Mathematische Zeitschrift (2024) 308:66
https://doi.org/10.1007/s00209-024-03620-7

Mathematische Zeitschrift

Concentration of ground state solutions for supercritical zero-mass (N, q)-equations of Choquard reaction

Liejun Shen[1] · **Vicenţiu D. Rădulescu**[1,2,3,4,5]

Received: 8 February 2024 / Accepted: 15 September 2024

Abstract

We study the following singularly perturbed (N, q)-equation of Choquard type

$$-\varepsilon^N \Delta_N u - \varepsilon^q \Delta_q u = \varepsilon^{\mu-N}\left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^\mu} dy\right)K(x)f(u),\ x \in \mathbb{R}^N,$$

where $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$ denotes the usual r-Laplacian operator with $r \in \{q, N\}$ and $1 < q < N$, $\varepsilon > 0$ is a sufficiently small parameter, $K \in C^0(\mathbb{R}^N)$ satisfies some technical assumptions, $0 < \mu < N$ and F is the primitive of f that fulfills a supercritical exponential growth in the Trudinger–Moser sense. Due to the new version of Trudinger–Moser type inequality introduced in Shen and Rădulescu (Zero-mass (N, q)-Laplacian equation with Stein-Weiss convolution part in $\mathbb{R}^N$: supercritical exponential case. submitted), we aim to derive the existence and concentration of ground state solutions for the given equation using variational method, where the concentrating phenomenon appears at the maximum point set of K as $\varepsilon \to 0^+$.

Keywords Zero-mass (N, q)-Laplacian equation · Nonlocal nonlinearity · Supercritical exponential growth · Trudinger–Moser inequality · Ground state · Variational method

Mathematics Subject Classification 35J60 · 35A23 · 35B06

✉ Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

Liejun Shen
ljshen@zjnu.edu.cn

1 Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, People's Republic of China

2 Faculty of Applied Mathematics, AGH University of Kraków, al. Mickiewicza 30, 30-059 Kraków, Poland

3 Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 3058/10, Brno 61600, Czech Republic

4 Department of Mathematics, University of Craiova, 200585 Craiova, Romania

5 Simion Stoilow Institute of Mathematics of the Romanian Academy, 010702 Bucharest, Romania

1 Introduction and main results

In this article, we are concerned with the existence and concentration of ground state solutions for the following singularly perturbed (N, q)-Laplacian equation of Choquard type

$$- \varepsilon^N \Delta_N u - \varepsilon^q \Delta_q u = \varepsilon^{\mu-N}\left(\int_{\mathbb{R}^N} \frac{K(y)F(u(y))}{|x-y|^\mu}dy\right)K(x)f(u),\ x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_r u = \mathrm{div}(|\nabla u|^{r-2}\nabla u)$ denotes the usual r-Laplacian operator with $r \in \{q, N\}$ and $1 < q < N$, $\varepsilon > 0$ is a sufficiently small parameter, $K \in C^0(\mathbb{R}^N)$ satisfies some technical assumptions, $0 < \mu < N$ and F is the primitive of f that fulfills a supercritical exponential growth in the Trudinger–Moser sense at infinity.

The problems like Eq. (1.1) are usually utilized to look for stationary solutions of time-dependent reaction-diffusion systems

$$\partial_t u = \mathrm{div}[D(u)\nabla u] + g(x, u),\ t > 0 \text{ and } x \in \mathbb{R}^N, \tag{1.2}$$

where $D(u) \triangleq \mathrm{div}(|\nabla u|^{p-2} + |\nabla u|^{q-2})$. There are wide range of applications on the systems in physics and its related sciences, such as biophysics, plasma physics and chemical reaction (see [18]). In fact, in the applications, the function u denotes a state variable and describes density or concentration of multi-component substances, $\mathrm{div}[D(u)\nabla u]$ represents the diffusion with a coefficient $D(u)$ and $g(x, u)$ is the reaction term related to source and loss mechanisms. Especially, $g(x, u)$ has a polynomial form associated with the unknown concentration symbolized by u.

An interesting phenomenon is that the operator involved in Eq. (1.1) is the so-called *double phase* operator behavior switches between two different elliptic situations. According to the pioneered work [67] considering such operators, the author introduced these classes to provide models of strongly anisotropic materials; see also the monograph of Zhikov, Kozlov and Olĕinik [68]. Meanwhile, Eq. (1.1) also known as *double phase* problem can be inspired by numerous models arising in mathematical physics. For example, it is closely related to the study of the well-known Born–Infeld equation

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-2|\nabla u|^2}}\right) = g(x, u),\ x \in \mathbb{R}^N,$$

that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density, see e.g. [14, 15]. Indeed, by means of the Taylor expansion formula, that is,

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^2}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}}x^{n-1} + \cdots \text{ for } |x| < 1.$$

One immediately derives Eq. (1.1) for $q = 2$ and $N = 4$ if taking $x = 2|\nabla u|^2$ and adopting the first order approximation. Furthermore, the following multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u$$

is driven by the n-th order approximation above.

For the convenience of the interested reader to acquaint more about the *double phase* problem, we prefer to suggest Mugnai and Papageorgiou [49], Liu and Dai [41], Ambrosio and Rădulescu [12], Papageorgiou, Rădulescu and Repovš [50], Zhang, Zhang and Rădulescu [65] and their references therein even if these references are far to be exhaustive. In fact, among the cited papers, the authors contemplated the existence of nontrivial solutions for the following elliptic equation

$$-\Delta_p u - \Delta_q u + V(x)|u|^{p-2}u + W(x)|u|^{q-2}u = g(x,u),\ x \in \mathbb{R}^N, \tag{1.3}$$

where $1 < q < p < N$ and $V,\ W : \mathbb{R}^N \to \mathbb{R}$ are external potentials. Very recently, Pomponio and Watanabe [53] considered Eq. (1.3) with $V \equiv 0$ and $W \equiv 0$ which can be called by the *zero-mass* case, where $N \geq 3$, $1 < q < p$ and $q < N$. Subsequently, Carvalho et al. [17] proposed a Trudinger–Moser inequality and established the existence of nontrivial solutions for a related work in the context of zero-mass (N, q)-Laplacian equation with $1 < q < N$ of the form

$$-\Delta_N u - \Delta_q u = g(u),\ x \in \mathbb{R}^N.$$

By introducing a singular version of Trudinger–Moser inequality corresponding to [17], the authors in [57] obtained the existence of ground state solutions for the following zero-mass (N, q)-Laplacian equation with $1 < q < N$ involving supercritical exponential growth

$$-\Delta_N u - \Delta_q u = \frac{1}{|x|^\beta}\left(\int_{\mathbb{R}^N} \frac{G(u)}{|y|^\beta |x-y|^\mu} dy\right) g(u),\ x \in \mathbb{R}^N,$$

where $\beta > 0$, $0 < \mu < N$ with $2\beta + \mu < N$ and G is the primitive of g.

Throughout this paper, inspired by [17, 53, 57], we shall define the work space below

$$E \triangleq \left\{u \in L^1_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^N dx < +\infty \text{ and} \int_{\mathbb{R}^N} |\nabla u|^q dx < +\infty\right\}$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\| = |\nabla u|_N + |\nabla u|_q,\ \forall u \in E,$$

where $|\cdot|_p$ stands for the usual norm associated with the Lebesgue space $L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$. By means of the Gagliardo–Nirenberg inequality and interpolation, one could deduce that the imbedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for all $p \in [q^*, +\infty)$, where $q^* = Nq/(N-q)$.

Indeed, without considering the q-Laplacian operator $-\Delta_q u$ in Eq. (1.1), it belongs to a class of Choquard equations

$$-\Delta u + u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u,\ x \in \mathbb{R}^N. \tag{1.4}$$

arising from the study of Bose–Einstein condensation. In the relevant physical case, by supposing $N = 3$, $\mu = 1$ and $p = 2$ in (1.4), Pekar [51] used it which is called by Choquard–Pekar equation to describe a polaron at rest in the quantum field theory. Choquard exploited it to characterization an electron trapped in its own hole as an approximation to the Hartree-Fock theory for a one component plasma [37]. Afterwards, Lieb [35] and Lions [39] obtained the existence and uniqueness of positive solutions to (1.4) by variational methods. The authors in [42, 45] verified the regularity, positivity and radial symmetry of the ground state solutions and established the decay property at infinity. We refer the reader to [1, 5, 7, 33, 44, 56, 59, 66] and particularly [46] for a very abundant and meaningful works of the Choquard equations.

Because of the appearance of convolution operator in (1.4), we would like to recall the Hardy–Littlewood–Sobolev (HLS in short) inequality as follows.

Proposition 1.1 (Hardy–Littlewood–Sobolev inequality [36, Theorem 4.3]). *Suppose that $m, r > 1$ and $0 < \mu < N$ with $1/m + \mu/N + 1/r = 2$, $\varphi \in L^m(\mathbb{R}^N)$ and $\psi \in L^r(\mathbb{R}^N)$. Then, there is a sharp constant $C = C(m, N, \mu, r) > 0$, independent of φ and ψ, such that*

$$\int_{\mathbb{R}^N} [|x|^{-\mu} * \varphi(x)]\psi(x)dx \leq C|\psi|_m|\psi|_r. \tag{1.5}$$

Applying the HLS inequality (1.5), one knows that $\int_{\mathbb{R}^N}[|x|^{-\mu}*(K(x)F(u))]K(x)F(u)dx$ is well-defined provided that $K(x)F(u) \in L^m(\mathbb{R}^N)$ for all $m > 1$ which is determined by $\frac{2}{m} + \frac{\mu}{N} = 2$. This means that we must make sure that

$$K(x)F(u) \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$$

Let us suppose particularly that $K \equiv 1$ and $F(u) = |u|^r$ for every $u \in E$, preserving the variational structure, by the continuous imbedding $E \hookrightarrow L^p(\mathbb{R}^N)$ with $p \in [q^*, +\infty)$, it must have that

$$r \geq \frac{(2N-\mu)q^*}{2N}.$$

However, in view of the nonlinearity f dealing with in the present paper is of supercritical growth, it would be far from enough since the spatial dimension of Eq. (1.1) is very special which prompts us to contemplate it carefully. Briefly speaking, we couldn't conclude that the imbedding $E \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous. To handle it in the limiting case, the celebrated Trudinger–Moser inequality [48, 52, 60] may be a good candidate as suitable substitute of the above imbedding inequality.

Inspired by the Trudinger–Moser type inequality, it says that a function $f(s)$ has *critical exponential growth* if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^{\frac{N}{N-1}}}} = \begin{cases} 0, & \forall\alpha > \alpha_0, \\ +\infty, & \forall\alpha < \alpha_0. \end{cases} \tag{1.6}$$

It should be pointed out that the definition above was extensively considered in the literatures, see e.g. [2–4, 16, 19, 26, 32, 34, 55, 64] for example.

In light of the work space E here, the classic version of Trudinger–Moser inequality are unapplicable. So, we shall introduce the particular case of [57, Theorem 1.2] (see e.g. [17, Theorem 1.1]) as follows.

Proposition 1.2 *Suppose that $1 < q < N$, then for every $\alpha > 0$ and $u \in E$, there holds*

$$\int_{\mathbb{R}^N} \left(e^{\alpha|u|^{\frac{N}{N-1}}} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!}|u|^{\frac{Nj}{N-1}} \right) dx < +\infty, \tag{1.7}$$

where $j_0 \triangleq \inf\{j \in \mathbb{N}^+ : j \geq q(N-1)/(N-q)\}$. Moreover, it holds that

$$\mathbb{S}(\alpha) \triangleq \sup_{u\in E, \|u\|\leq 1} \int_{\mathbb{R}^N} \left(e^{\alpha|u|^{\frac{N}{N-1}}} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!}|u|^{\frac{Nj}{N-1}} \right) dx < +\infty \tag{1.8}$$

for all $\alpha \leq \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\mathbb{S}(\alpha) = +\infty$ if $\alpha > \alpha_N$, where ω_{N-1} stands for the volume of the unit sphere S^{N-1}.

As one can observe that if u is a solution of zero-mass (N, q)-Laplacian equation (1.1) and $x^* \in \mathbb{R}^N$, then the function $v = u(\varepsilon x + x^*)$ solves

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^N} \frac{K(\varepsilon y + x^*)F(u(y))}{|x-y|^\mu} dy\right) K(\varepsilon x + x^*) f(u), \ x \in \mathbb{R}^N.$$

It infers that there is a convergence, as $\varepsilon \to 0^+$ of the family of solutions of Eq. (1.1), to the solution of its associated limiting equation

$$-\Delta_N u - \Delta_q u = [K(x^*)]^2 \left(\int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\mu} dy\right) f(u), \ x \in \mathbb{R}^N. \tag{1.9}$$

This phenomenon is regarded as the so-called semi-classical limiting for semilinear elliptic equations

$$-\varepsilon^2 \Delta u + V(x)u = g(u), \ x \in \mathbb{R}^N. \tag{1.10}$$

The reader could refer to [11, 13] for a detailed survey on such topic which should date back to the pioneering research work by Foler and Weinstein in [28]. Soon afterwards, Eq. (1.10) and its variants have been investigated extensively under different hypotheses on the potential and the nonlinearity, see e.g. [10, 21–24, 28, 29, 31, 54, 61] and the references therein.

Let us mention here that the Lyapunov-Schmidt reduction argument has been proved to be one of the most effective tools in the study of semiclassical problems for local Schrödinger equations like Eq. (1.10). Whereas, to our best knowledge, little is known about the uniqueness and non-degeneracy of the ground states of the limiting problem

$$-\Delta u + u = [|x|^{-\mu} * G(u)]g(u), \ x \in \mathbb{R}^N.$$

As a consequence, it is quite natural to ask whether the existence and concentration results for local Schrödinger equations still hold for the nonlocal equation with supercritical growth in the sense of Trudinger–Moser inequality. We anticipate that our problem exhibits three conspicuously interesting features:

(1) There are two distinct operators which generate a double phase associated energy;
(2) Due to the unboundedness of the whole space $\mathbb{R}^N$ and the supercritical exponential growth, there is a lack of the compactness property of the corresponding variational functional;
(3) The presences of convolution type term and the nonlinearity involving supercritical exponential growth together with the potential, the proofs combine dedicated analysis techniques including regular theory and topological tools.

Now, let us begin defining the supercritical exponential growth on f. Suppose that the nonlinearity f carries the form of type

$$f(t) = h(t)e^{\alpha|t|^\tau}, \ \forall t \in \mathbb{R} \tag{1.11}$$

for $\alpha > 0$ and $\tau \geq \frac{N}{N-1}$. Hereafter, we assume that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(h_1) $h \in C^0(\mathbb{R})$ with $h(t) \equiv 0$ for all $t \leq 0$ and $h(t) = o(t^{\sigma-1})$ as $t \to 0^+$, where $\sigma > \max\{N, q^*\}$;
(h_2) There exists a $\theta > \frac{N}{2}$ such that $0 < \theta H(t) \leq h(t)t$ for all $t > 0$, where $H(t) = \int_0^t h(s)\, ds$;
(h_3) There exist some $\delta \in (0, \frac{N}{N-1})$ and $\gamma, M > 0$ such that $0 \leq h(t) \leq Me^{\gamma t^\delta}$ for all $t \geq 0$.

Recalling the previous works in [8, 9], there are two ways to understand that the function h, defined in (1.11) together with (h_2), satisfies the so-called ***supercritical exponential growth*** in the following sense:

$$\text{(I) } \tau > N \text{ is arbitrary and } \alpha > 0 \text{ is fixed; (II) } \alpha > 0 \text{ is arbitrary and } \tau \geq N \text{ is fixed.} \tag{1.12}$$

Moreover, one could call Cases (I) and (II) in (1.12) to be the ***subcritical-supercritical exponential growth*** and ***critical-supercritical exponential growth***, respectively.

We study the existence of nontrivial solutions for the supercritical nonlocal equation with periodic potential, namely we consider the equation

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu} dy\right) Q(x) f(u),\ x \in \mathbb{R}^N. \tag{1.13}$$

Assume that the potential Q satisfies the conditions

- (Q_1) $Q \in C^0(\mathbb{R}^N)$ and $Q(x) \geq Q_0$ for all $x \in \mathbb{R}^N$ for some $Q_0 > 0$;
- (Q_2) Q is $\mathbb{Z}^N$-periodic, that is, $Q(x) = Q(x+y)$ for all $x \in \mathbb{R}^N$ and $y \in \mathbb{Z}^N$.

The first main result can be stated as follows.

Theorem 1.3 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1) - (h_3)$ and the potential Q requires $(Q_1) - (Q_2)$, then for each $\tau \geq N/(N-1)$, there is a $\alpha^* = \alpha^*(\tau) > 0$ such that Eq. (1.13) has a nontrivial solution in E for all $\alpha \in (0, \alpha^*)$. Moreover, if in addition we suppose that*

- (h_4) *there are constants $\xi > 0$ and $p > \frac{N}{2}$ such that $H(t) \geq \xi t^p$ for all $t \in [0, 1]$,*

then for each $\alpha > 0$, there exists a $\tau_* = \tau_*(\alpha) > N/(N-1)$ and $\xi_0 > 0$ such that Eq. (1.1) possesses a nontrivial solution in E for every $\tau \in [N/(N-1), \tau_*)$ and $\xi > \xi_0$.

To treat Eq. (1.13) variationally, we must assure that the variational functional $J : E \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(x)F(u(x))Q(y)F(u(y))}{|x-y|^\mu} dxdy$$

is well-defined in E and of class C^1. Unfortunately, it seems impossible because f has the supercritical exponential growth at infinity. So, we cannot look for critical points of J directly to consider Theorem 1.3 via Propositions 1.1 and 1.2. Motivated by [8, 9], given a fixed constant $R > 0$, we shall study an auxiliary equation which possesses a (sub)critical exponential growth. Speaking it clearly, introducing a cutoff function $f^{R,\bar{\delta}}$ which is

$$f^{R,\bar{\delta}}(t) = \begin{cases} 0, & t \leq 0, \\ h(t)e^{\alpha t^\tau}, & 0 \leq t \leq R, \\ h(t)e^{\alpha R^{\tau-\bar{\delta}} t^{\bar{\delta}}}, & t \geq R, \end{cases} \tag{1.14}$$

where

$$\bar{\delta} \triangleq \begin{cases} \delta, & \text{if the Case I in (1.12) is considered,} \\ N/(N-1), & \text{if the Case II in (1.12) is considered,} \end{cases}$$

then we would consider the following auxiliary equation

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^2} \frac{Q(y)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu} dy\right) Q(x) f^{R,\bar{\delta}}(u),\ x \in \mathbb{R}^N, \tag{1.15}$$

where and in the sequel $F^{R,\bar{\delta}}(t)=\int_0^t f^{R,\bar{\delta}}(s)ds$. Using (h_3), it is ready to observe that $f^{R,\bar{\delta}}$ admits a subcritical or critical exponential growth at infinity for all fixed $R>0$. So, the variational functional $J^{R,\bar{\delta}}:E\to\mathbb{R}$ below

$$J^{R,\bar{\delta}}(u)=\frac{1}{N}\int_{\mathbb{R}^N}|\nabla u|^N dx+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q dx\\ -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{Q(x)F^{R,\bar{\delta}}(u(x))Q(y)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu}dxdy$$

associated with Eq. (1.15) is well-defined and belongs to $C^1(E)$. Moreover, we should deduce that each critical point of it is a (weak) solution of Eq. (1.15). It could obviously conclude that if every mountain-pass type solution $u_R\in E$ of Eq. (1.15) satisfying $|u_R|_\infty\le R$, then u_R is a mountain-pass type solution of Eq. (1.13). Have it in mind, the reader is invited to acquaint that we shall establish such a solution u_R to derive the proof of Theorem 1.3. Thereby, it is necessary to prove the following result.

Theorem 1.4 *Let $1<q<N$ and $0<\mu<N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1)-(h_3)$ and the potential Q requires $(Q_1)-(Q_2)$, then for each fixed $R>0$, Eq. (1.15) with $\bar{\delta}=\delta$ admits a nontrivial solution in E. Moreover, if we suppose additionally that (h_4), then for each fixed $R>0$, there is a $\xi_0=\xi_0(R)>0$ dependent of R such that Eq. (1.15) with $\bar{\delta}=N/(N-1)$ possesses a nontrivial solution in E for all $\xi>\xi_0$.*

Let $Q(x)=|x|^{-\beta}$ with $0<\beta<N$ and $2\beta+\mu<N$ for every $x\in\mathbb{R}^N$ in Eq. (1.13), the authors in [57] investigated the existence of ground state solutions under the assumptions $(h_1)-(h_4)$ and

(h_5) The function $h\in C^1$ satisfies $t\mapsto h(t)/t^{\frac{N-2}{2}}$ is increasing on $t\in(0,+\infty)$.

Alternatively, we can never repeat the calculations in the cited paper to finish the proofs of Theorems 1.3 and 1.4. On the one hand, thanks to the fact that the imbedding $E\hookrightarrow L^p(\mathbb{R}^N;|x|^{-s}dx)\triangleq\{u\in L^p(\mathbb{R}^N):\int_{\mathbb{R}^N}|x|^{-s}|u|^p dx<+\infty\}$ for all $s\in(0,N)$ and $p\in[q^*,+\infty)$ is compact in [57, Lemma 2.1], one could easily recover the compactness of $J^{R,\bar{\delta}}$. On the other hand, if the condition (h_5) is absence, it could just show that the energy of nontrivial critical point of $J^{R,\bar{\delta}}$ is smaller than (or equal to) the mountain-pass level whence $(Q_1)-(Q_2)$ are satisfied. Considering these facts, we will establish the concentration-compactness principle with respect to the Trudinger–Moser inequality in Proposition 1.2 and it should be consistent with the space E.

Theorem 1.5 *Suppose that $1<q<N$ and let $\{u_n\}\subset E$ be a sequence satisfying $\|u_n\|\equiv 1$ and $u_n\rightharpoonup u\neq 0$ in E, then*

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^N}\left(e^{\bar{p}\alpha_N|u_n|^{\frac{N}{N-1}}}-\sum_{j=0}^{j_0-1}\frac{(\bar{p}\alpha_N)^j}{j!}|u_n|^{\frac{Nj}{N-1}}\right)dx<+\infty,\ \forall 0<\bar{p}<\bar{P}(u), \tag{1.16}$$

where the sharp constant $\bar{P}(u)$ is defined by

$$\bar{P}(u)=\begin{cases}\left(\frac{1}{1-\|u\|^N}\right)^{\frac{1}{N-1}}, & \text{if } \|u\|<1,\\ +\infty, & \text{if } \|u\|=1.\end{cases}$$

Remark 1.6 We emphasize that Theorem 1.5 should be viewed as the counterpart of [20, 27, 40, 58], but it is still new in our settings. Moreover, in contrast to [17, Theorem 1.2], even if we only consider Theorem 1.4 instead of Theorem 1.3, as far as we are concerned, there exist three main contributions:

- As one shall see later, we do not depend on the compact imbedding $E_r \hookrightarrow L^p(\mathbb{R}^N)$ for every $q^* < p < +\infty$, where $E_r = \{u \in E : u(x) = u(|x|)\}$. Conversely, this fact plays an extremely crucial role in [17]. Indeed, in order to circumvent the difficulty, we establish a new type of Lion's Vanishing lemma (see Lemma 2.5 below) corresponding to the work space E and then get the nontrivial solution by the periodicity of Q. Besides, we are trying our best to remove the periodic assumption on Q and replace it with a more general restriction, but it would be postponed in a further work;
- We conclude a unified approach to investigate the existence of nontrivial solutions for a class of zero-mass (N, q)-Laplacian equations, like (1.13), with subcritical and critical exponential growth with the help of mountain-pass theorem. Moreover, one could derive that the energy of such obtained nontrivial solution equals to the mountain-pass level due to Theorem 1.5 if the work space E is radially symmetric and (h_4) can be weakened to some extent;
- Because of the appearance of the convolution operator in Eq. (1.13), the considered problem can be seen as a nonlocal type of zero-mass (N, q)-Laplacian equation, which is an extension of the local problem studied in [17].

As a supplement of [57], whose detailed proof shall be left to the reader, we could follow Theorem 1.5 jointly with [57, Theorem 1.1] to derive the theorem below.

Corollary 1.7 *Suppose that* $1 < q < N$ *and* $0 < s < N$. *Let* $\{u_n\} \subset E$ *be a sequence satisfying* $\|u_n\| \equiv 1$ *and* $u_n \rightharpoonup u \neq 0$ *in* E, *then for the sharp constant* $\bar{P}(u)$ *appearing in Theorem 1.5, it holds that*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} |x|^{-s} \left(e^{\bar{p}\alpha_N |u_n|^{\frac{N}{N-1}}} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} |u_n|^{\frac{Nj}{N-1}} \right) dx < +\infty, \ \forall 0 < \bar{p} < \bar{P}(u).$$

Next, we focus on the existence and concentration results for Eq. (1.1). Let us suppose that the potential K satisfies the conditions

(K_1) $K \in C^0(\mathbb{R}^N)$ and $\max_{x\in\mathbb{R}^N} K(x) \triangleq K_0 \in (0, +\infty)$ achieves its maximum at some point $x \in \mathbb{R}^N$;

(K_2) $\limsup_{|x|\to\infty} K(x) \triangleq K_\infty \in (0, K_0)$ with $K(x) \geq K_\infty$, and the inequality is strict in a subset of positive Lebesgue measure.

It should be pointed out that the condition (K_2) comes essentially form [54]. Starting from here, we shall denote by

$$\Sigma = \left\{x \in \mathbb{R}^N : K(x) = K_0\right\}$$

the maximum points set of K in $\mathbb{R}^N$. Without loss of generality, we always suppose that the original point 0 belongs to Σ, namely $K(0) = K_0$. Thus, we can establish the following result.

Theorem 1.8 *Let* $1 < q < N$ *and* $0 < \mu < N$. *Suppose that the nonlinearity* f *defined in* (1.12) *satisfies* $(h_1) - (h_3)$ *with* (h_5) *and the potential* K *requires* $(K_1) - (K_2)$, *then for each* $\tau \geq N/(N-1)$, *there is a* $\bar{\alpha}^* = \bar{\alpha}^*(\tau) > 0$ *such that Eq.* (1.1) *admits a ground state solution* $u_\varepsilon \in E$ *for all* $\alpha \in (0, \bar{\alpha}^*)$ *and* $\varepsilon > 0$ *small enough. Furthermore, we obtain the following conclusions:*

(a) u_ε *possesses a maximum point* $\gamma_\varepsilon \in \mathbb{R}^N$ *such that, going to a subsequence if necessary,*

$$\lim_{\varepsilon \to 0^+} K(\gamma_\varepsilon) = K_0,$$

and $\gamma_\varepsilon \to x^* \in \Sigma$ *as* $\varepsilon \to 0^+$*;*

(b) If we set $\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + \gamma_\varepsilon)$*, going to a subsequence if necessary, we have* $\widetilde{u}_\varepsilon \to \widetilde{u}$ *in* E *as* $\varepsilon \to 0^+$ *and* $\widetilde{u}$ *is a ground state solution of Eq.* (1.9).

Moreover, if in addition we suppose (h_4)*, then for each* $\alpha > 0$*, there exists a* $\bar{\tau}_* = \bar{\tau}_*(\alpha) > N/(N-1)$ *and* $\bar{\xi}_0 > 0$ *such that Eq.* (1.1) *possesses a ground state solution* $u_\varepsilon \in E$ *for every* $\bar{\tau} \in [N/(N-1), \bar{\tau}_*)$*,* $\xi > \bar{\xi}_0$ *and* $\varepsilon > 0$ *small enough as well as the properties* $(a) - (b)$ *remain true in this situation.*

Recalling the discussions as before, to contemplate Theorem 1.8, we have to first take into account its auxiliary equation

$$-\varepsilon^N \Delta_N u - \varepsilon^q \Delta_q u = \varepsilon^{\mu - N}\left(\int_{\mathbb{R}^N} \frac{K(y)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu}dy\right)K(x)f^{R,\bar{\delta}}(u),\ x \in \mathbb{R}^N, \tag{1.17}$$

where $f^{R,\bar{\delta}}$ is defined in (1.14).

So, we are going to conclude the results below.

Theorem 1.9 *Let* $1 < q < N$ *and* $0 < \mu < N$*. Suppose that the nonlinearity* f *defined in* (1.12) *satisfies* $(h_1) - (h_3)$ *with* (h_5) *and the potential* K *requires* $(K_1) - (K_2)$*, then for each* $R > 0$*, Eq.* (1.17) *with* $\bar{\delta} = \delta$ *has a ground state solution* $u_\varepsilon \in E$ *for all* $\varepsilon > 0$ *small enough. Furthermore, we obtain the following conclusions:*

(a) u_ε *possesses a maximum point* $\gamma_\varepsilon \in \mathbb{R}^N$ *such that, going to a subsequence if necessary,*

$$\lim_{\varepsilon \to 0^+} K(\gamma_\varepsilon) = K_0,$$

and $\gamma_\varepsilon \to x^* \in \Sigma$ *for all fixed* $R > 0$ *as* $\varepsilon \to 0^+$*;*

(b) If we set $\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + \gamma_\varepsilon)$*, going to a subsequence if necessary, we have* $\widetilde{u}_\varepsilon \to \widetilde{u}$ *for all fixed* $R > 0$ *in* E *as* $\varepsilon \to 0^+$ *and* $\widetilde{u}$ *is a ground state solution of*

$$-\Delta_N u - \Delta_q u = [K(x^*)]^2\left(\int_{\mathbb{R}^N} \frac{F^{R,\delta}(u(y))}{|x-y|^\mu}dy\right)f^{R,\delta}(u),\ x \in \mathbb{R}^N. \tag{1.18}$$

Moreover, if we suppose additionally (h_4)*, then for each* $R > 0$*, there exists a* $\bar{\xi}_0(R) > 0$ *such that Eq.* (1.17) *with* $\bar{\delta} = N/(N-1)$ *admits a ground state solution* $u_\varepsilon^R \in E$ *for all* $\xi > \bar{\xi}_0(R)$ *and* $\varepsilon > 0$ *small enough. In addition, we still have the properties* $(a) - (b)$ *above by replacing* δ *with* $N/(N-1)$*.*

Performing the scaling $u(x) = v(\varepsilon x)$, one could observe that, to study Eq. (1.17), it is equivalent to consider the problem

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^N} \frac{K(\varepsilon y)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu}dy\right)K(\varepsilon x)f^{R,\bar{\delta}}(u),\ x \in \mathbb{R}^N, \tag{1.19}$$

whose variational functional $\mathcal{J}_\varepsilon^{R,\bar{\delta}} : E \to \mathbb{R}$ is defined by

$$\mathcal{J}_\varepsilon^{R,\bar{\delta}}(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx$$

$$-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\varepsilon x)F^{R,\bar{\delta}}(u(x))K(\varepsilon y)F^{R,\bar{\delta}}(u(y))}{|x-y|^{\mu}}dxdy.$$

A solution $u \in E$ of Eq. (1.19) is called by the ground state if it satisfies $\mathcal{J}_\varepsilon^{R,\bar{\delta}}(u) = m_\varepsilon^{R,\bar{\delta}}$, where

$$m_\varepsilon^{R,\bar{\delta}} \triangleq \inf_{v\in\mathcal{N}_\varepsilon^{R,\bar{\delta}}} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(v) \tag{1.20}$$

with $\mathcal{N}_\varepsilon^{R,\bar{\delta}}$ denoting the Nehari manifold

$$\mathcal{N}_\varepsilon^{R,\bar{\delta}} = \big\{u \in E\backslash\{0\} : (\mathcal{J}_\varepsilon^{R,\bar{\delta}})'(u)[u] = 0\big\}.$$

We note that, up to our best knowledge, it is the first time to deal with the semiclassical ground state solutions under the zero-mass (N, q)-Laplacian setting in the supercritical exponential case, even in the critical exponential case. Although it is standard to contemplate the singularly perturbed problems by the arguments introduced in [11, 13, 28], we should emphasize here that there exist two essential difficulties arising in Theorems 1.8 and 1.9. On the one hand, because of the appearance of critical-supercritical exponential case in Theorem 1.8, or critical exponential case in Theorem 1.9, we are confronted with the lack of compactness of $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$. To overcome it, we would follow [65, Lemma 4.4] to establish a compact lemma, see e.g. Lemma 4.6 below. On the other hand, since the operator $-\Delta_N u - \Delta_q u$ is nonhomogeneous and there is a competition interaction between it and the nonlocal nonlinearity having (super)critical exponential growth, we cannot apply directly the methods used in [30] to explore the regular result in our problem. Hence, we prefer to regard it as the most striking highlight in the present paper. In fact, we proceed as [57, Lemma 4.4] to deduce that every nontrivial solution of Eq. (1.19) belongs to $L^\infty(\mathbb{R}^N)$ and then derive its smoothness. Alternatively, the reader would discover some additional and unpleasant barriers in the proofs of Theorems 1.8 and 1.9.

The outline of the paper is organized as follows. In Sect. 2, we mainly present some preliminary results and obtain the proof of Theorem 1.5. Sections 3 and 4 are devoted to the proofs of Theorems 1.3–1.4 and 1.8–1.9, respectively.

Notations: From now on in the present paper, otherwise mentioned particularly, we shall adopt the following notations:

- $C, C_1, C_2, \ldots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(Z, \|\cdot\|_Z)$ be a Banach space with dual space $(Z^{-1}, \|\cdot\|_{Z^{-1}})$, and Ψ be functional on Z.
- The (C) sequence at a level $c \in \mathbb{R}$ ($(C)_c$ sequence in short) corresponding to Ψ means that $\Psi(x_n) \to c$ and $(1+\|\Psi'(x_n)\|_{X^{-1}})\|x_n\|_X \to 0$ in X^{-1} as $n \to \infty$, where $\{x_n\} \subset Z$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^N$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^N : |y-x| < \varrho\}$.
- For each Lebesgue measurable set $\Omega \subset \mathbb{R}^N$, $|\Omega|$ stands for the Lebesgue measure of Ω.
- Given a mensurable function u, we shall denote by u^+ and u^- its positive and negative parts respectively, given by

$$u^+(x) = \max\{u(x), 0\} \text{ and } u^-(x) = \min\{u(x), 0\}.$$

- $o_n(1)$ denotes the real sequences with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and "$\rightharpoonup$" stand for the strong and weak convergence in the related function spaces, respectively.

2 Variational framework and preliminaries

In this section, we shall formulate the variational structure and present some preliminary results for our problems. To begin it, let us show the following imbedding result.

Lemma 2.1 *Let $1 < q < N$, then the imbedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for all $q^* \leq p < +\infty$.*

Proof The proof is standard and we refer the reader to [18, 57]. □

For every fixed $R > 0$ and the cutoff function in (1.14), define the functional $\Psi^{R,\bar{\delta}} : E \to \mathbb{R}$ by

$$\Psi^{R,\bar{\delta}}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F^{R,\bar{\delta}}(u(x))F^{R,\bar{\delta}}(u(y))}{|x-y|^{\mu}}dxdy. \tag{2.1}$$

With the help of Lemma 2.1, we can establish the lemmas below.

Lemma 2.2 *Let $1 < q < N$ and suppose that the nonlinearity f in (1.11) satisfies (h_1) and (h_3), then the functional $\Psi^{R,\bar{\delta}}$ is well-defined in E and of class C^1 whose derivative is given by*

$$(\Psi^{R,\bar{\delta}}(u))'[v] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f^{R,\bar{\delta}}(u(x))v(x)F^{R,\bar{\delta}}(u(y))}{|x-y|^{\mu}}dxdy, \ \forall u, v \in E.$$

Moreover, if $u_n \rightharpoonup u$ in E as $n \to \infty$, up to a subsequence if necessary, it holds that

$$(\Psi^{R,\bar{\delta}})'(u_n)[\psi] \to (\Psi^{R,\bar{\delta}})'(u)[\psi] \text{ as } n \to \infty, \ \forall \psi \in C_0^{\infty}(\mathbb{R}^N). \tag{2.2}$$

If $\bar{\delta} = \frac{N}{N-1}$ in (2.2), we should suppose in addition that $\sup\limits_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{2(\gamma+\alpha R^{\tau-\frac{N}{N-1}})}\min\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\}$, where $\alpha_N, \gamma, \alpha, \sigma, N$ and μ are positive constants independent of $R > 0$.

Proof By (1.14), one sees that $f^{R,\bar{\delta}}$ admits the subcritical and critical exponential growth at infinity for $\bar{\delta} = \delta$ and $\bar{\delta} = \frac{N}{N-1}$, respectively. Thanks to (1.5) and (1.8), one could easily conclude the first part of this lemma and the details are left. If $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$ in (2.2), as explained before, it is the subcritical exponential case, thus the conclusion is immediate. Let us just consider the case $\bar{\delta} = \frac{N}{N-1}$ in (2.2). Applying (h_1) and (h_3) to (1.14), we could argue as in the proof of [57, Lemma 2.2] to find a constant $C(R) > 0$ (that depends on $R > 0$) such that

$$|f^{R,\frac{N}{N-1}}(t)| \leq |t|^{\sigma-1} + C(R)|t|^{\nu-1}\Phi_{\gamma+\alpha R^{\tau-\frac{N}{N-1}},j_0}(t), \ \forall t \in \mathbb{R}, \tag{2.3}$$

where $\nu > 1$ is arbitrary. We denote Ω by the support of ψ, then $|\Omega| < \infty$ and so there is a sufficiently large $\varrho > 0$ such that $\Omega \subset B_{\varrho/2}(0)$. Setting $\xi_n \triangleq |x|^{-\mu} * F^{R,\frac{N}{N-1}}(u_n)$ and $\xi_0 \triangleq |x|^{-\mu} * F^{R,\frac{N}{N-1}}(u)$, to get (2.2), we firstly claim that

Claim. ξ_n is uniformly bounded in $n \in \mathbb{N}$ and $\xi_n \to \xi_0$ a.e. in $\mathbb{R}^N$.

Verification. After some simple calculations, there holds

$$\int_{\mathbb{R}^N}\frac{F^{R,\frac{N}{N-1}}(u_n(y))}{|x-y|^{\mu}}dy \leq \int_{B_{\varrho}(0)}\frac{F(u_n(y))}{|x-y|^{\mu}}dy + \left(\frac{2}{\varrho}\right)^{\mu}\int_{B^c_{\varrho}(0)}F(u_n(y))dy$$

$$\leq \int_{B_\varrho(0)} F^{R,\frac{N}{N-1}}(u_n(y))dy + \int_{B_\varrho(0)\cap\{|x-y|<1\}} \frac{F^{R,\frac{N}{N-1}}(u_n(y))}{|x-y|^\mu}dy$$
$$+ \left(\frac{2}{\varrho}\right)^\mu \int_{B^c_\varrho(0)} F(u_n(y))dy. \tag{2.4}$$

Letting $\nu = \frac{\sigma+1}{2}$ in (2.3), by (1.8), we have

$$\int_{\mathbb{R}^N} |u_n|^{\frac{\sigma+1}{2}} \Phi_{\gamma+\alpha R^{\tau-\frac{N}{N-1}},j_0}(u_n)dx \leq |u_n|_\sigma^{\frac{\sigma+1}{2}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2\sigma}{\sigma-1}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}),j_0}(u_n)dx\right)^{\frac{\sigma-1}{2\sigma}}$$
$$= |u_n|_\sigma^{\frac{\sigma+1}{2}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2\sigma}{\sigma-1}\left(\gamma+\alpha R^{\tau-\frac{N}{N-1}}\right)\|u_n\|^{\frac{N}{N-1}},j_0}(u_n/\|u_n\|)dx\right)^{\frac{\sigma-1}{2\sigma}} \leq C|u_n|_\sigma^{\frac{\sigma+1}{2}}. \tag{2.5}$$

Here and in the sequel $\Phi_{\alpha,j_0}(t) \triangleq e^{\alpha|u|^{\frac{N}{N-1}}} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!}|u|^{\frac{Nj}{N-1}}$ for all $t \in \mathbb{R}$ as well as the inequality (see e.g. [63, Lemma 2.1]):

$$(\Phi_{\alpha,j_0}(t))^m \leq \Phi_{m\alpha,j_0}(t),\ \forall t \in \mathbb{R},\ \alpha > 0 \text{ and } m > 1.$$

It follows from the Hölder's inequality that

$$\int_{B_\varrho(0)\cap\{|x-y|<1\}} \frac{F^{R,\frac{N}{N-1}}(u_n(y))}{|x-y|^\mu}dy \leq C\left(\int_{B_\varrho(0)} [F^{R,\frac{N}{N-1}}(u_n)]^{\frac{N+\mu}{N-\mu}}dy\right)^{\frac{N-\mu}{N+\mu}}.$$

From this inequality, we choose $\nu = \frac{\sigma}{2}$ in (2.3) and exploit (1.8) to get

$$\int_{\mathbb{R}^N} |u_n|^{\frac{\sigma(N+\mu)}{2(N-\mu)}} \Phi_{\frac{N+\mu}{N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}),j_0}(u_n)dx$$
$$\leq |u_n|^{\frac{\sigma(N+\mu)}{2(N-\mu)}}_{\frac{\sigma(N+\mu)}{N-\mu}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2(N+\mu)}{N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}),j_0}(u_n)dx\right)^{\frac{1}{2}}$$
$$= |u_n|^{\frac{\sigma(N+\mu)}{2(N-\mu)}}_{\frac{\sigma(N+\mu)}{N-\mu}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2(N+\mu)}{N-\mu}\left(\gamma+\alpha R^{\tau-\frac{N}{N-1}}\right)\|u_n\|^{\frac{N}{N-1}},j_0}(u_n/\|u_n\|)dx\right)^{\frac{1}{2}} \leq C|u_n|^{\frac{\sigma(N+\mu)}{2(N-\mu)}}_{\frac{\sigma(N+\mu)}{N-\mu}}. \tag{2.6}$$

Combining (2.3), (2.4), (2.5), (2.6) and Lemma 2.1, we would easily see that ξ_n is uniformly bounded in $n \in \mathbb{N}$. Since $u_n \to u$ in $L^p(B_\varrho(0))$ for every $p > q^*$, with (2.5) and (2.6) in hands, we tend $n \to \infty$ by the generalized Vatali's Convergence Dominated theorem and then $\varrho \to +\infty$ in (2.4) to reach the Claim. With the help of the Claim above,

$$\left|(\Psi^{R,\frac{N}{N-1}}(u_n))'[\psi]\right| \leq \int_{\mathbb{R}^N} \left|\xi_n f^{R,\frac{N}{N-1}}(u_n)\psi\right|dx = \int_\Omega \left|\xi_n f^{R,\frac{N}{N-1}}(u_n)\psi\right|dx$$
$$\leq C\int_\Omega \left|f^{R,\frac{N}{N-1}}(u_n)\right||\psi|dx \leq C\left(\int_\Omega \left|f^{R,\frac{N}{N-1}}(u_n)\right|^2dx\right)^{\frac{1}{2}}\left(\int_\Omega |\psi|^2dx\right)^{\frac{1}{2}}$$

which together with (2.3) and (2.6) indicates that $\{\xi_n f^{R,\frac{N}{N-1}}(u_n)\psi\}$ is uniformly integrable. Adopting the Claim again, we can derive the proof of (2.2). The proof of this lemma is completed. □

In the following, we establish a nonlocal version of the Brézis–Lieb type lemma in E for nonlinearities of (sub)critical exponential growth.

Lemma 2.3 *Let $1 < q < N$ and suppose that the nonlinearity f in (1.11) satisfies (h_1) and (h_3). If $u_n \rightharpoonup u$ in E as $n \to \infty$, up to a subsequence if necessary, there holds*

$$\Psi^{R,\delta}(u_n) - \Psi^{R,\delta}(u_n - u) - \Psi^{R,\delta}(u) \to 0 \text{ as } n \to \infty. \tag{2.7}$$

Moreover, if in addition we assume that $\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{18(\gamma+\alpha R^{\tau-\frac{N}{N-1}})} \min\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\}$, there holds

$$\Psi^{R,\frac{N}{N-1}}(u_n) - \Psi^{R,\frac{N}{N-1}}(u_n - u) - \Psi^{R,\frac{N}{N-1}}(u) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Proof Since the proof of (2.7) is simple, we show (2.8) in detail. Obviously,

$$\begin{aligned}
2\big[\Psi^{R,\frac{N}{N-1}}(u_n) - \Psi^{R,\frac{N}{N-1}}(u_n - u)\big] &\triangleq \Xi_1 + \Xi_2 + \Xi_3 \\
&= \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u)\big] F^{R,\frac{N}{N-1}}(u_n - u)dx \\
&\quad + \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u)\big] F^{R,\frac{N}{N-1}}(u_n)dx \\
&\quad + \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u_n)\big]\big[F^{R,\frac{N}{N-1}}(u_n) - F^{R,\frac{N}{N-1}}(u_n - u) - F^{R,\frac{N}{N-1}}(u)\big]dx \\
&\quad + \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u_n - u)\big]\big[F^{R,\frac{N}{N-1}}(u_n) - F^{R,\frac{N}{N-1}}(u_n - u) - F^{R,\frac{N}{N-1}}(u)\big]dx.
\end{aligned}$$

We claim that $\{F^{R,\frac{N}{N-1}}(u_n)\}$ is uniformly bounded in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. In fact, exploiting $\nu = \frac{2N-\mu}{2N}\sigma > 1$ in (2.3) and arguing as (2.6), it suffices to show that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^\sigma \Phi_{\frac{2N}{2N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}),j_0}(u_n)dx \\
&\leq |u_n|^\sigma_{\frac{(N+\mu)\sigma}{\mu}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2(N+\mu)}{2N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}),j_0}(u_n)dx\right)^{\frac{N}{N+\mu}} \\
&\leq |u_n|^\sigma_{\frac{(N+\mu)\sigma}{\mu}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2(N+\mu)}{N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}})\|u_n\|^{\frac{N}{N-1}},j_0}(u_n/\|u_n\|)dx\right)^{\frac{N}{N+\mu}} \\
&\leq C|u_n|^\sigma_{\frac{(N+\mu)\sigma}{\mu}} \leq C.
\end{aligned}$$

In view of (1.5), one has $|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u) \in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$ jointly with $F^{R,\frac{N}{N-1}} \in C^0$ implies that

$$\Xi_1 \to \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\frac{N}{N-1}}(u)\big] F^{R,\frac{N}{N-1}}(u)dx = 2\Psi^{R,\frac{N}{N-1}}(u) \text{ as } n \to \infty.$$

From it and the claim, to get (2.8), it is enough to verify that

$$F^{R,\frac{N}{N-1}}(u_n) - F^{R,\frac{N}{N-1}}(u_n - u) - F^{R,\frac{N}{N-1}}(u) \to 0 \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N) \text{ as } n \to \infty, \tag{2.9}$$

whose proof has been left in the Appendix. The proof is completed. □

As a byproduct of Lemma 2.3, we immediately have the following results whose proof are omitted.

Lemma 2.4 *Let $1 < q < N$ and suppose that the nonlinearity f in (1.11) satisfies (h_1) and (h_3). If $u_n \to u$ in $L^p(\mathbb{R}^N)$ for some $p \in (q^*, +\infty)$ as $n \to \infty$, up to a subsequence if necessary, there holds*

$$\Psi^{R,\delta}(u_n) \to \Psi^{R,\delta}(u) \text{ as } n \to \infty. \tag{2.10}$$

Moreover, if we assume additionally that $\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \le \frac{\alpha_N}{18(\gamma+\alpha R^{\tau-\frac{N}{N-1}})} \min\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\}$, *then*

$$\Psi^{R,\frac{N}{N-1}}(u_n) \to \Psi^{R,\frac{N}{N-1}}(u) \text{ as } n \to \infty. \tag{2.11}$$

The following lemma which is crucial in the proofs of Theorems 1.3 and 1.4 is an another type of Vanishing lemma in [40].

Lemma 2.5 *Let $1 < q < N$ and $r > 0$. If $\{u_n\}$ is bounded in E and suppose that*

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^{q^*} dx = 0,$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $q^* < p < +\infty$.

Proof For all $p \in (q^*, +\infty)$, there exists a $s \in (q, N)$ which is very close to N so that $p < s^*$, where $s^* = \frac{Ns}{N-s}$. Thus, we have $q < s < N$ and $q^* < p < s^*$. Applying the Hölder's inequality to get

$$\begin{cases} \left(\int_{B_r(y)} |\nabla u_n|^s dx\right)^{\frac{1}{s}} \le \left(\int_{B_r(y)} |\nabla u_n|^q dx\right)^{\frac{1-\omega}{q}} \left(\int_{B_r(y)} |\nabla u_n|^N dx\right)^{\frac{\omega}{N}}, \\ \left(\int_{B_r(y)} |u_n|^p dx\right)^{\frac{1}{p}} \le \left(\int_{B_r(y)} |u_n|^{q^*} dx\right)^{\frac{1-\varpi}{q^*}} \left(\int_{B_r(y)} |u_n|^{s^*} dx\right)^{\frac{\varpi}{s^*}}, \end{cases} \tag{2.12}$$

where $\omega = \frac{s-q}{N-q} \cdot \frac{N}{s}$ and $\varpi = \frac{p-q^*}{s^*-q^*} \cdot \frac{s^*}{p}$. Combining (2.12) and the Sobolev's imbedding inequality, we have that

$$\begin{aligned} \int_{B_r(y)} |u_n|^p dx &\le C\left(\int_{B_r(y)} |u_n|^{q^*} dx\right)^{\frac{p(1-\varpi)}{q^*}} \left(\int_{B_r(y)} |\nabla u_n|^q dx\right)^{\frac{p(1-\omega)\varpi}{q}} \left(\int_{B_r(y)} |\nabla u_n|^N dx\right)^{\frac{p\omega\varpi}{N}} \\ &\le C\left(\int_{B_r(y)} |u_n|^{q^*} dx\right)^{\frac{p(1-\varpi)}{q^*}} \|u_n\|^{p\varpi-q} \int_{B_r(y)} |\nabla u_n|^q dx. \end{aligned}$$

Let us contemplate that $p \ge \frac{q(s^*-q^*)}{s^*} + q^*$, namely $\frac{q(s^*-q^*)}{s^*} + q^* \le p < s^*$ and so $p\varpi - q \ge 0$. Covering $\mathbb{R}^N$ by balls of radius r in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we obtian

$$\int_{\mathbb{R}^N} |u_n|^p dx \le C(N+1) \sup_{y\in\mathbb{R}^N} \left(\int_{B_r(y)} |u_n|^{q^*} dx\right)^{\frac{p(1-\varpi)}{q^*}} \|u_n\|^{p\varpi}.$$

Under the assumption of the lemma, it holds that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for each $\frac{q(s^*-q^*)}{s^*} + q^* \le p < s^*$. The remaining part is $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $q^* < p < \frac{q(s^*-q^*)}{s^*} + q^*$, but it is trivial by means of the Hölder's inequality again. So, the proof is completed. □

We conclude this section by showing the proof of Theorem 1.5.

Proof of Theorem 1.5 Due to $\|u\| \le \liminf_{n\to\infty} \|u_n\| \equiv 1$, we can split the proof into two cases.

Case 1: $\|u\| < 1$. Arguing it by contradiction that for some $0 < p_1 < \bar{P}(u)$, where $\bar{P}(u)$ is given by (1.16), there holds

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \Phi_{\alpha_N p_1, j_0}(u_n)dx = +\infty. \tag{2.13}$$

In light of a constant $L \in (0, +\infty)$ which is determined later and $v \in E$, set

$$G_L(v) = \begin{cases} L, & \text{if } v > L, \\ -L, & \text{if } v < -L, \\ v, & \text{if } |v| \le L, \end{cases} \text{ and } T_L(v) = v - G_L(v).$$

Plainly, there exists a constant $\epsilon \in (0, 1)$ such that $\big(p_1(1+\epsilon)^2\big)^{N-1} < \frac{1}{1-\|u\|^N}$. Obviously, $\|G_L(u)\| \to \|u\|$ as $L \to +\infty$, then one can choose a sufficiently large $L > 0$ such that

$$\big(p_1(1+\epsilon)^2\big)^{N-1} < \frac{1}{1-\|G_L(u)\|^N}. \tag{2.14}$$

We claim that

$$\limsup_{n\to\infty} \left[\left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N dx\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q dx\right)^{\frac{1}{q}}\right]^N < \left(\frac{1}{p_1(1+\epsilon)^2}\right)^{N-1}. \tag{2.15}$$

Otherwise, going to a subsequence of $\{T_L(u_n)\}$ if necessary, we have

$$\begin{aligned} \|T_L(u_n)\|^N &= \left[\left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^N dx\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} |\nabla T_L(u_n)|^q dx\right)^{\frac{1}{q}}\right]^N \\ &\ge \left(\frac{1}{p_1(1+\epsilon)^2}\right)^{N-1}, \ \forall n \in \mathbb{N}^+ \end{aligned}$$

which together with the fact $\nabla T_L(u_n)\nabla G_L(u_n) \equiv 0$ yields that

$$1 = \|u_n\|^N \ge \|T_L(u_n)\|^N + \|G_L(u_n)\|^N \ge \left(\frac{1}{p_1(1+\epsilon)^2}\right)^{N-1} + \|G_L(u_n)\|^N.$$

Since $\{G_L(u_n)\}$ is bounded in E and $G_L(u_n) \rightharpoonup G_L(u)$ in E, by using the above formula, we derive

$$\big(p_1(1+\epsilon)^2\big)^{N-1} \ge \frac{1}{1-\|G_L(u_n)\|^N}$$

which is in contradiction with (2.14). Thus, (2.15) holds true. Up to a subsequence if necessary, we can suppose that $\alpha_N p_1(1+\epsilon)^2\|T_L(u_n)\|^{\frac{N}{N-1}} < \alpha_N$ for all $n \in \mathbb{N}$. In view of (1.8), we obtain

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \int_{\Omega_{n,L}} \left(e^{\alpha_N p_1(1+\epsilon)^2|u_n-L|^{N/(N-1)}} - \sum_{j=0}^{j_0-1} \frac{(\alpha_N p_1(1+\epsilon)^2)^j}{j!}|u_n - L|^{(N-1)j/N}\right)dx \\ &\le \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} \Phi_{\alpha_N p_1(1+\epsilon)^2\|T_L(u_n)\|^{N/(N-1)}, j_0}\big(|T_L(u_n)|/\|T_L(u_n)\|\big)dx < +\infty, \end{aligned} \tag{2.16}$$

where $\Omega_{n,L} \triangleq \{x \in \mathbb{R}^N : |u_n(x)| \geq L\}$. By means of Lemma 2.1, we derive

$$|\Omega_{n,L}| \triangleq \int_{\Omega_{n,L}} dx \leq \frac{1}{L^{p^*}} \int_{\Omega_{n,L}} |u_n|^{p^*} dx \leq \frac{C\|u_n\|^{p^*}}{L^{p^*}} = \frac{C}{L^{p^*}} < +\infty, \tag{2.17}$$

where $C > 0$ is a constant independent of n by the imbedding of $E \hookrightarrow L^2(\mathbb{R}^{p^*})$. To get a contradiction, let us write

$$\int_{\mathbb{R}^N} \Phi_{\alpha_N p_1, j_0}(u_n)dx = \int_{\Omega_{n,L}} \Phi_{\alpha_N p_1, j_0}(u_n)dx + \int_{\Omega_{n,L}^c} \Phi_{\alpha_N p_1, j_0}(u_n)dx.$$

Combining (2.16) and (2.17), we apply the following two type Young's inequalities

$$|u_n|^{N/(N-1)} \leq (1+\epsilon)|u_n - L|^{N/(N-1)} + \Lambda(\epsilon)|L|^{N/(N-1)}$$

with $\Lambda(\epsilon) = (1+\epsilon)((1+\epsilon)^{N-1} - 1)^{-1/(N-1)}$ and

$$e^{a+b} \leq (1+\epsilon)^{-1} e^{(1+\epsilon)a} + \epsilon(1+\epsilon)^{-1} e^{(1+\epsilon^{-1})b}, \ \forall a, b > 0$$

to conclude that for all $n \in \mathbb{N}^+$

$$\begin{aligned}
\int_{\Omega_{n,L}} \Phi_{\alpha_N p_1, j_0}(u_n)dx &\leq \int_{\Omega_{n,L}} e^{\alpha_N p_1 (1+\epsilon)|u_n - L|^{N/(N-1)} + \alpha_N p_1 \Lambda(\epsilon)|L|^{N/(N-1)}} dx \\
&\leq \frac{1}{1+\epsilon} \int_{\Omega_{n,L}} e^{\alpha_N p_1 (1+\epsilon)^2 |u_n - L|^{N/(N-1)}} dx + \frac{\epsilon}{1+\epsilon} \int_{\Omega_{n,L}} e^{\alpha_N p_1 (1+\epsilon^{-1})\Lambda(\epsilon)|L|^{N/(N-1)}} dx \\
&\leq \frac{C}{1+\epsilon} \int_{\Omega_{n,L}} \Phi_{\alpha_N p_1 (1+\varepsilon)^2, j_0}(|u_n - L|)dx + \frac{\epsilon|\Omega_{n,L}|}{1+\epsilon} e^{\alpha_N p_1 (1+\epsilon^{-1})\Lambda(\epsilon)|L|^{N/(N-1)}} \\
&\leq C < +\infty.
\end{aligned}$$

On the other hand, since $Nj_0/(N-1) \geq p^*$, we apply Lemma 2.1 to get

$$\begin{aligned}
\int_{\Omega_{n,L}^c} \Phi_{\alpha_N p_1, j_0}(u_n)dx &= \int_{\{|u_n(x)| < L\}} \Phi_{\alpha_N p_1, j_0}(u_n)dx \\
&= \int_{\{|u_n(x)| < L\}} \sum_{j=j_0}^{\infty} \frac{(\alpha_N p_1 L^{N/(N-1)})^j}{j!} \left|\frac{u_n}{L}\right|^{Nj/(N-1)} dx \\
&\leq \sum_{j=j_0}^{\infty} \frac{(\alpha_N p_1 L^{N/(N-1)})^j}{j!} \int_{\mathbb{R}^N} \left|\frac{u_n}{L}\right|^{p^*} dx \\
&\leq C \left\|\frac{u_n}{L}\right\|^{p^*} \sum_{j=j_0}^{\infty} \frac{(\alpha_N p_1 L^{N/(N-1)})^j}{j!} \\
&\leq C L^{-p^*} \sum_{j=j_0}^{\infty} \frac{(\alpha_N p_1 L^{N/(N-1)})^j}{j!} \leq C < +\infty.
\end{aligned}$$

The above two formulas reveal a contradiction to (2.13). So, the theorem in this case holds true.

Case 2: $\|u\| = 1$. Since $u_n \rightharpoonup u$ in E, using the lower semicontinuity of norm, we can derive that $u_n \to u$ in E. Recalling that the Lebesgue theorem, there is a function $v \in E$ such that $|u_n| \leq v$ a.e. in $\mathbb{R}^2$ which together with (1.7) yields (1.16).

Next, we turn to focus on the sharpness of $\bar{P}(u)$, that is, there is a sequence $\{u_n\} \subset E$ satisfying $\|u_n\| \equiv 1$ and $u_n \rightharpoonup u \neq 0$ in E such that the supremum given by (1.16) is infinite for each $p' \geq \bar{P}(u)$. To this aim, for some constants $r > 0$ and $\varrho = 3r$, we define $\underline{w}_n(x)$ as

$$\underline{w}_n(x) \triangleq \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} N^{-\frac{N-1}{N}} n^{\frac{N-1}{N}}, & \text{if } 0 \leq |x| \leq re^{-\frac{n}{N}}, \\ N^{\frac{1}{N}} \log(r/|x|) n^{-\frac{1}{N}}, & \text{if } re^{-\frac{n}{N}} < |x| \leq r, \\ 0, & \text{if } |x| > r, \end{cases}$$

and $u \in E$ as

$$u \triangleq \begin{cases} \underline{A}, & \text{if } 0 \leq |x| \leq \frac{2}{3}\varrho, \\ 3\underline{A}\left(1 - \frac{|x|}{\varrho}\right), & \text{if } \frac{2}{3}\varrho < |x| \leq \varrho, \\ 0, & \text{if } |x| > \varrho, \end{cases}$$

respectively. Here, the constant $\underline{A} > 0$ is chosen in such a way that $\|u\| = \sigma < 1$. We set

$$\underline{u}_n = \sqrt[N]{1 - \sigma^N}\, \underline{w}_n + u \in E.$$

It is simple to calculate that

$$\int_{\mathbb{R}^N} |\nabla \underline{w}_n|^N dx = \frac{N}{\omega_{N-1} n} \int_{re^{-\frac{n}{N}} < |x| \leq r} \frac{1}{|x|^N} dx = \frac{N}{n} \int_{re^{-n/N}}^{r} \frac{1}{\rho} d\rho = 1,$$
$$0 \leq \int_{\mathbb{R}^N} |\nabla \underline{w}_n|^q dx = \frac{N^{q/N} r^{N-q}}{\omega_{N-1}^{(q-N)/N} n^{q/N} (N-q)} (1 - e^{-(N-q)n/N}) \triangleq \underline{\delta}_n \to 0 \text{ as } n \to \infty. \tag{2.18}$$

Since $B_r(0) \cap B^c_{2\varrho/3}(0) = \emptyset$ by $\varrho = 3r$, one has $\nabla \underline{w}_n \nabla u \equiv 0$ for all $x \in \mathbb{R}^N$ and then

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \underline{u}_n|^N dx &= (1 - \sigma^N) \int_{\mathbb{R}^N} |\nabla \underline{w}_n|^N dx + \int_{\mathbb{R}^N} |\nabla u|^N dx \\ &= (1 - \sigma^N) + \int_{\mathbb{R}^N} |\nabla u|^N dx \end{aligned} \tag{2.19}$$

By using (2.18), there is a constant $\tilde{C}_q > 0$ such that

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |\nabla \underline{u}_n|^q dx \right)^{\frac{N}{q}} &\leq \tilde{C}_q (1 - \sigma^N) \left(\int_{\mathbb{R}^N} |\nabla \underline{w}_n|^q dx \right)^{\frac{N}{q}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q dxx \right)^{\frac{N}{q}} \\ &= \tilde{C}_q (1 - \sigma^N) \underline{\delta}_n^{\frac{N}{q}} + \left(\int_{\mathbb{R}^N} |\nabla u|^q dx \right)^{\frac{N}{q}}. \end{aligned} \tag{2.20}$$

Thereby, it indicates that $\|\underline{u}_n\|^N \leq 1 + \tilde{C}_p (1 - \sigma^N) \underline{\delta}_n^{N/q} \triangleq 1 + \underline{\tau}_n$ with $\underline{\tau}_n \to 0$ by (2.18). Actually, one could also verify that $\|\underline{u}_n\| \to 1$ via (2.19) and (2.20). So, without loss of generality, we could suppose that $\|\underline{u}_n\| = 1 + \underline{\tau}_n$. Now, we define $u_n \triangleq \underline{u}_n/(1 + \underline{\tau}_n)^{1/N}$. Clearly,

$$\|u_n\| \equiv 1 \text{ and } u_n \rightharpoonup u \neq 0 \text{ as } n \to \infty.$$

Thereby, for all $\varepsilon_0 \geq 0$ and $p_{\varepsilon_0} = (1+\varepsilon_0)\bar{P}(u) = (1+\varepsilon_0)/(1-\sigma^N)^{1/(N-1)} \geq \bar{P}(u)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Phi_{\alpha_N p_{\varepsilon_0}, j_0}(u_n)dx = \int_{\mathbb{R}^N} \Phi_{\alpha_N(1+\varepsilon_0)/(1-\sigma^N)^{1/(N-1)}, j_0}(u_n)dx \\
&\geq \int_{B_{re^{-n/N}}(0)} e^{\alpha_N(1+\varepsilon_0)(1-\sigma^N)^{-1/(N-1)}(\underline{A}+(1-\sigma^N)^{1/N}\underline{w}_n)^{N/(N-1)}} dx \\
&\geq \int_{B_{re^{-n/N}}(0)} e^{\alpha_N C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^N)^{-1/(N-1)}(1+\underline{w}_n)^{N/(N-1)}} dx \\
&\geq e^{\alpha_N C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^N)^{-1/(N-1)}(1+n^{(N-1)/N})^{N/(N-1)}} |B_{re^{-n/N}}(0)| \\
&= \frac{\omega_{N-1}}{N} r^N e^{\alpha_N C_{\sigma,\underline{A}}(1+\varepsilon_0)(1-\sigma^N)^{-1/(N-1)}(1+n^{(N-1)/N})^{N/(N-1)}} e^{-n} \to +\infty \text{ as } n \to \infty,
\end{aligned}$$

where $C_{\sigma,\underline{A}} = \min\{(1-\sigma^N)^{1/(N-1)}, \underline{A}^{N/(N-1)}\} > 0$. The proof is completed. □

Remark 2.6 When verifying the sharpness of $\bar{P}(u)$ in the proof of Theorem 1.5, we contemplate the space E with norm $\|\cdot\|_* = \sqrt[N]{|\nabla\cdot|_N^N + |\nabla\cdot|_q^N}$ instead of $\|\cdot\| = |\nabla\cdot|_N + |\nabla\cdot|_q$. It would never cause an essential impact on the results since they are equivalent. Indeed,

$$\|\cdot\|_* \leq \|\cdot\| \leq 2^{\frac{N-1}{N}} \|\cdot\|_*.$$

3 The periodic problem (1.13): proofs of Theorems 1.3–1.4

In this section, we show the detailed proof of Theorems 1.3–1.4. When there is no misunderstanding, we shall always suppose that $1 < q < N$, $0 < \mu < N$, $(Q_1) - (Q_2)$ and $(h_1) - (h_4)$ throughout this section. Firstly, let us give some observations on the shape of the functional $J^{R,\bar{\delta}}$.

Lemma 3.1 *Let $1 < q < N$ and $R > 0$ be fixed, then there exists a constant $\zeta > 0$ such that*

$$m_\rho \triangleq \inf\left\{J^{R,\bar{\delta}}(u) : u \in E, \|u\| = \rho\right\} > 0, \ \forall \rho \in (0, \zeta], \tag{3.1}$$

and

$$n_\rho \triangleq \inf\left\{(J^{R,\bar{\delta}})'(u)[u] : u \in E, \|u\| = \rho\right\} > 0, \ \forall \rho \in (0, \zeta]. \tag{3.2}$$

Proof In view of Remark 2.6, we show that (3.1) and (3.2) hold under the norm $\|\cdot\|_*$. Using (1.5), (1.8) and arguing as calculations in the proof of Lemma 2.2, there is a constant $\overline{\varsigma} \in (0, 1)$ such that

$$\begin{aligned}
J^{R,\bar{\delta}}(u) \geq \frac{1}{N}\|u\|_*^N - C\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N\sigma}{2N-\mu}} dx\right)^{\frac{2N-\mu}{N}} \\
-C\left(\int_{\mathbb{R}^N} |u_n|^{\frac{4N\nu}{2N-\mu}} dx\right)^{\frac{2N-\mu}{2N}} \text{ whenever } \|u\| \leq \overline{\varsigma}.
\end{aligned}$$

Here, we are aware of the fact that $\|u\|_*^N = |\nabla u|_N^N + |\nabla u|_q^N \leq |\nabla u|_N^N + |\nabla u|_q^q$ provided $\|u\|_* < 1$. In view of $\sigma > \max\{q^*, N\}$ in (h_1), choosing $\nu = \sigma$ in (2.3), then we apply Lemma 2.1 to get

$$J^{R,\bar{\delta}}(u) \geq \frac{1}{N}\|u\|_*^N - C\|u\|_*^{2\sigma} \text{ whenever } \|u\| \leq \overline{\varsigma}.$$

So, there is a constant $\varsigma \in (0, \overline{\varsigma})$ such that (3.1) holds true. Since

$$(J^{R,\bar{\delta}})'(u)[u] = |\nabla u|_N^N + |\nabla u|_q^q - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(x)f^{R,\bar{\delta}}(u(x))u(x)Q(y)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu}dxdy,$$

we can obtain (3.2) as before. The proof is completed. □

Lemma 3.2 *Let $1 < q < N$ and $R > 0$ be fixed. Suppose that $u \in E\backslash\{0\}$ and consider $t > 0$, then we have*

$$J^{R,\bar{\delta}}(tu) \to -\infty \text{ as } t \to +\infty.$$

In particular, the functional $J^{R,\bar{\delta}}$ is not bounded from below.

Proof For any fixed positive function $u \in E\backslash\{0\}$ and $t > 1$, we have that

$$\frac{J(tu)}{t^N} \le \frac{1}{N}|\nabla u|_N^N + \frac{1}{q}|\nabla u|_q^q - \frac{1}{t^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{Q(x)F^{R,\bar{\delta}}(tu(x))Q(y)F^{R,\bar{\delta}}(tu(y))}{|x-y|^\mu}dxdy.$$

Due to (h_3) and Lemma 3.4 below, $F^{R,\bar{\delta}}(t) \ge Ct^\theta$ with $\theta > \frac{N}{2}$ for all sufficiently large $t > 0$. Hence, using the Fatou's lemma, we derive $J^{R,\bar{\delta}}(tu)/t^N \to -\infty$ as $t \to +\infty$, and the claim follows. □

Relying on Lemmas 3.1 and 3.2, we shall exploit the following critical point theorem without the (C) condition introduced in [30, 43] to find a (C) sequence for $J^{R,\bar{\delta}}$.

Proposition 3.3 *Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$ Gateaux differentiable for all $v \in X$, with G-derivative $\varphi'(v) \in X^{-1}$ continuous from the norm topology of X to the weak $*$ topology of X^{-1} and $\varphi(0) = 0$. Let S be a closed subset of X which disconnects (archwise) X. Let $v_0 = 0$ and $v_1 \in X$ be points belonging to distinct connected components of $\bar{X}\backslash X$. Suppose that*

$$\inf_S \varphi \ge \varrho > 0 \text{ and } \varphi(v_1) \le 0$$

and let $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) \text{ and } \gamma(1) = v_1\}$. Then

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \varphi(\gamma(t)) \ge \varrho > 0$$

and there is a $(C)_c$ sequence for φ.

Combining Lemmas 3.1 and 3.2 as well as Proposition 3.3, there is a sequence $\{u_n\} \subset E$ such that

$$J^{R,\bar{\delta}}(u_n) \to c^{R,\bar{\delta}} \text{ and } (1 + \|u_n\|)\|(J^{R,\bar{\delta}}(u_n))'\|_{E^{-1}} \to 0, \tag{3.3}$$

where

$$c^{R,\bar{\delta}} \triangleq \inf_{\gamma\in\Gamma^{R,\bar{\delta}}}\max_{t\in[0,1]} J^{R,\bar{\delta}}(\gamma(t)) > 0 \tag{3.4}$$

with $\Gamma^{R,\bar{\delta}} = \{\gamma \in C([0, 1], E) : \gamma(0) = 0 \text{ and } J^{R,\bar{\delta}}(\gamma(1)) < 0\}$.

Lemma 3.4 *The function $f^{R,\bar{\delta}}$ defined in (1.14) satisfies the Ambrosetti-Rabinowits condition with the constant $\theta > \frac{N}{2}$ appearing in (h_3), that is,*

$$0 < \theta F^{R,\bar{\delta}}(t) \le f^{R,\bar{\delta}}(t)t, \ \forall t > 0.$$

Proof We refer to reader to [8, 9] and omit the deatils. □

Lemma 3.5 *Let* $1 < q < N$ *and* $R > 0$ *be fixed, then every sequence* $\{\|u_n\|\}$ *satisfying* (3.3) *is uniformly bounded in* $n \in \mathbb{N}$*. Moreover, it holds that*

$$\|u_n\|^{\frac{N}{N-1}} \leq \max\left\{\left(\frac{2q\theta}{2\theta-q}\right)^{\frac{1}{N-1}}, \left(\frac{2q\theta}{2\theta-q}\right)^{\frac{N}{q(N-1)}}\right\} \left(\sqrt[N]{c^{R,\bar{\delta}}+o_n(1)}+\sqrt[q]{c^{R,\bar{\delta}}+o_n(1)}\right)^{\frac{N}{N-1}}. \tag{3.5}$$

Proof Firstly, we can invoke from (3.3) and Lemma 3.4 as well as (Q_1) that

$$\begin{aligned} c^{R,\bar{\delta}}+o_n(1) &= J^{R,\bar{\delta}}(u_n) - \frac{1}{2\theta}(J^{R,\bar{\delta}})'(u_n)[u_n] \\ &= \frac{2\theta-N}{2N\theta}|\nabla u_n|_N^N + \frac{2\theta-q}{2q\theta}|\nabla u_n|_q^q \\ &\quad + \frac{Q_0^2}{2\theta}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left[f^{R,\bar{\delta}}(u_n(x))u_n(x)-\theta F^{R,\bar{\delta}}(u_n(x))\right]F^{R,\bar{\delta}}(u_n(y))}{|x-y|^{\mu}}dxdy \\ &\geq \frac{2\theta-N}{2N\theta}|\nabla u_n|_N^N + \frac{2\theta-q}{2q\theta}|\nabla u_n|_q^q \geq \frac{2\theta-q}{2q\theta}\left(|\nabla u_n|_N^N+|\nabla u_n|_q^q\right) \end{aligned}$$

which reveals that $|\nabla u_n|_N \leq \sqrt[N]{\frac{2q\theta}{2\theta-q}c^{R,\bar{\delta}}+o_n(1)}$ and $|\nabla u_n|_q \leq \sqrt[q]{\frac{2q\theta}{2\theta-q}c^{R,\bar{\delta}}+o_n(1)}$. Taking $\|\cdot\| = |\cdot|_N + |\cdot|_q$ into account, we obtain

$$\|u_n\| \leq \sqrt[N]{\frac{2q\theta}{2\theta-q}c^{R,\bar{\delta}}+o_n(1)} + \sqrt[q]{\frac{2q\theta}{2\theta-q}c^{R,\bar{\delta}}+o_n(1)}$$

which yields the desired result. □

Before presenting the proof of Theorem 1.4, we need the following result.

Lemma 3.6 *Let* $1 < q < N$ *and* $R > 0$ *be fixed. Suppose, in addition, that* (h_5)*, then there exists a* $\xi_0 = \xi_0(R) > 0$ *such that for all* $\xi > \xi_0$

$$\sqrt[N]{c^{R,\frac{N}{N-1}}+1}+\sqrt[q]{c^{R,\frac{N}{N-1}}+1} \leq \left(\frac{\alpha_N \min\left\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\right\}}{2(\gamma+\alpha R^{\tau-\frac{N}{N-1}})}\right)^{\frac{N-1}{N}} \min\left\{\sqrt[N]{\frac{2\theta-q}{2q\theta}}, \sqrt[q]{\frac{2\theta-q}{2q\theta}}\right\}.$$

Proof Choosing a cutoff function $\varphi_0 \in C_0^\infty(B_1(0))$ satisfying $0 \leq \varphi_0 \leq 1$; $\varphi_0(x) \equiv 1$ if $|x| \leq 1/2$; $\varphi_0(x) \equiv 0$ if $|x| \geq 1$; and $|\nabla\varphi_0| \leq 1$ for every $x \in \mathbb{R}^N$. Recalling the definition of $f^{R,\frac{N}{N-1}}$ and (h_4), $F^{R,\frac{N}{N-1}}(t) \geq \xi t^p$ with $p > \frac{N}{2}$ for all $t \in [0, 1]$. Thus,

$$\begin{aligned} J^{R,\frac{N}{N-1}}(\varphi_0) &= \frac{1}{N}\int_{B_1(0)}|\nabla\varphi_0|^N dx + \frac{1}{q}\int_{B_1(0)}|\nabla\varphi_0|^q dx \\ &\quad - \frac{1}{2}\int_{B_1(0)}\int_{B_1(0)}\frac{Q(x)F^{R,\frac{N}{N-1}}(\varphi_0(x))Q(y)F^{R,\frac{N}{N-1}}(\varphi_0(y))}{|x-y|^{\mu}}dxdy \\ &< \frac{2\omega_{N-1}}{Nq} - \frac{\xi^2 Q_0^2}{4}\left(\int_{B_{1/2}(0)}|\varphi_0|^p dx\right)^2 \leq \frac{2\omega_{N-1}}{Nq} - \frac{\xi_1^2 Q_0^2\omega_{N-1}^2}{4^{N+1}N^2} = 0 \end{aligned} \tag{3.6}$$

if $\xi \geq \xi_1$, where $\xi_1 = Q_0^{-1}\sqrt{\frac{2N}{q\omega_{N-1}}}2^{N+1}$. In particular, invoking from (3.6) that

$$\frac{1}{N}\int_{B_1(0)} |\nabla\varphi_0|^N dx + \frac{1}{q}\int_{B_1(0)} |\nabla\varphi_0|^q dx < \frac{\xi_1^2 Q_0^2}{4}\left(\int_{B_{1/2}(0)} |\varphi_0|^p dx\right)^2. \tag{3.7}$$

Setting $\gamma_0^{R,\frac{N}{N-1}}(t) = t\varphi_0$, one has that $\gamma_0^{R,\frac{N}{N-1}} \in \Gamma^{R,\frac{N}{N-1}} = \{\gamma \in C([0,1], E) : \gamma(0) = 0, J^{R,\frac{N}{N-1}}(\gamma(1)) < 0\}$ by (3.7). Therefore, we have that

$$\begin{aligned}\max_{t\in[0,1]} J^{R,\frac{N}{N-1}}(t\varphi_0) &\leq \max_{t\in[0,1]}\left\{\frac{2\omega_{N-1}}{Nq}t^q - \frac{\xi^2 Q_0^2\omega_{N-1}^2}{4^{N+1}N^2}t^{2p}\right\}\\ &\leq \max_{t\geq 0}\left\{\frac{2\omega_{N-1}}{Nq}t^q - \frac{\xi^2 Q_0^2\omega_{N-1}^2}{4^{N+1}N^2}t^{2p}\right\}\\ &= \frac{(2p-q)\omega_{N-1}}{Nqp}\left[\frac{4^{N+1}N}{p\omega_{N-1}\xi^2 Q_0^2}\right]^{\frac{q}{2p-q}}.\end{aligned}$$

Due to the definition of $c^{R,\frac{N}{N-1}}$, we let the constant $\xi_0 = \xi_0(R) > \xi_1$ be such that $\max_{t\in[0,1]} J^{R,\frac{N}{N-1}}(t\varphi_0)$ satisfies the desired inequality for all $\xi > \xi_0$. The proof is completed. □

Proof of Theorem 1.4 Due to Lemmas 3.1 and 3.2 as well as Proposition 3.3, there is a sequence $\{u_n\} \subset E$ satisfying (3.3). Using Lemma 3.5, $\{\|u_n\|\}$ is uniformly bounded in $n \in \mathbb{N}$. Going to a subsequence if necessary, by adopting Lemma 2.1, there exists a $u \in E$ such that $u_n \rightharpoonup u$, $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for all $p \in [q^*, +\infty)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. We claim that $(J^{R,\bar{\delta}})'(u) = 0$ which is obvious when $\bar{\delta} = \delta$ by (2.2). Now, we shall contemplate the case $\bar{\delta} = \frac{N}{N-1}$. Combining Lemmas 3.5 and 3.6, we could conclude that the additional assumption in Lemma 2.2 is satisfied, so $(J^{R,\frac{N}{N-1}})'(u) = 0$. Finally, we finish the proof by showing $u \neq 0$. Arguing it indirectly, since $\{\|u_n\|\}$ is uniformly bounded, we have either $\{u_n\}$ is vanishing, that is, for any $r > 0$, it holds that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)} |u_n|^{p^*} dx = 0$$

or it is non-vanishing, i.e. there exist $r, r_0 > 0$ and a sequence $\{z_n\} \subset \mathbb{Z}^N$ such that

$$\lim_{n\to\infty}\int_{B_r(z_n)} |u_n|^{p^*} dx \geq r_0. \tag{3.8}$$

If $\{u_n\}$ is vanishing, thanks to Lemma 2.5, we can derive that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $q^* < p < +\infty$. Since the additional assumption in Lemma 2.2 holds true, using (1.5), (1.8) and (2.3), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[|x|^{-\mu} * (Q(x)F^{R,\frac{N}{N-1}}(u_n))\right]Q(x)F^{R,\frac{N}{N-1}}(u_n)dx = 0$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[|x|^{-\mu} * (Q(x)F^{R,\frac{N}{N-1}}(u_n))\right]Q(x)f^{R,\frac{N}{N-1}}(u_n)u_n dx = 0$$

which together with $J^{R,\frac{N}{N-1}}(u_n) \to c^{R,\frac{N}{N-1}}$ and $(J^{R,\frac{N}{N-1}})'(u_n)[u_n] \to 0$ indicate that $c^{R,\frac{N}{N-1}} \equiv 0$. It contradicts with (3.4). So, (3.8) holds true and we can define $v_n = u_n(\cdot - z_n)$. Thus

$$\int_{B_r(0)} |v_n|^{p^*} dx \geq \frac{r_0}{2}. \tag{3.9}$$

Due to (Q_2), both $J^{R,\bar{\delta}}$ and $(J^{R,\bar{\delta}})'$ are translation-invariant in $\mathbb{Z}^N$ and so $\{v_n\}$ is again a $(C)_{c^{R,\bar{\delta}}}$ sequence of $J^{R,\bar{\delta}}$. Then, up to a subsequence if necessary, $v_n \rightharpoonup v$ in E with $v \neq 0$ by (3.9). Repeating the arguments above, one knows that $(J^{R,\bar{\delta}})'(v) = 0$ finishing the proof. □

Next, we shall present the proof of Theorem 1.3. According to the discussions as before, we must take the uniform L^∞-estimate for the nontrivial solution obtained in Theorem 1.4. Let $u_R \in E$ be a nontrivial solution associated with Eq. (1.15), with the help of the definition of $f^{R,\bar{\delta}}$ which is defined as in (1.12), then it is a nontrivial solution for Eq. (1.13) provided $|u_R|_\infty \leq R$. So, the key idea is to find a constant $C_0 > 0$ which is independent of $R > 0$ satisfying $|u_R|_\infty \leq C_0$. To this aim. we must firstly verify that the constant $c^{R,\bar{\delta}}$ and $\xi_0(R)$ appearing in Theorem 1.4 do not depend on R. Let us recall the two cases in Theorem 1.4, to proceed it clearly, we would split it by two subsections: (I) $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$; (II) $\bar{\delta} = \frac{N}{N-1}$.

In the Cases (I) and (II), we shall choose $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$ and $\tau_* = \frac{N}{N-1} + \frac{1}{R} > 0$, respectively.

3.1 The case (I) in (1.12): $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$

In this Subsection, we suppose that the nonlinearity f defined in (1.11) requires $(h_1) - (h_3)$. Firstly, we have the following result.

Lemma 3.7 *If $1 < q < N$ and let $\{u_n\} \subset E$ be a (C) sequence of $J^{R,\delta}$ at the level $c^{R,\delta}$, then $\{\|u_n\|\}$ is uniformly bounded in $n \in \mathbb{N}$ and $R > 0$, that is, there is a constant $K > 0$ independent of $n \in \mathbb{N}$ and $R > 0$ such that*

$$\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq K < +\infty. \tag{3.10}$$

Proof We claim that there are constants $A_0 > 0$ and $c > 0$ independent of $R > 0$ such that

$$A_0 < c^{R,\delta} \leq c < +\infty, \ \forall R > 0. \tag{3.11}$$

Indeed, recalling the definition of $f^{R,\delta}$, one has that $F^{R,\delta}(t) \geq H(t)$ for all $t \in \mathbb{R}$ and so $J^{R,\delta}(u) \geq I(u)$ for all $u \in E$, where the variational functional $I : E \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{H(u(x))H(u(y))}{|x|^\beta |x-y|^\mu |y|^\beta} dx. \tag{3.12}$$

Choosing the constant $d > 0$ to be a mountain-pass level associated with I, the existence of such number follows from Lemmas 3.1 and 3.2. So $c^{R,\delta} \leq c$. Similarly, one can derive a constant $A_0 > 0$ satisfying $c^{R,\delta} \geq A_0$. Combining (3.5) and (3.11), we can get the desired result (3.10). The proof is completed. □

With the choice of $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$ in this subsection, there is a constant $C > 0$ independent of $R > 0$ such that

$$|f^{R,\delta}(t)| \leq |t|^{\sigma-1} + C|t|^{\nu-1}\Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(t) \text{ with } \Upsilon \triangleq \min\left\{\frac{N-\mu}{4N}, \frac{\sigma-N}{\sigma}\right\}, \ \forall t \in \mathbb{R}, \tag{3.13}$$

where $\nu > 1$ is arbitrary and $K > 0$ independent of $R > 0$ appearing in (3.10). In fact, motivated by [57, Lemma 2.2], for every $\alpha \in (0, \alpha^*)$ and $|t| \geq 1$, there are $M_2 \geq M_1 \geq M$ independent of $R > 0$ such that

$$|f^{R,\delta}(t)| \leq Me^{\gamma|t|^\delta} e^{\alpha^* R^{\tau-\delta}|t|^\delta} \leq Me^{(\gamma+1)|t|^\delta} \leq M_1 e^{\frac{\alpha_N \Upsilon}{K}|t|^{\frac{N}{N-1}}}$$
$$\leq M_2 \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(t) \leq M_2 |t|^{\nu-1} \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(t)$$

jointly with $f^{R,\delta}(t) = o(t^{\sigma-1})$ uniformly in R as $t \to 0$ by (h_1), we have (3.13) at once.

Remark 3.8 From now on, one could observe that both (3.10) and (3.13) are independent of $R > 0$. Moreover, by using (3.13) and (3.10), one sees that (2.2) with $\bar{\delta} = \delta$ holds true independently with respect to $R > 0$.

Lemma 3.9 *If* $1 < q < N$ *and let* $u_R \in E$ *be a nonnegative ground state solution of Eq.* (1.17) *with* $\bar{\delta} = \delta$ *established by Theorem 1.4 for all fixed* $R > 0$, *if* $\alpha^* = \frac{1}{R^{\tau-\delta}} > 0$, *then for all* $\alpha \in (0, \alpha^*)$ *and* $\tau \geq \frac{N}{N-1}$, *we have*

$$0 \leq \Pi_R(x) \triangleq |x|^{-\mu} * (Q(y)F^{R,\delta}(u_R)) \leq C_1, \ \forall R > 0.$$

for some $C_1 > 0$ *independent of* $R > 0$.

Proof Recalling the proof of Theorem 1.4, we deduce that the nonnegative solution $u_R \neq 0$ of Eq. (1.15) is established by looking for the weak limit of $\{u_n\} \subset E$ which is a (C) sequence of $J^{R,\delta}$ at the level $c^{R,\delta}$. Thereby, combining the Fatou's lemma and (3.10), $\|u_R\|^{\frac{N}{N-1}} \leq K$ for all $R > 0$, where $K > 0$ is a constant independent of $R > 0$. From (3.13) and $(Q_1) - (Q_2)$, there is a $C_2 > 0$ independent of $R > 0$ such that

$$\Pi_R(x) = \int_{\mathbb{R}^N} \frac{Q(y)F^{R,\delta}(u_R(y))}{|x-y|^\mu} dy \leq |Q|_\infty \int_{\mathbb{R}^N} \frac{|u_R(y)|^\sigma + C_1 |u_R(y)|^\nu \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(u_R(y))}{|x-y|^\mu} dy$$
$$= |Q|_\infty \int_{\mathbb{R}^N} \frac{|u_R(y)|^\sigma}{|x-y|^\mu} dy + C_2 |Q|_\infty \int_{\mathbb{R}^N} \frac{|u_R(y)|^\nu \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(u_R(y))}{|x-y|^\mu} dy$$
$$= |Q|_\infty \Pi^1 + C_2 |Q|_\infty \Pi^2.$$

The next goal of accomplishing the proof is, therefore, to verify that both Π^1 and Π^2 are uniformly bounded with respect to $R > 0$.

For Π^1, since $0 < \mu < N$, we can pick $t > 1$ such that $\beta t < N$ and $\mu t' < N$, where $\frac{1}{t} + \frac{1}{t'} = 1$. By means of the Holder's inequality and $\sigma \geq q^*$ by (h_1),

$$\Pi^1 = \int_{\mathbb{R}^N} \frac{|u_R(y)|^\sigma}{|x-y|^\mu} dy \leq \left(\int_{\mathbb{R}^N} |u_R(y)|^{t\sigma} dy \right)^{\frac{1}{t}} \left(\int_{\mathbb{R}^N} \frac{1}{|x-y|^{\mu t'}} dy \right)^{\frac{1}{t'}}$$
$$\leq C\|u_R\|^\sigma \leq CK^{\frac{(N-1)\sigma}{N}},$$

where $C > 0$ depends on the imbedding constant in Lemma 2.1.

For Π^2, we rewrite it as $\Pi^2 = \Pi^{2,1} + \Pi^{2,2}$. We continue to choose $\nu = \sigma \geq q^*$ by (h_1) and then

$$\Pi^{2,1} = \int_{|x-y|>1} \frac{|u_R(y)|^\nu \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(u_R(y))}{|x-y|^\mu} dy \leq \int_{\mathbb{R}^N} |u_R(y)|^\nu \Phi_{\frac{\alpha_N \Upsilon}{K}, j_0}(u_R(y)) dy$$

$$\leq \left(\int_{\mathbb{R}^N}|u_R(y)|^{2\nu}dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\Phi_{\frac{2\alpha_N\Upsilon}{K}\|u_n\|^{\frac{N}{N-1}},j_0}(u_R(y)/\|u_R\|)dy\right)^{\frac{1}{2}}$$
$$\leq C\|u_R\|^{\sigma}\mathbb{S}^{\frac{1}{2}}\big(\frac{1}{2}\alpha_N\big)\leq\tilde{C},$$

where the constant $\tilde{C}>0$ is independent of $R>0$ by exploiting (1.8) and Lemma 2.1. It follows some elementary calculations that $\big(\int_{|x-y|\leq 1}|x-y|^{-\frac{2N\mu}{N+\mu}}dy\big)^{\frac{N+\mu}{2N}}\leq C$ for a constant $C>0$ which is independent of $R>0$. Using (1.8) and Lemma 2.1 again,

$$\Pi^{2,2}=\int_{|x-y|\leq 1}\frac{|u_R(y)|^{\nu}\Phi_{\frac{\alpha_N\Upsilon}{K},j_0}(u_R(y))}{|x-y|^{\mu}}dy\leq C\bigg(\int_{\mathbb{R}^N}|u_R(y)|^{\frac{2N\nu}{N-\mu}}\Phi_{\frac{2N\alpha_N\Upsilon}{K(N-\mu)},j_0}(u_R(y))dy\bigg)^{\frac{N-\mu}{2N}}$$
$$\leq C\bigg(\int_{\mathbb{R}^N}|u_R(y)|^{\frac{4N\nu}{N-\mu}}dy\bigg)^{\frac{N-\mu}{4N}}\bigg(\int_{\mathbb{R}^N}\Phi_{\frac{4N\alpha_N\Upsilon}{K(N-\mu)}\|u_R\|^{\frac{N}{N-1}},j_0}(u_R(y)/\|u_R\|)dy\bigg)^{\frac{N-\mu}{4N}}$$
$$\leq\bar{C}\|u_R\|^{\sigma}\mathbb{S}^{\frac{N-\mu}{4N}}(\alpha_N)\leq\hat{C},$$

where $\hat{C}>0$ is independent of $R>0$. So, we can finish the proof of this lemma. □

Now, we shall exploit the Nash–Moser iteration procedure [47] to conclude the uniform L^{∞}-estimate of u_R which is a key point is this paper.

Lemma 3.10 *Under the assumptions of Lemma 3.9, for the nontrivial nonnegative solution of Eq. (1.15), there is a constant $C_0>0$ independent of $R>0$ such that $|u_R|_{\infty}\leq C_0$.*

Proof Let $L>0$ and $\vartheta>1$ determined later, we shall contemplate $u_{R,L}\triangleq(u_R)_L\in E$ and $z_{R,L}\triangleq u_Ru_{R,L}^{r(\vartheta-1)}\in E$, where $(u_R)_L=\min\{u_R,L\}$ and $r\in\{q,N\}$. Obviously, $\nabla u_R\nabla z_{R,L}\geq u_{R,L}^{r(\vartheta-1)}|\nabla u_R|^2$ in $\mathbb{R}^N$. Therefore, taking $z_{R,L}$ as a test function of Eq. (1.17), there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\big(|\nabla u_R|^N+|\nabla u_R|^q\big)u_{R,L}^{r(\vartheta-1)}dx&\leq\int_{\mathbb{R}^N}(|\nabla u_R|^{N-2}+|\nabla u_R|^{q-2})\nabla u_R\nabla z_{R,L}dx\\&=\int_{\mathbb{R}^N}\Pi_R(x)\frac{f^{R,\delta}(u_R(x))z_{R,L}(x)}{|x|^{\beta}}dx\\&\leq C_1\int_{\mathbb{R}^N}\frac{f^{R,\delta}(u_R(x))z_{R,L}(x)}{|x|^{\beta}}dx,\end{aligned}\tag{3.14}$$

where $C_1>0$ is a constant (appearing in Lemma 3.9) which is independent of $R>0$. Then, let us define $w_{R,L}=u_Ru_{R,L}^{\vartheta-1}$ which indicates that $|\nabla w_{R,L}|\leq\vartheta|\nabla u_R|u_{R,L}^{\vartheta-1}$. Using (3.14), it holds that

$$\begin{cases}\displaystyle\int_{\mathbb{R}^N}|\nabla w_{R,L}|^qdx\leq\vartheta^q\int_{\mathbb{R}^N}|\nabla u_R|^qu_{R,L}^{q(\vartheta-1)}dx\leq C_1\vartheta^q\int_{\mathbb{R}^N}\frac{f^{R,\delta}(u_R)u_Ru_{R,L}^{q(\vartheta-1)}}{|x|^{\beta}}dx,\\\displaystyle\int_{\mathbb{R}^N}|\nabla w_{R,L}|^Ndx\leq\vartheta^N\int_{\mathbb{R}^N}|\nabla u_R|^Nu_{R,L}^{N(\vartheta-1)}dx\leq C_1\vartheta^N\int_{\mathbb{R}^N}\frac{f^{R,\delta}(u_R)u_Ru_{R,L}^{N(\vartheta-1)}}{|x|^{\beta}}dx.\end{cases}$$

For all $\sigma^* > \sigma$, combining the above formulas and Lemma 2.1, we obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_{R,L}|^{\sigma^*}dx\right)^{\frac{N}{\sigma^*}} &\le C_2\big(|\nabla w_{R,L}|_q+|\nabla w_{R,L}|_N\big)^N \le C_3(|\nabla w_{R,L}|_q^N+|\nabla w_{R,L}|_N^N)\\ &\le C_4\vartheta^N\Bigg[\left(\int_{\mathbb{R}^N} f^{R,\delta}(u_R)u_R u_{R,L}^{q(\vartheta-1)}dx\right)^{\frac{N}{q}}\\ &\quad+\int_{\mathbb{R}^N} f^{R,\delta}(u_R)u_R u_{R,L}^{N(\vartheta-1)}dx\Bigg].\end{aligned} \tag{3.15}$$

With (3.15) in hands, we are left the detailed calculations in the Appendix to conclude that

$$\left(\int_{\mathbb{R}^N}|u_R|^{\chi\sigma\vartheta}dx\right)^{\frac{1}{\chi\sigma\vartheta}} \le C_5^{\frac{1}{\vartheta}}\vartheta^{\frac{1}{\vartheta}}\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{1}{\sigma\vartheta}} \tag{3.16}$$

where $C_5 > 0$ is independent of $R > 0$ and $\chi = \sigma^*/\sigma > 1$. As a special case of (3.16), there holds

$$\left(\int_{\mathbb{R}^N}|u_R|^{\chi\sigma}dx\right)^{\frac{1}{\chi\sigma}} \le C_5\left(\int_{\mathbb{R}^N}|u_R|^{\sigma}dx\right)^{\frac{1}{\sigma}}.$$

For $\vartheta = \chi^m$ with $m \in \mathbb{N}^+$ in (3.16), we derive

$$\left(\int_{\mathbb{R}^N}|u_R|^{\chi^{m+1}\sigma}dx\right)^{\frac{1}{\chi^{m+1}\sigma}} \le C_5^{\frac{1}{\chi^m}}\chi^{\frac{m}{\chi^m}}\left(\int_{\mathbb{R}^N}|u_R|^{\chi^m\sigma}dx\right)^{\frac{1}{\chi^m\sigma}}.$$

From it, proceeding this iteration procedure m times and multiplying these $m+1$ formulas,

$$\left(\int_{\mathbb{R}^N}|u_R|^{\chi^{m+1}\sigma}dx\right)^{\frac{1}{\chi^{m+1}\sigma}} \le C_5^{\sum_{j=0}^{m}\frac{1}{\chi^j}}\chi^{\sum_{j=1}^{m}\frac{j}{\chi^j}}\left(\int_{\mathbb{R}^N}|u_R|^{\sigma}dx\right)^{\frac{1}{\sigma}}.$$

Since $\sum_{j=0}^{\infty}\frac{1}{\chi^j} = \frac{\chi}{\chi-1}$ and $\sum_{j=1}^{\infty}\frac{j}{\chi^j} = \frac{\chi}{(\chi-1)^2}$, we could take the limit as $m \to +\infty$ to get the desired result. The proof is completed. □

3.2 The case (II) in (1.12): $\bar{\delta} = \frac{N}{N-1}$

In this Subsection, we always suppose that the nonlinearity f defined in (1.11) requires $(h_1) - (h_3)$ and (h_4).

With the choice of $\tau_* = \frac{N}{N-1} + \frac{1}{R} > 0$ in this subsection, we improve (2.3) in the sense: there is a constant $C' > 0$ independent of $R > e$ such that

$$|f^{R,\frac{N}{N-1}}(t)| \le |t|^{\sigma-1} + C'|t|^{\nu-1}\Phi_{\gamma+\alpha e^{e^{-1}},j_0}(t),\ \forall t \in \mathbb{R}. \tag{3.17}$$

Firstly, one can observe that $\lim\limits_{R\to+\infty} R^{\frac{1}{R}} = 1$ and the function $R^{\frac{1}{R}}$ is strictly decreasing in $R \in (e, +\infty)$, then $0 < R^{\frac{1}{R}} \le e^{\frac{1}{e}}$ for each $R \in (e, +\infty)$. For all $|t| \ge 1$, by means of (h_4), there are $M_2 > M_1 > M$ independent of $R > e$ such that

$$\begin{aligned}|f^{R,\frac{N}{N-1}}(t)| &\le Me^{\gamma|t|^\delta}e^{\alpha R^{\tau_*-\frac{N}{N-1}}|t|^{\frac{N}{N-1}}} = Me^{\gamma|t|^\delta}e^{\alpha R^{\frac{1}{R}}|t|^{\frac{N}{N-1}}} \le Me^{\gamma|t|^\delta}e^{\alpha e^{\frac{1}{e}}|t|^{\frac{N}{N-1}}}\\ &\le M_1e^{\gamma|t|^{\frac{N}{N-1}}}e^{\alpha e^{\frac{1}{e}}|t|^{\frac{N}{N-1}}} = M_1e^{\left(\gamma+\alpha e^{\frac{1}{e}}\right)|t|^{\frac{N}{N-1}}} \le M_2\Phi_{\gamma+\alpha e^{e^{-1}},j_0}(t)\end{aligned}$$

$$\leq M_1|t|^{\nu-1}\Phi_{\gamma+\alpha e^{e^{-1}},j_0}(t)$$

which together with $f^{R,\frac{N}{N-1}}(t)=o(t^{\sigma-1})$ uniformly in $R>0$ as $t\to 0$.

Lemma 3.11 *Let $1<q<N$ and suppose additionally that (h_4), then there are some constants $\bar{A}_0>0$ and $\xi_0>0$ independent of $R>e$ such that for all $\xi>\xi_0$, there holds $\bar{A}_0\leq c^{R,\frac{N}{N-1}}$ and*

$$\sqrt[N]{c^{R,\frac{N}{N-1}}+o_n(1)}+\sqrt[q]{c^{R,\frac{N}{N-1}}+o_n(1)}\leq\left(\frac{\alpha_N\min\{\frac{\sigma-1}{\sigma},\frac{N-\mu}{N+\mu}\}}{2(\gamma+\alpha e^{\frac{1}{e}})}\right)^{\frac{N-1}{N}}\min\left\{\sqrt[N]{\frac{2\theta-q}{2q\theta}},\sqrt[q]{\frac{2\theta-q}{2q\theta}}\right\}. \tag{3.18}$$

Proof Applying (3.17) to Lemma 3.1, one can find such a $\bar{A}_0>0$ and the details are left. By the definition of $f^{R,\frac{N}{N-1}}$, then $c^{R,\frac{N}{N-1}}\leq c$, where c is a mountain-pass level corresponding to the variational functional I defined by (3.12). Let φ_0 be as in Lemma 3.6,

$$\begin{aligned}J^{R,\frac{N}{N-1}}(\varphi_0)&=\frac{1}{N}\int_{B_1(0)}|\nabla\varphi_0|^N dx+\frac{1}{q}\int_{B_1(0)}|\nabla\varphi_0|^q dx\\&\quad-\frac{1}{2}\int_{B_1(0)}\int_{B_1(0)}\frac{Q(x)F^{R,\frac{N}{N-1}}(\varphi_0(x))Q(y)F^{R,\frac{N}{N-1}}(\varphi_0(y))}{|x-y|^{\mu}}dxdy\\&<\frac{2\omega_{N-1}}{Nq}-\frac{\xi^2Q_0^2}{4}\left(\int_{B_{1/2}(0)}|\varphi_0|^p dx\right)^2\leq\frac{2\omega_{N-1}}{Nq}-\frac{\xi_1^2Q_0^2\omega_{N-1}^2}{4^{N+1}N^2}=0\end{aligned} \tag{3.19}$$

if $\xi\geq\xi_1$, where $\xi_1=Q_0^{-1}\sqrt{\frac{2N}{q\omega_{N-1}}}2^{N+1}$. In particular, invoking from (3.19) that

$$\frac{1}{N}\int_{B_1(0)}|\nabla\varphi_0|^N dx+\frac{1}{q}\int_{B_1(0)}|\nabla\varphi_0|^q dx<\frac{\xi_1^2Q_0^2}{4}\left(\int_{B_{1/2}(0)}|\varphi_0|^p dx\right)^2. \tag{3.20}$$

Setting $\gamma_0^{R,\frac{N}{N-1}}(t)=t\varphi_0$, one has that $\gamma_0^{R,\frac{N}{N-1}}\in\Gamma^{R,\frac{N}{N-1}}=\{\gamma\in C([0,1],E):\gamma(0)=0,J^{R,\frac{N}{N-1}}(\gamma(1))<0\}$ by (3.20). Therefore, we have that

$$\begin{aligned}\max_{t\in[0,1]}J^{R,\frac{N}{N-1}}(t\varphi_0)&\leq\max_{t\in[0,1]}\left\{\frac{2\omega_{N-1}}{Nq}t^q-\frac{\xi^2\omega_{N-1}^2}{4^{N+1}N^2}t^{2p}\right\}\\&\leq\max_{t\geq 0}\left\{\frac{2\omega_{N-1}}{Nq}t^q-\frac{\xi^2\omega_{N-1}^2}{4^{N+1}N^2}t^{2p}\right\}\\&=\frac{(2p-q)\omega_{N-1}}{Nqp}\left[\frac{4^{N+1}N}{p\omega_{N-1}\xi^2}\right]^{\frac{q}{2p-q}}.\end{aligned}$$

In view of the definition of $c^{R,\frac{N}{N-1}}$, there exists a constant $\xi_0>\xi_1$ independent of $R>e$ such that $\max_{t\in[0,1]}J^{R,\frac{N}{N-1}}(t\varphi_0)$ satisfies the desired inequality for all $\xi>\xi_0$. The proof is completed. □

Remark 3.12 With Lemma 3.11 in hands, it holds that

$$\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{2\left(\gamma+\alpha e^{\frac{1}{e}}\right)} \min\left\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\right\}, \ \forall R > e, \tag{3.21}$$

where $\{u_n\} \subset E$ is a (C) sequence of $J^{R,\frac{N-1}{N}}$ at the level $c^{R,\frac{N-1}{N}}$. Indeed, using (3.5) and (3.18), it is obvious. Moreover, it follows from (3.17) and (3.21) that (2.2) holds independently with respect to $R > e$.

Lemma 3.13 *If $1 < q < N$ and let $u_R \in E$ be a nonnegative nontrivial solution of Eq. (1.15) with $\bar{\delta} = \frac{N}{N-1}$ established by Theorem 1.4 for all fixed $R > e$, if $\tau_* = \frac{N}{N-1} + \frac{1}{R} > 0$, then for all $\alpha > 0$ and $\tau \in [\frac{N}{N-1}, \tau_*)$, we have*

$$0 \leq \bar{\Pi}_R(x) \triangleq |x|^{-\mu} * (Q(y)F^{R,\frac{N}{N-1}}(u_R)) \leq C_1', \ \forall R > e.$$

for some $C_1' > 0$ independent of $R > e$.

Proof In light of Lemma 3.9, it suffices to prove that

$$\int_{\mathbb{R}^N} \frac{Q(y)|u_R(y)|^\nu \Phi_{\gamma+\alpha e^{e^{-1}},j_0}(u_R(y))}{|x-y|^\mu} dy \leq C_2' \tag{3.22}$$

for some $C_2' > 0$ independent of $R > e$. Firstly, due to the Fatou's lemma, $\|u_R\|^{\frac{N}{N-1}} \leq \frac{\alpha_N(\sigma-1)}{2\sigma(\gamma+\alpha e^{\frac{1}{e}})}$ and $\|u_R\|^{\frac{N}{N-1}} \leq \frac{\alpha_N(N-\mu)}{2(N+\mu)(\gamma+\alpha e^{\frac{1}{e}})}$ by (3.21) for every $R > e$. On the one hand, choosing $\nu = \frac{\sigma+1}{2}$ and then adopting (1.8) as well as Lemma 2.1,

$$\begin{aligned}
&\int_{|x-y|>1} \frac{Q(y)|u_R(y)|^\nu \Phi_{\gamma+\alpha e^{e^{-1}},j_0}(u_R(y))}{|x-y|^\mu} dy \leq |Q|_\infty \int_{\mathbb{R}^N} |u_R(y)|^\nu \Phi_{\gamma+\alpha e^{e^{-1}},j_0}(u_R(y)) dy \\
&\leq |Q|_\infty \left(\int_{\mathbb{R}^N} |u_R(y)|^\sigma dy\right)^{\frac{\sigma+1}{2\sigma}} \left(\int_{\mathbb{R}^N} \Phi_{\frac{2\sigma}{\sigma-1}(\gamma+\alpha e^{e^{-1}})\|u_n\|^{\frac{N}{N-1}},j_0}(u_R(y)/\|u_n\|) dy\right)^{\frac{\sigma-1}{2\sigma}} \\
&\leq C|Q|_\infty \|u_n\|^\sigma \mathbb{S}^{\frac{\sigma-1}{2\sigma}}(\alpha_N) \leq \tilde{C}_2'
\end{aligned}$$

for $\tilde{C}_2' > 0$ independent of $R > e$. On the one hand, since $\int_{|x-y|\leq 1} |x-y|^{-\frac{N+\mu}{2}} dy \leq C$, we derive

$$\begin{aligned}
&\int_{|x-y|\leq 1} \frac{Q(y)|u_R(y)|^\nu \Phi_{\gamma+\alpha e^{e^{-1}},j_0}(u_R(y))}{|x-y|^\mu} dy \\
&\leq C|Q|_\infty \left(\int_{\mathbb{R}^N} |u_R(y)|^{\frac{(N+\mu)\nu}{N-\mu}} \Phi_{\frac{(N+\mu)(\gamma+\alpha e^{e^{-1}})}{N-\mu},j_0}(u_R(y)) dy\right)^{\frac{N-\mu}{N+\mu}} \\
&\leq C|Q|_\infty \left(\int_{\mathbb{R}^N} |u_R(y)|^{\frac{2(N+\mu)\nu}{N-\mu}} dy\right)^{\frac{N-\mu}{2(N+\mu)}} \\
&\quad \left(\int_{\mathbb{R}^N} \Phi_{\frac{2(N+\mu)(\gamma+\alpha e^{e^{-1}})}{N-\mu}\|u_n\|^{\frac{N}{N-1}},j_0}(u_R(y)/\|u_n\|) dy\right)^{\frac{N-\mu}{2(N+\mu)}} \\
&\leq C|Q|_\infty \|u_n\|^\sigma \mathbb{S}^{\frac{N-\mu}{2(N+\mu)}}(\alpha_N) \leq \hat{C}_2'
\end{aligned}$$

for some $\hat{C}_2' > 0$ independent of $R > e$, where $\nu = \sigma$. Thus, we can finish the proof of this lemma. ◻

Arguing as a very similar calculations in Lemma 3.10, we could follow Lemma 3.13 to conclude the following L^∞-estimate with respect to u_R.

Lemma 3.14 *Under the assumptions of Lemma 3.13, for the nontrivial nonnegative solution of Eq.* (1.15)*, there is a constant $C_0' > 0$ independent of $R > e$ such that $|u_R|_\infty \le C_0'$.*

Now, we are in a position to present the proof of Theorem 1.3.

Proof of Theorem 1.3 Due to Theorem 1.4, we have established a nonnegative nontrivial solution for Eq. (1.15) under the suitable assumptions. Let us denote the obtained nontrivial solution by u_R. According to the explanations in Remarks 3.8 and 3.12, it follows from Lemmas 3.10 and 3.14 that we could choose $R = C_0$ and $R = \max\{C_0', e\}$ for $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$ and $\bar{\delta} = \frac{N}{N-1}$, respectively. In this situation, $\alpha^* = \frac{1}{C_0^{\tau-\delta}}$ and $\tau_* = \frac{N}{N-1} + \frac{1}{\max\{C_0', e\}}$. So, u_R is a nonnegative nontrivial solution of Eq. (1.13). The proof is completed. □

4 Existence and concentration for Eq. (1.1): proofs of Theorems 1.8 and 1.9

In this section, we mainly contemplate the existence and concentrating behavior of ground state solutions of Eq. (1.1). As explained in the previous section, we first pay our attention to Eq. (1.17). Actually, according to the classic theory in [11, 13, 28], we shall focus on Eq. (1.19). Before proceeding it, we introduce the following equation

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^N} \frac{BF^{R,\bar{\delta}}(u(y))}{|x-y|^\mu} dy\right) Bf^{R,\bar{\delta}}(u), \ x \in \mathbb{R}^N, \tag{4.1}$$

where $B > 0$ is an arbitrary constant. Let us denote $J_B^{R,\bar{\delta}} : E \to \mathbb{R}$ by the variational functional corresponding to Eq. (4.1)

$$\mathcal{J}_B^{R,\bar{\delta}}(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx \\ - \frac{B^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F^{R,\bar{\delta}}(u(x))F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu} dxdy.$$

and $m_B^{R,\bar{\delta}} \triangleq \inf_{v \in \mathcal{N}_B^{R,\bar{\delta}}} \mathcal{J}_B^{R,\bar{\delta}}(v)$ by its ground state energy with $\mathcal{N}_B^{R,\bar{\delta}} = \big\{u \in E\backslash\{0\} : (\mathcal{J}_B^{R,\bar{\delta}})'(u)[u] = 0\big\}$, respectively. In the sequel, we shall replace B with K_∞ and K_0 directly just for simplicity.

Repeating the arguments in the proof of Theorem 1.4, one could conclude that Eq. (4.1) admits a ground state solution if the assumption (h_5) is additionally satisfied. Indeed, we obtain the following result.

Lemma 4.1 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in* (1.12) *satisfies $(h_1)-(h_3)$ and (h_5), then for each fixed $R > 0$, Eq.* (4.1) *with $\bar{\delta} = \delta$ admits a ground state solution in E. Moreover, if in addition we suppose that (h_4), then for each fixed $R > 0$, there is a $\tilde{\xi}_0 = \tilde{\xi}_0(R) > 0$ dependent of R such that Eq.* (4.1) *with $\bar{\delta} = N/(N-1)$ possesses a ground state solution in E for all $\xi > \tilde{\xi}_0$.*

Proof Since the constant $B > 0$ trivially satisfies $(Q_1) - (Q_2)$ in Sect. 3, we could follow Theorem 1.4 to prove that Eq. (4.1) has a nontrivial solution $u_R \in E$ for all $R > 0$ under the assumptions above and so $u_R \in \mathcal{N}_B^{R,\bar{\delta}}$ implying that $\mathcal{J}_B^{R,\bar{\delta}}(u_R) \geq m_B^{R,\bar{\delta}}$. The remainder is to verify $\mathcal{J}_B^{R,\bar{\delta}}(u_R) \leq m_B^{R,\bar{\delta}}$.

On the one hand, we know that u_R is a weak limit of a (C) sequence $\{u_n\} \subset E$ of $\mathcal{J}_B^{R,\bar{\delta}}$ at the level

$$c_B^{R,\bar{\delta}} \triangleq \inf_{\gamma \in \Gamma_B^{R,\bar{\delta}}} \max_{t\in[0,1]} \mathcal{J}_B^{R,\bar{\delta}}(\gamma(t)) > 0$$

with $\Gamma_B^{R,\bar{\delta}} = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \mathcal{J}_B^{R,\bar{\delta}}(\gamma(1)) < 0\}$. In view of Lemma 3.4 and the Fatou's lemma,

$$\begin{aligned} c_B^{R,\bar{\delta}} &= \liminf_{n\to\infty} \mathcal{J}_B^{R,\bar{\delta}}(u_n) = \liminf_{n\to\infty} \left[\mathcal{J}_B^{R,\bar{\delta}}(u_n) - \frac{1}{2\theta}(\mathcal{J}_B^{R,\bar{\delta}})'(u_n)[u_n]\right] \\ &\geq \mathcal{J}_B^{R,\bar{\delta}}(u_R) - \frac{1}{2\theta}(\mathcal{J}_B^{R,\bar{\delta}})'(u_R)[u_R] = \mathcal{J}_B^{R,\bar{\delta}}(u_R). \end{aligned} \tag{4.2}$$

On the other hand, we show that $c_B^{R,\bar{\delta}} \leq m_B^{R,\bar{\delta}}$. In fact, thanks to (h_5), we can verify that

$$c_B^{R,\bar{\delta}} = m_B^{R,\bar{\delta}} = \inf_{v\in E\backslash\{0\}} \max_{t\geq 0} \mathcal{J}_B^{R,\bar{\delta}}(tv) > 0, \tag{4.3}$$

whose detailed proof is omitted and we should refer the interested reader to [56, 57]. As a consequence of (4.2) and (4.3), the desired result is obvious. The proof is completed. □

Lemma 4.2 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1)-(h_3)$ and (h_5), then for each fixed $R > 0$, we could derive the following properties on the Nehair manifold $\mathcal{N}_B^{R,\bar{\delta}}$ below.*

(i) Given a function $u \in E\backslash\{0\}$, there exists a unique constant $t_u > 0$ such that $t_u u \in \mathcal{N}_B^{R,\bar{\delta}}$ and $\mathcal{J}_B^{R,\bar{\delta}}(t_u u) = \max_{t\geq 0} \mathcal{J}_B^{R,\bar{\delta}}(tu)$;

(ii) For any $u \in \mathcal{N}_B^{R,\bar{\delta}}$, there exists $C^{R,\bar{\delta}} > 0$ independent of u such that $\|u\| \geq C^{R,\bar{\delta}} > 0$;

(iii) The functional $\mathcal{J}_B^{R,\bar{\delta}}$ is coercive on $\mathcal{N}_B^{R,\bar{\delta}}$, namely $\mathcal{J}_B^{R,\bar{\delta}}(u) \to \infty$ as $\|u\| \to \infty$ if $u \in \mathcal{N}_B^{R,\bar{\delta}}$;

(iv) If $\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{18(\gamma+\alpha R^{\tau-\frac{N}{N-1}})} \min\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\}$, $(\mathcal{J}_B^{R,\bar{\delta}})'(u_n)[u_n] \to 0$ and $\|u_n\| \to a_0 > 0$, then, passing to a subsequence if necessary, there exists a constant $t_n > 0$ such that

$$(\mathcal{J}_B^{R,\bar{\delta}})'(t_n u_n)[t_n u_n] = 0 \text{ and } t_n \to 1.$$

Proof (i). Proceeding as the proofs of Lemmas 3.1 and 3.2, one can easily find such a number $t_u > 0$ such that $t_u u \in \mathcal{N}_B^{R,\bar{\delta}}$ and $\mathcal{J}_B^{R,\bar{\delta}}(t_u u) = \max_{t\geq 0} \mathcal{J}_B^{R,\bar{\delta}}(tu)$. To get the uniqueness of such t_u, we argue it indirectly and suppose that there is $t_i > 0$ such that $t_i u \in \mathcal{N}_B^{R,\bar{\delta}}$ for $i \in \{1,2\}$. By [57, (3.8)],

$$\mathcal{J}_B^{R,\bar{\delta}}(u) - \mathcal{J}_B^{R,\bar{\delta}}(tu) - \frac{1-t^N}{N}(\mathcal{J}_B^{R,\bar{\delta}})'(u)[u] \geq \frac{qt^N - Nt^q + N - q}{Nq}|\nabla u|_q^q, \quad \forall t > 0 \text{ and } u \in E. \tag{4.4}$$

As a consequence of (4.4) and $t_i u \in \mathcal{N}_B^{R,\bar{\delta}}$ for $i \in \{1,2\}$, we obtain

$$\begin{cases} \mathcal{J}_B^{R,\bar{\delta}}(t_1 u) - \mathcal{J}_B^{R,\bar{\delta}}(t_2 u) \geq \frac{t_1^q}{Nq}\big[q\big(\frac{t_2}{t_1}\big)^N - N\big(\frac{t_2}{t_1}\big)^q + N - q\big]|\nabla u|_q^q, \\ \mathcal{J}_B^{R,\bar{\delta}}(t_2 u) - \mathcal{J}_B^{R,\bar{\delta}}(t_1 u) \geq \frac{t_2^q}{Nq}\big[q\big(\frac{t_1}{t_2}\big)^N - N\big(\frac{t_1}{t_2}\big)^q + N - q\big]|\nabla u|_q^q. \end{cases}$$

We would derive a contradiction by adding the above two formulas., which gives the desired result.

(ii). Supposing it by a contradiction, i.e. there exists a sequence $\{u_n\} \subset \mathcal{N}_B^{R,\bar{\delta}}$ such that $\|u_n\| \to 0$ as $n \to \infty$. Because $f^{R,\bar{\delta}}$ in (1.14) is of subcritical exponential growth if $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$, the proof would be very simple and we should omit it here. Let us consider the case $\bar{\delta} = \frac{N}{N-1}$. Choosing $\nu = \sigma$ in (2.3), then we apply (1.5) and Lemma 2.1 as well as the Hölder's inequality to derive

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{BF^{R,\frac{N}{N-1}}(u_n)Bf^{R,\frac{N}{N-1}}(u_n)u_n}{|x-y|^\mu}dxdy &\leq C\Big(\int_{\mathbb{R}^N} |f^{R,\bar{\delta}}(u_n)u_n|^{\frac{2N}{2N-\mu}}dx\Big)^{\frac{2N-\mu}{N}} \\ &\leq C|u_n|^{2\sigma}_{\frac{2N\sigma}{2N-\mu}} + C(R)|u_n|^{2\sigma}_{\frac{2N\sigma}{2N-\mu}}\Big(\int_{\mathbb{R}^N} \Phi_{\frac{4N}{2N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}})\|u_n\|^{\frac{N}{N-1}}, j_0}(u_n/\|u_n\|)dx\Big)^{\frac{2N-\mu}{2N}} \\ &\leq C(R)|u_n|^{2\sigma}_{\frac{2N\sigma}{2N-\mu}} \leq C(R)\|u_n\|^{2\sigma}, \end{aligned} \tag{4.5}$$

where the last second inequality holds since $\|u_n\| \to 0$ infers (1.8) is workable. Moreover, we can assume that $\|u_n\| \leq 1$ which implies that $|\nabla u_n|_q^N \leq |\nabla u_n|_q^q$, thus

$$\begin{aligned} \|u_n\|^N &= \big(|\nabla u_n|_N + |\nabla u_n|_q\big)^N \leq 2^{N-1}\big(|\nabla u_n|_N^N + |\nabla u_n|_q^N\big) \\ &\leq 2^{N-1}\big(|\nabla u_n|_N^N + |\nabla u_n|_q^q\big) \end{aligned} \tag{4.6}$$

Recalling $\{u_n\} \subset \mathcal{N}_B^{R,\frac{N}{N-1}}$, we could get a contradiction by (4.5) and (4.6) since $2\sigma > N$.

(iii). The proof is standard, see Lemma 3.5 for example.

(iv). By Point-(i), there exists a constant $t_n > 0$ such that $t_n u_n \in \mathcal{N}_B^{R,\bar{\delta}}$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u_n|^N dx + \frac{1}{t_n^{N-q}}\int_{\mathbb{R}^N} |\nabla u_n|^q dx \\ &= \frac{B^2}{t^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F^{R,\bar{\delta}}(t_n u_n(y))f^{R,\bar{\delta}}(t_n u_n(x))t_n u_n(x)}{|x-y|^\mu}dxdy \end{aligned}$$

We claim that $\{t_n\}$ is uniformly bounded. Otherwise, passing to a subsequence if necessary, we can assume $t_n \to +\infty$ and $t_n \geq 1$ for all $n \in \mathbb{N}$. Since $(\mathcal{J}_B^{R,\bar{\delta}})'(u_n)[u_n] = o_n(1)$, using (h_5), we obtain

$$\begin{aligned} 0 \geq B^2 \int_{\mathbb{R}^N} \Big[&\Big(|x|^{-\mu} * \frac{F^{R,\bar{\delta}}(u_n)}{|u_n|^{\frac{N}{2}}}\Big)\frac{f^{R,\bar{\delta}}(u_n)u_n}{|u_n|^{\frac{N}{2}}} \\ &- \Big(|x|^{-\mu} * \frac{F^{R,\bar{\delta}}(t_n u_n)}{|t_n u_n|^{\frac{N}{2}}}\Big)\frac{f^{R,\bar{\delta}}(t_n u_n)t_n u_n}{|t_n u_n|^{\frac{N}{2}}}\Big]|u_n|^N dx \\ = \Big(1 - \frac{1}{t_n^{N-q}}\Big)&|\nabla u_n|_q^q + o_n(1) = |\nabla u_n|_q^q + o_n(1) \geq o_n(1) \end{aligned}$$

yielding that $|\nabla u_n|_q^q \to 0$. Due to the Sobolev inequality, one gets $|u_n|_{q^*} \to 0$. Hence, all assumptions in Lemma 2.4 are satisfied, Combining (2.10)–(2.11) and $(\mathcal{J}_B^{R,\bar{\delta}})'(u_n)[u_n] = o_n(1)$, we could conclude that $|\nabla u_n|_N^N \to 0$, which is impossible since $\|u_n\| \to a_0 > 0$. By $t_n u_n \in \mathcal{N}_B^{R,\bar{\delta}}$, then $\|t_n u_n\| \geq C^{R,\bar{\delta}} > 0$ and so $\{t_n\}$ is uniformly bounded from below by a positive constant. Up to a subsequence if necessary, there is a $t_0 \in (0, +\infty)$ such that $t_n \to t_0$ as $n \to \infty$. If $0 < t_0 < 1$, without loss of generality, we can suppose that $0 < t_n < 1$ for all $n \in \mathbb{N}$. So, by adopting (h_5) and $(\mathcal{J}_B^{R,\bar{\delta}})'(u_n)[u_n] = o_n(1)$ again, there holds

$$\begin{aligned} 0 \leq B^2 \int_{\mathbb{R}^N} \Bigg[& \left(|x|^{-\mu} * \frac{F^{R,\bar{\delta}}(u_n)}{|u_n|^{\frac{N}{2}}}\right) \frac{f^{R,\bar{\delta}}(u_n)u_n}{|u_n|^{\frac{N}{2}}} \\ & - \left(|x|^{-\mu} * \frac{F^{R,\bar{\delta}}(t_n u_n)}{|t_n u_n|^{\frac{N}{2}}}\right) \frac{f^{R,\bar{\delta}}(t_n u_n)t_n u_n}{|t_n u_n|^{\frac{N}{2}}} \Bigg] |u_n|^N dx \\ = & \left(1 - \frac{1}{t_0^{N-q}}\right) |\nabla u_n|_q^q + o_n(1) \leq o_n(1), \end{aligned}$$

a contradiction as before. Alternatively, if $1 < t_0 < +\infty$, we can also suppose that $1 < t_n < +\infty$ for all $n \in \mathbb{N}$ which infers a contradiction. Consequently, it must hold that $t_0 \equiv 1$ showing the proof. □

Lemma 4.3 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1) - (h_3)$ and (h_5), let $\{u_n\}$ be a sequence such that $\{u_n\} \subset \mathcal{N}_B^{R,\bar{\delta}}$ and $\mathcal{J}_B^{R,\bar{\delta}}(u_n) \to m_B^{R,\bar{\delta}}$, then we can assume that $\{u_n\}$ is a (PS) sequence of $\mathcal{J}_B^{R,\bar{\delta}}$ at the level $m_B^{R,\bar{\delta}}$ ($(PS)_{m_B^{R,\bar{\delta}}}$ for short) for each fixed $R > 0$, namely $\mathcal{J}_B^{R,\bar{\delta}}(u_n) \to m_B^{R,\bar{\delta}}$ and $(\mathcal{J}_B^{R,\bar{\delta}}(u_n))' \to 0$ in E^{-1}.*

Proof Exploiting the Ekeland's variational principle in [62, Theorem 8.5], there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$(\mathcal{J}_B^{R,\bar{\delta}}(u_n))' = \lambda_n (\bar{\mathcal{J}}_B^{R,\bar{\delta}})'(u_n) + o_n(1), \tag{4.7}$$

where $\bar{\mathcal{J}}_B^{R,\bar{\delta}} : E \to \mathbb{R}$ is given by

$$\bar{\mathcal{J}}_B^{R,\bar{\delta}}(u) = |\nabla u|_N^N + |\nabla u|_q^q - B^2 \int_{\mathbb{R}^N} [|x|^{-\mu} * F^{R,\bar{\delta}}(u_n)] f^{R,\bar{\delta}}(u_n) u_n dx.$$

Since $\mathcal{J}_B^{R,\bar{\delta}}(u_n)[u_n] = 0$, we use (h_5) and Lemma 3.4 to derive

$$\begin{aligned} (\bar{\mathcal{J}}_B^{R,\bar{\delta}}(u_n))'[u_n] &= N|\nabla u_n|_N^N + q|\nabla u_n|_q^q - B^2 \int_{\mathbb{R}^N} \big[|x|^{-\mu} * (f^{R,\bar{\delta}}(u_n)u_n)\big] f^{R,\bar{\delta}}(u_n)u_n dx \\ &\quad - B^2 \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\bar{\delta}}(u_n)\big]\big[(f^{R,\bar{\delta}}(u_n))' u_n^2 + f^{R,\bar{\delta}}(u_n)u_n\big] dx \\ &\leq N|\nabla u_n|_N^N + q|\nabla u_n|_q^q - \frac{1}{2}(2\theta + N)B^2 \int_{\mathbb{R}^N} \big[|x|^{-\mu} * F^{R,\bar{\delta}}(u_n)\big] f^{R,\bar{\delta}}(u_n)u_n dx \\ &= \frac{N - 2\theta}{2} |\nabla u_n|_N^N + \frac{2q - N - 2\theta}{2} |\nabla u_n|_q^q \leq 0. \end{aligned} \tag{4.8}$$

Let us suppose that $\limsup_{n\to\infty} \bar{\mathcal{J}}_B^{R,\bar{\delta}}(u_n)[u_n] = l \leq 0$ and so $l < 0$. Otherwise, one knows that $\|u_n\| \to 0$ since $q < N < 2\theta$ in (4.8). Combining $l < 0$ and (4.7), we accomplish the proof of the lemma. □

Remark 4.4 We note that, by adopting $(K_1)-(K_2)$, the same conclusions in Lemmas 4.2 and 4.3 remain true for the Nehari manifold $\mathcal{N}_\varepsilon^{R,\bar{\delta}}$ associated with $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$ defined in (1.20). From now on, we shall employ Lemmas 4.2 and 4.3 for $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$ directly if there is no misunderstanding.

Now, let us concentrate ourself on the compactness of (PS) sequence of $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$. Proceeding as the proofs of Lemmas 3.1 and 3.2, there is a (PS) sequence $\{u_n\}$ of $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$ at the level $c_\varepsilon^{R,\bar{\delta}}$. Moreover, it is similar to (4.3) that

$$c_\varepsilon^{R,\bar{\delta}} = m_\varepsilon^{R,\bar{\delta}} = \inf_{v\in E\backslash\{0\}} \max_{t\geq 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tv) > 0, \tag{4.9}$$

In view of $(K_1)-(K_2)$, we can argue as Lemmas 3.5 and 3.6 to derive the following result which is crucial in our problems.

Lemma 4.5 *Let $1<q<N$ and $0<\mu<N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1)-(h_3)$ and (h_5) as well as $(K_1)-(K_2)$. Let $\{u_n\}$ be a $(PS)_{c_\varepsilon^{R,\bar{\delta}}}$ sequence for each fixed $R>0$, then $\{u_n\}$ is uniformly bounded in $n\in\mathbb{N}$. If in addition we assume that (h_4), there exists a $\bar{\xi}_0=\bar{\xi}_0(R)$ such that*

$$\sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{18\big(\gamma+\alpha R^{\tau-\frac{N}{N-1}}\big)} \min\left\{\frac{\sigma-1}{\sigma}, \frac{N-\mu}{N+\mu}\right\} \tag{4.10}$$

for all $\xi>\bar{\xi}_0$.

Next, we start verifying the $(PS)_{c_\varepsilon^{R,\bar{\delta}}}$ condition of $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$, that is, each sequence $(PS)_{c_\varepsilon^{R,\bar{\delta}}}$ sequence $\{u_n\}$ admits a strongly convergent subsequence.

Lemma 4.6 *Let $1<q<N$ and $0<\mu<N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1)-(h_3)$ and (h_5) as well as $(K_1)-(K_2)$. Let $\{u_n\}$ be a $(PS)_{c_\varepsilon^{R,\bar{\delta}}}$ sequence with $u_n\rightharpoonup u$ in E for each fixed $R>0$, then one of the alternative holds: either $u_n\to u$ in E along a subsequence, or $c_\varepsilon^{R,\bar{\delta}}-\mathcal{J}_\varepsilon^{R,\bar{\delta}}(u)\geq m_{K_\infty}^{R,\bar{\delta}}$. Moreover, we shall suppose additionally that (4.10) if $\bar{\delta}=\frac{N}{N-1}$.*

Proof Denoting $v_n \triangleq u_n-u$ and assume that $v_n\nrightarrow 0$ in E. Without loss of generality, we could suppose that $v_n\neq 0$ for all $n\in\mathbb{N}$. According to Lemma 4.2-(i), there exists a unique $t_n>0$ such that $t_nv_n\in\mathcal{N}_{K_\infty}^{R,\bar{\delta}}$. Because the case $\bar{\delta}=\delta\in(0,\frac{N}{N-1})$ is much simpler, we just show the case $\bar{\delta}=\frac{N}{N-1}$. Let us divide the proof into intermediate steps.

Step 1. $\{t_n\}$ is uniformly bounded in $\mathbb{R}$. Indeed, we derive it trivially by following the calculations in the proof of Lemma 4.2-(iv).

Step 2. $\{\Psi^{R,\frac{N}{N-1}}(v_n)\}$ given by (2.1) is uniformly bounded in $\mathbb{R}$ and $\Psi|_{|x|<\varrho}^{R,\frac{N}{N-1}}(v_n)=o_n(1)$. For each fixed $\varrho>0$, passing to a subsequence if necessary, we can suppose that $v_n\to 0$ in $L^p(B_\varrho(0))$ for all $p>p^*$. By (4.10), after very similar calculations in (4.5), we obatin

$$\Psi|_{|x|<\varrho}^{R,\frac{N}{N-1}}(v_n)=\int_{|x|<\rho}\frac{F^{R,\frac{N}{N-1}}(v_n(x))F^{R,\bar{\delta}}(v_n(y))}{|x-y|^\mu}dxdy\leq C(R)|u_n|^{2\sigma}_{\frac{2N\sigma}{2N-\mu}}\to 0 \text{ as } n\to\infty.$$

Step 3. Conclusion.

Firstly, adopting the definition of K_∞, for all $\epsilon>0$, there is a $\varrho=\varrho(\epsilon)>0$ such that

$$K(\varepsilon x)\leq K_\infty+\epsilon,\ \forall |x|\geq\varrho.$$

From this inequality, using the Step 2 to obtain

$$\int_{\mathbb{R}^N} \frac{K_\infty F^{R,\frac{N}{N-1}}(v_n(x)) K_\infty F^{R,\frac{N}{N-1}}(v_n(y)) - K(\varepsilon x) F^{R,\frac{N}{N-1}}(v_n(x)) K(\varepsilon y) F^{R,\frac{N}{N-1}}(v_n(y))}{|x-y|^\mu} dx$$
$$\geq 2\int_{\mathbb{R}^N} \frac{[K_\infty - K(\varepsilon x)] F^{R,\frac{N}{N-1}}(v_n(x)) K_0 F^{R,\frac{N}{N-1}}(v_n(y))}{|x-y|^\mu} dxdy$$
$$= 2\int_{|x|<\rho} \frac{[K_\infty - K(\varepsilon x)] F^{R,\frac{N}{N-1}}(v_n(x)) K_0 F^{R,\frac{N}{N-1}}(v_n(y))}{|x-y|^\mu} dxdy - 4K_0 \Psi^{R,\frac{N}{N-1}}(v_n)\epsilon$$
$$\geq o_n(1) - C(R)\epsilon.$$

Similarly, we can conclude that

$$\int_{\mathbb{R}^N} \frac{K_\infty f^{R,\frac{N}{N-1}}(v_n(x)) v_n(x) K_\infty F^{R,\frac{N}{N-1}}(v_n(y))}{|x-y|^\mu} dx$$
$$\geq \int_{\mathbb{R}^N} \frac{K(\varepsilon x) f^{R,\frac{N}{N-1}}(v_n(x)) v_n(x) K(\varepsilon y) F^{R,\frac{N}{N-1}}(v_n(y))}{|x-y|^\mu} dx + o_n(1) - C(R)\epsilon.$$

In view of (4.4), we apply the above two formulas as well as Step 1 and Lemma 2.3 to get

$$m_{K_\infty}^{R,\frac{N}{N-1}} \leq \mathcal{J}_{K_\infty}^{R,\frac{N}{N-1}}(t_n v_n) \leq \mathcal{J}_{K_\infty}^{R,\frac{N}{N-1}}(v_n) - \frac{1-t_n^N}{N}\big(\mathcal{J}_{K_\infty}^{R,\frac{N}{N-1}}\big)'(v_n)[v_n]$$
$$\leq \mathcal{J}_\varepsilon^{R,\frac{N}{N-1}}(v_n) + o_n(1) - C(R)\epsilon = c_\varepsilon^{R,\frac{N}{N-1}} - \mathcal{J}_\varepsilon^{R,\bar{\delta}}(u) + o_n(1) - C(R)\epsilon.$$

Letting $n \to \infty$ and then $\epsilon \to 0^+$, we would accomplish the proof of this lemma. □

Due to the appearances of critical exponential growth and convolution operator in the variational functional $J_\varepsilon^{R,\bar{\delta}}$, the proof of Lemma 4.6 seems much more complicated than the counterpart in [65], but the reader could find that our method is definitely comprehensible and delicate. It depends on the significant inequality (4.4) which enables us to avoid considering the relation between $\limsup_{n\to\infty} t_n$ and 1. So, it is a powerful tool when the monotone assumption (h_5) is satisfied

As we know, if $\{u_n\}$ is a $(PS)_{c_\varepsilon^{R,\delta}}$ sequence with $u_n \rightharpoonup u$ in E, it deduces that $(\mathcal{J}_\varepsilon^{R,\delta})'(u) = 0$ by (2.2). Suppose additionally that (4.10) if $\bar{\delta} = \frac{N}{N-1}$, then $(\mathcal{J}_\varepsilon^{R,\frac{N-1}{N}})'(u) = 0$ by (2.2) again. Therefore, under the assumption of Lemma 4.6 and Lemma 3.4, we have

$$\mathcal{J}_\varepsilon^{R,\bar{\delta}}(u) = \mathcal{J}_\varepsilon^{R,\bar{\delta}}(u) - \frac{1}{2\theta}(\mathcal{J}^{R,\bar{\delta}})'(u)[u] \geq 0. \tag{4.11}$$

With (4.11) in hands, to derive $u_n \to u$ in E by Lemma 4.6, it suffices to conclude that $c_\varepsilon^{R,\bar{\delta}} \leq m_{K_\infty}^{R,\bar{\delta}}$. To reach it, we prove the following lemma.

Lemma 4.7 *Let $1 < q < N$ and $0 < \mu < N$. Under the assumptions of Lemma 4.6, then we have that $\lim_{\varepsilon\to 0} c_\varepsilon^{R,\bar{\delta}} = m_{K_0}^{R,\bar{\delta}}$ for each fixed $R > 0$.*

Proof By (K_1) and (4.3) with $B = K_0$, we derive $c_\varepsilon^{R,\bar{\delta}} \geq c_{K_0}^{R,\bar{\delta}} = m_{K_0}^{R,\bar{\delta}}$ for all $\varepsilon > 0$ which indicates that $\liminf_{\varepsilon\to 0} c_\varepsilon^{R,\bar{\delta}} \geq m_{K_0}^{R,\bar{\delta}}$. So, we end the proof by showing $\limsup_{\varepsilon\to 0} c_\varepsilon^{R,\bar{\delta}} \leq m_{K_0}^{R,\bar{\delta}}$.

In view of Lemma 4.1, there exists a $v_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ such that $J_{K_0}^{R,\bar{\delta}}(v_0) = m_{K_0}^{R,\bar{\delta}}$. Let $\varphi(x) : \mathbb{R}^N \to [0,1]$ be a cut-off function satisfying $\varphi(x) \equiv 1$ when $|x| \leq 1$, $\varphi(x) \equiv 0$ when $|x| \geq 2$

and $|\varphi'(x)| \le 2$ on $\mathbb{R}^N$. For each $\varrho > 0$, we define $v_\varrho(x) \triangleq \varphi(x/\varrho)v_0(x)$. By the definition of φ and the Lebesgue theorem, one has $v_\varrho \to v_0$ in E as $\varrho \to \infty$. Using Lemma 4.2-(i), there exists a $t_{\varepsilon,\varrho} > 0$ such that $t_{\varepsilon,\varrho} v_\varrho \in \mathcal{N}_\varepsilon^{R,\bar{\delta}}$ and

$$\mathcal{J}_\varepsilon^{R,\bar{\delta}}(t_{\varepsilon,\varrho}v_\varrho) = \max_{t\ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tt_{\varepsilon,\varrho}v_\varrho) = \max_{t\ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tv_\varrho). \tag{4.12}$$

Employing a very similar calculations in the proof of Lemma 4.2-(iv), $\{t_{\varepsilon,\varrho}\}$ is uniformly bounded from above and below in $\varepsilon > 0$, up to a subsequence if necessary, we may suppose that $\lim_{\varepsilon\to 0^+} t_{\varepsilon,\varrho} = t_\varrho \in (0, +\infty)$. Recalling the facts $\operatorname{supp} v_\varrho \subset B_{2\varrho}(0)$ and $\lim_{\varepsilon\to 0^+} K(\varepsilon x) = K(0) = K_0$ for all $x \in B_{2\varrho}(0)$, we proceed as the proof of Step 2 in the proof of Lemma 4.6 to derive

$$\begin{aligned}
\mathcal{J}_\varepsilon^{R,\bar{\delta}}(t_{\varepsilon,\varrho}v_\varrho) &= \frac{t_{\varepsilon,\varrho}^N}{N}\int_{B_{2\varrho}(0)} |\nabla v_\varrho|^N dx + \frac{t_{\varepsilon,\varrho}^q}{q}\int_{B_{2\varrho}(0)} |\nabla v_\varrho|^q dx \\
&\quad - \frac{1}{2}\int_{B_{2\varrho}(0)}\int_{B_{2\varrho}(0)} \frac{K(\varepsilon y)F^{R,\bar{\delta}}(t_{\varepsilon,\varrho}v_\varrho)K(\varepsilon x)f^{R,\bar{\delta}}(t_{\varepsilon,\varrho}v_\varrho)t_{\varepsilon,\varrho}v_\varrho}{|x-y|^\mu} dxdy \\
&\to \frac{t_\varrho^N}{N}\int_{B_{2\varrho}(0)} |\nabla v_\varrho|^N dx + \frac{t_\varrho^q}{q}\int_{B_{2\varrho}(0)} |\nabla v_\varrho|^q dx \\
&\quad - \frac{1}{2}\int_{B_{2\varrho}(0)}\int_{B_{2\varrho}(0)} \frac{K(0)F^{R,\bar{\delta}}(t_\varrho v_\varrho)K(0)f^{R,\bar{\delta}}(t_\varrho v_\varrho)t_{\varepsilon,\varrho}v_\varrho}{|x-y|^\mu} dxdy \ (\text{as } \varepsilon \to 0^+) \\
&= \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_\varrho v_\varrho).
\end{aligned} \tag{4.13}$$

Similar to (4.13) and by using $t_{\varepsilon,\varrho}v_\varrho \in \mathcal{N}_\varepsilon^{R,\bar{\delta}}$, there holds

$$t_\varrho^N \int_{\mathbb{R}^N} |\nabla v_\varrho|^N dx + t_\varrho^q \int_{\mathbb{R}^N} |\nabla v_\varrho|^q dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K_0 F^{R,\bar{\delta}}(t_\varrho v_\varrho)K_0 f^{R,\bar{\delta}}(t_\varrho v_\varrho)t_{\varepsilon,\varrho}v_\varrho}{|x-y|^\mu} dxdy$$

which implies that $t_\varrho v_\varrho \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ and hence $\mathcal{J}_{K_0}^{R,\bar{\delta}}(t_\varrho v_\varrho) = \max_{t\ge 0}\mathcal{J}_{K_0}^{R,\bar{\delta}}(tt_\varrho v_\varrho) = \max_{t\ge 0}\mathcal{J}_{K_0}^{R,\bar{\delta}}(tv_\varrho)$. On the other hand, because the facts $v_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$, $t_\varrho v_\varrho \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ and $v_\varrho \to v_0$ in E as $\varrho \to \infty$ are clear, we have that $t_\varrho \to 1$ as $\varrho \to \infty$ by Lemma 4.2-(iv). As a consequence, it holds that

$$\|t_\varrho v_\varrho - v_0\| \le |t_\varrho - 1| \cdot \|v_\varrho\| + \|v_\varrho - v_0\| \to 0 \text{ as } \varrho \to \infty.$$

It follows from (4.9) and (4.12)–(4.13) that

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} c_\varepsilon^{R,\bar{\delta}} &= \limsup_{\varepsilon\to 0^+} \inf_{v\in E\setminus\{0\}} \max_{t\ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tv) \le \limsup_{\varepsilon\to 0^+} \max_{t\ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tv_\varrho) \\
&= \limsup_{\varepsilon\to 0^+} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(t_{\varepsilon,\varrho}v_\varrho) = \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_\varrho v_\varrho).
\end{aligned}$$

Letting $\varrho \to \infty$ and adopting the above two formulas, we have $\limsup_{\varepsilon\to 0^+} c_\varepsilon^{R,\bar{\delta}} \le m_{K_0}^{R,\bar{\delta}}$ finishing the proof of this lemma. □

Proposition 4.8 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1) - (h_3)$ and (h_5) as well as $(K_1) - (K_2)$. For each fixed $R > 0$, there is a sufficiently small $\varepsilon^1 > 0$ such that Eq. (1.19) with $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$ admits a nonnegative ground state solution for all $\varepsilon \in (0, \varepsilon^1)$. If in addition (h_4), there exists a $\bar{\xi}_0 > 0$ such that Eq. (1.19) with $\bar{\delta} = \frac{N}{N-1}$ has a nonnegative ground state solution for all $\varepsilon \in (0, \varepsilon^1)$ and $\xi > \bar{\xi}_0$.*

Proof For simplicity, we shall take into account the cases $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$ and $\bar{\delta} = \frac{N}{N-1}$ in a unified way. Let $\{u_n\}$ be a $(PS)_{c_\varepsilon^{R,\bar{\delta}}}$ sequence of $\mathcal{J}_\varepsilon^{R,\bar{\delta}}$. Due to Lemma 4.5, up to a subsequence if necessary, there exists a $u \in E$ such that $u_n \rightharpoonup u$. Combining (2.2) and (4.10), there holds $(\mathcal{J}_\varepsilon^{R,\bar{\delta}})'(u) = 0$. By means of (4.11), thanks to Lemma 4.6 and (4.9), we are done provided $c_\varepsilon^{R,\bar{\delta}} < m_{K_\infty}^{R,\bar{\delta}}$. It follows from Lemma 4.7 that for all $\epsilon > 0$, there is a $\varepsilon^1 > 0$ such that $c_\varepsilon^{R,\bar{\delta}} < m_{K_0}^{R,\bar{\delta}} + \epsilon$ for all $\varepsilon \in (0, \varepsilon^1)$. We claim that $m_{K_0}^{R,\bar{\delta}} < m_{K_\infty}^{R,\bar{\delta}}$. Indeed, there is a $u_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ such that $\mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) = m_{K_0}^{R,\bar{\delta}} > 0$ by Lemma 4.1. Via Lemma 4.2-(i), there exists a $t_0 > 0$ such that $t_0 u_0 \in \mathcal{N}_{K_\infty}^{R,\bar{\delta}}$ satisfying $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 u_0) = \max_{t \geq 0} \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t u_0)$. Therefore,

$$m_{K_\infty}^{R,\bar{\delta}} \geq \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 u_0) \geq \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(u_0) = \mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) + \frac{K_0^2 - K_\infty^2}{2} \Psi^{R,\bar{\delta}}(u_0) > \mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) = m_{K_0}^{R,\bar{\delta}}$$

showing the claim. The proof is completed by choosing $\epsilon = m_{K_\infty}^{R,\bar{\delta}} - m_{K_0}^{R,\bar{\delta}} > 0$. □

Afterwards, we shall investigate the concentrating behavior of nonnegative ground state solution obtained in Proposition 4.8. For short, from now on until the end of this article, all of the assumptions in Theorem 1.9 would not exhibit any longer.

As a consequence of Proposition 4.8, there exists $\varepsilon^1 > 0$ such that for any $\varepsilon \in (0, \varepsilon^1)$, Eq. (1.17) possesses a nonnegative ground state solution $u_\varepsilon(x) = v_\varepsilon(\varepsilon/x) \in E$ satisfying $\mathcal{J}_\varepsilon^{R,\bar{\delta}}(v_\varepsilon) = c_\varepsilon^{R,\bar{\delta}} > 0$, where v_ϵ is a nonnegative ground state solution of Eq. (1.19). Before contemplating the concentrating behavior of $\tilde{u}_\varepsilon$, we need the following key lemma.

Lemma 4.9 *If $\{u_n\} \subset \mathcal{N}_{K_0}^{R,\bar{\delta}}$ satisfies $\mathcal{J}_{K_0}^{R,\bar{\delta}}(u_n) \to m_{K_0}^{R,\bar{\delta}}$ and $u_n \rightharpoonup u_0 \neq 0$ in E as $n \to \infty$, then $u_n \to u_0$ in E along a subsequence. In particular, $(\mathcal{J}_{K_0}^{R,\bar{\delta}})'(u_0) = 0$ in E and $\mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) = m_{K_0}^{R,\bar{\delta}}$.*

Proof Employing Lemma 4.3, we could suppose that $\{u_n\}$ is a (PS) sequence of $\mathcal{J}_{K_0}^{R,\bar{\delta}}$ at the level $m_{K_0}^{R,\bar{\delta}}$. Thanks to (h_4), we are able to make sure that Lemma 4.5 holds true. Using (2.2) and (4.10), we obtain that $(\mathcal{J}_{K_0}^{R,\bar{\delta}})'(u_0) = 0$.

On the other hand, one has $u_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ since $u_0 \neq 0$ and so

$$\begin{aligned} m_{K_0}^{R,\bar{\delta}} &\leq \mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) - \frac{1}{2\theta}(\mathcal{J}_{K_0}^{R,\bar{\delta}})'(u_0)[u_0] \\ &\leq \liminf_{n\to\infty} \left[\mathcal{J}_{K_0}^{R,\bar{\delta}}(u_n) - \frac{1}{2\theta}(\mathcal{J}_{K_0}^{R,\bar{\delta}})'(u_n)[u_n] \right] = m_{K_0}^{R,\bar{\delta}} \end{aligned}$$

yielding that $u_n \to u_0$ in E and then $\mathcal{J}_{K_0}^{R,\bar{\delta}}(u_0) = m_{K_0}^{R,\bar{\delta}}$. The proof is completed. □

Recalling the definition of v_ε, that is, $\mathcal{J}_\varepsilon^{R,\bar{\delta}}(v_\varepsilon) = c_\varepsilon^{R,\bar{\delta}} > 0$ and $(\mathcal{J}_\varepsilon^{R,\bar{\delta}})'(v_\varepsilon) = 0$. With the help of (4.10), we can follow the proof of (3.8) to find a a family $\{\bar{y}_\varepsilon\} \subset \mathbb{R}^N$ and $\bar{r}, \bar{r}_0 > 0$ such that

$$\int_{B_{\bar{r}}(\bar{y}_\varepsilon)} |v_\varepsilon|^{p^*} dx \geq \bar{r}_0 > 0. \tag{4.14}$$

Lemma 4.10 *The family $\{\varepsilon \bar{y}_\varepsilon\}$ constructed in (4.14) is uniformly bounded in ε. Furthermore if we take x^* as the limit of the sequence of $\{\varepsilon_n \bar{y}_{\varepsilon_n}\}$, then one has $x^* \in \Sigma$, where $\{\varepsilon_n \bar{y}_{\varepsilon_n}\}$ is a subsequence of $\{\varepsilon \bar{y}_\varepsilon\}$.*

Proof Arguing it by contradiction, we suppose that $\varepsilon_n \to 0$ and $|\varepsilon_n \bar{y}_{\varepsilon_n}| \to +\infty$ as $n \to \infty$. Let us take $\bar{y}_n \triangleq \bar{y}_{\varepsilon_n}$ and $v_n \triangleq v_{\varepsilon_n}$ for simplicity and define $w_n(\cdot) \triangleq v_n(\cdot + \bar{y}_n) \geq 0$, then

$$-\Delta_N w_n - \Delta_q w_n = \left(\int_{\mathbb{R}^N} \frac{K(\varepsilon_n y + \varepsilon_n \bar{y}_n) F^{R,\bar{\delta}}(w_n(y))}{|x-y|^\mu} dy \right) K(\varepsilon_n x + \varepsilon_n \bar{y}_n) f^{R,\bar{\delta}}(w_n),\ x \in \mathbb{R}^N, \quad (4.15)$$

and from (4.14), one has

$$\int_{B_{\bar{r}}(0)} |w_n|^{p^*} dx \geq \bar{r}_0 > 0. \quad (4.16)$$

Obviously, $\|w_n\| = \|v_n\|$, thereby $\{w_n\}$ is uniformly bounded in $n \in \mathbb{N}$ and $w_n \rightharpoonup w_0$ in E along a subsequence. Moreover $w_0 \geq 0$ and we can see that $w_0 \neq 0$ from (4.16). According to Lemma 4.2-(i), there exists a $t_n > 0$ such that $t_n w_n \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ and so $\liminf_{n\to\infty} \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) \geq m_{K_0}^{R,\bar{\delta}}$. On the other hand, as a consequence of (K_1) and Lemma 4.7,

$$\begin{aligned}
\mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) &\leq \frac{t_n^N}{N} |\nabla w_n|_N^N + \frac{t_n^q}{q} |\nabla w_n|_q^q \\
&\quad - \int_{\mathbb{R}^{2N}} \frac{K(\varepsilon_n x + \varepsilon_n \bar{y}_n) F^{R,\bar{\delta}}(t_n w_n) K(\varepsilon_n x + \varepsilon_n \bar{y}_n) F^{R,\bar{\delta}}(t_n w_n)}{|x-y|^\mu} dxdy \\
&= \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(t_n v_n) \leq \max_{t \geq 0} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(t v_n) = \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(v_n) = c_{\varepsilon_n}^{R,\bar{\delta}} = m_{K_0}^{R,\bar{\delta}} + o_n(1),
\end{aligned}$$

which indicates that $\limsup_{n\to\infty} \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) = m_{K_0}^{R,\bar{\delta}}$ and so $\lim_{n\to\infty} \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) = m_{K_0}^{R,\bar{\delta}}$. Proceeding as the proof of Lemma 4.2-(iv), there is a $t_0 > 0$ such that $\lim_{n\to\infty} t_n = t_0 > 0$ along a subsequence. Adopting the uniqueness of the weak limit, we derive $t_n w_n \rightharpoonup t_0 w_0 \neq 0$ in E. In summary, we have concluded that $t_n w_n \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$, $\lim_{n\to\infty} \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) = m_{K_0}^{R,\bar{\delta}}$ and $t_n w_n \rightharpoonup t_0 w_0 \neq 0$. By means of Lemma 4.9, one gets $t_n w_n \to t_0 w_0 \neq 0$ in E which implies that $t_0 w_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$. Using Fatou's lemma and (h_5), one has

$$\begin{aligned}
m_{K_0}^{R,\bar{\delta}} &\leq \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_0 w_0) < \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 w_0) = \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 w_0) - \frac{1}{N} (\mathcal{J}_{K_0}^{R,\bar{\delta}})'(t_0 w_0)[t_0 w_0] \\
&= \frac{N-q}{qN} t_0^N \int_{\mathbb{R}^N} |\nabla w_0|^q dx \\
&\quad + \frac{1}{N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K_0 f^{R,\bar{\delta}}(t_0 w_0(x)) t_0 w_0(x) K_0 F^{R,\bar{\delta}}(t_0 w_0(y))}{|x-y|^\mu} dxdy \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K_\infty F^{R,\bar{\delta}}(t_0 w_0(x)) K_\infty F^{R,\bar{\delta}}(t_0 w_0(y))}{|x-y|^\mu} dxdy \\
&\leq \liminf_{n\to\infty} \left\{ \frac{N-q}{qN} t_n^N \int_{\mathbb{R}^N} |\nabla w_n|^q dx \right. \\
&\quad + \frac{1}{N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K_0 f^{R,\bar{\delta}}(t_n w_n(x)) t_n w_n(x) K_0 F^{R,\bar{\delta}}(t_n w_n(y))}{|x-y|^\mu} dxdy \\
&\quad \left. - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{K(\varepsilon_n x + \varepsilon_n \bar{y}_n) F^{R,\bar{\delta}}(t_n w_n(x)) K(\varepsilon_n y + \varepsilon_n \bar{y}_n) F^{R,\bar{\delta}}(t_n w_n(y))}{|x-y|^\mu} dxdy \right\} \\
&= \liminf_{n\to\infty} \left[\mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(t_n v_n) - \frac{1}{N} (\mathcal{J}_{K_0}^{R,\bar{\delta}})'(t_n w_n)[t_n w_n] \right] = \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(t_n v_n)
\end{aligned}$$

$$\leq \liminf_{n\to\infty} \max_{t\geq 0} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(tv_n) = \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(v_n) = \liminf_{n\to\infty} c_{\varepsilon_n}^{R,\bar{\delta}} = m_{K_0}^{R,\bar{\delta}},$$

which is impossible, where we have used the formula whose detailed proof is left in the Appendix

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(\varepsilon_n x+\varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(t_n w_n(x))K(\varepsilon_n y+\varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(t_n w_n(y))}{|x-y|^\mu} dxdy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K_\infty F^{R,\bar{\delta}}(t_0 w_0(x))K_\infty F^{R,\bar{\delta}}(t_0 w_0(y))}{|x-y|^\mu} dxdy. \end{aligned} \tag{4.17}$$

So, $\{\varepsilon \bar{y}_\varepsilon\}$ is bounded in $\mathbb{R}^N$. Passing to a subsequence if necessary, we could suppose that $\varepsilon_n \bar{y}_{\varepsilon_n} \to x^*$ in $\mathbb{R}^N$ as $n \to \infty$. Finally, we verify that $x^* \in \Sigma$. To the end, let us denote the variational functional $J_{x^*}^{R,\bar{\delta}}$ by $J_B^{R,\bar{\delta}}$ with $B = K(x^*) > 0$. If $x^* \notin \Sigma$, then $K(x^*) < K_0$ by the definition of Σ in (K_2). In this situation, arguing as before,

$$\begin{aligned} m_{K_0}^{R,\bar{\delta}} &\leq \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_0 w_0) < \mathcal{J}_{x^*}^{R,\bar{\delta}}(t_0 w_0) \leq \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 w_0) - \frac{1}{N}(\mathcal{J}_{K_0}^{R,\bar{\delta}})'(t_0 w_0)[t_0 w_0] \\ &\leq \liminf_{n\to\infty} \max_{t\geq 0} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(tv_n) = \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}^{R,\bar{\delta}}(v_n) = \liminf_{n\to\infty} c_{\varepsilon_n}^{R,\bar{\delta}} = m_{K_0}^{R,\bar{\delta}}, \end{aligned}$$

a contradiction. The proof is completed. □

Lemma 4.11 *Up to a subsequence if necessary, $w_n \to w_0$ in E as $n \to \infty$, where $\{w_n\}$ and w_0 are given by Lemma 4.10. Furthermore, there exists a $\varepsilon^2 > 0$ such that $\lim_{|x|\to\infty} w_\varepsilon(x) = 0$ uniformly in $\varepsilon \in (0, \varepsilon^2)$.*

Proof In view of the proof of Lemma 4.10, that is, $t_n w_n \to t_0 w_0 \neq 0$ in E and $t_n \to t_0$ with $t_0 > 0$, then we can conclude that

$$t_0\|w_n - w_0\| = \|t_0 w_n - t_n w_n + t_n w_n - t_0 w_0\| \leq |t_n - t_0|\cdot\|w_n\| + \|t_n w_n - t_0 w_0\| \to 0 \tag{4.18}$$

which implies that $w_n \to w_0$ in E as $n \to \infty$. By (4.15), one has

$$-\Delta_N w_0 - \Delta_q w_0 = \left(\int_{\mathbb{R}^N} \frac{K(x^*)F^{R,\bar{\delta}}(w_n(y))}{|x-y|^\mu} dy\right) K(x^*) f^{R,\bar{\delta}}(w_0),\ x \in \mathbb{R}^N,$$

shaoing that $w_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$. Recalling $t_0 w_0 \in \mathcal{N}_{K_0}^{R,\bar{\delta}}$ by Lemma 4.10 and using Lemma 4.2-(i), we have $t_0 \equiv 1$. Again Lemma 4.10 exhibits us that $\lim_{n\to\infty} \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_n w_n) = m_{K_0}^{R,\bar{\delta}}$, then $\mathcal{J}_{K_0}^{R,\bar{\delta}}(w_0) = \mathcal{J}_{K_0}^{R,\bar{\delta}}(t_0 w_0) = m_{K_0}^{R,\bar{\delta}}$ which reveals that w_0 is a ground state solution of Eq. (4.18). We claim that $|w_n|_\infty$ is uniformly bounded in $n \in \mathbb{N}$. Indedd, due to (4.15), $\{w_n\}$ is a sequence of solutions of

$$-\Delta_N u - \Delta_q u = \left(\int_{\mathbb{R}^N} \frac{K(\varepsilon_n y+\varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(u(y))}{|x-y|^\mu} dy\right) K(\varepsilon_n x+\varepsilon_n \bar{y}_n) f^{R,\bar{\delta}}(u),\ x \in \mathbb{R}^N, \tag{4.19}$$

Since $K_\infty \leq K(\varepsilon_n x+\varepsilon_n \bar{y}_n) \leq K_0$ for any $x \in \mathbb{R}^N$ and $f^{R,\bar{\delta}}(t) \geq 0$ for all $t \in \mathbb{R}$, proceeding as the very similar calculations in Lemmas 3.10 and 3.14 to look for a constant $C(R) > 0$ independent of $n \in \mathbb{N}$ such that $|w_n|_{\chi^{m+1}\sigma} \leq C(R)|w_n|_\sigma$ for every $m \in \mathbb{N}^+$. Thus, we derive the claim by $m \to +\infty$. Moreover, thanks to the results found in the works of DiBenedetto [25] and Lieberman [38], there holds that $w_\varepsilon \in C^{1,\bar{\gamma}}(\mathbb{R}^N)$ for some $\bar{\gamma} \in (0, 1)$. Finally, we shall focus on verifying the decay property at infinity of w_ε when $\varepsilon > 0$ is sufficiently small. For simplicity, we continue to use the notation w_n.

Let $x_0 \in \mathbb{R}^N$, $\varrho_0 > 1$, $0 < t < s < 1 < \varrho_0$ and $\eta \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \le \eta \le 1, \ \operatorname{supp} \eta \subset B_s(x_0), \ \eta \equiv 1 \text{ on } B_t(x_0) \text{ and } |\nabla \eta| \le \frac{2}{s-t}.$$

For all $\zeta \ge 1$, we define $A_{n,\zeta,\varrho} = \{x \in B_\varrho(x_0) : w_n(x) > \zeta\}$ and

$$Q_n = \int_{A_{n,\zeta,s}} (|\nabla u_n|^q + |\nabla u_n|^N)\eta^N dx.$$

Therefore, taking $\eta_n = \eta^N(w_n - \zeta)^+$ as a test function of Eq. (4.19), there holds

$$\begin{aligned} N &\int_{A_{n,\zeta,s}} (|\nabla w_n|^{N-2} + |\nabla w_n|^{q-2})\eta^{N-1}(w_n - \zeta)^+ \nabla w_n \nabla \eta dx \\ &+ \int_{A_{n,\zeta,s}} (|\nabla w_n|^N + |\nabla w_n|^q)\eta^N dx \\ &\le K_0^2 \int_{A_{n,\zeta,s}} [|x|^{-\mu} * F^{R,\bar{\delta}}(w_n)] f^{R,\bar{\delta}}(w_n)\eta^N(w_n - \zeta)^+ dx \\ &\le C(R) \int_{A_{n,\zeta,s}} f^{R,\bar{\delta}}(w_n)\eta^N(w_n - \zeta)^+ dx, \end{aligned}$$

where we have adopted (5.3) below and argue as the Claim in Lemma 2.2. Because $|w_n|_\infty$ is uniformly bounded, letting $\nu = \sigma$ in (2.3), we obtain

$$\int_{A_{n,\zeta,s}} f^{R,\bar{\delta}}(w_n)\eta^N(w_n - \zeta)^+ dx \le C(R)\bigg(\int_{A_{n,\zeta,s}} |(w_n - \zeta)^+|^\sigma dx + \zeta^\sigma |A_{n,\zeta,s}|\bigg)$$

which implies that

$$\begin{aligned} Q_n \le N &\int_{A_{n,\zeta,s}} (|\nabla w_n|^{N-2} + |\nabla w_n|^{q-2})\eta^{N-1}(w_n - \zeta)^+ |\nabla w_n||\nabla \eta| dx \\ &+ C(R)\bigg(\int_{A_{n,\zeta,s}} |(w_n - \zeta)^+|^\sigma dx + \zeta^\sigma |A_{n,\zeta,s}|\bigg). \end{aligned}$$

Exploiting the definition of η and Young's inequality, we can infer that

$$\int_{A_{n,\zeta,t}} |\nabla w_n|^N dx \le Q_n \le C(R)\bigg(\int_{A_{n,\zeta,s}} \left|\frac{w_n - \zeta}{s-t}\right|^\sigma dx, + \zeta^\sigma |A_{n,\zeta,s}|\bigg).$$

where $C(R)$ does not depend on η and $\zeta \ge 1$. So far, we could argue as [6, Lemma 3.5] step by step to accomplish the proof of this lemma. □

We are in a position to present the proof of Theorem 1.9.

Proof of Theorem 1.9 Setting $\varepsilon^* = \min\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$, then it follows from Proposition 4.8 that Eq. (1.19) admits at least a nonnegative ground state solution v_ε for every $\varepsilon \in (0, \varepsilon^*)$, and so $u_\varepsilon = v_\varepsilon(\cdot/\varepsilon)$ is nonnegative ground state solution to Eq. (1.17) for each $\varepsilon \in (0, \varepsilon^*)$. Next, we shall complete the proof one by one:

(a) Due to the proof of Lemma 4.11 and $w_\varepsilon(x) = v_\epsilon(x + \bar{y}_\varepsilon)$, there exists a $\varrho > 0$ such that $w_\varepsilon(x)$ has a global maximum point $k_\varepsilon \in B_\varrho(0)$ and hence we can deduce that the global maximum point of v_ε is given by $z_\varepsilon \triangleq \bar{y}_\varepsilon + k_\varepsilon$. Notice that $u_\varepsilon(x) = v_\varepsilon(x/\varepsilon)$, one could easily derive that $u_\varepsilon(x)$ possesses a global maximum point $\gamma_\varepsilon = \varepsilon z_\varepsilon$. Since $k_\varepsilon \in B_\varrho(0)$ has been chosen and Lemma 4.10 tells us that $\varepsilon \bar{y}_\varepsilon \to x^*$ as $\varepsilon \to 0^+$, it holds that $\gamma_\varepsilon = \varepsilon z_\varepsilon \to x^*$. Recalling $K(x) \in C(\mathbb{R}^N)$ and $x^* \in \Sigma$, we have

$$\lim_{\varepsilon \to 0^+} K(\gamma_\varepsilon) = K(x^*) = K_0.$$

(b) It follows from the above facts that

$$\widetilde{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + \gamma_\varepsilon) = u_\epsilon(\varepsilon x + \varepsilon z_\varepsilon) = v_\varepsilon(x + z_\varepsilon) = v_\varepsilon(x + \bar{y}_\varepsilon + k_\varepsilon) = w_\varepsilon(x + k_\varepsilon).$$

Since $|k_\varepsilon| \le \varrho$, we may suppose that $k_\varepsilon \to k_0$ along a subsequence as $\varepsilon \to 0^+$. In view of the proof of Lemma 4.11, there holds $\widetilde{u}_\varepsilon \to w_0(x + k_0) \triangleq \widetilde{u}$ in E. Thanks to the translation-invariance of $\mathcal{J}_{K_0}^{R,\bar{\delta}}$ and $(\mathcal{J}_{K_0}^{R,\bar{\delta}})'$, we could conclude that $\widetilde{u}$ is a ground state solution of Eq. (1.18). The proof is completed. □

With Theorem 1.9 in hands, now we can give the proof of Theorem 1.8.

Proof of Theorem 1.8 Inspired by the proof of Theorem 1.3, it suffices show that the solution v_ε of Eq. (1.19) obtained in Theorem 1.9 is uniformly bounded in $R > 0$. To reach it, by choosing some suitable $\bar{\alpha}^*$ and $\bar{\tau}_*$, we could verify that $f^{R,\bar{\delta}}$ admits the similar growth condition to (3.13) and (3.17), respectively. Then, the remaining parts would be trivial and we omit the details. □

As one can find that Theorems 1.8 and 1.9 only exhibit the existence of ground state solutions, it is natural to wonder that whether Eqs. (1.1) and (1.17) always have ground state solutions. Actually, it would not be true if we slightly modify the conditions $(K_1) - (K_2)$. More precisely, let us replace them with the assumption below

- (K) $K \in C^0(\mathbb{R}^N)$ satisfies $0 < K(x) \le K_\infty$ for every $x \in \mathbb{R}^N$ and the strict inequality holds true on a positive measure subset.

Firstly, we derive the following surprising result.

Lemma 4.12 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1) - (h_3)$ and (h_5), then for each fixed $R > 0$, $m_{K_\infty}^{R,\bar{\delta}} = c_\varepsilon^{R,\bar{\delta}}$ for every $\varepsilon > 0$. Moreover, if $\bar{\delta} = N/(N-1)$, we should suppose in addition (h_4) with $\xi > 0$ sufficiently large.*

Proof For all $\epsilon > 0$, there exists $u_\epsilon \in \mathcal{N}_\varepsilon^{R,\bar{\delta}}$ such that $c_\varepsilon^{R,\bar{\delta}} \le \mathcal{J}_\varepsilon^{R,\bar{\delta}}(u_\epsilon) < c_\varepsilon^{R,\bar{\delta}} + \epsilon$. In particular, one deduces $\mathcal{J}_\varepsilon^{R,\bar{\delta}}(u_\epsilon) = \max_{t \ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tu_\epsilon)$ by Lemma 4.2-(i). Using Lemma 4.2-(i) again, there is $t_\epsilon > 0$ such that $t_\epsilon u_\epsilon \in \mathcal{N}_{K_\infty}^{R,\bar{\delta}}$ and $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_\epsilon u_\epsilon) = \max_{t \ge 0} \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t t_\epsilon u_\epsilon) = \max_{t \ge 0} \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(tu_\epsilon)$. By (K),

$$m_{K_\infty}^{R,\bar{\delta}} \le \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_\epsilon u_\epsilon) = \max_{t \ge 0} \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(tu_\epsilon) \le \max_{t \ge 0} \mathcal{J}_\varepsilon^{R,\bar{\delta}}(tu_\epsilon) = \mathcal{J}_\varepsilon^{R,\bar{\delta}}(u_\epsilon) < c_\varepsilon^{R,\bar{\delta}} + \epsilon$$

indicating that $m_{K_\infty}^{R,\bar{\delta}} \le c_\varepsilon^{R,\bar{\delta}}$ by tending $n \to \infty$. Next, we show that $c_\varepsilon^{R,\bar{\delta}} \le m_{K_\infty}^{R,\bar{\delta}}$ for all $\varepsilon > 0$.

By Lemma 4.1, the problem (4.1) with $B = K_\infty$ has a ground state solution $u_\infty \in \mathcal{N}_{K_\infty}^{R,\bar{\delta}}$ verifying $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(u_\infty) = m_{K_\infty}^{R,\bar{\delta}}$. Let $\{\bar{x}_n\} \subset \mathbb{R}^N$ satisfy $|\bar{x}_n| \to \infty$ as $n \to \infty$ and set $u_n(x) \triangleq u_\infty(x - \bar{x}_n)$. Thanks to the translation-invariance of $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}$ and $(\mathcal{J}_{K_\infty}^{R,\bar{\delta}})'$, we conclude that $u_n \in \mathcal{N}_{K_\infty}^{R,\bar{\delta}}$ and $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(u_n) = m_{K_\infty}^{R,\bar{\delta}}$. By means of Lemma 4.2-(i), there exists $t_n > 0$ such that $t_n u_n \in \mathcal{N}_\varepsilon^{R,\bar{\delta}}$. In view of Lemma 4.5, we can show that $\{u_n\}$ satisfies (4.10). So, up to a subsequence if necessary, it holds that $\lim_{n\to\infty} t_n = 1$ by Lemma 4.2-(iv). Arguing as the calculations in Subsect. 5.3 in the Appendix, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K_\infty F^{R,\bar{\delta}}(t_n v_\infty(x)) K_\infty F^{R,\bar{\delta}}(t_n v_\infty(y))}{|x-y|^\mu} dxdy$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(\varepsilon x + \varepsilon \bar{x}_n) F^{R,\bar{\delta}}(t_n v_\infty(x)) (\varepsilon y + \varepsilon \bar{x}_n) F^{R,\bar{\delta}}(t_n v_\infty(y))}{|x-y|^\mu} dxdy.$$

It follows from all the discussions above that

$$\begin{aligned}\mathcal{J}_\varepsilon^{R,\bar{\delta}}(t_n v_n) &= \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_n v_\infty) + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K_\infty F^{R,\bar{\delta}}(t_n v_\infty(x)) K_\infty F^{R,\bar{\delta}}(t_n v_\infty(y))}{|x-y|^\mu} dxdy \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(\varepsilon x + \varepsilon \bar{x}_n) F^{R,\bar{\delta}}(t_n v_\infty(x)) (\varepsilon y + \varepsilon \bar{x}_n) F^{R,\bar{\delta}}(t_n v_\infty(y))}{|x-y|^\mu} dxdy \\ &\quad \to J_\infty(v_\infty) = m_\infty,\end{aligned}$$

showing that $c_\varepsilon^{R,\bar{\delta}} \le m_{K_\infty}^{R,\bar{\delta}}$. Therefore, we can derive $c_\varepsilon^{R,\bar{\delta}} = m_{K_\infty}^{R,\bar{\delta}}$. The proof is completed. □

Proposition 4.13 *Let $1 < q < N$ and $0 < \mu < N$. Suppose that the nonlinearity f defined in (1.12) satisfies $(h_1) - (h_3)$ and (h_5), then for each fixed $R > 0$, Eq. (1.19) with $\bar{\delta} = \delta$ does not admit ground state solution for all $\varepsilon > 0$ in E. Moreover, if we suppose additionally that (h_4), then for each fixed $R > 0$, there is a $\xi_0 = \xi_0(R) > 0$ dependent of R such that Eq. (1.19) with $\bar{\delta} = N/(N-1)$ does not possess ground state solution in E for all $\xi > \xi_0$ and $\varepsilon > 0$.*

Proof Arguing it by contradiction, we could suppose that there exist $\varepsilon_0 > 0$ and $u_0 \in E$ such that $\mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(v_0) = c_{\varepsilon_0}^{R,\bar{\delta}}$ and $(\mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}})'(v_0) = 0$. Using Lemma 4.2-(i), one knows $\mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(v_0) = \max_{t\ge 0} \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(tv_0)$. Taking Lemma 4.2-(i) into account again, there is a constant $t_0 > 0$ such that $t_0 v_0 \in \mathcal{N}_{K_\infty}^{R,\bar{\delta}}$. Recalling Lemma 4.12, that is, $m_{K_\infty}^{R,\bar{\delta}} = c_{\varepsilon_0}^{R,\bar{\delta}}$, one has

$$\begin{aligned}m_{K_\infty}^{R,\bar{\delta}} &\le \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 v_0) \le \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(t_0 v_0) \le \max_{t\ge 0} \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(tv_0) \\ &= \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(v_0) = c_{\varepsilon_0}^{R,\bar{\delta}} = m_{K_\infty}^{R,\bar{\delta}}\end{aligned}$$

showing that $\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 v_0) \le \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(t_0 v_0)$. While, we shall exploit (K) to get

$$\begin{aligned}\mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 v_0) &= \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(t_0 v_0) - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K_\infty F^{R,\bar{\delta}}(t_0 v_0(x)) K_\infty F^{R,\bar{\delta}}(t_0 v_0(y))}{|x-y|^\mu} dxdy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{K(\varepsilon_0 x) F^{R,\bar{\delta}}(t_0 v_0(x)) K(\varepsilon_0 y) F^{R,\bar{\delta}}(t_0 w_0(y))}{|x-y|^\mu} dxdy \\ &< \mathcal{J}_{\varepsilon_0}^{R,\bar{\delta}}(t_0 v_0) = \mathcal{J}_{K_\infty}^{R,\bar{\delta}}(t_0 v_0),\end{aligned}$$

a contradiction. The proof is completed. □

Remark 4.14 As one can find in Proposition 4.13, we have showed that Eq. (1.17) does not have ground state solution. However, one cannot demonstrate that there is no ground state solution for Eq. (1.1). As far as we are concerned, it is interesting to consider it further.

5 Appendix

In this section, we mainly accomplish the detailed proofs of some facts that are left in the previous sections.

5.1 The validity of (2.9)

Define $v_n \triangleq u_n - u$ and $w_n = |v_n| + |u|$, then we have that

$$\begin{aligned}\int_{\mathbb{R}^N} (|v_n|^{\sigma-1}|u| + |u|^{\sigma})^{\frac{2N}{2N-\mu}} dx &\le C_1\bigg(\int_{\mathbb{R}^N} |v_n|^{\frac{2N(\sigma-1)}{2N-\mu}} |u|^{\frac{2N}{2N-\mu}} dx + \int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg)\\ &\le C_1\bigg[\bigg(\int_{\mathbb{R}^N} |v_n|^{\frac{2N\sigma}{2N-\mu}} dx\bigg)^{\frac{\sigma-1}{\sigma}} \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg)^{\frac{1}{\sigma}} + \int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg]\\ &\le C_1\bigg[\Big(2\sup_{n\in\mathbb{N}} \|u_n\|\Big)^{\frac{2N(\sigma-1)}{2N-\mu}} \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg)^{\frac{1}{\sigma}} + \int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg]\end{aligned}$$

and $\sup_{n\in\mathbb{N}} \|w_n\|^{\frac{N}{N-1}} \le 3^{\frac{N}{N-1}} \sup_{n\in\mathbb{N}} \|u_n\|^{\frac{N}{N-1}}$ which is applied in (1.8) to get

$$\begin{aligned}&\int_{\mathbb{R}^N} (|v_n|^{\sigma-1}|u| + |u|^{\sigma})^{\frac{2N}{2N-\mu}} \Phi_{\frac{2N}{2N-\mu}(\gamma+\alpha R^{\tau-\frac{N}{N-1}}), j_0}(|w_n|) dx\\ &\le \bigg(\int_{\mathbb{R}^N} (|v_n|^{\sigma-1}|u| + |u|^{\sigma})^{\frac{2N}{N-\mu}} dx\bigg)^{\frac{N-\mu}{2N-\mu}} \bigg(\int_{\mathbb{R}^N} \Phi_{2(\gamma+\alpha R^{\tau-\frac{N}{N-1}})\|w_n\|^{\frac{N}{N-1}}, j_0}(|w_n|/\|w_n\|) dx\bigg)^{\frac{N}{2N-\mu}}\\ &\le C_2\bigg[\bigg(\int_{\mathbb{R}^N} |v_n|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{(N-\mu)(\sigma-1)}{(2N-\mu)\sigma}} \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{N-\mu}{(2N-\mu)\sigma}} + \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{N-\mu}{2N-\mu}}\bigg]\\ &\le C_2\bigg[\Big(\sup_{n\in\mathbb{N}} \|v_n\|\Big)^{\frac{2N(\sigma-1)}{2N-\mu}} \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{N-\mu}{(2N-\mu)\sigma}} + \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{N-\mu}{2N-\mu}}\bigg].\end{aligned}$$

Letting $\nu = \sigma$ in (2.3), one has

$$\begin{aligned}\Big|F^{R,\frac{N}{N-1}}(u_n) - F^{R,\frac{N}{N-1}}(v_n)\Big| &\le \int_0^1 |f^{R,\frac{N}{N-1}}(v_n + tu)u| dt\\ &\le C(|v_n|^{\sigma-1}|u| + |u|^{\sigma})\\ &\quad + C(R)(|v_n|^{\nu-1}|u| + |u|^{\nu})\Phi_{\gamma+\alpha R^{\tau-\frac{N}{N-1}}, j_0}(|w_n|)\end{aligned}$$

which together with the above two formulas implies that

$$\begin{aligned}&\int_{\mathbb{R}^N} |F^{R,\frac{N}{N-1}}(u_n) - F^{R,\frac{N}{N-1}}(v_n)|^{\frac{2N}{2N-\mu}} dx\\ &\le C_3\bigg[\bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx\bigg)^{\frac{1}{\sigma}} + \int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{2N-\mu}} dx + \bigg(\int_{\mathbb{R}^N} |u|^{\frac{2N\sigma}{N-\mu}} dx\bigg)^{\frac{N-\mu}{(2N-\mu)\sigma}}\end{aligned}$$

$$+\left(\int_{\mathbb{R}^N}|u|^{\frac{2N\sigma}{N-\mu}}dx\right)^{\frac{N-\mu}{2N-\mu}}\Bigg]$$

From this inequality, we could exploit the generalized Vitali's Dominated Convergence theorem to finish the proof of (2.9).

5.2 The validity of (3.16)

We begin verifying the validity of (3.16) which is a crucial gradient in the L^∞-estimate of nontrivial solution. It infers from the Hölder's inequality that

$$\left(\int_{\mathbb{R}^N}u_R^\sigma u_{R,L}^{q(\vartheta-1)}dx\right)^{\frac{N}{q}}\le\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}|u_R|^{\sigma}dx\right)^{\frac{(\sigma-q)N}{\sigma q}}$$

and we apply the constant $\Upsilon>0$ in (3.13) together with (1.8) to deduce that

$$\left(\int_{\mathbb{R}^N}|u_R|^q\Phi_{\frac{\alpha_N\Upsilon}{K},j_0}(u_R)u_{R,L}^{q(\vartheta-1)}dx\right)^{\frac{N}{q}}\le\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}\Phi_{\frac{\alpha_N\Upsilon\sigma}{K(\sigma-q)},j_0}(u_R)dx\right)^{\frac{(\sigma-q)N}{\sigma q}}$$
$$=\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}\Phi_{\frac{\alpha_N\Upsilon\sigma}{K(\sigma-q)}\|u_n\|^{\frac{N}{N-1}},j_0}(u_R/\|u_R\|)dx\right)^{\frac{(\sigma-q)N}{\sigma q}}.$$

Letting $\nu=q$ in (3.13), by the above two formulas as well as Lemma 2.1 and (3.10), we obtain

$$\left(\int_{\mathbb{R}^N}f^{R,\delta}(u_R)u_Ru_{R,L}^{q(\vartheta-1)}dx\right)^{\frac{N}{q}}\le C\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\tag{5.1}$$

for some $C>0$ independent of $R>0$. Similarly, we can derive

$$\int_{\mathbb{R}^N}u_R^\sigma u_{R,L}^{N(\vartheta-1)}dx\le\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}|u_R|^{\sigma}dx\right)^{\frac{\sigma-N}{\sigma}}$$

and

$$\int_{\mathbb{R}^N}|u_R|^\nu\Phi_{\frac{\alpha_N\Upsilon}{K},j_0}(u_R)u_{R,L}^{N(\vartheta-1)}dxdx\le\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}\Phi_{\frac{\alpha_N\Upsilon\sigma}{K(\sigma-N)},j_0}(u_R)dx\right)^{\frac{\sigma-N}{\sigma}}$$
$$=\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\left(\int_{\mathbb{R}^N}\Phi_{\frac{\alpha_N\Upsilon\sigma}{K(\sigma-N)}\|u_n\|^{\frac{N}{N-1}},j_0}(u_R/\|u_n\|)dx\right)^{\frac{\sigma-N}{\sigma}}$$

which together with $\nu=N$ in (3.13) indicate that

$$\int_{\mathbb{R}^N}f^{R,\delta}(u_R)u_Ru_{R,L}^{N(\vartheta-1)}dx\le C\left(\int_{\mathbb{R}^N}|u_R|^{\sigma\vartheta}dx\right)^{\frac{N}{\sigma}}\tag{5.2}$$

for some $C>0$ independent of $R>0$. With the help of (5.1) and (5.2), we immediately get (3.16) by tending $L\to+\infty$ in (3.15).

Remark 5.1 The estimate which is vear similar to (3.16) in the proof of Lemma 3.14 would occur, thanks to $\xi_0>0$ which is independent of $R>e$ in Lemma 3.11, we could make sure that $\|u_n\|$ is sufficiently small. In other words, by means of some trivial adjustments in Remark 3.12, all of the calculations in Subsection 3.2 are true and we can still get the desired estimate.

5.3 The validity of (4.17)

Let us denote $\bar{w}_n = t_n w_n$ and $\bar{w}_0 = t_0 w_0$ for short. In view of the proof of Lemma 4.10, we deduce that $\{\bar{w}_n\}$ is a minimizing sequence of $m^{R,\bar{\delta}}_{K_0}$ and $\bar{w}_n \to \bar{w}_0 \neq 0$ in E. Because the case $\bar{\delta} = \delta \in (0, \frac{N}{N-1})$ is simple, we only consider the case $\bar{\delta} = \frac{N}{N-1}$. Exploiting Lemma 4.5, there holds

$$\sup_{n\in\mathbb{N}} \|\bar{w}_n\|^{\frac{N}{N-1}} \leq \frac{\alpha_N}{18(\gamma + \alpha R^{\tau - \frac{N}{N-1}})} \min\left\{\frac{\sigma - 1}{\sigma}, \frac{N-\mu}{N+\mu}\right\}. \tag{5.3}$$

So, we can apply it jointly with (1.8) and (2.3) with $\nu = \sigma$ in (1.5) to get

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F^{R,\frac{N}{N-1}}(\bar{w}_n(y))}{|x-y|^\mu}dy\right)F^{R,\frac{N}{N-1}}(\bar{w}_n(x))dx \leq \left(\int_{\mathbb{R}^N}|\bar{w}_n|^{\frac{2N\sigma}{2N-\mu}}dx\right)^{\frac{2N-\mu}{N}}.$$

Recalling $\bar{w}_n \to \bar{w}_0$ in $L^{\frac{2N\sigma}{2N-\mu}}(\mathbb{R}^N)$, for all $\epsilon > 0$, there is a sufficiently large $\varrho > 0$ such that

$$\int_{|x|\geq\varrho}\left(\int_{\mathbb{R}^N}\frac{F^{R,\frac{N}{N-1}}(\bar{w}_n(y))}{|x-y|^\mu}dy\right)F^{R,\frac{N}{N-1}}(\bar{w}_n(x))dx \leq C(R)\epsilon$$

Given $\epsilon > 0$, for all $y \in B_\varrho(0)$, there is $n_0 \in \mathbb{N}^+$ such that $K(\varepsilon_n y + \varepsilon_n \bar{y}_n) - K_\infty \leq \epsilon$ for $n \geq n_0$, thus

$$\int_{|x|<\varrho}\left(\int_{\mathbb{R}^N}\frac{[K(\varepsilon_n y + \varepsilon_n \bar{y}_n) - K_\infty]F^{R,\frac{N}{N-1}}(\bar{w}_n(y))}{|x-y|^\mu}dy\right)K_0F^{R,\frac{N}{N-1}}(\bar{w}_n(x))dx \leq C(R)\epsilon.$$

As a consequence, there holds

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{K(\varepsilon_n x + \varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(\bar{w}_n(x))K(\varepsilon_n y + \varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(\bar{w}_n(y)) - K_\infty F^{R,\bar{\delta}}(\bar{w}_n(x))K_\infty F^{R,\bar{\delta}}(\bar{w}_n(y))}{|x-y|^\mu}dxdy\\ &\leq 2\int_{\mathbb{R}^{2N}}\frac{[K(\varepsilon_n x + \varepsilon_n \bar{y}_n) - K_\infty]F^{R,\frac{N}{N-1}}(\bar{w}_n(x))K_0F^{R,\frac{N}{N-1}}(\bar{w}_n(y))}{|x-y|^\mu}dxdy \leq C(R)\epsilon.\end{aligned}$$

On the other hand, $\tilde{w}_n = \bar{w}_n - \bar{w}_0 \to 0$ in $L^p(\mathbb{R}^N)$ for all $p > q^*$, using a very similar arguments in Step 2 of Lemma 4.6, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K_\infty F^{R,\bar{\delta}}(\tilde{w}_n(x))K_\infty F^{R,\bar{\delta}}(\tilde{w}_n(y))}{|x-y|^\mu}dxdy = 0.$$

From them, letting $n \to \infty$ and then $\epsilon \to 0^+$, it must have that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K(\varepsilon_n x + \varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(\bar{w}_n(x))K(\varepsilon_n y + \varepsilon_n \bar{y}_n)F^{R,\bar{\delta}}(\bar{w}_n(y))}{|x-y|^\mu}dxdy\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K_\infty F^{R,\bar{\delta}}(\bar{w}_n(x))K_\infty F^{R,\bar{\delta}}(\bar{w}_n(y))}{|x-y|^\mu}dxdy\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{K_\infty F^{R,\bar{\delta}}(\bar{w}_0(x))K_\infty F^{R,\bar{\delta}}(\bar{w}_0(y))}{|x-y|^\mu}dxdy\end{aligned}$$

showing the desired result.

Acknowledgements The first author would like to express his sincere thanks to Professor Claudianor Oliveira Alves for the insightful discusions. L.J. Shen is partially supported by NNSF of China (12201565). The research of V.D. Rădulescu was supported by the grant "Nonlinear Differential Systems in Applied Sciences" of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8/22.

Data availability No datasets were generated or analysed during the current study.

References

1. Ackermann, N.: On a periodic Schrödinger equation with nonlocal superlinear part. Math. Z. **248**, 423–443 (2004)
2. Adimurthi, K., Sandeep, A.: singular Moser–Trudinger embedding and its applications. NoDEA Nonlinear Differ. Equ. Appl. **13**(5–6), 585–603 (2007)
3. Adimurthi, S.L.: Yadava, Multiplicity results for semilinear elliptic equations in bounded domain of $\mathbb{R}^2$ involving critical exponent. Ann. Sci. Norm. Super. Pisa **17**, 481–504 (1990)
4. Adimurthi, Y.Y.: An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its Applications. Int. Math. Res. Not. IMRN **13**, 2394–2426 (2010)
5. Aghajani, A., Kinnunen, J.: Supersolutions to nonautonomous Choquard equations in general domains. Adv. Nonlinear Anal. **12**(1), 20230107, 21 (2023)
6. Alves, C.O., da Silva, A.R.: Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz–Sobolev space. J. Math. Phys. **57**, 111502 (2016)
7. Alves, C.O., N'obrega, A.B., Yang, M.: Multi-bump solutions for Choquard equation with deepening potential well. Calc. Var. Partial Differ. Equ. **55**(3), 28 (2016)
8. Alves, C.O., Shen, L.: On existence of solutions for some classes of elliptic problems with supercritical exponential growth. Math. Z. **306**(2), 29, 21 (2024)
9. Alves, C.O., Shen, L.: On a class of strongly indefinite Schrödinger equations with Stein-Weiss convolution parts and supercritical exponential growth in $\mathbb{R}^2$. J. Fixed Point Theory Appl. **26**, 52 (2024). https://doi.org/10.1007/s11784-024-01143-6 https://doi.org/10.1007/s11784-024-01143-6
10. Ambrosetti, A., Badiale, M., Cingolani, S.: Semiclassical states of nonlinear Schrödinger equations. Arch. Ration. Mech. Anal. **140**, 285–300 (1997)
11. Ambrosetti, A., Malchiodi, A.: Concentration phenomena for nonlinear Schrödinger equations: recent results and new perspectives. In: Berestycki, H., Bertsch, M., Browder, F.E., Nirenberg, L., Peletier, L.A., Véron, L. (eds.), Perspectives in Nonlinear Partial Differential Equations, in: Contemp. Math., vol. 446, Amer. Math. Soc., Providence, RI, pp. 19–30 (2007)
12. Ambrosio, V., Rădulescu, V.D.: Fractional double-phase patterns: concentration and multiplicity of solutions. J. Math. Pures Appl. (9) **142**, 101–145 (2020)
13. Benci, V., Fortunato, D.: Variational Methods in Nonlinear Field Equations. Solitary Waves, Hylomorphic Solitons and Vortices, Springer Monographs in Mathematics, Springer, Cham, xviii+250 pp (2014)
14. Bonheure, D., d'Avenia, P., Pomponio, A.: On the electrostatic Born–Infeld equation with extended charges. Commun. Math. Phys. **346**, 877–906 (2016)
15. Bonheure, D., Iacopetti, A.: On the regularity of the minimizers of the electrostatic Born–Infeld energy. Arch. Ration. Mech. Anal. **232**, 697–725 (2019)
16. Cao, D.: Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$. Commun. Partial Differ. Equ. **17**, 407–435 (1992)
17. Carvalho, J.L., Figueiredo, G.M., Furtado, M.F., Medeiros, E.: On a zero-mass (N, q)-Laplacian equation in $\mathbb{R}^N$ with exponential critical growth. Nonlinear Anal. **213**, 112488, 14 (2021)
18. Cherfils, L., Il'yasov, V.: On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian. Commun. Pure Appl. Anal. **1**, 9–22 (2005)

19. de Figueiredo, D.G., Miyagaki, O.H., Ruf, B.: Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range. Calc. Var. Partial Differ. Equ. **3**, 139–153 (1995)
20. de Souza, M., do Ó, J.M.: A sharp Trudinger-Moser type inequality in $\mathbb{R}^2$. Trans. Am. Math. Soc. **366**, 4513–4549 (2014)
21. del Pino, M., Felmer, P.: Multipeak bound states of nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **15**, 127–149 (1998)
22. del Pino, M., Felmer, P.: Local Mountain Pass for semilinear elliptic problems in unbounded domains. Calc. Var. Partial Differ. Equ. **4**, 121–137 (1996)
23. Ding, Y.H., Lin, F.H.: Solutions of perturbed Schrödinger equations with critical nonlinearity. Calc. Var. Partial Differ. Equ. **30**, 231–249 (2007)
24. Ding, Y.H., Liu, X.Y.: Semiclassical solutions of Schrödinger equations with magnetic fields and critical nonlinearities. Manuscripta Math. **140**, 51–82 (2013)
25. DiBenedetto, E.: $C^{1,\gamma}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1985)
26. do Ó, J.M.: N-Laplacian equations in $\mathbb{R}^N$ with critical growth. Abstr. Appl. Anal. **2**, 301–315 (1997)
27. do Ó, J.M., de Souza, M., de Medeiros, E., Severo, U.: An improvement for the Trudinger–Moser inequality and applications. J. Differ. Equ. **256**(4), 1317–1349 (2014)
28. Floer, A., Weinstein, A.: Nonspreading wave pachets for the packets for the cubic Schrödinger with a bounded potential. J. Funct. Anal. **69**, 397–408 (1986)
29. Gui, C., Wei, J.: Multiple interior peak solutions for some singularly perturbed Neumann problems. J. Differ. Equ. **158**, 1–27 (1999)
30. He, C., Li, G.: The regularity of weak solutions to nonlinear scalar field elliptic equations containing p&q-Laplacians. Ann. Acad. Sci. Fenn. Math. **33**(2), 337–371 (2008)
31. Jeanjean, L., Tanaka, K.: Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities. Calc. Var. Partial Differ. Equ. **21**, 287–318 (2004)
32. Lam, N., Lu, G.: Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$. J. Funct. Anal. **262**(3), 1132–1165 (2012)
33. Lenzmann, E.: Uniqueness of ground states for pseudorelativistic Hartree equations. Anal. PDE **2**, 1–27 (2009)
34. Li, Y., Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^N$. Indiana Univ. Math. J. **57**, 451–480 (2008)
35. Lieb, E.H.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Stud. Appl. Math. **57**, 93–105 (1977)
36. Lieb, E.H., Loss, M.: Analysis. In: Graduate Studies in Mathematics, AMS, Providence, Rhode island (2001)
37. Lieb, E.H., Simon, B.: The Hartree–Fock theory for Coulomb systems. Commun. Math. Phys. **53**, 185–194 (1977)
38. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)
39. Lions, P.L.: The Choquard equation and related questions. Nonlinear Anal. **4**, 1063–1073 (1980)
40. Lions, P.L.: The concentration-compactness principle in the calculus of variations. The limit case. I, Rev. Mat. Iberoam **1**, 145–201 (1985)
41. Liu, W., Dai, G.: Existence and multiplicity results for double phase problem. J. Differ. Equ. **265**(9), 4311–4334 (2018)
42. Ma, L., Zhao, L.: Classification of positive solitary solutions of the nonlinear Choquard equation. Arch. Ration. Mech. Anal. **195**, 455–467 (2010)
43. Mawhin, J., Willem, M.: Critical point theory and Hamiltonian system. Springer-Verlag, New York/Berlin (1989)
44. Moroz, I.M., Penrose, R., Tod, P.: Spherically–symmetric solutions of the Schrödinger-Newton equations. Class. Quantum Gravity **15**, 2733–2742 (1998)
45. Moroz, V., Van Schaftingen, J.: Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. J. Funct. Anal. **265**, 153–184 (2013)
46. Moroz, V., Van Schaftingen, J.: A guide to the Choquard equation. J. Fixed Point Theory Appl. **19**, 773–813 (2017)
47. Moser, J.: A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. Commun. Pure Appl. Math. **13**, 457–468 (1960)
48. Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J., **20**, 1077–1092 (1970/1971)
49. Mugnai, D., Papageorgiou, N.S.: Wang's multiplicity result for superlinear (p, q)-equations without the Ambrosetti–Rabinowitz condition. Trans. Am. Math. Soc. **366**, 4919–4937 (2014)

50. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Anisotropic (p, q)-equations with gradient dependent reaction. Nonlinearity **34**(8), 5319–5343 (2021)
51. Pekar, S.I.: Untersuchung über die Elektronentheorie der Kristalle. Akademie Verlag, Berlin (1954)
52. Pohožaev, S.I.: The Sobolev embedding in the case $pl = n$. In: Proc. Tech. Sci. Conf. on Adv. Sci., Research 1964–1965, pp. 158–170. Mathematics Section, Moscow (1965)
53. Pomponio, A., Watanabe, T.: Some quasilinear elliptic equations involving multiple p-Laplacians. Indiana Univ. Math. J. **67**(6), 2199–2224 (2018)
54. Rabinowitz, P.: On a class of nonlinear Schrödinger equations. Z. Angew. Math. Phys. **43**, 270–291 (1992)
55. Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$. J. Funct. Anal. **219**, 340–367 (2005)
56. Shen, L., Rădulescu, V.D., Yang, M.: Planar Schrödinger–Choquard equations with potentials vanishing at infinity: the critical case. J. Differ. Equ. **329**, 206–254 (2022)
57. Shen, L., Rădulescu, V.D.: Zero-mass (N, q)-Laplacian equation with Stein-Weiss convolution part in $\mathbb{R}^N$: supercritical exponential case. submitted
58. Shen, L.: Zero-mass gauged Schrödinger equations with supercritical exponential growth. J. Differ. Equ. **393**, 204–237 (2024)
59. Sun, X., Song, Y., Liang, S., Zhang, B.: Critical Kirchhoff equations involving the p-sub-Laplacians operators on the Heisenberg group. Bull. Math. Sci. **13**(2), 2250006, 26 (2023). https://doi.org/10.1142/S1664360722500060
60. Trudinger, N.S.: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–484 (1967)
61. Wang, X.: On concentration of positive bound states of nonlinear Schrödinger equations. Commun. Math. Phys. **53**, 229–244 (1993)
62. Willem, M.: Minimax Theorems. Birkhäuser Verlag (1996)
63. Yang, Y.: Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space. J. Funct. Anal. **262**(4), 1679–1704 (2012)
64. Yang, Y., Zhu, X.: Blow-up analysis concerning singular Trudinger–Moser inequalities in dimension two. J. Funct. Anal. **272**, 3347–3374 (2017)
65. Zhang, J., Zhang, W., Rădulescu, V.D.: Double phase problems with competing potentials: concentration and multiplication of ground states. Math. Z. **301**(4), 4037–4078 (2022)
66. Zhang, W., Yuan, S., Wen, L.: Existence and concentration of ground-states for fractional Choquard equation with indefinite potential. Adv. Nonlinear Anal. **11**(1), 1552–1578 (2022)
67. Zhikov, V.V.: On Lavrentiev's phenomenon. Russ. J. Math. Phys. **3**, 249–269 (1995)
68. Zhikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer, Berlin (1994)