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Orlicz-Sobolev versus Hölder local minimizers for nonlinear Robin problems

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Abstract. We establish regularity results for weak solutions of Robin problems driven by the well-known Orlicz g-Laplacian operator given by

$$\begin{cases} -\Delta_g u = f(x,u), & x \in \Omega \\ a(|\nabla u|)\dfrac{\partial u}{d\nu} + b(x)|u|^{p-2}u = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Delta_g u := \operatorname{div}(a(|\nabla u|)\nabla u)$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with C^2-boundary $\partial\Omega$, $\frac{\partial u}{d\nu} = \nabla u \cdot \nu$, ν is the unit exterior vector on $\partial\Omega$, $p > 0$, $b \in C^{1,\theta}(\partial\Omega)$ with $\theta \in (0,1)$ and $\inf_{x \in \partial\Omega} b(x) > 0$. Specifically, using a suitable variation of the Moser iteration technique, we prove that every weak solution of the problem (P) is bounded. Moreover, we combine this result with the Lieberman regularity theorem, to show that every $C^1(\overline{\Omega})$-local minimizer is also a $W^{1,G}(\Omega)$-local minimizer for the corresponding energy functional of problem (P).

1. Introduction

In this paper, we study the boundedness regularity for a weak solution and the relationship between the Hölder local minimizer and the Orlicz-Sobolev local min-

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imizer for the corresponding energy functional of the following Robin problem:

$$\begin{cases} -\Delta_g u = f(x,u), & \text{on } \Omega \\ a(|\nabla u|)\dfrac{\partial u}{d\nu} + b(x)|u|^{p-2}u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ with C^2-boundary $\partial\Omega$, $\Delta_g u := \operatorname{div}(a(|\nabla u|)\nabla u)$ is the Orlicz g-Laplacian operator, $\frac{\partial u}{d\nu} = \nabla u.\nu$, ν is the unit exterior vector on $\partial\Omega$, $p > 0$, $b \in C^{1,\theta}(\partial\Omega)$ with $\theta \in (0,1)$ and $\inf\limits_{x\in\partial\Omega} b(x) > 0$ and the function $a(|t|)t$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. In the right side of problem (P) there is a Carathéodory function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, that is $x \longmapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and $s \longmapsto f(x,s)$ continuous for a.e. $x \in \Omega$.

Due to the nature of the non-homogeneous differential operator g-Laplacian, we shall work in the framework of Orlicz and Orlicz-Sobolev spaces. The study of variational problems in the classical Sobolev and Orlicz-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasi-conformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory (see [4–7,11]).

It is worthwhile to mention that the Orlicz-Sobolev space is a generalization of the classical Sobolev space. Hence, several properties of the Sobolev spaces have been extended to the Orlicz-Sobolev spaces. To the best of our knowledge, there is a lack of some regularity results concerning the problem (P). Precisely, the boundedness of a weak solution and the relationship between the Orlicz-Sobolev and Hölder local minimizers for the corresponding energy functional of (P). Those results are crucial in some methods of the existence and multiplicity of solutions for the problem (P).

The question of the boundedness, regularity, and the relationship between the Sobolev and Hölder local minimizers for certain C^1-functionals have been treated by many authors [3,8,9,11,13,15,16,18–20,25–27,29,31–33] and references therein. In [25], G. M. Lieberman treated the regularity result up to the boundary for the weak solutions of the following problem

$$-\Delta_p u = f(x,u),\ x \in \Omega \tag{E}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\alpha}$-boundary. Precisely, under some assumptions on the structure of the p-Laplacian operator and on the non-linear term f, he proved that every bounded (i.e. $u \in L^\infty(\Omega)$) weak solution of the problem (E) (with Dirichlet or Neumann boundary conditions) belongs to $C^{1,\beta}(\overline{\Omega})$. In [26], G. M. Lieberman, extended the results obtained in [25] to the Orlicz g-Laplacian operator. In [9], X. L. Fan, established the same results gave in [25] for the variable exponent Sobolev spaces (p being variable). Note that all the results cited in [9,25,26] require that the weak solution belongs to $L^\infty(\Omega)$. The boundedness result for weak solutions in the Dirichlet case can be deduced from Theorem 7.1 of Ladyzhenskaya-Uraltseva [24] (problems with standard growth conditions) and

Theorem 4.1 of Fan-Zhao [10] (problems with non-standard growth conditions). For the Neumann case, the boundedness result is deduced from Proposition 3.1 of Gasiǹski-Papageorgiou [13] (problems with sub-critical growth conditions).

To the best of our knowledge, there is only one paper (see [11]) devoted to the boundedness result of weak solutions to problems driven by the Orlicz g-Laplacian operator. Precisely, in [11], F. Fang and Z. Tan, with sub-critical growth conditions, proved that every weak solution of problems with Dirichlet boundary conditions belongs to $L^{\infty}(\Omega)$. The approaches used by Fang and Tan in [11] for the boundedness result don't work in our case (Robin boundary condition) since they require that $u_{|\partial\Omega}$ is bounded (u being the weak solution). To overcome this difficulty, we apply a suitable variation of the Moser iteration technique.

The question of the relationship between the Sobolev and Hölder local minimizers for certain functionals has taken the attention of many authors [3,8,11,13,15–17,19,20,22,28,32,33] and references therein. In [8], Brezis and Nirenberg have proved a famous theorem which asserts that the local minimizers in the space C^1 are also local minimizers in the space H^1 for certain variational functionals. A result of this type was later extended to the space $W_0^{1,p}(\Omega)$ (Dirichlet boundary condition), with $1 < p < \infty$, by Garcia Azorera-Manfredi-Peral Alonso [16] (see also Guo-Zhang [19], where $2 \leq p$). The $W_n^{1,p}(\Omega)$-version (Neumann boundary condition) of the result can be found in Motreanu-Motreanu-Papageorgiou [28]. Moreover, this theorem has been extended to the $p(x)$-Laplacian equations (see [13]), non-smooth functionals (see [3,22,32]), and singular equations with critical terms (see [17]).

As far as we know, there is only one paper (see [11]) devoted to the result of Brezis and Nirenberg in the Orlicz case. Precisely, in [11], F. Fang and Z. Tan proved a boundedness regularity result and established the relation between the $C^1(\overline{\Omega})$ and $W_0^{1,G}(\Omega)$ minimizers for an Orlicz problem with Dirichlet boundary condition. Since our problem (P) is with Robin boundary condition, many approaches used in [11] don't work.

The main novelty of our work is the study of the boundedness regularity for weak solutions of problem (P) and the relationship between the Orlicz-Sobolev and Hölder local minimizers for the energy functional of problem (P). The non-homogeneity of the g-Laplacian operator brings us several difficulties in order to get the boundedness of a weak solution to the Robin Problem (P).

This paper is organized as follows. In Section 2, we recall the basic properties of the Orlicz Sobolev spaces and the Orlicz Laplacian operator, and we state the main hypotheses on the data of our problem. Section 3 deals with two regularity results. In the first we prove that every weak solution of problem (P) belongs to $L^s(\Omega)$, for all $1 \leq s < \infty$. In the second we show that every solution of problem (P) is bounded. In the last Section, we establish the relationship between the local $C^1(\overline{\Omega})$-minimizer and the local $W^{1,G}(\Omega)$-minimizer for the corresponding energy functional.

2. Preliminaries

To deal with problem (P), we use the theory of Orlicz-Sobolev spaces since problem (P) contains a non-homogeneous function $a(.)$ in the differential operator. Therefore, we start with some basic concepts of Orlicz-Sobolev spaces, and we set the hypotheses on the non-linear term f. For more details on the Orlicz-Sobolev spaces see [1,2,11,12,23,30] and the references therein.

The function $a : (0, +\infty) \to (0, +\infty)$ is a function such that the mapping, defined by

$$g(t) := \begin{cases} a(|t|)t, \text{ if } t \neq 0, \\ 0, \qquad \text{if } t = 0, \end{cases}$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$G(t) := \int_0^t g(s)\,\mathrm{d}s, \quad \forall\, t \in \mathbb{R},$$

G is an N-function, i.e. Young function satisfying: G is even, positive, continuous and convex function. Moreover, $G(0) = 0$, $\frac{G(t)}{t} \to 0$ as $t \to 0$ and $\frac{G(t)}{t} \to +\infty$ as $t \to +\infty$ (see [23, Lemma 3.2.2, p. 128]).

In order to construct an Orlicz-Sobolev space setting for problem (P), we impose the following class of assumptions on G, a and g:

(G) (g_1) : $a(t) \in C^1(0, +\infty)$, $a(t) > 0$ and $a(t)$ is an increasing function for $t > 0$.
(g_2) : $1 < p < g^- := \inf_{t>0} \frac{g(t)t}{G(t)} \leq g^+ := \sup_{t>0} \frac{g(t)t}{G(t)} < N$.
(g_3) : $0 < g^- - 1 = a^- := \inf_{t>0} \frac{g'(t)t}{g(t)} \leq g^+ - 1 = a^+ := \sup_{t>0} \frac{g'(t)t}{g(t)}$.
(g_4) : $\displaystyle\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}\mathrm{d}t = \infty$ and $\displaystyle\int_0^1 \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}\mathrm{d}t < \infty$.

The conjugate N-function of G, is defined by

$$\tilde{G}(t) = \int_0^t \tilde{g}(s)\,\mathrm{d}s,$$

where $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is given by $\tilde{g}(t) = \sup\{s : \ g(s) \leq t\}$. If g is continuous on $\mathbb{R}$, then $\tilde{g}(t) = g^{-1}(t)$ for all $t \in \mathbb{R}$. Moreover, we have

$$st \leq G(s) + \tilde{G}(t), \tag{2.1}$$

which is known as the Young inequality. Equality in (2.1) holds if and only if either $t = g(s)$ or $s = \tilde{g}(t)$. In our case, since g is continuous, we have

$$\tilde{G}(t) = \int_0^t g^{-1}(s)\,\mathrm{d}s.$$

The functions G and $\tilde{G}$ are complementary N-functions.

We say that G satisfies the Δ_2-condition, if there exists $C > 0$, such that

$$G(2t) \leq CG(t), \ \text{for all } t > 0. \tag{2.2}$$

We want to remark that assumption (g_2) and (2.2) are equivalent (see [23, Theorem 3.4.4, p. 138] and [12]).

If G_1 and G_2 are two N-functions, we say that G_1 grow essentially more slowly than G_2 ($G_1 \prec\prec G_2$ in symbols), if and only if for every positive constant k, we have

$$\lim_{t\to+\infty}\frac{G_1(kt)}{G_2(t)}=0. \tag{2.3}$$

Another important function related to the N-function G, is the Sobolev conjugate function G_* defined by

$$G_*^{-1}(t)=\int_0^t\frac{G^{-1}(s)}{s^{\frac{N+1}{N}}}\,\mathrm{d}s,\ t>0$$

(see [23, Definition 7.2.1, p. 352]).

If G satisfies the Δ_2-condition, then G_* also satisfies the Δ_2-condition. Namely, there exist $g_*^-=\frac{Ng^-}{N-g^-}$ and $g_*^+=\frac{Ng^+}{N-g^+}$ such that

$$g^+<g_*^-:=\inf_{t>0}\frac{g_*(t)t}{G_*(t)}\le\frac{g_*(t)t}{G_*(t)}\le g_*^+:=\sup_{t>0}\frac{g_*(t)t}{G_*(t)}<+\infty,\ \text{for all } t>0 \tag{2.4}$$

(see [12, Lemma 2.4, p. 240]).

The Orlicz space $L^G(\Omega)$ is the vectorial space of measurable functions $u:\Omega\to\mathbb{R}$ such that

$$\rho(u)=\int_\Omega G(|u(x)|)\,\mathrm{d}x<\infty.$$

$L^G(\Omega)$ is a Banach space under the Luxemburg norm

$$\|u\|_{(G)}=\inf\left\{\lambda>0\ :\ \rho(\frac{u}{\lambda})\le 1\right\}.$$

For Orlicz spaces, the Hölder inequality reads as follows

$$\int_\Omega uv\mathrm{d}x\le\|u\|_{(G)}\|v\|_{(\tilde{G})},\quad \text{for all } u\in L^G(\Omega)\ \text{ and } u\in L^{\tilde{G}}(\Omega).$$

Next, we introduce the Orlicz-Sobolev space. We denote by $W^{1,G}(\Omega)$ the Orlicz-Sobolev space defined by

$$W^{1,G}(\Omega):=\left\{u\in L^G(\Omega):\ \frac{\partial u}{\partial x_i}\in L^G(\Omega),\ i=1,...,N\right\}.$$

$W^{1,G}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_G=\|u\|_{(G)}+\|\nabla u\|_{(G)}.$$

Another equivalent norm is

$$\|u\|=\inf\left\{\lambda>0\ :\ \mathcal{K}(\frac{u}{\lambda})\le 1\right\},$$

where

$$\mathcal{K}(u) = \int_\Omega G(|\nabla u(x)|)\mathrm{d}x + \int_\Omega G(|u(x)|)\,\mathrm{d}x. \tag{2.5}$$

If G and its complementary function $\tilde{G}$ satisfied the Δ_2-condition, then $W^{1,G}(\Omega)$ is Banach, separable and reflexive space. For that, in our work, we also assume that $\tilde{G}$ satisfies the Δ_2-condition.

In the sequel, we give general results related to the N-function and the Orlicz, Orlicz-Sobolev spaces.

Lemma 2.1. *(see [30]). Let G and H be N-functions, such that H grows essentially more slowly than G_* (where G_* is the Sobolev conjugate function of G).*

(1) If $\displaystyle\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}\mathrm{d}t = \infty$ *and* $\displaystyle\int_0^1 \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}\mathrm{d}t < \infty$*, then the embedding* $W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$ *is compact and the embedding* $W^{1,G}(\Omega) \hookrightarrow L^{G_*}(\Omega)$ *is continuous.*

(2) If $\displaystyle\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}\mathrm{d}t} < \infty$*, then the embedding* $W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$ *is compact and the embedding* $W^{1,G}(\Omega) \hookrightarrow L^\infty(\Omega)$ *is continuous.*

Lemma 2.2. *(see [12])*

Let G be an N-function satisfying $(g_1)-(g_3)$ *such that* $G(t) = \int_0^t g(s)\,\mathrm{d}s = \int_0^t a(|s|)s\,\mathrm{d}s$. *Then*

(1) $\min\{t^{g^-}, t^{g^+}\}G(1) \le G(t) \le \max\{t^{g^-}, t^{g^+}\}G(1)$, *for all* $0 < t$*;*
(2) $\min\{t^{g^--1}, t^{g^+-1}\}g(1) \le g(t) \le \max\{t^{g^--1}, t^{g^+-1}\}g(1)$, *for all* $0 < t$*;*
(3) $\min\{t^{g^--2}, t^{g^+-2}\}a(1) \le a(t) \le \max\{t^{g^--2}, t^{g^+-2}\}a(1)$, *for all* $0 < t$*;*
(4) $\min\{t^{g^-}, t^{g^+}\}G(z) \le G(tz) \le \max\{t^{g^-}, t^{g^+}\}G(z)$, *for all* $0 < t$ *and* $z \in \mathbb{R}$*;*
(5) $\min\{t^{g^--1}, t^{g^+-1}\}g(z) \le g(tz) \le \max\{t^{g^--1}, t^{g^+-1}\}g(z)$, *for all* $0 < t$ *and* $z \in \mathbb{R}$*;*
(6) $\min\{t^{g^--2}, t^{g^+-2}\}a(|\eta|) \le a(|t\eta|) \le \max\{t^{g^--2}, t^{g^+-2}\}a(|\eta|)$, *for all* $0 < t$ *and* $\eta \in \mathbb{R}^N$.

Lemma 2.3. *(See [12]). Let G be an N-function satisfying* (g_2) *such that* $G(t) = \int_0^t g(s)\,\mathrm{d}s$. *Then*

(1) if $\|u\|_{(G)} < 1$ *then* $\|u\|_{(G)}^{g^+} \le \rho(u) \le \|u\|_{(G)}^{g^-}$*;*
(2) if $\|u\|_{(G)} \ge 1$ *then* $\|u\|_{(G)}^{g^-} \le \rho(u) \le \|u\|_{(G)}^{g^+}$*;*
(3) if $\|u\| < 1$ *then* $\|u\|^{g^+} \le \mathcal{K}(u) \le \|u\|^{g^-}$*;*
(4) if $\|u\| \ge 1$ *then* $\|u\|^{g^-} \le \mathcal{K}(u) \le \|u\|^{g^+}$.

Lemma 2.4. *Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Then the embedding* $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ *is compact provided* $1 \le r < p^*$*, where* $p^* = \frac{Np}{N-p}$ *if* $p < N$ *and* $p^* := +\infty$ *otherwise.*

Lemma 2.5. *Assume that Ω is a bounded domain and has a Lipschitz boundary $\partial\Omega$. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact provided $1 \le r < p^*$.*

Theorem 2.6. *The Orlicz-Sobolev space $W^{1,G}(\Omega)$ is continuously and compactly embedded in the classical Lebesgue spaces $L^r(\Omega)$ and $L^r(\partial\Omega)$ for all $1 \le r < g_*^-$.*

Proof. By help of the assumption (g_2), the Orlicz-Sobolev space $W^{1,G}(\Omega)$ is continuously embedded in the classical Sobolev space $W^{1,g^-}(\Omega)$. In light of Lemmas 2.4 and 2.5, we deduce that $W^{1,g^-}(\Omega)$ is compactly embedded in $L^r(\Omega)$ and $L^r(\partial\Omega)$ for all $1 \le r < g_*^-$. Hence, $W^{1,G}(\Omega)$ is continuously and compactly embedded in the classical Lebesgue space $L^r(\Omega)$ and $L^r(\partial\Omega)$ for all $1 \le r < g_*^-$. □

Lemma 2.7. *[11, Lemma 3.2, p. 354]*

(1) If $a(t)$ is increasing for $t > 0$, there exists constant d_1 depending on $g^-,\ g^+$, such that

$$|a(|\eta|)\eta - a(|\xi|)\xi| \le d_1|\eta - \xi|a(|\eta| + |\xi|), \tag{2.6}$$

for all $\eta,\ \xi \in \mathbb{R}^N$.

(2) If $a(t)$ is decreasing for $t > 0$, there exists constant d_2 depending on $g^-,\ g^+$, such that

$$|a(|\eta|)\eta - a(|\xi|)\xi| \le d_2 g(|\eta - \xi|), \tag{2.7}$$

for all $\eta,\ \xi \in \mathbb{R}^N$.

Lemma 2.8. *Let G be an N-function satisfying (g_1)–(g_3) such that $G(t) = \int_0^t g(s)\,\mathrm{d}s = \int_0^t a(|s|)s\,\mathrm{d}s$. Then for every $\xi, \eta \in \mathbb{R}^N$, we have*

$$\langle a(|\eta|)\eta - a(|\xi|)\xi, \eta - \xi\rangle_{\mathbb{R}^N} \ge 0$$

where $\langle . \rangle_{\mathbb{R}^N}$ is the inner product on $\mathbb{R}^N$.

Proof. Let $\eta, \xi \in \mathbb{R}^N$. Since G is convex, we have

$$G(|\eta|) \le G\left(\left|\frac{\eta + \xi}{2}\right|\right) + \langle a(|\eta|)\eta, \frac{\eta - \xi}{2}\rangle_{\mathbb{R}^N}$$

and

$$G(|\xi|) \le G\left(\left|\frac{\eta + \xi}{2}\right|\right) + \langle a(|\xi|)\xi, \frac{\xi - \eta}{2}\rangle_{\mathbb{R}^N}.$$

Adding the above two relations, we find that

$$\frac{1}{2}\langle a(|\eta|)\eta - a(|\xi|)\xi, \eta - \xi\rangle_{\mathbb{R}^N} \ge G(|\eta|) + G(|\xi|) - 2G\left(\left|\frac{\eta + \xi}{2}\right|\right) \quad \text{for all } \eta, \xi \in \mathbb{R}^N. \tag{2.8}$$

On the other hand, the convexity and the monotonicity of G give

$$G\left(\left|\frac{\eta + \xi}{2}\right|\right) \le \frac{1}{2}\left[G(|\eta|) + G(|\xi|)\right] \quad \text{for all } \eta, \xi \in \mathbb{R}^N. \tag{2.9}$$

From (2.8) and (2.9), we get

$$\langle a(|\eta|)\eta - a(|\xi|)\xi, \eta - \xi\rangle_{\mathbb{R}^N} \geq 0, \quad \text{for all } \eta, \xi \in \mathbb{R}^N.$$

The proof is now complete. □

Definitions 2.9. (See [2])

We say that $u \in W^{1,G}(\Omega)$ is a weak solution for problem (P) if

$$\int_\Omega a(|\nabla u|)\nabla u.\nabla v \mathrm{d}x + \int_{\partial\Omega} b(x)|u|^{p-2}uv\mathrm{d}\gamma = \int_\Omega f(x,u)v\mathrm{d}x, \ \forall v \in W^{1,G}(\Omega) \tag{2.10}$$

where $\mathrm{d}\gamma$ is the measure on the boundary $\partial\Omega$.

The energy functional corresponding to problem (P) is the C^1-functional $J : W^{1,G}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega G(|\nabla u|)\mathrm{d}x + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p \mathrm{d}\gamma - \int_\Omega F(x,u)\mathrm{d}x, \tag{2.11}$$

for all $u \in W^{1,G}(\Omega)$. Where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

Definitions 2.10. (1) We say that $u_0 \in W^{1,G}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, if we can find $r_0 > 0$ such that

$$J(u_0) \leq J(u_0 + v), \ \text{for all } v \in C^1(\overline{\Omega}) \text{ with } \|v\|_{C^1(\overline{\Omega})} \leq r_0.$$

(2) We say that $u_0 \in W^{1,G}(\Omega)$ is a local $W^{1,G}(\Omega)$-minimizer of J, if we can find $r_1 > 0$ such that

$$J(u_0) \leq J(u_0 + v), \ \text{for all } v \in W^{1,G}(\Omega) \text{ with } \|v\| \leq r_1.$$

Now, we set the assumption on the non-linear term f as follows.

(H) $f(x,0) = 0$ and there exist an odd increasing homomorphism $h \in C^1(\mathbb{R},\mathbb{R})$, and a positive function $\widehat{a}(t) \in L^\infty(\Omega)$ such that

$$|f(x,t)| \leq \widehat{a}(x)(1 + h(|t|)), \ \ \forall\, t \in \mathbb{R}, \ \forall x \in \overline{\Omega}$$

and

$$G \prec\prec H \prec\prec G_*,$$

$$1 < g^+ < h^- := \inf_{t>0} \frac{h(t)t}{H(t)} \leq h^+ := \sup_{t>0} \frac{h(t)t}{H(t)} \leq \frac{g_*^-}{g^-},$$

$$1 < h^- - 1 := \inf_{t>0} \frac{h^{'}(t)t}{h(t)} \leq h^+ - 1 := \sup_{t>0} \frac{h^{'}(t)t}{h(t)},$$

where

$$H(t) := \int_0^t h(s)\, \mathrm{d}s,$$

is an N-function.

Remark 2.11. Some assertions in Lemma 2.2 are remain valid for the N-function H and the function h

(1) $\min\{t^{h^-}, t^{h^+}\}H(1) \leq H(t) \leq \max\{t^{h^-}, t^{h^+}\}H(1)$, for all $0 < t$;
(2) $\min\{t^{h^--1}, t^{h^+-1}\}h(1) \leq h(t) \leq \max\{t^{h^--1}, t^{h^+-1}\}h(1)$, for all $0 < t$;
(3) $\min\{t^{h^-}, t^{h^+}\}H(z) \leq H(tz) \leq \max\{t^{h^-}, t^{h^+}\}H(z)$, for all $0 < t$ and $z \in \mathbb{R}$;
(4) $\min\{t^{h^--1}, t^{h^+-1}\}h(z) \leq h(tz) \leq \max\{t^{h^--1}, t^{h^+-1}\}h(z)$, for all $0 < t$ and $z \in \mathbb{R}$.

The main results of this paper are:

Theorem 2.12. *Under the assumptions* (G) *and* (H)*, if* $u \in W^{1,G}(\Omega)$ *is a non-trivial weak solution of problem (P), then* $u \in L^\infty(\Omega)$ *and* $\|u\|_\infty \leq M = M(\|\widehat{a}\|_\infty, h(1), g^-, |\Omega|, \|u\|_{h^+})$.

Theorem 2.13. *Under the assumptions* (G) *and* (H)*, if* $u_0 \in W^{1,G}(\Omega)$ *is a local* $C^1(\overline{\Omega})$*-minimizer of* J*, then* $u_0 \in C^{1,\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0, 1)$ *and* u_0 *is also a local* $W^{1,G}(\Omega)$*-minimizer of* J.

3. Boundedness results for weak solutions of problem (P)

In this section, by using the Moser iteration technique, we prove a result concerning the boundedness regularity for the problem (P). Our method, inspired by the work of Gasiǹski and Papageorgiou [13]. Considering the following problem

$$\begin{cases} -\mathrm{div}(\mathcal{A}(x, \nabla u)) = \mathcal{B}(x, u), \text{ in } \Omega \\ \\ \mathcal{A}(x, \nabla u).\nu + \psi(x, u) = 0, \text{ in } \partial\Omega \end{cases} \tag{A}$$

where Ω is a bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with C^2-boundary, $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$. We assume that problem (A) satisfies the following growth conditions:

$$\mathcal{A}(x, \eta)\eta \geq G(|\eta|), \quad \text{for all } x \in \Omega \text{ and } \eta \in \mathbb{R}^N, \tag{3.12}$$

$$\mathcal{A}(x, \eta) \leq c_0 g(|\eta|) + c_1, \quad \text{for all } x \in \Omega \text{ and } \eta \in \mathbb{R}^N, \tag{3.13}$$

$$\mathcal{B}(x, t) \leq c_2(1 + h(|t|)), \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}, \tag{3.14}$$

$$\psi(x, t) \geq 0, \quad \text{for all } x \in \partial\Omega \text{ and } t \in \mathbb{R}_+, \tag{3.15}$$

where c_0, c_1, c_2 are positive constant and h is defined in assumption (H).

We say that $u \in W^{1,G}(\Omega)$ is a weak solution of problem (A) if

$$\int_\Omega \mathcal{A}(x, \nabla u)\nabla v \mathrm{d}x + \int_{\partial\Omega} \psi(x, u)v \mathrm{d}\gamma = \int_\Omega \mathcal{B}(x, u)v \mathrm{d}x, \quad \text{for all } v \in W^{1,G}(\Omega). \tag{3.16}$$

Let us state the following useful result

Proposition 3.1. *Suppose that* (G), (H) *and (3.12)-(3.15) are satisfied. Then, if* $u \in W^{1,G}(\Omega)$ *is a non-trivial weak solution of problem (A),* u *belongs to* $L^s(\Omega)$ *for every* $1 \leq s < \infty$.

Proof. Let $u \in W^{1,G}(\Omega)$ be a non-trivial weak solution of problem (A), $u^+ := \max\{u, 0\} \in W^{1,G}(\Omega)$ and $u^- := \max\{-u, 0\} \in W^{1,G}(\Omega)$. Since $u = u^+ - u^-$, without loss of generality we may assume that $u \geq 0$.

We set, recursively

$$p_{n+1} = \widehat{g} + \frac{\widehat{g}}{g^-}\left(\frac{p_n - h^+}{h^+}\right), \quad \text{for all } n \geq 0,$$

such that

$$p_0 = \widehat{g} = g_*^- = \frac{Ng^-}{N - g^-} \quad (\text{recall that } g^- \leq g^+ < N).$$

It is clear that the sequence $\{p_n\}_{n\geq 0} \subseteq \mathbb{R}_+$ is increasing. Put $\theta_n = \dfrac{p_n - h^+}{h^+} > 0$, $\{\theta_n\}_{n\geq 0}$ is an increasing sequence.

Let

$$u_k = \min\{u, k\} \in W^{1,G}(\Omega) \cap L^\infty(\Omega), \text{ for all } k \geq 1 \;\; (\text{since } u_k \leq k, \text{ for all } k \geq 1).$$

In (3.16), we act with $u_k^{\theta_n+1} \in W^{1,G}(\Omega)$, to obtain

$$\int_\Omega \mathcal{A}(x, \nabla u).\nabla u_k^{\theta_n+1}\mathrm{d}x + \int_{\partial\Omega} \psi(x, u)u_k^{\theta_n+1}\mathrm{d}\gamma = \int_\Omega \mathcal{B}(x, u)u_k^{\theta_n+1}\mathrm{d}x.$$

It follows, by conditions (3.12), (3.15) and Lemma 2.2, that

$$\begin{aligned}
(\theta_n + 1)\int_{\{|\nabla u_k|\leq 1\}} u_k^{\theta_n} G(|\nabla u_k|)\mathrm{d}x &+ (\theta_n + 1)G(1)\int_{\{|\nabla u_k|>1\}} u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x \\
&\leq (\theta_n + 1)\int_\Omega u_k^{\theta_n} G(|\nabla u_k|)\mathrm{d}x \\
&\leq (\theta_n + 1)\int_\Omega u_k^{\theta_n}\left[\mathcal{A}(x, \nabla u).\nabla u_k\right]\mathrm{d}x \\
&\leq \int_\Omega \mathcal{A}(x, \nabla u).\nabla u_k^{\theta_n+1}\mathrm{d}x \\
&\leq \int_\Omega \mathcal{B}(x, u)u_k^{\theta_n+1}\mathrm{d}x.
\end{aligned} \tag{3.17}$$

Therefore

$$(\theta_n + 1)G(1)\int_{\{|\nabla u_k|>1\}} u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x \leq \int_\Omega \mathcal{B}(x, u)u_k^{\theta_n+1}\mathrm{d}x, \tag{3.18}$$

this gives,

$$\begin{aligned}
&(\theta_n + 1)G(1)\int_\Omega u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x \\
&= (\theta_n + 1)G(1)\left[\int_{\{|\nabla u_k|>1\}} u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x + \int_{\{|\nabla u_k|\leq 1\}} u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x\right] \\
&\leq \int_\Omega \mathcal{B}(x, u)u_k^{\theta_n+1}\mathrm{d}x + (\theta_n + 1)G(1)\int_{\{|\nabla u_k|\leq 1\}} u_k^{\theta_n}|\nabla u_k|^{g^-}\mathrm{d}x
\end{aligned}$$

$$\leq \int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + (\theta_n+1)G(1)\int_\Omega u_k^{\theta_n}\mathrm{d}x. \tag{3.19}$$

Thus

$$\int_\Omega u_k^{\theta_n}|\nabla u_k|^{g^-}\,\mathrm{d}x \leq \frac{1}{(\theta_n+1)G(1)}\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + \int_\Omega u_k^{\theta_n}\mathrm{d}x. \tag{3.20}$$

Since $\theta_n \leq p_n$, and by the continuous embedding $L^{p_n}(\Omega) \hookrightarrow L^{\theta_n}(\Omega)$, then

$$\int_\Omega u_k^{\theta_n}\mathrm{d}x \leq |\Omega|^{1-\frac{\theta_n}{p_n}}\|u_k\|_{p_n}^{\theta_n}, \quad \text{for all } k \geq 1. \tag{3.21}$$

Combining (3.20) and (3.21), we infer that

$$\int_\Omega u_k^{\theta_n}|\nabla u_k|^{g^-}\,\mathrm{d}x \leq \frac{1}{(\theta_n+1)G(1)}\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + |\Omega|^{1-\frac{\theta_n}{p_n}}\|u_k\|_{p_n}^{\theta_n}. \tag{3.22}$$

Let us observe that

$$\nabla u_k^{\frac{\theta_n+g^-}{g^-}} = \nabla u_k^{(\frac{\theta_n}{g^-}+1)} = (\frac{\theta_n}{g^-}+1)u_k^{\frac{\theta_n}{g^-}}\nabla u_k$$

and

$$\left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-} = \left(\frac{\theta_n}{g^-}+1\right)^{g^-} u_k^{\theta_n}\,|\nabla u_k|^{g^-}.$$

Integrating over Ω, we get

$$\int_\Omega \left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x = \left(\frac{\theta_n}{g^-}+1\right)^{g^-}\int_\Omega u_k^{\theta_n}\,|\nabla u_k|^{g^-}\,\mathrm{d}x. \tag{3.23}$$

Putting together (3.22) and (3.23), we conclude that

$$\begin{aligned}
&\int_\Omega \left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x \\
&\leq \left(\frac{\theta_n}{g^-}+1\right)^{g^-}\left[\frac{1}{(\theta_n+1)G(1)}\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + |\Omega|^{1-\frac{\theta_n}{p_n}}\|u_k\|_{p_n}^{\theta_n}\right] \\
&\leq (\theta_n+1)^{g^-}\left[\frac{1}{(\theta_n+1)G(1)}\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + |\Omega|^{1-\frac{\theta_n}{p_n}}\|u_k\|_{p_n}^{\theta_n}\right] \\
&\leq C_0\left(\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + \left(1+\|u_k\|_{p_n}^{p_n}\right)\right),
\end{aligned} \tag{3.24}$$

where $C_0 = (\theta_n + 1)^{g^-}\left(\frac{1}{(\theta_n + 1)G(1)} + |\Omega|^{1-\frac{\theta_n}{p_n}}\right) > 0$.

On the other side, using the condition (3.14) and Remark 2.11, we see that

$$\begin{aligned}
&\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x\\
&\le c_2\int_\Omega (1+h(|u|))\,u_k^{\theta_n+1}\mathrm{d}x\\
&\le c_2\int_\Omega \left(1+h(1)\max\{|u|^{h^--1},|u|^{h^+-1}\}\right)u_k^{\theta_n+1}\mathrm{d}x\\
&\le c_2\left(\|u_k\|_{\theta_n+1}^{\theta_n+1}+h(1)\int_\Omega \max\{|u|^{h^--1},|u|^{h^+-1}\}u_k^{\theta_n+1}\mathrm{d}x\right)\\
&\le c_2\left[\|u_k\|_{\theta_n+1}^{\theta_n+1}+h(1)\left(\int_{\{u\le 1\}} u^{h^--1}u_k^{\theta_n+1}\mathrm{d}x+\int_{\{u>1\}} u^{h^+-1}u_k^{\theta_n+1}\mathrm{d}x\right)\right]\\
&\le c_2\left[(1+h(1))\,\|u_k\|_{\theta_n+1}^{\theta_n+1}+h(1)\int_\Omega u^{h^+-1}u_k^{\theta_n+1}\mathrm{d}x\right]\\
&\le c_2\left[(1+h(1))\,\|u_k\|_{\theta_n+1}^{\theta_n+1}+h(1)\|u\|_{h^+}^{h^+-1}\|u_k\|_{(\theta_n+1)h^+}^{\theta_n+1}\right]\\
&\qquad\qquad(\text{Hölder with } h^+ \text{ and } (h^+)'=\frac{h^+}{h^+-1})\\
&= c_2\left[(1+h(1))\,\|u_k\|_{\theta_n+1}^{\theta_n+1}+h(1)\|u\|_{h^+}^{h^+-1}\|u_k\|_{p_n}^{\theta_n+1}\right]\\
&\le c_2\left[(1+h(1))\,|\Omega|^{1-\frac{\theta_n+1}{p_n}}\|u_k\|_{p_n}^{\theta_n+1}+h(1)\|u\|_{h^+}^{h^+-1}\|u_k\|_{p_n}^{\theta_n+1}\right]\\
&\qquad\qquad(\text{since } L^{p_n}(\Omega)\hookrightarrow L^{\theta_n+1}(\Omega))\\
&\le c_2\left[(1+h(1))\,|\Omega|^{1-\frac{1}{h^+}}+h(1)\|u\|_{h^+}^{h^+-1}\right]\|u_k\|_{p_n}^{\theta_n+1}\\
&\qquad\qquad(\text{since } \frac{\theta_n+1}{p_n}=\frac{1}{h^+})\\
&\le C_1\left(1+\|u_k\|_{p_n}^{p_n}\right),
\end{aligned}\tag{3.25}$$

where $C_1 = c_2\left[(1+h(1))\,|\Omega|^{1-\frac{1}{h^+}}+h(1)\|u\|_{h^+}^{h^+-1}\right] > 0$. In (3.25), we used the fact that $\theta_n + 1 < (\theta_n + 1)h^+ = p_n$.

Using (3.24) and (3.25), we find

$$\begin{aligned}
\int_\Omega \left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x+\int_\Omega \left|u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x &\le C_2\left(1+\|u_k\|_{p_n}^{p_n}\right)+\int_\Omega \left|u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x\\
&\le C_2\left(1+\|u_k\|_{p_n}^{p_n}\right)+|\Omega|^{1-\frac{\theta_n+g^-}{p_n}}\|u_k\|_{p_n}^{\theta_n+g^-}\\
&\le \left(C_2+|\Omega|^{1-\frac{\theta_n+g^-}{p_n}}\right)\left(1+\|u_k\|_{p_n}^{p_n}\right)\\
&= C_3\left(1+\|u_k\|_{p_n}^{p_n}\right),
\end{aligned}\tag{3.26}$$

where $C_2 = C_0(C_1 + 1)$ and $C_3 = C_2 + |\Omega|^{1-\frac{\theta_n+g^-}{p_n}}$.

The inequality (3.26) gives

$$\left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{W^{1,g^-}(\Omega)}^{g^-} \leq C_3 \left(1 + \|u_k\|_{p_n}^{p_n}\right). \tag{3.27}$$

Recall that $p_{n+1} = \widehat{g} + \frac{\widehat{g}}{g^-}\theta_n$ and so

$$\frac{\theta_n + g^-}{g^-} = \frac{p_{n+1}}{\widehat{g}}. \tag{3.28}$$

Since $g^- < \widehat{g} = \frac{Ng^-}{N-g^-} = g_*^-$, then the embedding $W^{1,g^-}(\Omega) \hookrightarrow L^{\widehat{g}}(\Omega)$ is continuous.

Hence, there is $C_4 > 0$ such that

$$\left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{\widehat{g}}^{g^-} \leq C_4 \left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{W^{1,g^-}(\Omega)}^{g^-}. \tag{3.29}$$

Combining (3.27), (3.28) and (3.29), we obtain

$$\|u_k\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{g}} g^-} \leq C_5 \left(1 + \|u_k\|_{p_n}^{p_n}\right), \tag{3.30}$$

where $C_5 = C_4 C_3$. Next, let $k \to +\infty$ in (3.30) and applying the monotone convergence theorem, we find that

$$\|u\|_{p_{n+1}}^{\frac{p_{n+1}}{\widehat{g}} g^-} \leq C_5 \left(1 + \|u\|_{p_n}^{p_n}\right). \tag{3.31}$$

Since $p_0 = \widehat{g}$ and the embeddings $W^{1,G}(\Omega) \hookrightarrow W^{1,g^-}(\Omega) \hookrightarrow L^{\widehat{g}}(\Omega)$ are continuous, from (3.31), we get

$$u \in L^{p_n}(\Omega), \quad \text{for all } n \geq 0. \tag{3.32}$$

Note that $p_n \to +\infty$ as $n \to +\infty$. Indeed, suppose that the sequence $\{p_n\}_{n \geq 0} \subseteq [\widehat{g}, +\infty)$ is bounded. Then we have $p_n \longrightarrow \widehat{p} \geq \widehat{g}$ as $n \to +\infty$. By definition we have

$$p_{n+1} = \widehat{g} + \frac{\widehat{g}}{g^-} \left(\frac{p_n - h^+}{h^+}\right) \quad \text{for all } n \geq 0,$$

with $p_0 = \widehat{g}$, so

$$\widehat{p} = \widehat{g} + \frac{\widehat{g}}{g^-} \left(\frac{\widehat{p} - h^+}{h^+}\right),$$

thus

$$0 \leq \widehat{p} \left(\frac{\widehat{g}}{g^- h^+} - 1\right) = \widehat{g} \left(\frac{1}{g^-} - 1\right) < 0$$

which gives us a contradiction since $g^-h^+ \leq \widehat{g} = g_*^-$ (see assumption (H)). Recall that for any measurable function $u : \Omega \longrightarrow \mathbb{R}$, the set

$$S_u = \left\{p \geq 1 : \|u\|_p < +\infty\right\}$$

is an interval. Hence, $S_u = [1, +\infty)$ (see (3.32)) and

$$u \in L^s(\Omega), \quad \text{for all } s \geq 1. \tag{3.33}$$

This ends the proof. □

In the following, we prove that, if $u \in W^{1,G}(\Omega)$ is a weak solution of problem (A) such that $u \in L^s(\Omega)$ for all $1 \leq s < \infty$, then u is a bounded function.

Proposition 3.2. *Assume that* (G), (H) *and (3.12)-(3.15) hold. Let* $u \in W^{1,G}(\Omega)$ *be a non-trivial weak solution of problem (A) such that* $u \in L^s(\Omega)$ *for all* $1 \leq s < \infty$*, then* $u \in L^\infty(\Omega)$ *and* $\|u\|_\infty \leq M = M(c_2, h(1), g^-, |\Omega|, \|u\|_{h^+})$.

Proof. Let $u \in W^{1,G}(\Omega)$ be a non-trivial weak solution of problem (A), $u^+ := \max\{u, 0\} \in W^{1,G}(\Omega)$ and $u^- := \max\{-u, 0\} \in W^{1,G}(\Omega)$. Since $u = u^+ - u^-$, we may assume without loss of generality that $u \geq 0$.

Let $\sigma_0 = \widehat{g} = g_*^- = \frac{Ng^-}{N-g^-}$ and we define by a recursively way

$$\sigma_{n+1} = \left(\frac{\sigma_n}{h^+} - 1 + g^-\right)\frac{\widehat{g}}{g^-}, \quad \text{for all } n \geq 0.$$

We have that the sequence $\{\sigma_n\}_{n\geq 0} \subseteq [\widehat{g}, +\infty)$ is increasing and $\sigma_n \longrightarrow +\infty$ as $n \to +\infty$. Arguing as in the proof of Proposition 3.1, with $\theta_n = \frac{\sigma_n}{h^+} - 1$ and $u_k^{\theta_n+1} \in W^{1,G}(\Omega) \cap L^\infty(\Omega)$ as a test function in (3.16). So, we find the following estimation

$$\int_\Omega \left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-} \mathrm{d}x \leq (\theta_n+1)^{g^-}\left[\frac{1}{(\theta_n+1)G(1)}\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x + \int_\Omega u_k^{\theta_n}\mathrm{d}x\right] \tag{3.34}$$

Using the assumption (3.14), (3.33), Remark 2.11 and Hölder inequality (with h^+ and $(h^+)' = \frac{h^+}{h^+-1}$), we deduce that

$$\begin{aligned}
\int_\Omega \mathcal{B}(x,u)u_k^{\theta_n+1}\mathrm{d}x &= \int_\Omega \mathcal{B}(x,u)u_k^{\frac{\sigma_n}{h^+}}\,\mathrm{d}x \\
&\leq c_2\int_\Omega \left(1 + h(1)\max\left\{u^{h^--1}, u^{h^+-1}\right\}\right)u_k^{\frac{\sigma_n}{h^+}}\,\mathrm{d}x \\
&\leq c_2\int_\Omega \left((1+h(1)) + h(1)u^{h^+-1}\right)u_k^{\frac{\sigma_n}{h^+}}\,\mathrm{d}x \ (\text{ since } h^- \leq h^+) \\
&\leq c_2\left[(1+h(1))\int_\Omega u_k^{\frac{\sigma_n}{h^+}}\,\mathrm{d}x + h(1)\int_\Omega u^{h^+-1}u_k^{\frac{\sigma_n}{h^+}}\,\mathrm{d}x\right] \\
&\leq c_2\left[(1+h(1))\|u_k\|_{\frac{\sigma_n}{h^+}}^{\frac{\sigma_n}{h^+}} + h(1)\left(\int_\Omega u^{h^+}\mathrm{d}x\right)^{\frac{h^+-1}{h^+}}\left(\int_\Omega u_k^{\sigma_n}\mathrm{d}x\right)^{\frac{1}{h^+}}\right]
\end{aligned}$$

$$\leq c_2\left((1+h(1))|\Omega|^{1-\frac{1}{h^+}}\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}+h(1)\|u\|_{h^+}^{h^+-1}\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}\right)$$
$$\text{(since } L^{\sigma_n}(\Omega)\hookrightarrow L^{\frac{\sigma_n}{h^+}}(\Omega)\text{)}$$
$$\leq c_2\left((1+h(1))|\Omega|^{1-\frac{1}{h^+}}+h(1)\|u\|_{h^+}^{h^+-1}\right)\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}$$
$$\leq C_6\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}} \tag{3.35}$$

for all $n\in\mathbb{N}$, where $C_6=c_2\left((1+h(1))|\Omega|^{1-\frac{1}{h^+}}+h(1)\|u\|_{h^+}^{h^+-1}\right)$.

Using the fact that $L^{\sigma_n}(\Omega)\hookrightarrow L^{\frac{\sigma_n}{h^+}-1}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega u_k^{\theta_n}\mathrm{d}x &= \int_\Omega u_k^{\frac{\sigma_n}{h^+}-1}\mathrm{d}x\\ &\leq |\Omega|^{1-\frac{1}{h^+}+\frac{1}{\sigma_n}}\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1}\\ &= C_7\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1},\quad \text{for all } n\in\mathbb{N},\end{aligned} \tag{3.36}$$

where $C_7(n)=|\Omega|^{1-\frac{1}{h^+}+\frac{1}{\sigma_n}}$.

By Hölder's inequality (with exponents h^+ and $(h^+)'=h^+/(h^+-1)$) and the embedding $L^{h^+}(\Omega)\hookrightarrow L^{\frac{h^+(g^--1)}{h^+-1}}(\Omega)$ (since $g^-<h^+$), we infer that

$$\begin{aligned}\int_\Omega\left|u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x &= \int_\Omega u_k^{\theta_n+1}u_k^{g^--1}\mathrm{d}x\\ &\leq \int_\Omega u_k^{\theta_n+1}u^{g^--1}\mathrm{d}x\ (\text{since } u_k\leq u,\ \text{for all } k\geq 1)\\ &\leq \left(\int_\Omega u^{\frac{h^+(g^--1)}{h^+-1}}\mathrm{d}x\right)^{\frac{h^+-1}{h^+}}\left(\int_\Omega u_k^{(\theta_n+1)h^+}\mathrm{d}x\right)^{\frac{1}{h^+}}\\ &\leq |\Omega|^{\frac{h^+-g^-}{h^+}}\|u\|_{h^+}^{g^--1}\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}\\ &= C_8\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}},\quad \text{for all } n\in\mathbb{N},\end{aligned} \tag{3.37}$$

where $C_8=|\Omega|^{\frac{h^+-g^-}{h^+}}\|u\|_{h^+}^{g^--1}$.

Putting together (3.34), (3.35), (3.36) and (3.37), we find that

$$\begin{aligned}&\int_\Omega\left|\nabla u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x+\int_\Omega\left|u_k^{\frac{\theta_n+g^-}{g^-}}\right|^{g^-}\mathrm{d}x\\ &\leq(\theta_n+1)^{g^-}\left[\left(\frac{C_6}{(\theta_n+1)G(1)}+C_8\right)\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}+C_7\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1}\right]\\ &\leq(\theta_n+1)^{g^-}\left[(C_6+C_8)\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}+C_7\|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1}\right],\quad(\text{since } (\theta_n+1)G(1)\geq 1)\end{aligned} \tag{3.38}$$

for all $n \in \mathbb{N}$. Since $g^- < \widehat{g} = \frac{Ng^-}{N-g^-} = g_*^-$, then the embedding $W^{1,G}(\Omega) \hookrightarrow W^{1,g^-}(\Omega) \hookrightarrow L^{\widehat{g}}(\Omega)$ are continuous. Moreover, there is $C_9 > 0$ such that

$$\left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{\widehat{g}}^{g^-} \leq C_9 \left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{W^{1,g^-}(\Omega)}^{g^-}, \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

From (3.38) and (3.39), we obtain

$$\left\| u_k^{\frac{\theta_n+g^-}{g^-}} \right\|_{\widehat{g}}^{g^-} \leq C_9 \left(\theta_n + 1\right)^{g^-} \left[\left(C_6 + C_8\right) \|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}} + C_7 \|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1} \right] \tag{3.40}$$

for all $n \in \mathbb{N}$. From the definition of the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$, we have $\dfrac{\sigma_{n+1}}{\widehat{g}} = \dfrac{\theta_n + g^-}{g^-}$.

It follows, by (3.40), that

$$\|u_k\|_{\sigma_{n+1}}^{\sigma_{n+1}\frac{g^-}{\widehat{g}}} \leq \left(\theta_n + 1\right)^{g^-} C_9 \left[\left(C_6 + C_8\right) \|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}} + C_7 \|u_k\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1} \right] \tag{3.41}$$

for all $n \in \mathbb{N}$.

Let $k \longrightarrow +\infty$ in (3.41), and using the monotone convergence theorem , we get

$$\|u\|_{\sigma_{n+1}}^{\sigma_{n+1}\frac{g^-}{\widehat{g}}} \leq \left(\theta_n + 1\right)^{g^-} C_9 \left[\left(C_6 + C_8\right) \|u\|_{\sigma_n}^{\frac{\sigma_n}{h^+}} + C_7 \|u\|_{\sigma_n}^{\frac{\sigma_n}{h^+}-1} \right] \tag{3.42}$$

We distinguish two cases.

Case 1: If $\left\{n \in \mathbb{N},\ \|u\|_{\sigma_n} \leq 1\right\}$ is unbounded. Then, without loss of generality, we may assume that

$$\|u\|_{\sigma_n} \leq 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.43}$$

Hence,

$$\|u\|_\infty \leq 1$$

since, $\sigma_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and $u \in L^s(\Omega)$ for all $s \geq 1$. So, we are done with $M = 1$.

Case 2: If $\left\{n \in \mathbb{N},\ \|u\|_{\sigma_n} \leq 1\right\}$ is bounded. Then, without loss of generality, we can suppose that

$$\|u\|_{\sigma_n} > 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.44}$$

From (3.42) and (3.44), we find that

$$\|u\|_{\sigma_{n+1}}^{\sigma_{n+1}\frac{g^-}{\widehat{g}}} \leq C_{10} \|u\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}, \quad \text{for all } n \in \mathbb{N} \tag{3.45}$$

where $C_{10}(n) = \left(\theta_n + 1\right)^{g^-} C_9 \left(C_6 + C_7 + C_8\right)$.

We want to remark that

$C_9 \left(C_6 + C_7 + C_8\right)$

$$= C_9\left[c_2\left((1+h(1))|\Omega|^{1-\frac{1}{h^+}} + h(1)\|u\|_{h^+}^{h^+-1}\right) + |\Omega|^{\frac{h^+-g^-}{h^+}}\|u\|_{h^+}^{g^--1} + |\Omega|^{1-\frac{1}{h^+}+\frac{1}{\sigma_n}}\right]$$
$$\leq C_9\left[c_2\left((1+h(1))|\Omega|^{1-\frac{1}{h^+}} + h(1)\|u\|_{h^+}^{h^+-1}\right) + |\Omega|^{\frac{h^+-g^-}{h^+}}\|u\|_{h^+}^{g^--1} + |\Omega| + 1\right]$$
$$= C_{11}, \quad \text{for all } n \in \mathbb{N}. \tag{3.46}$$

Hence $C_{11} > 0$ is independent of n. Moreover, we have that

$$(\theta_n + 1)^{g^-} = (\frac{\sigma_n}{h^+})^{g^-} \leq (\sigma_n)^{g^-} \leq (\sigma_{n+1})^{g^-}, \quad \text{for all } n \in \mathbb{N}. \tag{3.47}$$

From (3.45), (3.46) and (3.47), we obtain

$$\|u\|_{\sigma_{n+1}}^{\sigma_{n+1}\frac{g^-}{\widehat{g}}} \leq (\sigma_{n+1})^{g^-} C_{11}\|u\|_{\sigma_n}^{\frac{\sigma_n}{h^+}}, \quad \text{for all } n \in \mathbb{N}. \tag{3.48}$$

Therefore, from [14, Theorem 6.2.6, p. 737], we find that

$$\|u\|_{\sigma_{n+1}} \leq M, \quad \text{for all } n \in \mathbb{N} \tag{3.49}$$

for some $M\left(c_2, h(1), g^-, |\Omega|, \|u\|_{h^+}\right) \geq 0$.

On the other hand, by the hypotheses of the proposition, we have that

$$u \in L^s(\Omega), \quad \text{for all } 1 \leq s < \infty. \tag{3.50}$$

Exploiting (3.49),(3.50) and the fact that $\sigma_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$, we deduce that

$$\|u\|_\infty \leq M.$$

This ends the proof. □

Proof of Theorem 2.12. Let

$$\mathcal{A}(x,\eta) = a(|\eta|)\eta, \qquad \text{for all } x \in \Omega \text{ and } \eta \in \mathbb{R}^N$$
$$\mathcal{B}(x,t) = f(x,t), \qquad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}$$
$$\psi(x,t) = b(x)|t|^{p-2}t, \text{ for all } x \in \partial\Omega \text{ and } t \in \mathbb{R}$$

in problem (A). Then, $\mathcal{A}, \mathcal{B}$ and ψ satisfy the growth conditions (3.12)-(3.15) and the problem (A) turns to (P). By the Propositions 3.1 and 3.2, we conclude that every weak solution $u \in W^{1,G}(\Omega)$ of problem (P) belongs to $L^\infty(\Omega)$ and $\|u\|_\infty \leq M = M(c_2 = \|\widehat{a}\|_\infty, h(1), g^-, |\Omega|, \|u\|_{h^+})$. This ends the proof. □

4. $W^{1,G}(\Omega)$ versus $C^1(\overline{\Omega})$ local minimizers

In this section, using the regularity theory of Lieberman [26], we extend the result of Brezis and Nirenberg's [8] to the problem (P).

Proposition 4.1. *let $u_0 \in W^{1,G}(\Omega)$ be a local $C^1(\overline{\Omega})$-minimizer of J (see Definition 2.10), then u_0 is a weak solution for problem (P) and $u_0 \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$.*

Proof. By hypothesis u_0 is a local $C^1(\overline{\Omega})$-minimizer of J, for every $v \in C^1(\overline{\Omega})$ and $t > 0$ small enough, we have $J(u_0) \leq J(u_0 + tv)$. Hence,

$$0 \leq \langle J^{'}(u_0), v\rangle \quad \text{for all } v \in C^1(\overline{\Omega}). \tag{4.51}$$

Since $C^1(\overline{\Omega})$ is dense in $W^{1,G}(\Omega)$, from (4.51) we infer that $J^{'}(u_0) = 0$. Namely,

$$\begin{aligned} &\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x + \int_{\partial\Omega} b(x)|u_0|^{p-2}u_0 v \mathrm{d}\gamma \\ &= \int_\Omega f(x, u_0)v \mathrm{d}x, \quad \text{for all } v \in W^{1,G}(\Omega). \end{aligned} \tag{4.52}$$

By the nonlinear Green's identity, we get

$$\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x = -\int_\Omega \mathrm{div}(a(|\nabla u_0|)\nabla u_0).v \mathrm{d}x + \int_{\partial\Omega} a(|\nabla u_0|)\nabla u_0.\nu\ v \mathrm{d}\gamma, \tag{4.53}$$

for all $v \in W^{1,G}(\Omega)$. It follows that,

$$\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x = -\int_\Omega \mathrm{div}(a(|\nabla u_0|)\nabla u_0).v \mathrm{d}x, \quad \text{for all } v \in W_0^{1,G}(\Omega). \tag{4.54}$$

Hence, by (4.52)

$$-\int_\Omega \mathrm{div}(a(|\nabla u_0|)\nabla u_0).v \mathrm{d}x = \int_\Omega f(x, u_0)v \mathrm{d}x, \quad \text{for all } v \in W_0^{1,G}(\Omega),$$

which gives,

$$-\,\mathrm{div}(a(|\nabla u_0(x)|)\nabla u_0(x)) = f(x, u_0(x)), \quad \text{for almost } x \in \Omega. \tag{4.55}$$

From (4.52), (4.53) and (4.55), we obtain

$$\left\langle a(|\nabla u_0|)\frac{\partial u_0}{\partial \nu} + b(x)|u_0|^{p-2}u_0, v\right\rangle_{\partial\Omega} = 0 \quad \text{for all } v \in W^{1,G}(\Omega). \tag{4.56}$$

It follows that

$$a(|\nabla u_0|)\frac{\partial u_0}{\partial \nu} + b(x)|u_0|^{p-2}u_0 = 0 \quad \text{on } \partial\Omega.$$

So, $u_0 \in W^{1,G}(\Omega)$ is a weak solution for the problem (P). From Theorem 2.12, we have that $u_0 \in L^\infty(\Omega)$.

We define $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $B : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} A(x, \eta) = a(|\eta|)\eta; \\ B(x, t) = f(x, t); \\ \phi(x, t) = b(x)|t|^{p-2}t. \end{cases} \tag{4.57}$$

It is easy to show that, for $x, y \in \overline{\Omega},\ \eta \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N,\ t \in \mathbb{R}$, the following estimations hold:

$$A(x, 0) = 0, \tag{4.58}$$

$$\sum_{i,j=1}^{N} \frac{\partial (A)_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j \geq \frac{g(|\eta|)}{|\eta|}|\xi|^2, \tag{4.59}$$

$$\sum_{i,j=1}^{N} \left|\frac{\partial (A)_j}{\partial \eta_i}(x, \eta)\right| |\eta| \leq c(1 + g(|\eta|)), \tag{4.60}$$

$$|A(x, \eta) - A(y, \eta)| \leq c(1 + g(|\eta|))(|x - y|^{\theta}),\ \text{for some } \theta \in (0, 1), \tag{4.61}$$

$$|B(x, t)| \leq c\,(1 + h(|t|))\,. \tag{4.62}$$

Indeed: inequalities (4.58) , (4.61) and (4.62) are evident.

For $x \in \overline{\Omega},\ \eta \in \mathbb{R}^N\backslash\{0\},\ \xi \in \mathbb{R}^N$, we have

$$D_\eta(A(x, \eta))\xi = a(|\eta|)\xi + a^{'}(|\eta|)\frac{\langle \eta, \xi\rangle_{\mathbb{R}^N}}{|\eta|}\eta \tag{4.63}$$

and

$$\langle D_\eta(A(x, \eta))\xi, \xi\rangle_{\mathbb{R}^N} = a(|\eta|)\langle \xi, \xi\rangle_{\mathbb{R}^N} + a^{'}(|\eta|)\frac{\left[\langle \eta, \xi\rangle_{\mathbb{R}^N}\right]^2}{|\eta|} \tag{4.64}$$

where $\langle, \rangle_{\mathbb{R}^N}$ is the inner product in $\mathbb{R}^N$. Hence, we have the following derivative

$$D_\eta(a(|\eta|)\eta) = \frac{a^{'}(|\eta|)}{|\eta|}\eta\eta^T + a(|\eta|)I_N = a(|\eta|)\left(I_N + \frac{a^{'}(|\eta|)|\eta|}{a(|\eta|)}\frac{1}{|\eta|^2}M_N(\eta, \eta)\right) \tag{4.65}$$

for all $\eta \in \mathbb{R}^N\backslash\{0\}$, where η^T is the transpose of η, I_N is the unit matrix in $M_N(\mathbb{R})$ and

$$M_N(\eta, \eta) = \eta\eta^T = \begin{pmatrix} \eta_1^2 & \eta_1\eta_2 & \cdots & \eta_1\eta_N \\ \eta_2\eta_1 & \eta_2^2 & \cdots & \eta_2\eta_N \\ \vdots & \vdots & \ddots & \vdots \\ \eta_N\eta_1 & \eta_N\eta_2 & \cdots & \eta_N^2 \end{pmatrix} \tag{4.66}$$

for all $\eta \in \mathbb{R}^N$.

Note that, for all $\eta \in \mathbb{R}^N$, we have

$$\|M_N(\eta,\eta)\|_{\mathbb{R}^N} = \sum_{i,j=1}^{N} |\eta_i \eta_j| = \left(\sum_{i=1}^{N} |\eta_i|\right)^2 \leq N \sum_{i=1}^{N} |\eta_i|^2 = N|\eta|^2 \tag{4.67}$$

where $\|.\|_{\mathbb{R}^N}$ is a norm on $M_N(\mathbb{R})$.

From (4.64) and assumption (g_3), we have

$$\begin{aligned}\sum_{i,j=1}^{N} \frac{\partial (A)_j}{\partial \eta_i}(x,\eta)\xi_i\xi_j &= \langle D_\eta(A(x,\eta))\xi,\xi\rangle \\ &= a(|\eta|)\langle \xi,\xi\rangle_{\mathbb{R}^N} + a^{'}(|\eta|)\frac{\left[\langle \eta,\xi\rangle_{\mathbb{R}^N}\right]^2}{|\eta|} \\ &= a(|\eta|)\left[\langle \xi,\xi\rangle_{\mathbb{R}^N} + \frac{a^{'}(|\eta|)|\eta|}{a(|\eta|)}\frac{\left[\langle \eta,\xi\rangle_{\mathbb{R}^N}\right]^2}{|\eta|^2}\right] \\ &\geq \frac{g(|\eta|)}{|\eta|}|\xi|^2 \end{aligned}\tag{4.68}$$

for all $x \in \overline{\Omega},\ \eta \in \mathbb{R}^N\backslash\{0\},\ \xi \in \mathbb{R}^N$.

Moreover, from (4.65), (4.67) and assumption (g_3), we find that

$$\begin{aligned}\sum_{i,j=1}^{N} \left|\frac{\partial (A)_j}{\partial \eta_i}(x,\eta)\right| |\eta| &= \|D_\eta(A(x,\eta))\|_{\mathbb{R}^N}|\eta| \\ &\leq \left(\|I_N\|_{\mathbb{R}^N} + \frac{a^{'}(|\eta|)|\eta|}{a(|\eta|)}\frac{1}{|\eta|^2}\|M_N(\eta,\eta)\|_{\mathbb{R}^N}\right) g(|\eta|) \\ &\leq \left(1 + \frac{a^{'}(|\eta|)|\eta|}{a(|\eta|)}\right) N g(|\eta|) \\ &\leq a^+ N g(|\eta|) \\ &\leq a^+ N(1 + g(|\eta|)) \end{aligned}\tag{4.69}$$

for all $x \in \overline{\Omega},\ \eta \in \mathbb{R}^N\backslash\{0\}$.

The non-linear regularity result of Lieberman [26, p. 320] implies the existence of $\alpha \in (0,1)$ and $M_0 \geq 0$ such that

$$u_0 \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_0\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_0.$$

This ends the proof. □

Proposition 4.2. *Under the assumptions (G) and (H), if $u_0 \in W^{1,G}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J (see Definition 2.10), then $u_0 \in W^{1,G}(\Omega)$ is also a local $W^{1,G}(\Omega)$-minimizer of J (see Definition 2.10).*

Proof. Let u_0 be a local $C^1(\overline{\Omega})$-minimizer of J, then, by Proposition 4.1, we have

$$u_0 \in L^\infty(\Omega) \text{ and } u_0 \in C^{1,\alpha}(\overline{\Omega}) \text{ for some } \alpha \in (0,1). \tag{4.70}$$

To prove that u_0 is a local $W^{1,G}(\Omega)$-minimizer of J, we argue by contradiction. Suppose that u_0 is not a local $W^{1,G}(\Omega)$-minimizer of J. Let $\varepsilon \in (0,1)$ and define

$$B(u_0,\varepsilon) = \left\{ v \in W^{1,G}(\Omega) : \mathcal{K}(v-u_0) \le \varepsilon \right\},$$

recall that $\mathcal{K}(v-u_0) = \int_\Omega G(|\nabla(v-u_0)|)\mathrm{d}x + \int_\Omega G(|v-u_0|)\mathrm{d}x$.

We consider the following minimization problem:

$$m_\varepsilon = \inf \{ J(v) : v \in B(u_0,\varepsilon) \}. \tag{4.71}$$

By the hypothesis of contradiction and assumption (H), we have

$$-\infty < m_\varepsilon < J(u_0). \tag{4.72}$$

The set $B(u_0,\varepsilon)$ is bounded, closed and convex subset of $W^{1,G}(\Omega)$ and is a neighbourhood of $u_0 \in W^{1,G}(\Omega)$. Since $f(x,t)$ satisfies the assumption (H), the functional $J : W^{1,G}(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous. So, From the Weierstrass theorem there exist $v_\varepsilon \in B(u_0,\varepsilon)$ such that $m_\varepsilon = J(v_\varepsilon)$. Moreover, by (4.72), we deduce that $v_\varepsilon \neq 0$.

Now, using the Lagrange multiplier rule [21, p. 35], we can find $\lambda_\varepsilon \ge 0$ such that

$$\langle J'(v_\varepsilon), v\rangle + \lambda_\varepsilon \langle \mathcal{K}'(v_\varepsilon - u_0), v\rangle = 0 \text{ for all } v \in W^{1,G}(\Omega),$$

which implies

$$\begin{aligned}
&\langle J'(v_\varepsilon), v\rangle + \lambda_\varepsilon \langle \mathcal{K}'(v_\varepsilon - u_0), v\rangle \\
&= \int_\Omega a(|\nabla v_\varepsilon|)\nabla v_\varepsilon.\nabla v \mathrm{d}x + \int_{\partial\Omega} b(x)|v_\varepsilon|^{p-2} v_\varepsilon v \mathrm{d}\gamma \\
&+ \lambda_\varepsilon \int_\Omega a(|\nabla(v_\varepsilon - u_0)|)\nabla(v_\varepsilon - u_0).\nabla v \mathrm{d}x - \int_\Omega f(x,v_\varepsilon) v \mathrm{d}x \\
&+ \lambda_\varepsilon \int_\Omega a(|v_\varepsilon - u_0|)(v_\varepsilon - u_0) v \mathrm{d}x \\
&= 0
\end{aligned} \tag{4.73}$$

for all $v \in W^{1,G}(\Omega)$.

In the other side, from Proposition 4.1, we see that $u_0 \in W^{1,G}(\Omega)$ is a weak solution for the problem (P). Hence,

$$\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x + \int_{\partial\Omega} b(x)|u_0|^{p-2} u_0 v \mathrm{d}\gamma - \int_\Omega f(x,u_0) v \mathrm{d}x = 0 \tag{4.74}$$

for all $v \in W^{1,G}(\Omega)$.

Next, we have to show that v_ε belongs to $L^\infty(\Omega)$ and hence to $C^{1,\alpha}(\overline{\Omega})$. We distinguish three cases.

Case 1: If $\lambda_\varepsilon = 0$ with $\varepsilon \in (0, 1]$, we find that v_ε solves the Robin boundary value problem (P). As in Proposition 4.1, we prove that $v_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and there is $M_1 \geq 0$ (independent of ε) such that

$$\|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_1.$$

Case 2: If $0 < \lambda_\varepsilon \leq 1$ with $\varepsilon \in (0, 1]$. Multiplying (4.74) by $\lambda_\varepsilon > 0$ and adding (4.73), we get

$$\begin{aligned}
&\int_\Omega a(|\nabla v_\varepsilon|)\nabla v_\varepsilon.\nabla v \mathrm{d}x \\
&+ \lambda_\varepsilon \int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x + \lambda_\varepsilon \int_\Omega a(|\nabla (v_\varepsilon - u_0)|)\nabla (v_\varepsilon - u_0).\nabla v \mathrm{d}x \\
&+ \lambda_\varepsilon \int_{\partial\Omega} b(x)|u_0|^{p-2}u_0 v \mathrm{d}\gamma + \int_{\partial\Omega} b(x)|v_\varepsilon|^{p-2}v_\varepsilon v \mathrm{d}\gamma \\
&= \lambda_\varepsilon \int_\Omega f(x, u_0)v \mathrm{d}x + \int_\Omega f(x, v_\varepsilon)v \mathrm{d}x - \lambda_\varepsilon \int_\Omega a(|v_\varepsilon - u_0|)(v_\varepsilon - u_0)v \mathrm{d}x
\end{aligned} \tag{4.75}$$

for all $v \in W^{1,G}(\Omega)$.

Let $A_\varepsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $B_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\phi_\varepsilon : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{cases}
A_\varepsilon(x, \eta) = a(|\eta|)\eta + \lambda_\varepsilon a(|\eta - \nabla u_0|)(\eta - \nabla u_0) + \lambda_\varepsilon a(|\nabla u_0|)\nabla u_0; \\
B_\varepsilon(x, t) = f(x, t) + \lambda_\varepsilon f(x, u_0) - \lambda_\varepsilon a(|t - u_0|)(t - u_0); \\
\phi_\varepsilon(x, t) = b(x)\left(|t|^{p-2}t + \lambda_\varepsilon |u_0|^{p-2}u_0\right).
\end{cases} \tag{4.76}$$

It is clear that $A_\varepsilon \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$. Hence, the equation (4.75) is the weak formulation of the following Robin boundary value problem

$$\begin{cases}
-\mathrm{div}(A_\varepsilon(x, \nabla v_\varepsilon)) = B_\varepsilon(x, v_\varepsilon) \text{ on } \Omega, \\
A_\varepsilon(x, \nabla v_\varepsilon).\nu + \phi_\varepsilon(x, v_\varepsilon) = 0 \quad \text{on } \partial\Omega,
\end{cases}$$

where ν is the inner normal to $\partial\Omega$.

From Lemma 2.8 and assumption (G), for $\eta \in \mathbb{R}^n$ and $x \in \Omega$, we have

$$\begin{aligned}
\langle A_\varepsilon(x, \eta), \eta\rangle_{\mathbb{R}^N} &= \langle a(|\eta|)\eta, \eta\rangle_{\mathbb{R}^N} + \lambda_\varepsilon \langle a(|\eta - \nabla u_0|)(\eta - \nabla u_0), \eta - \nabla u_0 \\
&\quad - (-\nabla u_0)\rangle_{\mathbb{R}^N} - \lambda_\varepsilon \langle a(| - \nabla u_0|)(-\nabla u_0), \eta - \nabla u_0 - (-\nabla u_0)\rangle_{\mathbb{R}^N} \\
&\geq g(|\eta|)|\eta| \\
&\geq G(|\eta|)
\end{aligned} \tag{4.77}$$

and

$$|A_\varepsilon(x, \eta)| \leq a(|\eta|)|\eta| + \lambda_\varepsilon a(|\eta - \nabla u_0|)|\eta - \nabla u_0| + \lambda_\varepsilon a(|\nabla u_0|)|\nabla u_0|$$

$$\leq g(|\eta|) + g(|\eta - \nabla u_0|) + g(|\nabla u_0|) \text{ (since } 0 < \lambda_\varepsilon \leq 1)$$
$$\leq g(|\eta|) + g(|\eta| + |\nabla u_0|) + g(|\nabla u_0|)$$
$$\leq c_0 g(|\eta|) + c_1 \text{ (using Lemma 2.2 and the monotonicity of } g). \tag{4.78}$$

Then, A_ε, B_ε and ϕ_ε satisfy the corresponding growth conditions (3.12)-(3.15). So, using the Propositions 3.1 and 3.2, we obtain that $v_\varepsilon \in L^\infty(\Omega)$.

It remains, using the regularity theorem of Lieberman, to show that $v_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. So, we need to prove that A_ε and B_ε satisfy the corresponding (4.58)-(4.62). The inequalities (4.58) and (4.62) are evident. The inequality (4.61) follows from Lemma 2.7 and the fact that ∇u_0 is Hölder continuous.

As in (4.63) and (4.64), we have

$$D_\eta(a(|\eta - \nabla u_0|)(\eta - \nabla u_0))\xi$$
$$= a(|\eta - \nabla u_0|)\xi + a'(|\eta - \nabla u_0|)\frac{\langle \eta - \nabla u_0, \xi\rangle_{\mathbb{R}^N}}{|\eta - \nabla u_0|}(\eta - \nabla u_0) \tag{4.79}$$

and

$$\langle D_\eta(a(|\eta - \nabla u_0|)(\eta - \nabla u_0))\xi, \xi\rangle_{\mathbb{R}^N}$$
$$= a(|\eta - \nabla u_0|)\langle \xi, \xi\rangle_{\mathbb{R}^N} + a'(|\eta - \nabla u_0|)\frac{\left[\langle \eta - \nabla u_0, \xi\rangle_{\mathbb{R}^N}\right]^2}{|\eta - \nabla u_0|} \tag{4.80}$$

for all $x \in \overline{\Omega}$, $\eta \in \mathbb{R}^N \setminus \{\nabla u_0\}$, $\xi \in \mathbb{R}^N$.

Exploiting (4.68), (4.80) and assumption (g_3), we infer that

$$\sum_{i,j=1}^{N} \frac{\partial (A_\varepsilon)_j}{\partial \eta_i}(x, \eta)\xi_i\xi_j = \langle D_\eta(A)(x, \eta)\xi, \xi\rangle_{\mathbb{R}^N}$$
$$+ \lambda_\varepsilon a(|\eta - \nabla u_0|)\left(\langle \xi, \xi\rangle_{\mathbb{R}^N} + \frac{a'(|\eta - \nabla u_0|)|\eta - \nabla u_0|}{a(|\eta - \nabla u_0|)}\frac{\left[\langle \eta - \nabla u_0, \xi\rangle_{\mathbb{R}^N}\right]^2}{|\eta - \nabla u_0|^2}\right)$$
$$\geq \langle D_\eta(A)(x, \eta)\xi, \xi\rangle_{\mathbb{R}^N}$$
$$\geq \frac{g(|\eta|)}{|\eta|}|\xi|^2 \tag{4.81}$$

for all $x \in \overline{\Omega}$, $\eta \in \mathbb{R}^N \setminus \{\nabla u_0\}$, $\xi \in \mathbb{R}^N$.

Note that the derivative of A_ε has the form

$$D_\eta(A_\varepsilon(x, \eta)) = D_\eta(A(x, \eta)) + \lambda_\varepsilon a(|\eta - \nabla u_0|)$$
$$\left(I_N + \frac{a'(|\eta - \nabla u_0|)|\eta - \nabla u_0|}{a(|\eta - \nabla u_0|)}\frac{1}{|\eta - \nabla u_0|^2}M_N(\eta - \nabla u_0, \eta - \nabla u_0)\right) \tag{4.82}$$

for all $x \in \overline{\Omega},\ \eta \in \mathbb{R}^N \backslash \{\nabla u_0\}$, where $M_N(\eta - \nabla u_0, \eta - \nabla u_0)$ is defined in (4.66). As in (4.67), we have

$$\|M_N(\eta - \nabla u_0, \eta - \nabla u_0)\|_{\mathbb{R}^N} \leq N|\eta - \nabla u_0|^2. \tag{4.83}$$

In light of (4.69), (4.82), (4.83) and assumption (g_3), we see that

$$\begin{aligned}
\sum_{i,j=1}^{N} \left| \frac{\partial (A_\varepsilon)_j}{\partial \eta_i}(x,\eta) \right| |\eta| &= \|D_\eta(A_\varepsilon(x,\eta))\|_{\mathbb{R}^N} |\eta| \\
&\leq a^+ N a(|\eta|)|\eta| + \lambda_\varepsilon a(|\eta - \nabla u_0|)|\eta| \|I_N\|_{\mathbb{R}^N} \\
&+ \lambda_\varepsilon a(|\eta - \nabla u_0|)|\eta| \\
&\left(\frac{a^{'}(|\eta - \nabla u_0|)|\eta - \nabla u_0|}{a(|\eta - \nabla u_0|)} \frac{\|M_N(\eta - \nabla u_0, \eta - \nabla u_0)\|_{\mathbb{R}^N}}{|\eta - \nabla u_0|^2} \right) \\
&\leq a^+ N a(|\eta|)|\eta| + \lambda_\varepsilon a^+ N a(|\eta - \nabla u_0|)|\eta| \\
&\leq a^+ N |\eta| \left(a(|\eta|) + a(|\eta - \nabla u_0|) \right) \\
&\leq c(1 + g(|\eta|))
\end{aligned} \tag{4.84}$$

for all $x \in \overline{\Omega},\ \eta \in \mathbb{R}^N \backslash \{\nabla u_0\}$.

So, from the regularity theorem of Lieberman [26, p. 320], we can find $\alpha \in (0, 1)$ and $M_2 > 0$, both independent from ε, such that

$$v_\varepsilon \in C^{1,\alpha}(\overline{\Omega}), \quad \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_2 \ \text{ for all } \varepsilon \in (0, 1]. \tag{4.85}$$

Case 3: If $1 < \lambda_\varepsilon$ with $\varepsilon \in (0, 1]$. Multiplying (4.74) with -1, setting $y_\varepsilon = v_\varepsilon - u_0$ in (4.73) and adding, we get

$$\begin{aligned}
&\int_\Omega a(|\nabla(y_\varepsilon + u_0)|)\nabla(y_\varepsilon + u_0).\nabla v \mathrm{d}x \\
&- \int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla v \mathrm{d}x + \lambda_\varepsilon \int_\Omega a(|\nabla y_\varepsilon|)\nabla y_\varepsilon.\nabla v \mathrm{d}x \\
&- \int_{\partial\Omega} b(x)|u_0|^{p-2}u_0 v \mathrm{d}\gamma + \int_{\partial\Omega} b(x)|y_\varepsilon + u_0|^{p-2}(y_\varepsilon + u_0)v \mathrm{d}\gamma \\
&= \int_\Omega f(x, y_\varepsilon + u_0)v \mathrm{d}x - \int_\Omega f(x, u_0)v \mathrm{d}x - \lambda_\varepsilon \int_\Omega a(|y_\varepsilon|)y_\varepsilon v \mathrm{d}x
\end{aligned} \tag{4.86}$$

for all $v \in W^{1,G}(\Omega)$.

Defining again $\tilde{A}_\varepsilon : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$, $\tilde{B}_\varepsilon : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{\phi}_\varepsilon : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases}
\tilde{A}_\varepsilon(x,\eta) = a(|\eta|)\eta + \frac{1}{\lambda_\varepsilon} a(|\eta + \nabla u_0|)(\eta + \nabla u_0) - \frac{1}{\lambda_\varepsilon} a(|\nabla u_0|)\nabla u_0; \\
\tilde{B}_\varepsilon(x,t) = \frac{1}{\lambda_\varepsilon}\left[f(x, t + u_0) - f(x, u_0) \right] - a(|t|)t; \\
\tilde{\phi}_\varepsilon(x,t) = \frac{1}{\lambda_\varepsilon} b(x) \left(|t + u_0|^{p-2}(t + u_0) - |u_0|^{p-2}u_0 \right).
\end{cases} \tag{4.87}$$

It is clear that $A_\varepsilon \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$. Rewriting (4.86), we find the following equation

$$\begin{cases} -\text{div}(\tilde{A}_\varepsilon(x, \nabla y_\varepsilon)) = \tilde{B}_\varepsilon(x, y_\varepsilon) \text{ on } \Omega, \\ \\ \tilde{A}_\varepsilon(x, \nabla y_\varepsilon).\nu + \tilde{\phi}_\varepsilon(x, y_\varepsilon) = 0 \text{ on } \partial\Omega, \end{cases}$$

where ν is the inner normal to $\partial\Omega$.

Again, from Propositions 3.1 and 3.2, we conclude that $y_\varepsilon \in L^\infty(\Omega)$. By the same arguments used in case 2, we prove that $\tilde{A}_\varepsilon$ and $\tilde{B}_\varepsilon$ satisfy the corresponding inequalities (4.58)-(4.62). So, the regularity theorem of Lieberman [26, p. 320] implies the existence of $\alpha \in (0, 1)$ and $M_3 \geq 0$ both independent of ε such that

$$y_\varepsilon \in C^{1,\alpha}(\overline{\Omega}), \quad \text{and } \|y_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_3.$$

Since $y_\varepsilon = v_\varepsilon - u_0$ and $u_0 \in C^{1,\alpha}(\overline{\Omega})$, we infer that

$$v_\varepsilon \in C^{1,\alpha}(\overline{\Omega}), \quad \text{and } \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_3.$$

Let $\varepsilon_n \searrow 0$ as $n \longrightarrow +\infty$. Therefore, in the three cases, we have the same uniform $C^{1,\alpha}(\overline{\Omega})$ bounds for the sequence $\{v_{\varepsilon_n}\}_{n\geq 1} \subseteq W^{1,G}(\Omega)$. Hence, the Arzelà-Ascoli theorem guarantees that, up to a subsequence,

$$v_{\varepsilon_n} \to v \quad \text{in} \quad C^1(\overline{\Omega}) \tag{4.88}$$

for some $v \in C^1(\overline{\Omega})$.

Recalling that $\|v_{\varepsilon_n} - u_0\|^{g^+} \leq \varepsilon_n$, for all $n \in \mathbb{N}$. So,

$$v_{\varepsilon_n} \longrightarrow u_0 \text{ in } W^{1,G}(\Omega). \tag{4.89}$$

Therefore, from (4.88) and (4.89), we obtain $v_{\varepsilon_n} \to u_0$ in $C^1(\overline{\Omega})$. So, for n sufficiently large, say $n \geq n_0$, we have $\|v_{\varepsilon_n} - u_0\|_{C^1(\overline{\Omega})} \leq r_0$ (where $r_0 > 0$ is defined in Definition 2.10), which provides

$$J(u_0) \leq J(v_{\varepsilon_n}) \text{ for all } n \geq n_0. \tag{4.90}$$

On the other hand, we have

$$J(v_{\varepsilon_n}) < J(u_0) \text{ for all } n \in \mathbb{N}. \tag{4.91}$$

Comparing (4.90) and (4.91), we reach a contradiction. This proves that u_0 is a local $W^{1,G}(\Omega)$-minimizer of J. This ends the proof. □

Proof of Theorem 2.13:. The proof follows by applying Propositions 4.1 and 4.2. □

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Declarations

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