right), \quad \forall t, s \in \mathbb{R}. \end{aligned} \quad (3.20)$$

Both (3.16) and (3.18) imply that $U_n \in H_{\text{rad}}^1(\mathbb{R}^N)$.

Lemma 3.3. *Let $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$, $\mu > 0$, and $c \in (0, c_0)$. Then there holds:*

$$M_\mu(c) < m_\mu(c) + \frac{2+\alpha}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}}, \quad (3.21)$$

if one of the following conditions holds:

- (i) $N = 3, 4$ with $\alpha, \beta \in (0, N)$, and $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$;
- (ii) $N = 5$ with $\alpha \in [1, 2) \cup [4, 5)$, $\beta \in (0, N)$, and $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$;
- (iii) $N = 5$ with $\alpha \in (0, 1) \cup [2, 4)$, $\beta \in (0, \frac{7}{4})$, $4\beta - 5\alpha < 2$, and

$$\max\left\{\frac{7+2\beta}{6}, \frac{8+\beta}{6}, \frac{8+2\beta-\alpha}{6}\right\} < p < \frac{7+\beta}{5};$$

- (iv) $N \geq 6$ with $\alpha \in (0, N)$, $\beta \in (0, N-2)$, $4\beta - N\alpha < 2N - 8$, and

$$\max\left\{\frac{N+\beta-2}{N-2}, \frac{2N-2+\beta}{2N-4}, \frac{2N-2+2\beta-\alpha}{2N-4}\right\} < p < \frac{N+\beta+2}{N}.$$

Proof. By Lemma 2.3 and (3.2), we can deduce that

$$\Phi_\mu(u_c) = m_\mu(c), \quad u_c(x) > 0, \quad \forall x \in \mathbb{R}^N, \quad (3.22)$$

$$-\lambda_c \|u_c\|_2^2 = \mu \frac{(N-2)p - (N+\beta)}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx, \quad (3.23)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_n dx &= -\lambda_c \int_{\mathbb{R}^N} u_c U_n dx + \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \\ &\quad + \mu \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-2} u_c U_n dx. \end{aligned} \quad (3.24)$$

Set $b := \inf_{|x| \leq 1} u_c(x)$ and $B := \sup_{|x| \leq 2} u_c(x)$. Then $0 < b \leq B < +\infty$. Hence, it follows from (3.15) that

$$\int_{\mathbb{R}^N} u_c U_n dx \leq B \int_{|x| \leq 2} U_n dx = O\left(\frac{1}{n^{(N-2)/2}}\right), \quad n \rightarrow \infty, \quad (3.25)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\beta * u_c^p)(u_c^{p-1} U_n) dx &\leq C_{N,\beta} \left(\int_{\mathbb{R}^N} |u_c^p|^{\frac{2N}{N+\beta}} dx \right)^{\frac{N+\beta}{2N}} \left(\int_{\mathbb{R}^N} |u_c^{p-1} U_n|^{\frac{2N}{N+\beta}} dx \right)^{\frac{N+\beta}{2N}} \\ &\leq C_2 \left(\int_{|x| \leq 2} U_n^{\frac{2N}{N+\beta}} dx \right)^{\frac{N+\beta}{2N}} \\ &= C_2 \left[N\theta(N) K_N^{\frac{2N}{N+\beta}} \int_0^1 \left(\frac{n}{1+n^2 r^2} \right)^{\frac{N(N-2)}{N+\beta}} r^{N-1} dr \right. \\ &\quad \left. + N\theta(N) K_N^{\frac{2N}{N+\beta}} \left(\frac{n}{1+n^2} \right)^{\frac{N(N-2)}{N+\beta}} \int_1^2 (2-r)^{\frac{2N}{N+\beta}} r^{N-1} dr \right]^{\frac{N+\beta}{2N}} \\ &= C_2 \left[N\theta(N) K_N^{\frac{2N}{N+\beta}} n^{\frac{N(N-2)}{N+\beta}-N} \int_0^n \frac{s^{N-1}}{(1+s^2)^{\frac{N(N-2)}{N+\beta}}} ds \right. \\ &\quad \left. + N\theta(N) K_N^{\frac{2N}{N+\beta}} \left(\frac{n}{1+n^2} \right)^{\frac{N(N-2)}{N+\beta}} \int_0^1 s^{\frac{2N}{N+\beta}} (2-s)^{N-1} ds \right]^{\frac{N+\beta}{2N}} \\ &= O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right), \end{aligned} \quad (3.26)$$

where

$$\chi_\beta = \begin{cases} 1, & \text{if } \beta = N-4, \\ 0, & \text{if } \beta \neq N-4. \end{cases} \quad (3.27)$$

For any $t > 0$,

$$\begin{aligned} \|u_c + tU_n\|_2^2 &= \|u_c\|_2^2 + t^2 \|U_n\|_2^2 + 2t \int_{\mathbb{R}^N} u_c U_n dx \\ &= c + 2t \int_{\mathbb{R}^N} u_c U_n dx + t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right], \quad n \rightarrow \infty. \end{aligned} \quad (3.28)$$

Let $\tau := \|u_c + tU_n\|_2 / \sqrt{c}$. Then

$$\tau^2 = 1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right], \quad n \rightarrow \infty. \quad (3.29)$$

We define

$$W_{n,t}(x) := \tau^{(N-2)/2} [u_c(\tau x) + tU_n(\tau x)]. \quad (3.30)$$

Then

$$\|\nabla W_{n,t}\|_2^2 = \|\nabla(u_c + tU_n)\|_2^2, \quad \|W_{n,t}\|_2^2 = \tau^{-2} \|u_c + tU_n\|_2^2 = c, \quad (3.31)$$

$$\int_{\mathbb{R}^N} (I_\beta * |W_{n,t}|^p) |W_{n,t}|^p dx = \tau^{p(N-2)-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * |u_c + tU_n|^p) |u_c + tU_n|^p dx, \quad (3.32)$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |W_{n,t}|^{\frac{N+\alpha}{N-2}}) |W_{n,t}|^{\frac{N+\alpha}{N-2}} dx = \int_{\mathbb{R}^N} (I_\alpha * |u_c + tU_n|^{\frac{N+\alpha}{N-2}}) |u_c + tU_n|^{\frac{N+\alpha}{N-2}} dx. \quad (3.33)$$

It is easy to verify the following inequalities:

$$(a+b)^q \geq \begin{cases} a^q + qa^{q-1}b, & \text{if } q \geq 1, a > 0, b > 0; \\ a^q + b^q, & \text{if } q \geq 1, a > 0, b > 0; \\ a^q + qa^{q-1}b + b^q, & \text{if } q \geq 2, a > 0, b > 0; \\ a^q + qa^{q-1}b + qab^{q-1} + b^q, & \text{if } q \geq 3, a > 0, b > 0. \end{cases} \quad (3.34)$$

From (3.34), we can derive that

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\beta * |W_{n,t}|^p) |W_{n,t}|^p dx \\ &= \tau^{p(N-2)-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * |u_c + tU_n|^p) |u_c + tU_n|^p dx \\ &\geq \tau^{p(N-2)-(N+\beta)} \int_{\mathbb{R}^N} [I_\beta * (|u_c|^p + tp|u_c|^{p-1}U_n)] (|u_c|^p + tp|u_c|^{p-1}U_n) dx \\ &= \tau^{p(N-2)-(N+\beta)} \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx + 2pt \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1}U_n dx \right. \\ &\quad \left. + p^2t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1}U_n)] |u_c|^{p-1}U_n dx \right) \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\beta * |W_{n,t}|^p) |W_{n,t}|^p dx \\ &\geq \tau^{p(N-2)-(N+\beta)} \int_{\mathbb{R}^N} [I_\beta * (|u_c|^p + t^pU_n^p)] (|u_c|^p + t^pU_n^p) dx \\ &= \tau^{p(N-2)-(N+\beta)} \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx + 2t^p \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) U_n^p dx \right. \\ &\quad \left. + t^{2p} \int_{\mathbb{R}^N} (I_\beta * U_n^p) U_n^p dx \right). \end{aligned} \quad (3.36)$$

When $N = 3$ with $\alpha \in (0, 3)$, or $N = 4$ with $\alpha \in [2, 4)$, or $N = 5$ with $\alpha \in [4, 5)$, it follows that $\frac{N+\alpha}{N-2} \geq 3$. Combining this with (3.34), we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\alpha * |W_{n,t}|^{\frac{N+\alpha}{N-2}}) |W_{n,t}|^{\frac{N+\alpha}{N-2}} dx \\ &\geq \int_{\mathbb{R}^N} \left[I_\alpha * \left(u_c^{\frac{N+\alpha}{N-2}} + \frac{N+\alpha}{N-2} u_c^{\frac{2+\alpha}{N-2}} (tU_n) + \frac{N+\alpha}{N-2} u_c (tU_n)^{\frac{2+\alpha}{N-2}} + (tU_n)^{\frac{N+\alpha}{N-2}} \right) \right] \\ &\quad \times \left(u_c^{\frac{N+\alpha}{N-2}} + \frac{N+\alpha}{N-2} u_c^{\frac{2+\alpha}{N-2}} (tU_n) + \frac{N+\alpha}{N-2} u_c (tU_n)^{\frac{2+\alpha}{N-2}} + (tU_n)^{\frac{N+\alpha}{N-2}} \right) dx \\ &= \int_{\mathbb{R}^N} (I_\alpha * u_c^{\frac{N+\alpha}{N-2}}) u_c^{\frac{N+\alpha}{N-2}} dx + \left(\frac{N+\alpha}{N-2} \right)^2 t^2 \int_{\mathbb{R}^N} [I_\alpha * (u_c^{\frac{2+\alpha}{N-2}} U_n)] (u_c^{\frac{2+\alpha}{N-2}} U_n) dx \\ &\quad + \left(\frac{N+\alpha}{N-2} \right)^2 t^{\frac{4+2\alpha}{N-2}} \int_{\mathbb{R}^N} [I_\alpha * (u_c U_n^{\frac{2+\alpha}{N-2}})] (u_c U_n^{\frac{2+\alpha}{N-2}}) dx + t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\ &\quad + \frac{2(N+\alpha)}{N-2} t \int_{\mathbb{R}^N} (I_\alpha * u_c^{\frac{N+\alpha}{N-2}}) (u_c^{\frac{2+\alpha}{N-2}} U_n) dx + \frac{2(N+\alpha)}{N-2} t^{\frac{2+\alpha}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * u_c^{\frac{N+\alpha}{N-2}}) (u_c U_n^{\frac{2+\alpha}{N-2}}) dx \\ &\quad + 2t^{\frac{N+\alpha}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * u_c^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx + 2 \left(\frac{N+\alpha}{N-2} \right)^2 t^{\frac{2+\alpha}{N-2}+1} \int_{\mathbb{R}^N} [I_\alpha * (u_c^{\frac{2+\alpha}{N-2}} U_n)] (u_c U_n^{\frac{2+\alpha}{N-2}}) dx \\ &\quad + \frac{2(N+\alpha)}{N-2} t^{\frac{N+\alpha}{N-2}+1} \int_{\mathbb{R}^N} [I_\alpha * (u_c^{\frac{2+\alpha}{N-2}} U_n)] U_n^{\frac{N+\alpha}{N-2}} dx + \frac{2(N+\alpha)}{N-2} t^{\frac{N+2\alpha+2}{N-2}} \int_{\mathbb{R}^N} [I_\alpha * (u_c U_n^{\frac{2+\alpha}{N-2}})] U_n^{\frac{N+\alpha}{N-2}} dx. \end{aligned} \quad (3.37)$$

When $N = 4$ with $\alpha \in (0, 4)$, or $N = 5$ with $\alpha \in [1, 5)$, it follows that $\frac{N+\alpha}{N-2} \geq 2$. According to this fact and (3.34), we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} (I_\alpha * |W_{n,t}|^{\frac{N+\alpha}{N-2}}) |W_{n,t}|^{\frac{N+\alpha}{N-2}} dx \\
& \geq \int_{\mathbb{R}^N} \left[I_\alpha * \left(|u_c|^{\frac{N+\alpha}{N-2}} + \frac{N+\alpha}{N-2} t |u_c|^{\frac{2+\alpha}{N-2}} U_n + |tU_n|^{\frac{N+\alpha}{N-2}} \right) \right] \\
& \quad \times \left(|u_c|^{\frac{N+\alpha}{N-2}} + \frac{N+\alpha}{N-2} t |u_c|^{\frac{2+\alpha}{N-2}} U_n + |tU_n|^{\frac{N+\alpha}{N-2}} \right) dx \\
& = \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}} dx + t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^3} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\
& \quad + \frac{(N+\alpha)^2}{(N-2)^2} t^2 \int_{\mathbb{R}^N} [I_\alpha * (|u_c|^{\frac{2+\alpha}{N-2}} U_n)] |u_c|^{\frac{2+\alpha}{N-2}} U_n dx + \frac{2(N+\alpha)}{N-2} t \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{2+\alpha}{N-2}} U_n dx \\
& \quad + 2t^{\frac{N+\alpha}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx + \frac{2(N+\alpha)}{N-2} t^{\frac{N+\alpha}{N-2}+1} \int_{\mathbb{R}^N} [I_\alpha * (|u_c|^{\frac{2+\alpha}{N-2}} U_n)] U_n^{\frac{N+\alpha}{N-2}} dx. \quad (3.38)
\end{aligned}$$

Set

$$T_* := \left(\frac{N+\alpha}{N-2} \right)^{\frac{N-2}{4+2\alpha}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{-\frac{N(N-2)}{4(N+\alpha)}} \mathcal{S}_\alpha^{-\frac{(N-2)\alpha}{4(2+\alpha)}}.$$

Then

$$\begin{cases} 0 < \frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} \leq \frac{2+\alpha}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}, & 0 < t < T_*, \\ \frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} \leq 0, & t \geq T_*. \end{cases} \quad (3.39)$$

Case 1: $N = 3$ with $\alpha, \beta \in (0, 3)$; $N = 4$ with $\alpha \in [2, 4), \beta \in (0, 4)$; $N = 5$ with $\alpha \in [4, 5), \beta \in (0, 5)$. In this case, $\frac{N+\alpha}{N-2} \geq 3$. From (1.3), (3.17)-(3.20), (3.22)-(3.35), and (3.37), we get

$$\begin{aligned}
& \Phi_\mu(W_{n,t}) \\
& = \frac{1}{2} \|\nabla u_c\|_2^2 + \frac{t^2}{2} \|\nabla U_n\|_2^2 + t \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_n dx \\
& \quad - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_c + tU_n|^{\frac{N+\alpha}{N-2}}) |u_c + tU_n|^{\frac{N+\alpha}{N-2}} dx \\
& \quad - \frac{\mu}{2p} \tau^{(N-2)p-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * |u_c + tU_n|^p) |u_c + tU_n|^p dx \\
& \leq \frac{1}{2} \|\nabla u_c\|_2^2 + \frac{t^2}{2} \|\nabla U_n\|_2^2 + t \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \right. \\
& \quad \left. + \mu \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-2} u_c U_n dx - \lambda_c \int_{\mathbb{R}^N} u_c U_n dx \right) \\
& \quad - \frac{N-2}{2(N+\alpha)} \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}} dx + t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \right. \\
& \quad \left. + \frac{2(N+\alpha)}{N-2} t \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \right. \\
& \quad \left. + \frac{2(N+\alpha)}{N-2} t^{\frac{N+2\alpha+2}{N-2}} \int_{\mathbb{R}^N} [I_\alpha * (u_c U_n^{\frac{\alpha+2}{N-2}})] U_n^{\frac{N+\alpha}{N-2}} dx \right) \\
& \quad - \frac{\mu}{2p} \tau^{(N-2)p-(N+\beta)} \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx + 2pt \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \right. \\
& \quad \left. + p^2 t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1} U_n)] |u_c|^{p-1} U_n dx \right) \\
& = \Phi_\mu(u_c) + \frac{t^2}{2} \|\nabla U_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\
& \quad + \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 O\left(\frac{\xi(n)}{n^2}\right) \right]^{\frac{(N-2)p-(N+\beta)}{2}} \right\} \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \\
& \quad + \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 O\left(\frac{\xi(n)}{n^2}\right) \right]^{\frac{(N-2)p-(N+\beta)}{2}} \right\} \mu t \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx
\end{aligned}$$

$$\begin{aligned}
& + \left(\mu \frac{(N-2)p - (N+\beta)}{2pc} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \right) t \int_{\mathbb{R}^N} u_c U_n dx \\
& - \frac{\mu p}{2} \tau^{(N-2)p - (N+\beta)} t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1} U_n)] |u_c|^{p-1} U_n dx \\
& - t^{\frac{N+2\alpha+2}{N-2}} \int_{\mathbb{R}^N} [I_\alpha * (u_c U_n^{\frac{\alpha+2}{N-2}})] U_n^{\frac{N+\alpha}{N-2}} dx \\
& \leq \Phi_\mu(u_c) + \frac{t^2}{2} \|\nabla U_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\
& + \frac{(N+\beta) - (N-2)p}{4p} \mu t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \right) \\
& + \frac{(N+\beta) - (N-2)p}{c} \mu t^2 \left(\int_{\mathbb{R}^N} u_c U_n dx \right) \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \right) \\
& + \frac{(N+\beta) - (N-2)p}{2} \mu t^3 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \right) \\
& - \frac{\mu p}{2} \tau^{(N-2)p - (N+\beta)} t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1} U_n)] |u_c|^{p-1} U_n dx \\
& - t^{\frac{N+2\alpha+2}{N-2}} \int_{\mathbb{R}^N} [I_\alpha * (u_c U_n^{\frac{\alpha+2}{N-2}})] U_n^{\frac{N+\alpha}{N-2}} dx \\
& \leq m_\mu(c) + \left[\frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} \right] \\
& + t^2 \left[O\left(\frac{1}{n^{N-2}}\right) \right] + t^{\frac{2(N+\alpha)}{N-2}} \left[O\left(\frac{1}{n^{(N+\alpha)/2}}\right) \right] + t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \\
& + t^2 \left[O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \left[O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \\
& + t^3 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \left[O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \\
& - t^2 \left[O\left(\frac{1}{n^{\beta+2}}\right) \right] - t^{\frac{N+2\alpha+2}{N-2}} \left[O\left(\frac{1}{n^{(N-2)/2}}\right) \right]. \tag{3.40}
\end{aligned}$$

It follows from (3.39) and (3.40) that there exists $\bar{n}_1 \in \mathbb{N}$ such that

$$\sup_{t>0} \Phi_\mu(W_{\bar{n},t}) < m_\mu(c) + \frac{2+\alpha}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}. \tag{3.41}$$

Case 2: $N = 4$ with $\alpha \in (0, 2)$, $\beta \in (0, 4)$; $N = 5$ with $\alpha \in [1, 2)$, $\beta \in (0, 5)$. In this case, $\frac{N+\alpha}{N-2} \geq 2$. From (1.3), (3.17)-(3.20), (3.22)-(3.35), and (3.38), we get

$$\begin{aligned}
& \Phi_\mu(W_{n,t}) \\
& \leq \frac{1}{2} \|\nabla u_c\|_2^2 + \frac{t^2}{2} \|\nabla U_n\|_2^2 + t \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \right. \\
& + \mu \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-2} u_c U_n dx - \lambda_c \int_{\mathbb{R}^N} u_c U_n dx \Big) \\
& - \frac{N-2}{2(N+\alpha)} \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}} dx + t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \right. \\
& + \frac{2(N+\alpha)}{N-2} t \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \\
& + \frac{2(N+\alpha)}{N-2} t^{\frac{N+\alpha}{N-2}+1} \int_{\mathbb{R}^N} [I_\alpha * (|u_c|^{\frac{2+\alpha}{N-2}} U_n)] U_n^{\frac{N+\alpha}{N-2}} dx \Big) \\
& - \frac{\mu}{2p} \tau^{(N-2)p - (N+\beta)} \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx + 2pt \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \right. \\
& + p^2 t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1} U_n)] |u_c|^{p-1} U_n dx \Big)
\end{aligned}$$

$$\begin{aligned}
&= \Phi_\mu(u_c) + \frac{t^2}{2} \|\nabla U_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\
&\quad + \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 O\left(\frac{\xi(n)}{n^2}\right) \right]^{\frac{(N-2)p-(N+\beta)}{2}} \right\} \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \\
&\quad + \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 O\left(\frac{\xi(n)}{n^2}\right) \right]^{\frac{(N-2)p-(N+\beta)}{2}} \right\} \mu t \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \\
&\quad + \left(\mu \frac{(N-2)p-(N+\beta)}{2pc} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \right) t \int_{\mathbb{R}^N} u_c U_n dx \\
&\quad - \frac{\mu p}{2} \tau^{(N-2)p-(N+\beta)} t^2 \int_{\mathbb{R}^N} [I_\beta * (|u_c|^{p-1} U_n)] |u_c|^{p-1} U_n dx \\
&\quad - t^{\frac{N+\alpha}{N-2}+1} \int_{\mathbb{R}^N} [I_\alpha * (|u_c|^{\frac{2+\alpha}{N-2}} U_n)] U_n^{\frac{N+\alpha}{N-2}} dx \\
&\leq m_\mu(c) + \left[\frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} \right] \\
&\quad + t^2 \left[O\left(\frac{1}{n^{N-2}}\right) \right] + t^{\frac{2(N+\alpha)}{N-2}} \left[O\left(\frac{1}{n^{(N+\alpha)/2}}\right) \right] + t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \\
&\quad + t^2 \left[O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \left[O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \\
&\quad + t^3 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \left[O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \\
&\quad - t^2 \left[O\left(\frac{1}{n^{\beta+2}}\right) \right] - t^{\frac{N+\alpha}{N-2}+1} \left[O\left(\frac{1}{n^{(2+\alpha)/2}}\right) \right]. \tag{3.42}
\end{aligned}$$

It follows from (3.39) and (3.42) that there exists $\bar{n}_2 \in \mathbb{N}$ such that (3.41) holds.

Case 3: $N = 5$ with $\alpha \in (0, 1) \cup [2, 4)$, $\beta \in (0, \frac{7}{4})$, $4\beta - 5\alpha < 2$, $\max\left\{\frac{7+2\beta}{6}, \frac{8+\beta}{6}, \frac{8+2\beta-\alpha}{6}\right\} < p < \frac{7+\beta}{5}$; $N \geq 6$ with $\alpha \in (0, N)$, $\beta \in (0, N-2)$, $4\beta - N\alpha < 2N-8$, $\max\left\{\frac{N+\beta-2}{N-2}, \frac{2N-2+\beta}{2N-4}, \frac{2N-2+2\beta-\alpha}{2N-4}\right\} < p < \frac{N+\beta+2}{N}$. From (1.3), (3.17)-(3.20), (3.22)-(3.34), and (3.36), we get

$$\begin{aligned}
&\Phi_\mu(W_{n,t}) \\
&\leq \frac{1}{2} \|\nabla u_c\|_2^2 + \frac{t^2}{2} \|\nabla U_n\|_2^2 + t \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx \right. \\
&\quad \left. + \mu \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-2} u_c U_n dx - \lambda_c \int_{\mathbb{R}^N} u_c U_n dx \right) \\
&\quad - \frac{N-2}{2(N+\alpha)} \left(\int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}} dx + t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \right) \\
&\quad - \frac{\mu}{2p} \tau^{(N-2)p-(N+\beta)} \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx + t^{2p} \int_{\mathbb{R}^N} (I_\beta * U_n^p) U_n^p dx \right) \\
&= \Phi_\mu(u_c) + \frac{t^2}{2} \|\nabla U_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx \\
&\quad + \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_n dx + t^2 O\left(\frac{\xi(n)}{n^2}\right) \right]^{\frac{(N-2)p-(N+\beta)}{2}} \right\} \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \\
&\quad + \left(\mu \frac{(N-2)p-(N+\beta)}{2pc} \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \right) t \int_{\mathbb{R}^N} u_c U_n dx \\
&\quad + t \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx + \mu t \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \\
&\quad - \frac{\mu p}{2} \tau^{(N-2)p-(N+\beta)} t^{2p} \int_{\mathbb{R}^N} (I_\beta * U_n^p) U_n^p dx \\
&\leq \Phi_\mu(u_c) + \frac{t^2}{2} \|\nabla U_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * U_n^{\frac{N+\alpha}{N-2}}) U_n^{\frac{N+\alpha}{N-2}} dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N+\beta)-(N-2)p}{4p} \mu t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \left(\int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^p dx \right) \\
& + t \int_{\mathbb{R}^N} (I_\alpha * |u_c|^{\frac{N+\alpha}{N-2}}) |u_c|^{\frac{N+\alpha}{N-2}-2} u_c U_n dx + \mu t \int_{\mathbb{R}^N} (I_\beta * |u_c|^p) |u_c|^{p-1} U_n dx \\
& - \frac{\mu p}{2} \tau^{(N-2)p-(N+\beta)} t^{2p} \int_{\mathbb{R}^N} (I_\beta * U_n^p) U_n^p dx \\
\leq & m_\mu(c) + \left[\frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} \right] \\
& + t^2 \left[O\left(\frac{1}{n^{N-2}}\right) \right] + t^{\frac{2(N+\alpha)}{N-2}} \left[O\left(\frac{1}{n^{(N+\alpha)/2}}\right) \right] + t^2 \left[O\left(\frac{\xi(n)}{n^2}\right) \right] \\
& + t \left[O\left(\frac{1}{n^{(\alpha+2)/2}}\right) + \chi_\alpha O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] \\
& + t \left[O\left(\frac{1}{n^{(\beta+2)/2}}\right) + \chi_\beta O\left(\frac{(\ln n)^{\frac{N-2}{N}}}{n^{(N-2)/2}}\right) + O\left(\frac{1}{n^{(N-2)/2}}\right) \right] - t^{2p} \left[O\left(\frac{1}{n^{(N+\beta)-(N-2)p}}\right) \right]. \quad (3.43)
\end{aligned}$$

It follows from (3.39) and (3.43) that there exists $\bar{n}_3 \in \mathbb{N}$ such that (3.41) holds.

Next, we prove that (3.21) holds. Let $\bar{n} = \max\{\bar{n}_1, \bar{n}_2, \bar{n}_3\}$. By (3.28), (3.30), and (3.31), we have

$$W_{\bar{n},t}(x) := \tau^{(N-2)/2} [u_c(\tau x) + t U_{\bar{n}}(\tau x)], \quad \|W_{\bar{n},t}\|_2^2 = c \quad (3.44)$$

and

$$\|\nabla W_{\bar{n},t}\|_2^2 = \|\nabla(u_c + t U_{\bar{n}})\|_2^2 = \|\nabla u_c\|_2^2 + t^2 \|\nabla U_{\bar{n}}\|_2^2 + 2t \int_{\mathbb{R}^N} \nabla u_c \cdot \nabla U_{\bar{n}} dx, \quad (3.45)$$

where where

$$\tau^2 = \|u_c + t U_{\bar{n}}\|_2^2 / c = 1 + \frac{2t}{c} \int_{\mathbb{R}^N} u_c U_{\bar{n}} dx + t^2 \|U_{\bar{n}}\|_2^2. \quad (3.46)$$

It follows from (3.40), (3.42), (3.43), (3.44), (3.45), and (3.46) that $W_{\bar{n},t} \in \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ for all $t \geq 0$, $W_{\bar{n},0} = u_c$, and $\Phi_\mu(W_{\bar{n},t}) < 2m_\mu(c)$ for large $t > 0$. Thus, there exist $\bar{t} > 0$ such that

$$\Phi_\mu(W_{\bar{n},\bar{t}}) < 2m_\mu(c). \quad (3.47)$$

Let $\gamma_{\bar{n}}(t) := W_{\bar{n},t\bar{t}}$. Then $\gamma_{\bar{n}} \in \Gamma_c$ defined by (3.10). Hence, it follows from (3.9) and (3.41) that (3.21) holds. $\square$

Lemma 3.4. *The function $c \mapsto m_\mu(c)$ is nonincreasing on $(0, c_0)$.*

Proof. For any $c_1, c_2 \in (0, c_0)$ and $c_2 > c_1 > 0$, it follows that there exists $\{u_n\} \subset V_{c_1}$ such that

$$m_\mu(c_1) \leq \Phi_\mu(u_n) < m_\mu(c_1) + \frac{1}{n}. \quad (3.48)$$

Let $\zeta := \frac{c_2}{c_1} > 1$ and $v_n(x) := \zeta^{\frac{2-N}{4}} u_n(\zeta^{-\frac{1}{2}} x)$. Then

$$\|v_n\|_2^2 = \zeta \|u_n\|_2^2 = c_2, \quad \|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2, \quad (3.49)$$

$$\int_{\mathbb{R}^N} (I_\beta * |v_n|^p) |v_n|^p dx = \zeta^{\frac{N+\beta-(N-2)p}{2}} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx, \quad (3.50)$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx. \quad (3.51)$$

From (3.49), (3.50), (3.51), and (3.48), we have

$$\begin{aligned}
m_\mu(c_2) & \leq \Phi_\mu(v_n) \\
& = \frac{1}{2} \|\nabla v_n\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |v_n|^p) |v_n|^p dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\
&\quad - \frac{\mu}{2p} \zeta^{\frac{N+\beta-(N-2)p}{2}} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \\
&\leq \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \\
&= \Phi_\mu(u_n) < m_\mu(c_1) + \frac{1}{n},
\end{aligned}$$

which shows that $m_\mu(c_2) \leq m_\mu(c_1)$ by letting $n \rightarrow \infty$. $\square$

Proof of Theorem 1.2. By Lemma 3.2, we know that there exists a sequence $\{u_n\} \subset \mathcal{S}_c \cap H_{\text{rad}}^1(\mathbb{R}^N)$ such that

$$\Phi_\mu(u_n) \rightarrow M_\mu(c) > 0, \quad \Phi_\mu|'_{\mathcal{S}_c}(u_n) \rightarrow 0, \quad \mathcal{P}_\mu(u_n) \rightarrow 0. \quad (3.52)$$

To show the convergence of $\{u_n\}$ in $H_{\text{rad}}^1(\mathbb{R}^N)$, we proceed in three steps.

Step 1: $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^N)$ is bounded.

According to Lemma 2.2, we have

$$\int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \leq \frac{p}{\|Q_p\|_2^{2p-2}} \|\nabla u_n\|_2^{2p\gamma_p} \|u_n\|_2^{2p-2p\gamma_p}. \quad (3.53)$$

It follows from (3.52) and (3.53) that

$$\begin{aligned}
M_\mu(c) + o_n(1) &= \Phi_\mu(u_n) = \Phi_\mu(u_n) - \frac{N-2}{2(N+\alpha)} \mathcal{P}_\mu(u_n) + o_n(1) \\
&= \frac{\alpha+2}{2(N+\alpha)} \|\nabla u_n\|_2^2 - \frac{\mu}{2p} \left(1 - \frac{[Np - (N+\beta)](N-2)}{2(N+\alpha)} \right) \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \\
&\geq \frac{\alpha+2}{2(N+\alpha)} \|\nabla u_n\|_2^2 - \frac{\mu}{2p} \left(1 - \frac{[Np - (N+\beta)](N-2)}{2(N+\alpha)} \right) \frac{pc^{p-p\gamma_p}}{\|Q_p\|_2^{2p-2}} \|\nabla u_n\|_2^{2p\gamma_p}.
\end{aligned}$$

Since $\frac{N+\beta}{N} < p < \frac{N+\beta+2}{N}$, it is easy to see $2p\gamma_p < 2$, and then $\{u_n\}$ is bounded in $H_{\text{rad}}^1(\mathbb{R}^N)$.

Step 2: $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^N)$ has a non-trivial weak limit.

Since $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^N)$ is a bounded sequence, by the compact embedding of $H_{\text{rad}}^1(\mathbb{R}^N)$ into $L^s(\mathbb{R}^N)$, there exists a $u \in H_{\text{rad}}^1(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_{\text{rad}}^1(\mathbb{R}^N)$, $u_n \rightarrow u$ strongly in $L^s(\mathbb{R}^N)$, $s \in (2, 2^*)$.

Let us assume now, by contradiction, that u is trivial. Then, by Lemma 2.1,

$$\int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \leq \mathcal{C}_{N,\beta} \|u_n\|_{2Np/(N+\beta)}^{2p} = o_n(1), \quad (3.54)$$

and since

$$\mathcal{P}_\mu(u_n) = \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx - \mu \gamma_p \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx = o_n(1),$$

we deduce that

$$\mathcal{S}_\alpha \left[\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \right]^{\frac{N-2}{N+\alpha}} \leq \|\nabla u_n\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1). \quad (3.55)$$

We distinguish the two cases

$$\text{either (i) } \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \rightarrow 0 \quad \text{or (ii) } \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \rightarrow \ell_1 > 0.$$

If (i) holds then, in view of (3.55), we have that $\|\nabla u_n\|_2^2 \rightarrow 0$, which implies that

$$0 < M_\mu(c) + o_n(1) = \Phi_\mu(u_n) = \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx$$

$$-\frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx = o_n(1).$$

This is a contradiction. If (ii) holds, we deduce from (3.55) that

$$\|\nabla u_n\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1) \geq \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} + o_n(1). \quad (3.56)$$

From (3.54) and (3.56) we deduce that

$$\begin{aligned} \Phi_\mu(u_n) &= \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1) \\ &= \frac{\alpha+2}{2(N+\alpha)} \|\nabla u_n\|_2^2 + o_n(1) \geq \frac{\alpha+2}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} + o_n(1). \end{aligned}$$

But, since $m_\mu(c) < 0$, we have

$$\Phi_\mu(u_n) = M_\mu(c) + o_n(1) < m_\mu(c) + \frac{\alpha+2}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} + o_n(1) < \frac{\alpha+2}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

This is a contradiction.

Step 3: $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^N)$ strongly converges.

Since $\{u_n\}$ is bounded, from Lemma 2.6, we know that

$$\Phi_\mu|'_{\mathcal{S}_c}(u_n) \rightarrow 0 \Leftrightarrow \Phi'_\mu(u_n) + \lambda_n u_n \rightarrow 0,$$

where

$$\lambda_n = -\frac{1}{c} \langle \Phi'_\mu(u_n), u_n \rangle.$$

Thus, for any $w \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} o_n(1) &= \langle \Phi'_\mu(u_n) + \lambda_n u_n, w \rangle \\ &= \int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla w + \lambda_n u_n w) dx - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}-2} u_n w dx \\ &\quad - \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^{p-2} u_n w dx, \end{aligned} \quad (3.57)$$

where $o_n(1) \rightarrow 0$, $n \rightarrow \infty$ and

$$-\lambda_n c = \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx + o_n(1). \quad (3.58)$$

In particular, $\{\lambda_n\} \subset \mathbb{R}$ is bounded and, up to a subsequence, $\lambda_n \rightarrow \lambda$. From Lemma 2.5, for any $w \in H^1(\mathbb{R}^N)$, we can deduce

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^{p-2} u_n w dx = \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^{p-2} u w dx \quad (3.59)$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}-2} u_n w dx = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}-2} u w dx. \quad (3.60)$$

Now, passing to the limit in (3.57) by weak convergence and combining (3.59) with (3.60), we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^N} (\nabla u \cdot \nabla w + \lambda u w) dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}-2} u w dx \\ &\quad - \mu \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^{p-2} u w dx = 0, \quad \forall w \in H^1(\mathbb{R}^N). \end{aligned} \quad (3.61)$$

Hence, by Lemma 2.3, we have $\mathcal{P}_\mu(u) = 0$ and

$$\lambda = \frac{[N + \beta - (N - 2)p]\mu}{2p\|u\|_2^2} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx > 0. \quad (3.62)$$

Let $v_n := u_n - u$. Then $v_n \rightharpoonup 0$ weakly in $H_{\text{rad}}^1(\mathbb{R}^N)$, $v_n \rightarrow 0$ strongly in $L^s(\mathbb{R}^N)$, and $v_n \rightarrow 0$ a.e. in $\mathbb{R}^N$. Thus

$$\|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1). \quad (3.63)$$

Arguing as in [8, Lemma 2.4.], we can prove that

$$\begin{aligned} & \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\ &= \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx + \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1) \end{aligned} \quad (3.64)$$

and

$$\int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx = \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx + \int_{\mathbb{R}^N} (I_\beta * |v_n|^p) |v_n|^p dx + o_n(1). \quad (3.65)$$

In particular,

$$\Phi_\mu(u_n) = \Phi_\mu(u) + \Phi_\mu(v_n) + o_n(1) \quad (3.66)$$

and

$$\mathcal{P}_\mu(u_n) = \mathcal{P}_\mu(u) + \mathcal{P}_\mu(v_n) + o_n(1). \quad (3.67)$$

Here again we distinguish the two cases

$$\text{either (i) } \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx \rightarrow 0 \text{ or (ii) } \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx \rightarrow \ell_2 > 0.$$

Assuming that (ii) holds, and since $\mathcal{P}_\mu(u) = 0$, we deduce from (3.67) and (1.15) that

$$\mathcal{S}_\alpha \left[\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx \right]^{\frac{N-2}{N+\alpha}} \leq \|\nabla v_n\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1).$$

Then, reasoning as in Step 2, it follows that

$$\Phi_\mu(v_n) \geq \frac{\alpha + 2}{2(N + \alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} + o_n(1).$$

According to Lemma 3.4, we know that $c \mapsto m_\mu(c)$ is nonincreasing. Using $\mathcal{P}_\mu(u) = 0$ and since, by property of the weak limit, $\|u\|_2^2 \leq c$, we get that

$$\Phi_\mu(u) \geq m_\mu(\|u\|_2^2) \geq m_\mu(c).$$

Thus

$$\Phi_\mu(u_n) = \Phi_\mu(u) + \Phi_\mu(v_n) + o_n(1) \geq m_\mu(c) + \frac{\alpha + 2}{2(N + \alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} + o_n(1),$$

which contradicts our assumption on $M_\mu(c)$.

It remains to show that if (i) holds then $\{u_n\} \subset H_{\text{rad}}^1(\mathbb{R}^N)$ converges strongly. Since (i) holds, we get from (3.64) that $\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \rightarrow \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx$. Noting that $u_n \rightarrow u$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$, and using (3.65) together with Lemma 2.1, we can deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx = \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx.$$

Choosing $w = u_n$ in (3.57) and $w = u$ in (3.61), we deduce that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx + \lambda_n \|u_n\|_2^2 \right) \\ &= 0 = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx + \lambda \|u\|_2^2. \end{aligned}$$

Since $\lambda_n \rightarrow \lambda$, we obtain that

$$\lim_{n \rightarrow \infty} (\|\nabla u_n\|_2^2 + \lambda \|u_n\|_2^2) = \lim_{n \rightarrow \infty} (\|\nabla u_n\|_2^2 + \lambda_n \|u_n\|_2^2) = \|\nabla u\|_2^2 + \lambda \|u\|_2^2. \quad (3.68)$$

By (3.62) and (3.68), we conclude that $u_n \rightarrow u$ strongly in $H_{\text{rad}}^1(\mathbb{R}^N)$.

Since $u_n \rightarrow u$ strongly in $H_{\text{rad}}^1(\mathbb{R}^N)$, combining with (3.52), (3.61), and (3.62), we can deduce that

$$\Phi_\mu(u) = M_\mu(c) > 0, \quad \Phi'_\mu(u) + \lambda u = 0, \quad \mathcal{P}_\mu(u) = 0, \quad \|u\|_2^2 = c, \quad \lambda > 0.$$

□

4 L^2 -critical and L^2 -supercritical perturbation cases

In this subsection, we always assume that $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$ and shall prove Theorems 1.3 and 1.4. Let

$$h_1(t) := \frac{1-t^2}{2} - \frac{N-2}{2(N+\alpha)} \left(1 - t^{\frac{2(N+\alpha)}{N-2}} \right), \quad t > 0 \quad (4.1)$$

and

$$h_2(t) := \frac{[Np - (N+\beta)]\mu}{4p} (1-t^2) - \frac{\mu}{2p} \left(1 - t^{Np-(N+\beta)} \right), \quad t > 0. \quad (4.2)$$

It is easy to see that $h_1(t) > h_1(1) = 0$ and $h_2(t) \geq h_2(1) = 0$ for all $t \in (0, 1) \cup (1, +\infty)$.

Lemma 4.1. *Let $N \geq 3, c > 0, \mu > 0$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then there holds*

$$\begin{aligned} \Phi_\mu(u) &= \Phi_\mu(t^{N/2}u_t) + \frac{1-t^2}{2} \mathcal{P}_\mu(u) + h_1(t) \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx \\ &\quad + h_2(t) \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx, \quad \forall u \in \mathcal{S}_c, \quad t > 0. \end{aligned} \quad (4.3)$$

The above lemma can be proved by a straightforward calculation. From Lemma 4.1, we derive the following corollary.

Corollary 4.2. *Let $N \geq 3, c > 0, \mu > 0$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then for any $u \in \mathcal{M}_c$, there holds*

$$\Phi_\mu(u) = \max_{t>0} \Phi_\mu(t^{N/2}u_t). \quad (4.4)$$

Using a standard argument, we can establish the following lemma.

Lemma 4.3. *Let $N \geq 3, c > 0, \mu > 0$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then for any $u \in \mathcal{S}_c$, there exists a unique $t_u > 0$ such that $t_u^{N/2}u_{t_u} \in \mathcal{M}_c$.*

By Corollary 4.2 and Lemma 4.3, we get the following lemma.

Lemma 4.4. *Let $N \geq 3, c > 0, \mu > 0$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then*

$$\hat{m}_\mu(c) := \inf_{u \in \mathcal{M}_c} \Phi_\mu(u) = \inf_{u \in \mathcal{S}_c} \max_{t>0} \Phi_\mu(t^{N/2}u_t). \quad (4.5)$$

In what follows, we set

$$\begin{cases} \xi(N, p) := \frac{\|Q_p\|_2^{2p-2}}{2c^{(\beta+2)/N}}, & \text{if } p = \frac{N+\beta+2}{N}, \\ \xi(N, p) := +\infty, & \text{if } \frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}. \end{cases}$$

Lemma 4.5. *Let $N \geq 3, c > 0, 0 < \mu < \xi(N, p)$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then*

(i) there exists $\vartheta_c > 0$ such that $\Phi_\mu(u) > 0$ and $\mathcal{P}_\mu(u) > 0$ if $u \in A_{2\vartheta_c}$, and

$$0 < \sup_{u \in A_{\vartheta_c}} \Phi_\mu(u) < \inf \{ \Phi_\mu(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c \} =: \hat{\kappa}, \quad (4.6)$$

where

$$A_{\vartheta_c} := \{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq \vartheta_c\} \text{ and } A_{2\vartheta_c} := \{u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq 2\vartheta_c\}; \quad (4.7)$$

(ii) $\hat{\Gamma}_c := \{\gamma \in \mathcal{C}([0, 1], \mathcal{S}_c) : \|\nabla \gamma(0)\|_2^2 \leq \vartheta_c, \Phi_\mu(\gamma(1)) < 0\} \neq \emptyset$ and

$$\hat{M}_\mu(c) := \inf_{\gamma \in \hat{\Gamma}_c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \hat{\kappa} > \max_{\gamma \in \hat{\Gamma}_c} \max \{ \Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1)) \}. \quad (4.8)$$

Proof. (i) We distinguish the following two cases on p .

Case 1: $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$. In this case, $1 < p\gamma_p = p \frac{Np-(N+\beta)}{2p} < \frac{N+\beta}{N-2}$. Given that the value of $\frac{\alpha+2}{N-2} > 0$ and $p\gamma_p - 1 > 0$, there exists $\vartheta_c > 0$ small enough such that

$$\mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}}(2\vartheta_c)^{\frac{\alpha+2}{N-2}} + \frac{6\mu}{\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} (2\vartheta_c)^{p\gamma_p-1} < \frac{1}{4}. \quad (4.9)$$

By (1.3), (1.11), (1.15), (2.4), and (4.9), one has

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|\nabla u\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \\ &\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} \|\nabla u\|_2^{2p\gamma_p-2} \right) \\ &\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} (2\vartheta_c)^{p\gamma_p-1} \right) \\ &> 0, \quad \forall u \in A_{2\vartheta_c}, \end{aligned}$$

$$\begin{aligned} \mathcal{P}(u) &= \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu\gamma_p \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \\ &\geq \|\nabla u\|_2^2 \left(1 - \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} - \frac{\mu\gamma_p p}{\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} \|\nabla u\|_2^{2p\gamma_p-2} \right) \\ &\geq \|\nabla u\|_2^2 \left(1 - \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} - \frac{\mu\gamma_p p}{\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} (2\vartheta_c)^{p\gamma_p-1} \right) \\ &> 0, \quad \forall u \in A_{2\vartheta_c}, \end{aligned}$$

$$\Phi(u) \geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \vartheta_c^{\frac{\alpha+2}{N-2}} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} \vartheta_c^{p\gamma_p-1} \right) > 0, \quad \forall u \in A_{\vartheta_c},$$

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|\nabla u\|_2^2 - \frac{N-2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \\ &\leq \frac{1}{2} \|\nabla u\|_2^2 \leq \frac{1}{2} \vartheta_c, \quad \forall u \in A_{\vartheta_c}, \end{aligned}$$

and

$$\begin{aligned}
\Phi(u) &\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} \|\nabla u\|_2^{2p\gamma_p-2} \right) \\
&= \vartheta_c \left(1 - \frac{N-2}{N+\alpha} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} (2\vartheta_c)^{p\gamma_p-1} \right) \\
&\geq \frac{3}{4} \vartheta_c, \quad \forall u \in \{\Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c\}.
\end{aligned}$$

Case 2: $p = \frac{N+\beta+2}{N}$. In this case, $1 = p\gamma_p = p \frac{Np-(N+\beta)}{2p}$. Since $\frac{\alpha+2}{N-2} > 0$ and $0 < \mu < \frac{\|Q_p\|_2^{2p-2}}{2c^{(\beta+2)/N}}$, there exists $\vartheta_c > 0$ small enough such that

$$\mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} < \frac{1}{4} \left(\frac{1}{2} - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} \right). \quad (4.10)$$

By (1.3), (1.11), (1.15), (2.4), and (4.10), one has

$$\begin{aligned}
\Phi(u) &\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} \right) \\
&\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} \right) \\
&> 0, \quad \forall u \in A_{2\vartheta_c},
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}(u) &\geq \|\nabla u\|_2^2 \left(1 - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} \right) \\
&\geq \|\nabla u\|_2^2 \left(1 - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} \right) \\
&> 0, \quad \forall u \in A_{2\vartheta_c},
\end{aligned}$$

$$\Phi(u) \geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \vartheta_c^{\frac{\alpha+2}{N-2}} \right) > 0, \quad \forall u \in A_{\vartheta_c},$$

$$\Phi(u) \leq \frac{1}{2} \|\nabla u\|_2^2 \leq \frac{1}{2} \vartheta_c, \quad \forall u \in A_{\vartheta_c},$$

and

$$\begin{aligned}
\Phi(u) &\geq \|\nabla u\|_2^2 \left(\frac{1}{2} - \frac{\mu}{2\|Q_p\|_2^{2p-2}} c^{(1-\gamma_p)p} \|\nabla u\|_2^{2p\gamma_p-2} - \frac{N-2}{2(N+\alpha)} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} \|\nabla u\|_2^{\frac{2\alpha+4}{N-2}} \right) \\
&= \vartheta_c \left(1 - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} - \frac{N-2}{N+\alpha} \mathcal{S}_\alpha^{-\frac{N+\alpha}{N-2}} (2\vartheta_c)^{\frac{\alpha+2}{N-2}} \right) \\
&\geq \vartheta_c \left[\frac{1}{2} + \frac{3}{4} \left(\frac{1}{2} - \frac{\mu}{\|Q_p\|_2^{2p-2}} c^{(\beta+2)/N} \right) \right], \quad \forall u \in \{\Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c\}.
\end{aligned}$$

(ii) For any given $w \in \mathcal{S}_c$, we have $\|t^{N/2}w_t\|_2^2 = \|w\|_2^2$, and so $t^{N/2}w_t \in \mathcal{S}_c$ for every $t > 0$. Then (1.3) yields

$$\begin{aligned}
\Phi_\mu(t^{N/2}w_t) &= t^2 \left(\frac{1}{2} \|\nabla w\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}-2} \int_{\mathbb{R}^N} (I_\alpha * |w|^{\frac{N+\alpha}{N-2}}) |w|^{\frac{N+\alpha}{N-2}} dx \right. \\
&\quad \left. - \frac{\mu t^{2p\gamma_p-2}}{2p} \int_{\mathbb{R}^N} (I_\beta * |w|^p) |w|^p dx \right) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.
\end{aligned}$$

Thus we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\left\| \nabla \left(t_1^{N/2} w_{t_1} \right) \right\|_2^2 = t_1^2 \|\nabla w\|_2^2 \leq \vartheta_c \quad \text{and} \quad \Phi_\mu \left(t_2^{N/2} w_{t_2} \right) < 0.$$

Let $\gamma_0(t) := [t_1 + (t_2 - t_1)t]^{N/2} w_{t_1 + (t_2 - t_1)t}$. Then

$$\|\nabla \gamma_0(0)\|_2^2 = \left\| \nabla \left(t_1^{N/2} w_{t_1} \right) \right\|_2^2 \leq \vartheta_c \quad \text{and} \quad \Phi_\mu(\gamma_0(1)) = \Phi_\mu \left(t_2^{N/2} w_{t_2} \right) < 0.$$

Consequently $\gamma_0 \in \hat{\Gamma}_c$, and so $\hat{\Gamma}_c \neq \emptyset$.

For any $\gamma \in \hat{\Gamma}_c$, since $\Phi_\mu(u) > 0$ when $u \in A_{2\vartheta_c}$ and $\Phi_\mu(\gamma(1)) < 0$, we can deduce $\|\nabla \gamma(1)\|_2^2 > 2\vartheta_c$. Now using the intermediate value theorem, for any $\gamma \in \hat{\Gamma}_c$, there exists $t_0 \in (0, 1)$, depending on γ , such that $\|\nabla \gamma(t_0)\|_2^2 = 2\vartheta_c$ and

$$\max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \Phi_\mu(\gamma(t_0)) \geq \inf \{ \Phi_\mu(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c \},$$

which, together with the arbitrariness of $\gamma \in \hat{\Gamma}_c$, implies

$$\hat{M}_\mu(c) = \inf_{\gamma \in \hat{\Gamma}_c} \max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \inf \{ \Phi_\mu(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = 2\vartheta_c \}. \quad (4.11)$$

Hence, (4.8) follows directly from (4.6) and (4.11), and the proof is completed. $\square$

Lemma 4.6. *Let $N \geq 3, c > 0$, $\mu > 0$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then*

$$\hat{M}_\mu(c) = \hat{m}_\mu(c). \quad (4.12)$$

Proof. We first prove that $\hat{M}_\mu(c) \leq \hat{m}_\mu(c)$. For any $u \in \mathcal{M}_c$, there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\left\| \nabla \left(t_1^{N/2} u_{t_1} \right) \right\|_2^2 = t_1^2 \|\nabla u\|_2^2 \leq \vartheta_c, \quad \Phi_\mu \left(t_2^{N/2} u_{t_2} \right) < 0.$$

Let $\hat{\gamma}(t) := [(1-t)t_1 + tt_2]^{N/2} u_{(1-t)t_1 + tt_2}$. Then

$$\|\hat{\gamma}(t)\|_2^2 = \|u\|_2^2 = c, \quad \|\nabla \hat{\gamma}(0)\|_2^2 = \left\| \nabla \left(t_1^{N/2} u_{t_1} \right) \right\|_2^2 \leq \vartheta_c, \quad \Phi_\mu(\hat{\gamma}(1)) = \Phi_\mu \left(t_2^{N/2} u_{t_2} \right) < 0.$$

Hence $\hat{\gamma} \in \hat{\Gamma}_c$. By (4.4) and the definition of $\hat{M}_\mu(c)$, we have

$$\hat{M}_\mu(c) \leq \max_{t \in [0, 1]} \Phi_\mu(\hat{\gamma}(t)) \leq \max_{t > 0} \Phi_\mu(t^{N/2} u_t) = \Phi_\mu(u),$$

and so $\hat{M}_\mu(c) \leq \hat{m}_\mu(c)$.

On the other hand, by (4.3) with $t \rightarrow 0$, we have $\mathcal{P}_\mu(u) \leq 2\Phi_\mu(u)$ for $u \in \mathcal{S}_c$, which implies

$$\mathcal{P}_\mu(\gamma(1)) \leq 2\Phi_\mu(\gamma(1)) < 0, \quad \forall \gamma \in \hat{\Gamma}_c.$$

Since $\|\gamma(0)\|_2^2 \leq \vartheta_c < 2\vartheta_c$, by (i) of Lemma 4.5, we have $\mathcal{P}_\mu(\gamma(0)) > 0$. Hence, for any $\gamma \in \hat{\Gamma}_c$, there exists $t_0 \in (0, 1)$ such that

$$\mathcal{P}_\mu(\gamma(t_0)) = 0,$$

that is $\gamma(t_0) \in \mathcal{M}_c$. Therefore

$$\max_{t \in [0, 1]} \Phi_\mu(\gamma(t)) \geq \Phi_\mu(\gamma(t_0)) \geq \inf_{u \in \mathcal{M}_c} \Phi_\mu(u) = \hat{m}_\mu(c), \quad \forall \gamma \in \hat{\Gamma}_c,$$

and so $\hat{M}_\mu(c) \geq \hat{m}_\mu(c)$ due to the arbitrariness of γ . Therefore, $\hat{M}_\mu(c) = \hat{m}_\mu(c)$ for any $c > 0$, and the proof is completed. $\square$

Similar to Lemma 3.2, we can prove the following lemma with Lemmas 4.5 and 4.6.

Lemma 4.7. *Let $N \geq 3, c > 0$, $0 < \mu < \xi(N, p)$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. There exists a sequence $\{u_n\} \subset \mathcal{S}_c$ such that*

$$\Phi_\mu(u_n) \rightarrow \hat{M}_\mu(c) > 0, \quad \Phi_\mu|'_{\mathcal{S}_c}(u_n) \rightarrow 0, \quad \text{and} \quad \mathcal{P}_\mu(u_n) \rightarrow 0. \quad (4.13)$$

To deal with the lack of compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$, we next collect some properties of $\hat{m}(c)$ as follows.

Lemma 4.8. *Let $N \geq 3, c > 0$, $0 < \mu < \xi(N, p)$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. Then the function $c \mapsto \hat{m}_\mu(c)$ is nonincreasing on $(0, +\infty)$. In particular, if $\hat{m}_\mu(c_1)$ is achieved, then $\hat{m}_\mu(c_1) > \hat{m}_\mu(c_2)$ for any $c_2 > c_1$.*

Proof. For any $c_2 > c_1 > 0$, it follows from the definition of $\hat{m}_\mu(c_1)$ that there exists $\{u_n\} \subset \mathcal{M}_{c_1}$ such that

$$\Phi_\mu(u_n) < \hat{m}_\mu(c_1) + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (4.14)$$

Let $\zeta := \frac{c_2}{c_1} > 1$ and $v_n(x) := \zeta^{\frac{2-N}{4}} u_n(\zeta^{-\frac{1}{2}} x)$. Then $\|v_n\|_2^2 = c_2$ and $\|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2$. By Lemma 4.3 there exists $t_n > 0$ such that $t_n^{N/2}(v_n)_{t_n} \in \mathcal{M}_{c_2}$. Then it follows from (1.3), (4.14), and Corollary 4.2 that

$$\begin{aligned} \hat{m}_\mu(c_2) &\leq \Phi_\mu\left(t_n^{N/2}(v_n)_{t_n}\right) \\ &= \frac{t_n^2}{2} \|\nabla v_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t_n^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{\frac{N+\alpha}{N-2}}) |v_n|^{\frac{N+\alpha}{N-2}} dx \\ &\quad - \frac{\mu t_n^{2p\gamma_p}}{2p} \int_{\mathbb{R}^N} (I_\beta * |v_n|^p) |v_n|^p dx \\ &= \frac{t_n^2}{2} \|\nabla u_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t_n^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\ &\quad - \frac{\mu t_n^{2p\gamma_p}}{2p} \zeta^{\frac{(N+\beta)-(N-2)p}{2}} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \\ &< \Phi_\mu\left(t_n^{N/2}(u_n)_{t_n}\right) \leq \Phi_\mu(u_n) < \hat{m}_\mu(c_1) + \frac{1}{n}, \end{aligned} \quad (4.15)$$

which shows that $\hat{m}_\mu(c_2) \leq \hat{m}_\mu(c_1)$ by letting $n \rightarrow \infty$.

If $\hat{m}_\mu(c_1)$ is achieved, i.e., there exists $\tilde{u} \in \mathcal{M}_{c_1}$ such that $\Phi_\mu(\tilde{u}) = \hat{m}_\mu(c_1)$. By the same argument as in (4.15), we can obtain that $\hat{m}_\mu(c_2) < \hat{m}_\mu(c_1)$. $\square$

Next, we give a precise estimation for the energy level $\hat{M}_\mu(c)$ given by (4.8) when $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$. To this end, for any fixed $c > 0$, we choose $0 < \sigma < 1$ and $R_n > n^\sigma$ to be such that

$$\begin{aligned} c &= N\theta(N)K_N^2 \left\{ \frac{1}{n^2} \int_0^{n^{1+\sigma}} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds + \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \right. \\ &\quad \times \frac{2R_n^{N+2} - [(N+1)(N+2)R_n^2 - 2N(N+2)R_n n^\sigma + N(N+1)n^{2\sigma}]n^{N\sigma}}{N(N+1)(N+2)(R_n - n^\sigma)^2} \Big\}. \end{aligned} \quad (4.16)$$

Note that

$$\int_0^{n^{1+\sigma}} \frac{s^2}{1+s^2} ds = n^{1+\sigma} - \arctan n^{1+\sigma}, \quad (4.17)$$

$$\int_0^{n^{1+\sigma}} \frac{s^3}{(1+s^2)^2} ds = \frac{1}{2} \log(1+n^{2(1+\sigma)}) - \frac{n^{2(1+\sigma)}}{2(1+n^{2(1+\sigma)})}, \quad (4.18)$$

and

$$\int_0^{n^{1+\sigma}} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds \leq \frac{1}{N} + \frac{1}{N-4} - \frac{1}{(N-4)n^{(N-4)(1+\sigma)}}, \quad \forall N \geq 5. \quad (4.19)$$

From (4.16), (4.17), (4.18), and (4.19), one can deduced that

$$\lim_{n \rightarrow \infty} \frac{R_n}{n^{(N-2)(1+2\sigma)/N}} = \left[\frac{(N+1)(N+2)c}{2\theta(N)K_N^2} \right]^{\frac{1}{N}}. \quad (4.20)$$

Now, we define function $\tilde{U}_n(x) := \tilde{\Theta}_n(|x|)$, where

$$\tilde{\Theta}_n(r) = K_N \begin{cases} \left(\frac{n}{1+n^2 r^2} \right)^{\frac{N-2}{2}}, & 0 \leq r < n^\sigma; \\ \left(\frac{n}{1+n^{2+2\sigma}} \right)^{\frac{N-2}{2}} \frac{R_n-r}{R_n-n^\sigma}, & n^\sigma \leq r < R_n; \\ 0, & r \geq R_n. \end{cases} \quad (4.21)$$

Computing directly, we have

$$\begin{aligned} \|\tilde{U}_n\|_2^2 &= \int_{\mathbb{R}^N} |\tilde{U}_n|^2 dx = N\theta(N) \int_0^{+\infty} |\tilde{\Theta}_n(r)|^2 r^{N-1} dr \\ &= N\theta(N)K_N^2 \left[\int_0^{n^\sigma} \left(\frac{n}{1+n^2 r^2} \right)^{N-2} r^{N-1} dr + \int_{n^\sigma}^{R_n} \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \frac{(R_n-r)^2}{(R_n-n^\sigma)^2} r^{N-1} dr \right] \\ &= N\theta(N)K_N^2 \left\{ \frac{1}{n^2} \int_0^{n^{1+\sigma}} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds + \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \right. \\ &\quad \times \frac{2R_n^{N+2} - [(N+1)(N+2)R_n^2 - 2N(N+2)R_n n^\sigma + N(N+1)n^{2\sigma}]n^{N\sigma}}{N(N+1)(N+2)(R_n-n^\sigma)^2} \Big\} \\ &= c, \end{aligned} \quad (4.22)$$

$$\begin{aligned} \|\nabla \tilde{U}_n\|_2^2 &= \int_{\mathbb{R}^N} |\nabla \tilde{U}_n|^2 dx = \int_0^{+\infty} |\nabla \tilde{\Theta}_n|^2 N\theta(N) r^{N-1} dr \\ &= N\theta(N)K_N^2 \left[(N-2)^2 \int_0^{n^\sigma} \frac{n^{N+2} r^{N+1}}{(1+n^2 r^2)^N} dr + \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \int_{n^\sigma}^{R_n} \frac{r^{N-1}}{(R_n-n^\sigma)^2} dr \right] \\ &= N\theta(N)K_N^2 \left[(N-2)^2 \int_0^{n^{1+\sigma}} \frac{s^{N+1}}{(1+s^2)^N} ds + \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \frac{R_n^N - n^{N\sigma}}{N(R_n-n^\sigma)^2} \right] \\ &= S^{\frac{N}{2}} + N\theta(N)K_N^2 \left[-(N-2)^2 \int_{n^{1+\sigma}}^{+\infty} \frac{s^{N+1}}{(1+s^2)^N} ds + \left(\frac{n}{1+n^{2+2\sigma}} \right)^{N-2} \frac{R_n^N - n^{N\sigma}}{N(R_n-n^\sigma)^2} \right] \\ &= (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} + O\left(\frac{1}{n^{2(1+2\sigma)(N-2)/N}} \right), \quad n \rightarrow \infty, \end{aligned} \quad (4.23)$$

$$\begin{aligned} &\int_{\mathbb{R}^N} (I_\alpha * |\tilde{U}_n|^{\frac{N+\alpha}{N-2}}) |\tilde{U}_n|^{\frac{N+\alpha}{N-2}} dx \\ &\geq \int_{\mathbb{R}^N} (I_\alpha * |U|^{\frac{N+\alpha}{N-2}}) |U|^{\frac{N+\alpha}{N-2}} dx \\ &\quad - 2\mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \int_{B_{n^\sigma}(0)} \frac{\left(\frac{n}{1+n^2|x|^2} \right)^{\frac{N+\alpha}{2}} \left(\frac{n}{1+n^2|y|^2} \right)^{\frac{N+\alpha}{2}}}{|x-y|^{N-\alpha}} dx dy \\ &\quad - \mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \frac{\left(\frac{n}{1+n^2|x|^2} \right)^{\frac{N+\alpha}{2}} \left(\frac{n}{1+n^2|y|^2} \right)^{\frac{N+\alpha}{2}}}{|x-y|^{N-\alpha}} dx dy \\ &= (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} - 2D_1 - D_2, \end{aligned} \quad (4.24)$$

$$\begin{aligned}
D_1 &= \mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \int_{B_{n^\sigma}(0)} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{N+\alpha}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{N+\alpha}{2}}}{|x-y|^{N-\alpha}} dx dy \\
&\leq \mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \mathcal{C}_{N,\alpha}(N\theta(N))^{\frac{N+\alpha}{N}} \left(\int_{n^{1+\sigma}}^{\infty} \frac{s^{N-1}}{(1+s^2)^N} ds \right)^{\frac{N+\alpha}{2N}} \left(\int_0^{n^{1+\sigma}} \frac{s^{N-1}}{(1+s^2)^N} ds \right)^{\frac{N+\alpha}{2N}} \\
&= O\left(\frac{1}{n^{(N+\alpha)(1+\sigma)/2}} \right), \quad n \rightarrow \infty,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
D_2 &= \mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \int_{\mathbb{R}^N \setminus B_{n^\sigma}(0)} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{N+\alpha}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{N+\alpha}{2}}}{|x-y|^{N-\alpha}} dx dy \\
&\leq \mathcal{A}_{N,\alpha} K_N^{\frac{2(N+\alpha)}{N-2}} \mathcal{C}_{N,\alpha}(N\theta(N))^{\frac{N+\alpha}{N}} \left(\int_{n^{1+\sigma}}^{\infty} \frac{s^{N-1}}{(1+s^2)^N} ds \right)^{\frac{N+\alpha}{N}} \\
&= O\left(\frac{1}{n^{(N+\alpha)(1+\sigma)}} \right), \quad n \rightarrow \infty,
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} (I_\beta * |\tilde{U}_n|^t) |\tilde{U}_n|^s dx \\
&\geq \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \int_{B_{n^{1+\sigma}}(0)} \int_{B_{n^{1+\sigma}}(0)} \frac{\left(\frac{1}{1+|z|^2}\right)^{\frac{(N-2)t}{2}} \left(\frac{1}{1+|w|^2}\right)^{\frac{(N-2)s}{2}}}{|z-w|^{N-\beta}} dz dw \\
&\geq \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \int_{B_1(0)} \int_{B_1(0)} \frac{\left(\frac{1}{1+|z|^2}\right)^{\frac{(N-2)t}{2}} \left(\frac{1}{1+|w|^2}\right)^{\frac{(N-2)s}{2}}}{|z-w|^{N-\beta}} dz dw \\
&\geq \mathcal{A}_{N,\beta} K_N^{t+s} \frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \frac{1}{2^{(N-\beta)+(N-2)(t+s)/2}} \left(\int_{B_1(0)} 1 dz \right)^2 \\
&= O\left(\frac{1}{n^{(N+\beta)-(N-2)(t+s)/2}} \right), \quad \forall t, s \in \mathbb{R}.
\end{aligned} \tag{4.27}$$

The combination of (4.22) and (4.23) yields that $\tilde{U}_n \in \mathcal{S}_c$ for all $n \in \mathbb{N}$. Using the above estimates, we will prove the following lemma.

Lemma 4.9. *Let $c > 0$ and $\mu > 0$. Then there exists $\bar{n} \in \mathbb{N}$ such that*

$$\hat{M}_\mu(c) \leq \sup_{t>0} \Phi_\mu(t^{N/2}(\tilde{U}_{\bar{n}})_t) < \frac{2+\alpha}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}, \tag{4.28}$$

if one of the following conditions holds:

- (i) $N = 3$ with $\alpha \in (0, 3)$, $\beta \in (0, 1)$, $\frac{5+\beta}{3} < p < 3 + \beta$, and $\mu > 0$;
- (ii) $N = 3$ with $\alpha \in (0, 3)$, $\beta \in (1, 3)$, $\frac{5+\beta}{3} < p \leq 1 + \beta$, and $\mu > \mu^* > 0$, with μ^* sufficiently large;
- (iii) $N = 3$ with $\alpha \in (0, 3)$, $\beta \in [1, 3)$, $1 + \beta < p < 3 + \beta$, and $\mu > 0$;
- (iv) $N \geq 4$ with $\alpha, \beta \in (0, N)$, $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$, and $\mu > 0$.

Proof. From (1.3), (4.23), (4.24), (4.25), (4.26), and (4.27), we have

$$\begin{aligned}
\Phi_\mu(t^{N/2}(\tilde{U}_n)_t) &= \frac{t^2}{2} \|\nabla \tilde{U}_n\|_2^2 - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{U}_n|^{\frac{N+\alpha}{N-2}}) |\tilde{U}_n|^{\frac{N+\alpha}{N-2}} dx \\
&\quad - \frac{\mu}{2p} t^{Np-(N+\beta)} \int_{\mathbb{R}^N} (I_\beta * |\tilde{U}_n|^p) |\tilde{U}_n|^p dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{t^2}{2} \left[(\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} \mathcal{S}_\alpha^{\frac{N}{2}} + O\left(\frac{1}{n^{2(1+2\sigma)(N-2)/N}}\right) \right] \\
&\quad - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} \left[(\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N+\alpha}{2}} - O\left(\frac{1}{n^{(N+\alpha)(1+\sigma)/2}}\right) \right] \\
&\quad - \frac{\mu}{2p} t^{Np-(N+\beta)} \left[O\left(\frac{1}{n^{N+\beta-(N-2)p}}\right) \right] \\
&= \left[\frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{N}{2}} \right] \mathcal{S}_\alpha^{\frac{N}{2}} \\
&\quad + t^2 \left[O\left(\frac{1}{n^{2(1+2\sigma)(N-2)/N}}\right) \right] + t^{\frac{2(N+\alpha)}{N-2}} \left[O\left(\frac{1}{n^{(N+\alpha)(1+\sigma)/2}}\right) \right] \\
&\quad - t^{Np-(N+\beta)} \mu \left[O\left(\frac{1}{n^{N+\beta-(N-2)p}}\right) \right]. \tag{4.29}
\end{aligned}$$

It can be easily verified that

$$\frac{2(1+2\sigma)(N-2)}{N} \leq \frac{(N+\alpha)(1+\sigma)}{2}, \quad \forall \sigma \in (0, 1). \tag{4.30}$$

(i) If $N = 3, \alpha \in (0, 3), \beta \in (0, 1)$, and $\frac{5+\beta}{3} \leq p < 3 + \beta$, we then choose $\sigma = \frac{3+\beta}{4}$ in (4.21). Then, from (4.29), we have

$$\begin{aligned}
\Phi_\mu(t^{N/2}(\tilde{U}_n)_t) &\leq \left[\frac{t^2}{2} (\mathcal{A}_{3,\alpha} \mathcal{C}_{3,\alpha})^{\frac{3}{2(3+\alpha)}} - \frac{1}{2(3+\alpha)} t^{2(3+\alpha)} (\mathcal{A}_{3,\alpha} \mathcal{C}_{3,\alpha})^{\frac{3}{2}} \mathcal{S}_\alpha^{\frac{\alpha}{2}} \right] \mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + t^2 \left[O\left(\frac{1}{n^{(5+\beta)/3}}\right) \right] + t^{2(3+\alpha)} \left[O\left(\frac{1}{n^{(3+\alpha)(7+\beta)/8}}\right) \right] \\
&\quad - t^{3p-(3+\beta)} \left[O\left(\frac{1}{n^{3+\beta-p}}\right) \right]. \tag{4.31}
\end{aligned}$$

Note that

$$3 + \beta - p \leq \frac{4+2\beta}{3} < \frac{5+\beta}{3}, \quad \text{if } \beta \in (0, 1) \text{ and } \frac{5+\beta}{3} \leq p < 3 + \beta. \tag{4.32}$$

Hence, it follows from (3.39), (4.30), (4.31), and (4.32) that there exists $\bar{n} \in \mathbb{N}$ such that (4.28) holds for every $\mu > 0$.

(ii) If $N = 3, \alpha \in (0, 3), \beta \in [1, 3)$, and $\frac{5+\beta}{3} \leq p \leq 1 + \beta$, we then choose $0 < \sigma < 1$ in (4.21). Hence, by (3.39) and (4.29), for any fixed $\bar{n} \in \mathbb{N}$, there exists $\mu^* > 0$ large enough such that (4.28) holds for $\mu > \mu^*$.

(iii) If $N = 3, \alpha \in (0, 3), \beta \in [1, 3)$, and $1 + \beta < p < 3 + \beta$, we then choose $\frac{11+3\beta-3p}{8} \leq \sigma < 1$ in (4.21). Then, from (4.29), we have

$$\begin{aligned}
\Phi_\mu(t^{3/2}(\tilde{U}_n)_t) &\leq \left[\frac{t^2}{2} (\mathcal{A}_{3,\alpha} \mathcal{C}_{3,\alpha})^{\frac{3}{2(3+\alpha)}} - \frac{1}{2(3+\alpha)} t^{2(3+\alpha)} (\mathcal{A}_{3,\alpha} \mathcal{C}_{3,\alpha})^{\frac{3}{2}} \mathcal{S}_\alpha^{\frac{\alpha}{2}} \right] \mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + t^2 \left[O\left(\frac{1}{n^{2(1+2\sigma)/3}}\right) \right] + t^{2(3+\alpha)} \left[O\left(\frac{1}{n^{(3+\alpha)(1+\sigma)/2}}\right) \right] \\
&\quad - t^{3p-(3+\beta)} \left[O\left(\frac{1}{n^{3+\beta-p}}\right) \right]. \tag{4.33}
\end{aligned}$$

Note that

$$3 + \beta - p < \frac{5+\beta-p}{2} \leq \frac{2(1+2\sigma)}{3}, \quad \text{if } 1 + \beta < p < 3 + \beta. \tag{4.34}$$

Hence, it follows from (3.39), (4.30), (4.33), and (4.34) that there exists $\bar{n} \in \mathbb{N}$ such that (4.28) holds for every $\mu > 0$.

(iv) If $N \geq 4, \alpha, \beta \in (0, N)$, and $\frac{N+\beta+2}{N} \leq p < \frac{N+\beta}{N-2}$, we then choose $\sigma = \frac{\beta+3N-4}{6(N-2)} \in (0, 1)$ in (4.21). Then, from (4.29), we have

$$\Phi_\mu(t^{N/2}(\tilde{U}_n)_t) \leq \left[\frac{t^2}{2} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N(N-2)}{2(N+\alpha)}} - \frac{N-2}{2(N+\alpha)} t^{\frac{2(N+\alpha)}{N-2}} (\mathcal{A}_{N,\alpha} \mathcal{C}_{N,\alpha})^{\frac{N}{2}} \mathcal{S}_\alpha^{\frac{\alpha}{2}} \right] \mathcal{S}_\alpha^{\frac{N}{2}}$$

$$\begin{aligned}
& + t^2 \left[O \left(\frac{1}{n^{(12N+2\beta-20)/3N}} \right) \right] + t^{\frac{2(N+\alpha)}{N-2}} \left[O \left(\frac{1}{n^{(N+\alpha)(9N+\beta-16)/12(N-2)}} \right) \right] \\
& - t^{Np-(N+\beta)} \left[O \left(\frac{1}{n^{N+\beta-(N-2)p}} \right) \right].
\end{aligned} \tag{4.35}$$

Note that

$$N + \beta - (N - 2)p \leq \frac{2\beta + 4}{N} < \frac{12N + 2\beta - 20}{3N}, \text{ if } N \geq 4 \text{ and } \frac{N + \beta + 2}{N} \leq p < \frac{N + \beta}{N - 2}. \tag{4.36}$$

Hence, it follows from (3.39), (4.30), (4.35), and (4.36) that there exists $\bar{n} \in \mathbb{N}$ such that (4.28) holds for every $\mu > 0$. $\square$

Proof of Theorems 1.3 and 1.4. In view of Lemmas 4.7 and 4.9, there exists $\{u_n\} \subset \mathcal{S}_c$ such that

$$\|u_n\|_2^2 = c, \quad \Phi_\mu(u_n) \rightarrow \hat{M}_\mu(c) < \frac{2 + \alpha}{2(N + \alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}, \quad \Phi_\mu|_{\mathcal{S}_c}'(u_n) \rightarrow 0, \text{ and } \mathcal{P}_\mu(u_n) \rightarrow 0, \tag{4.37}$$

which, together with (1.3) and (1.11) implies that

$$\begin{aligned}
\hat{M}_\mu(c) + o_n(1) = \Phi_\mu(u_n) &= \frac{1}{2} \|\nabla u_n\|_2^2 - \frac{N - 2}{2(N + \alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\
&\quad - \frac{\mu}{2p} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx
\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}
o_n(1) = \mathcal{P}_\mu(u_n) &= \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\
&\quad - \frac{Np - (N + \beta)}{2p} \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx.
\end{aligned} \tag{4.39}$$

It follows from (4.38) and (4.39) that

$$\begin{aligned}
\hat{M}_\mu(c) + o_n(1) = \Phi_\mu(u_n) - \frac{1}{2} \mathcal{P}_\mu(u_n) &= \frac{\alpha + 2}{2(N + \alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\
&\quad + \mu \frac{Np - (N + \beta + 2)}{4p} \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx.
\end{aligned} \tag{4.40}$$

When $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$, we have $\left\{ \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \right\}$ and $\left\{ \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \right\}$ are bounded. By (4.39), one has

$$\|\nabla u_n\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + \frac{Np - (N + \beta)}{2p} \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx + o_n(1).$$

This shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ when $\frac{N+\beta+2}{N} < p < \frac{N+\beta}{N-2}$. When $p = \frac{N+\beta+2}{N}$, by (4.40), we have $\left\{ \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \right\}$ is bounded. From (4.39) and Lemma 2.2, one has

$$\begin{aligned}
\|\nabla u_n\|_2^2 &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + \frac{Np - (N + \beta)}{2p} \mu \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx + o_n(1) \\
&\leq \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + \frac{\mu}{\|Q_p\|_2^{2p-2}} \|\nabla u_n\|_2^2 \|u_n\|_2^{(2\beta+4)/N} + o_n(1) \\
&= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + \frac{\mu c^{(\beta+2)/N}}{\|Q_p\|_2^{2p-2}} \|\nabla u_n\|_2^2 + o_n(1).
\end{aligned} \tag{4.41}$$

Note that $0 < \mu < \frac{\|Q_p\|_2^{2p-2}}{2c^{(\beta+2)/N}}$ when $p = \frac{N+\beta+2}{N}$. Then from (4.41) we can deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ when $p = \frac{N+\beta+2}{N}$.

Let $\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx$. We show that $\delta > 0$. Otherwise, in light of Lions' concentration compactness principle [39, Lemma 1.21], $u_n \rightarrow 0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$. And then

$$\int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \leq \mathcal{C}_{N,\beta} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2Np}{N+\beta}} dx \right)^{\frac{N+\beta}{N}} = o_n(1), \quad \frac{N+\beta}{N} < p < \frac{N+\beta}{N-2}.$$

From (4.40), one can get

$$\frac{\alpha+2}{2(N+\alpha)} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx = \hat{M}_\mu(c) + o_n(1). \quad (4.42)$$

Combining with (4.39), we know that

$$\begin{aligned} \|\nabla u_n\|_2^2 &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + \mu \gamma_p \int_{\mathbb{R}^N} (I_\beta * |u_n|^p) |u_n|^p dx \\ &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx + o_n(1) = \frac{2(N+\alpha)}{\alpha+2} \hat{M}_\mu(c) + o_n(1). \end{aligned} \quad (4.43)$$

Hence, it follows from (1.15), (4.42), and (4.43) that

$$\begin{aligned} \frac{2(N+\alpha)}{\alpha+2} \hat{M}_\mu(c) + o_n(1) &= \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{\frac{N+\alpha}{N-2}}) |u_n|^{\frac{N+\alpha}{N-2}} dx \\ &\leq \left(\frac{\|\nabla u_n\|_2^2}{\mathcal{S}_\alpha} \right)^{\frac{N+\alpha}{N-2}} = \left(\frac{\frac{2(N+\alpha)}{\alpha+2} \hat{M}_\mu(c)}{\mathcal{S}_\alpha} \right)^{\frac{N+\alpha}{N-2}} + o_n(1). \end{aligned} \quad (4.44)$$

By (4.44), we have

$$\frac{\alpha+2}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}} \leq \hat{M}_\mu(c),$$

which contradicts $\hat{M}_\mu(c) < \frac{\alpha+2}{2(N+\alpha)} \mathcal{S}_\alpha^{\frac{N+\alpha}{2+\alpha}}$. Thus $\delta > 0$. Without loss of generality, we may assume the existence of $y_n \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n|^2 dx > \frac{\delta}{2}.$$

Let $\hat{u}_n(x) := u_n(x + y_n)$. Then we have

$$\|\hat{u}_n\|_2^2 = c, \quad \mathcal{P}_\mu(\hat{u}_n) \rightarrow 0, \quad \Phi_\mu(\hat{u}_n) \rightarrow \hat{M}_\mu(c), \quad \text{and} \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \frac{\delta}{2}.$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } H^1(\mathbb{R}^N), \quad \hat{u}_n \rightarrow \hat{u} \text{ in } L_{\text{loc}}^s(\mathbb{R}^N) \text{ for } s \in [1, 2^*), \quad \hat{u}_n \rightarrow \hat{u} \text{ a.e. in } \mathbb{R}^N.$$

By (4.37) and Lemma 2.6, one has

$$\begin{aligned} &\int_{\mathbb{R}^N} (\nabla \hat{u}_n \cdot \nabla w + \lambda_n \hat{u}_n w) dx - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}_n|^{\frac{N+\alpha}{N-2}}) |\hat{u}_n|^{\frac{N+\alpha}{N-2}-2} \hat{u}_n w dx \\ &- \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}_n|^p) |\hat{u}_n|^{p-2} \hat{u}_n w dx = o_n(1), \quad \forall w \in H^1(\mathbb{R}^N), \end{aligned} \quad (4.45)$$

where

$$\begin{aligned} \lambda_n &= -\frac{1}{\|\hat{u}_n\|_2^2} \langle \Phi'_\mu(\hat{u}_n), \hat{u}_n \rangle \\ &= -\frac{1}{c} \left[\|\nabla \hat{u}_n\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}_n|^{\frac{N+\alpha}{N-2}}) |\hat{u}_n|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}_n|^p) |\hat{u}_n|^p dx \right]. \end{aligned} \quad (4.46)$$

Since $\{\hat{u}_n\}$ is bounded in $H^1(\mathbb{R}^N)$, it follows from (4.46) that $\{|\lambda_n|\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that $\lambda_n \rightarrow \lambda_c$. By (4.45), Lemma 2.5, and some standard arguments, we can deduce

$$\begin{aligned} & \int_{\mathbb{R}^N} (\nabla \hat{u} \cdot \nabla w + \lambda_c \hat{u} w) dx - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}|^{\frac{N+\alpha}{N-2}}) |\hat{u}|^{\frac{N+\alpha}{N-2}-2} \hat{u} w dx \\ & - \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}|^p) |\hat{u}|^{p-2} \hat{u} w dx = 0, \quad \forall w \in H^1(\mathbb{R}^N). \end{aligned} \quad (4.47)$$

And then, by Lemma 2.3, one has

$$\mathcal{P}_\mu(\hat{u}) = \|\nabla \hat{u}\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}|^{\frac{N+\alpha}{N-2}}) |\hat{u}|^{\frac{N+\alpha}{N-2}} dx - \frac{Np - (N + \beta)}{2p} \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}|^p) |\hat{u}|^p dx = 0. \quad (4.48)$$

Combining (4.47) with (4.48), it is easy to deduce that

$$-\lambda_c \|\hat{u}\|_2^2 = \frac{(N-2)p - (N + \beta)}{2p} \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}|^p) |\hat{u}|^p dx < 0. \quad (4.49)$$

Set $\hat{c} := \|\hat{u}\|_2^2$. Then $0 < \hat{c} \leq c$ and (4.48) shows that $\hat{u} \in \mathcal{M}_{\hat{c}}$. From (4.38), (4.39), Lemma 4.8, and Fatou's lemma, one has

$$\begin{aligned} \hat{M}_\mu(c) &= \lim_{n \rightarrow \infty} \left[\Phi_\mu(\hat{u}_n) - \frac{1}{2} \mathcal{P}_\mu(\hat{u}_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}_n|^{\frac{N+\alpha}{N-2}}) |\hat{u}_n|^{\frac{N+\alpha}{N-2}} dx \right. \\ &\quad \left. + \frac{Np - (N + \beta + 2)}{4p} \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}_n|^p) |\hat{u}_n|^p dx \right] \\ &\geq \frac{2 + \alpha}{2(N + \alpha)} \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}|^{\frac{N+\alpha}{N-2}}) |\hat{u}|^{\frac{N+\alpha}{N-2}} dx + \frac{Np - (N + \beta + 2)}{4p} \mu \int_{\mathbb{R}^N} (I_\beta * |\hat{u}|^p) |\hat{u}|^p dx \\ &= \Phi_\mu(\hat{u}) - \frac{1}{2} \mathcal{P}_\mu(\hat{u}) = \Phi_\mu(\hat{u}) \\ &\geq \hat{m}_\mu(\hat{c}) \geq \hat{m}_\mu(c) = \hat{M}_\mu(c), \end{aligned}$$

which implies

$$\Phi_\mu(\hat{u}) = \hat{m}_\mu(\hat{c}) = \hat{m}_\mu(c).$$

This shows $\hat{m}_\mu(c)$ is achieved. In view of Lemma 4.8, $\hat{c} = c$. Thus,

$$\|\hat{u}\|_2^2 = c, \quad \Phi_\mu(\hat{u}) = \hat{m}_\mu(c), \quad \text{and} \quad \mathcal{P}_\mu(\hat{u}) = 0. \quad (4.50)$$

Both (4.47), (4.49), and (4.50) imply the conclusions of Theorems 1.3 and 1.4 hold. $\square$

5 The case when $\frac{N+\beta}{N} < p < \frac{N+\beta}{N-2}$ and $\mu \leq 0$

Proof of Theorem 1.5. Assume that $(u, \lambda) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ is a solution of equation (1.1). Then it follows from (1.1) and the Pohozaev type identity that

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu \frac{Np - (N + \beta)}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx = 0 \quad (5.1)$$

and

$$\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N-2}}) |u|^{\frac{N+\alpha}{N-2}} dx - \mu \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx + \lambda \|u\|_2^2 = 0. \quad (5.2)$$

Combining (5.1) and (5.2), we have

$$0 > -\lambda c = -\lambda \|u\|_2^2 = \mu \frac{(N-2)p - (N + \beta)}{2p} \int_{\mathbb{R}^N} (I_\beta * |u|^p) |u|^p dx \geq 0,$$

which is a contradiction. $\square$

Declarations

Conflict of Interest

The authors declare that there is no Conflict of Interest. We also declare that this manuscript has no associated data.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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