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ORIGINAL ARTICLE

Planar Choquard equations with critical exponential reaction and Neumann boundary condition

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Abstract

We study the existence of positive weak solutions for the following problem:

$$-\Delta u + \alpha(x)u = \left(\int_{\Omega} \frac{F(y,u)}{|x-y|^{\mu_1}}\, dy\right) f(x,u) \ \text{ in } \Omega,$$

$$\frac{\partial u}{\partial \eta} + \beta u = \left(\int_{\partial\Omega} \frac{G(y,u)}{|x-y|^{\mu_2}}\, d\nu\right) g(x,u) \ \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\alpha(x)$ is a bounded measurable function on Ω, β is nonnegative real number, η is the unit outer normal to $\partial\Omega$, $\mu_1 \in (0,2)$, and $\mu_2 \in (0,1)$. The functions f and g have critical exponential growth, while F and G are their primitives. The proofs combine the constrained minimization method with energy methods and topological tools.

KEYWORDS

Cherrier inequality, Hardy–Littlewood–Sobolev inequality, Moser–Trudinger inequality, Neumann problem, Nehari manifold

1 | INTRODUCTION

The purpose of this paper is to study a class of Neumann problems involving Choquard-type critical growth exponential nonlinearities. We consider the following problem:

$$(P)\begin{cases} -\Delta u + \alpha(x)u &= \left(\int\limits_{\Omega} \frac{F(y,u)}{|x-y|^{\mu_1}}\, dy\right) f(x,u), \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} + \beta u &= \left(\int\limits_{\partial\Omega} \frac{G(y,u)}{|x-y|^{\mu_2}}\, dv\right) g(x,u) \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^2$ having a smooth boundary, $\alpha(x)$ is a bounded measurable function on Ω, $\beta \geq 0$ is a real number, $\mu_1 \in (0,2)$, $\mu_2 \in (0,1)$ and η denotes the unit outer normal to $\partial\Omega$. Here, the functions f and g have exponential critical growth in Ω and $\partial\Omega$ respectively and $F(x,t) = \int\limits_0^t f(x,s)ds$ and $G(x,t) = \int\limits_0^t g(x,s)ds$ are the primitive of f and g, respectively. The main interest in studying this kind of problem is the presence of critical exponents both in the equation and in the nonlinear boundary condition.

A central role in the analysis developed in this paper is played by the fractional integral

$$(T_\mu \phi)(x) = \int_{\mathbb{R}^N} \frac{\phi(y)}{|x-y|^\mu}\, dy, \ \ 0 < \mu < N.$$

Weighted L^p estimates for T_μ is a fundamental problem of harmonic analysis, with a wide range of applications. Starting from the classical one-dimensional case studied by Hardy and Littlewood, an exhaustive analysis has been made on the admissible classes of weights and ranges of indices. An important contribution is due to Lieb [24], who applied the Riesz rearrangement inequalities to prove that the best constant for the classical Hardy–Littlewood–Sobolev inequality can be achieved by some extremals. Lieb also classified the solutions of the integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{u(y)^{\frac{N+\mu}{N-\mu}}}{|x-y|^{N-\tau}} dy, \ \ x \in \mathbb{R}^N \tag{1.1}$$

as an open problem. In fact, Equation (1.1) arises as an Euler–Lagrange equation for a functional under a constraint in the context of the Hardy–Littlewood–Sobolev inequality and is closely related to the following Lane–Emden fractional equation:

$$(-\Delta)^{\frac{\tau}{2}} u = u^{\frac{N+\mu}{N-\mu}}, \ \ x \in \mathbb{R}^N. \tag{1.2}$$

This area of study has been a focal point for decades, resulting in a substantial body of literature addressing problems with both sub-critical and critical nonlinearities.

Various aspects of these problems have been explored, including existence, uniqueness, regularity of solutions, qualitative behavior, and applications in physics. Among these studies, considerable attention has been given to the Neumann problems with a homogeneous boundary condition, characterized by $\frac{\partial u}{\partial \eta} = 0$. For instance, Lin et al. [27] established the existence of least energy solutions for the sub-critical case, while Adimurthi and Yadava [3] investigated the existence results for the critical case. For more details, see [2,22,33,39-41], and references therein.

We now redirect our attention to the consideration of nonhomogeneous boundary conditions. In higher dimensions, for a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, Adimurthi and Yadava in [5] established the existence of solutions for the Neumann problem with critical nonlinearity on the boundary. Additionally, Pierotti and Terracini [33] studied the problem with critical nonlinearity both in the equation and in the nonlinear boundary condition. With the help of the mean curvature of $\partial\Omega$ along with some geometrical condition, they obtained the existence of solutions on a bounded domain $\Omega \subset \mathbb{R}^N$, where $(N \geq 4)$. Adimurthi and Yadava [4] studied these class of problems in $N = 2$ with exponential critical growth in both the equation and in the nonlinear boundary condition, and established the existence of positive solutions by concentrating the Moser's function (2.1) at a boundary point. Also see, [15,28,35] for more such results and their subsequent generalizations.

We elaborate more on problems for $N = 2$ case, where the nonlinearity has an exponential critical growth. It is well known by now that in this case the critical growth is governed by the Trudinger–Moser inequality, which states that if Ω is a bounded domain in $\mathbb{R}^2$ then for all $\alpha > 0$ and $u \in H_0^1(\Omega)$, we have $e^{\alpha u^2} \in L^1(\Omega)$. Moreover, there exists a positive constant C such that the following inequality holds:

$$\sup_{\|\nabla u\|_2=1} \int_\Omega e^{\alpha|u|^2}\, dx \leq C|\Omega|, \quad \text{if } \alpha \leq 4\pi$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. This inequality is optimal, in the sense that for any growth $e^{\alpha|u|^2}$ with $\alpha > 4\pi$ the corresponding supremum is infinite. For further details, refer to [29,37]. Numerous papers have been devoted to the study of elliptic problems involving exponential-type nonlinearities, which draw motivation from the Trudinger–Moser inequality. For more comprehensive information on these topics, refer to [1,4,16-18,34] and for work on whole space $\mathbb{R}^2$, refer to [8, 14], alongside other relevant references.

On the other hand, the study of existence results for Choquard-type equations has attracted a lot of attention in recent years due to its applications in the study of models in quantum mechanics, Bose–Einstein condensation, and nonlinear optics. This type of problem has numerous applications in physical models. In particular, one of the applications of Choquard equations was given by Pekar [32]. He proposed the following problem:

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\, dy \right) u \text{ in } \mathbb{R}^3 \tag{1.3}$$

for the modeling of quantum polaron. Lieb [23] investigated it in the context of an approximation to the Hartree–Fock theory of one-component plasma. The author proved the existence and uniqueness of Equation (1.3), up to translations, of the ground state. For a thorough overview of the subject, readers can refer to [6,11,19-21,30,36,38,42,43], and references therein. For the study of Choquard nonlinearity with the Trudinger-Moser inequality in the entire domain $\mathbb{R}^N$, one can see [7, 9, 10].

An important question arises in two-dimensional setting ($N=2$) for Choquard nonlinearity with Neumann boundary conditions. As the critical growth of the nonlinearity in this case is exponential arising from the Trudinger-Moser inequality, which adds to its significance. However, there is no research addressing the Neumann problem with critical Choquard nonlinearity.

In this paper, we aim to contribute to the existing literature by investigating the problems with critical Choquard-type nonlinearities, both in the equation and the nonlinear boundary condition. Specifically, following the approach in [4], we establish the existence of positive solutions to this class of problems. To the best of our knowledge, this is the first article to address the existence of a solution for the nonhomogeneous Neumann problem featuring a doubly critical exponential convolution nonlinearity both within the equation and at the boundary. The main contribution of this study is the identification of the first critical level below which the Palais–Smale sequences are compact. Due to the presence of convolution-type nonlinearities in the domain and on the boundary, we found that these levels depend on μ_i. Then, we use a sequence of Moser functions concentrating on the boundary to perform the blow-up analysis while proving the existence of min–max sequences below the threshold level. Finally, with the help of the Nehari constraint minimization we establish the existence result.

We now state our main result:

Theorem 1.1. *Let (f, g) be a function of critical growth on $(\Omega, \partial\Omega)$ and the operator $-\Delta + \alpha(x)$ is positive. Further assume that*

(i) $\overline{\lim_{t\to\infty}} \inf_{x\in\overline{\Omega}} h(x,t)t = \overline{\lim_{t\to\infty}} \inf_{y\in\partial\Omega} k(y,t)t = \infty$
(ii) $f'(x, 0) = g'(y, 0) = 0$, *for all* $(x, y) \in \Omega \times \partial\Omega$.

Then, problem (P) has a positive solution.

Solutions of (P) correspond to critical points in $H^1(\Omega)$ of the energy functional $J : H^1(\Omega) \to \mathbb{R}$ defined as

$$\begin{aligned} J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \alpha(x)u^2 \, dx + \frac{\beta}{2} \int_{\partial\Omega} u^2 \, d\sigma - \frac{1}{2} \iint_{\Omega\times\Omega} \frac{F(x,u)F(y,u)}{|x-y|^{\mu_1}} \, dxdy \\ - \frac{1}{2} \iint_{\partial\Omega\times\partial\Omega} \frac{G(x,u)G(y,u)}{|x-y|^{\mu_2}} \, d\sigma d\nu. \end{aligned} \tag{1.4}$$

Observe that the functional $J \in C^1$, that is, for any $\phi \in H^1(\Omega)$

$$\begin{aligned} \langle J'(u), \phi \rangle = \int_{\Omega} \nabla u \nabla \phi \, dx + \int_{\Omega} \alpha(x) u\phi \, dx + \beta \int_{\partial\Omega} u\phi \, d\sigma - \iint_{\Omega\times\Omega} \frac{F(y,u)f(x,u)\phi(x)}{|x-y|^{\mu_1}} \, dxdy \\ - \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)\phi(x)}{|x-y|^{\mu_2}} \, d\sigma d\nu. \end{aligned} \tag{1.5}$$

A standard methodology to obtain solutions using variational techniques consists of looking for minimizers of the functional. We employ the technique of the artificial constraint introduced by Nehari [31]. The main idea of the proof is as follows. Define

$$\partial B(\Omega) = \{u \in H^1(\Omega)\backslash\{0\} : \langle J'(u), u \rangle = 0\}, \tag{1.6}$$

$$\frac{a(\Omega)^2}{2} = \inf\{J(u) : u \in \partial B(\Omega)\}, \tag{1.7}$$

then the minimizer of Equation (1.7) is a solution to the problem (P). Now by considering the optimal constant from Cherrier's work [12,13] and by considering the convolution exponential nonlinearity both in the equation and the boundary, one expects that J satisfies the Palais–Smale condition within within certain range, which is $\left(-\infty, \min\left\{\frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta}\right\}\right)$. Therefore, the main objective is to prove that $\frac{a(\Omega)^2}{2} < \min\left\{\frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta}\right\}$, to obtain a minimizer of Equation (1.7).

Remark 1.1. The conclusions of the paper can be generalized to establish the existence of positive weak solution for the following N-Laplacian problem with the co-normal boundary condition

$$\begin{aligned} -\Delta_N u + \alpha(x)|u|^{N-2}u = \left(\int_{\Omega} \frac{F(y,u)}{|x-y|^{\mu_1}} \, dy \right) f(x,u), \quad \text{in } \Omega, \\ |\nabla u|^{N-2}\frac{\partial u}{\partial \eta} + \beta|u|^{N-2}u = \left(\int_{\partial\Omega} \frac{G(y,u)}{|x-y|^{\mu_2}} \, d\nu \right) g(x,u) \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Delta_N = \nabla \cdot \left(|\nabla u|^{N-2}\nabla u\right)$, $\mu_1 \in (0,N)$, $\mu_2 \in (0, N-1)$, Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N > 2$ and f and g has a growth $e^{t^{N/N-1}}$ as $t \to \infty$.

Throughout the paper, we make use of the following notations:

- For any $u \in H^1(\Omega)$, we set

$$\|F(u)\|_{0,\Omega} := \iint_{\Omega\times\Omega} \frac{F(x,u)F(y,u)}{|x-y|^{\mu_1}} \, dxdy, \quad \|G(u)\|_{0,\partial\Omega} := \iint_{\partial\Omega\times\partial\Omega} \frac{G(x,u)G(y,u)}{|x-y|^{\mu_2}} \, d\sigma d\nu.$$

- the letters C, C_i denote various positive constants possibly different in various places.

The paper is organized as follows. In Section 2, we recall some definitions and provide some technical lemmas that will help prove our Theorem 1.1. In Section 1.1, we prove the main result of the paper.

2 | PRELIMINARIES

In this section, we will first recall some results that are required to establish the variational framework of the problem (P). Afterward, we will establish some essential lemmas.

Consider the functional space $H^1(\Omega)$ as the usual Sobolev space defined as

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega |\nabla u|^2 \, dx < \infty \right\},$$

and

$$\|u\|^2 = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \alpha(x) u^2 \, dx + \beta \int_{\partial\Omega} u^2 \, d\sigma$$

defines an equivalent norm on $H^1(\Omega)$. Thus, Equation (1.4) becomes

$$J(u) = \frac{\|u\|^2}{2} - \frac{\|F(u)\|_{0,\Omega}}{2} - \frac{\|G(u)\|_{0,\partial\Omega}}{2}.$$

The functions $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g \in C^1(\partial\Omega \times \mathbb{R}, \mathbb{R})$, we assume that $f(x,t) \equiv 0$ for $t \leq 0$ and $g(x,t) \equiv 0$ for $t \leq 0$. The functions f and g are said to be functions of critical growth on Ω and $\partial\Omega$, respectively, if it satisfies the following hypotheses for all $x \in \overline{\Omega}$ and $y \in \partial\Omega$:

($\mathbf{h_1}$) $f(x,t) > 0$ and $g(y,t) > 0$ for all $t > 0$;
($\mathbf{h_2}$) $f'(x,t)t > f(x,t)$, and $g'(y,t)t > g(y,t)$ for all $t > 0$;
($\mathbf{h_3}$) There exists constant $M > 0$ such that $F(x,t) \leq M(1 + f(x,t))$, and $G(y,t) \leq M(1 + g(y,t))$;

There exists functions $h \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $k \in C^1(\partial\Omega \times \mathbb{R}, \mathbb{R})$ and positive constants b, θ such that for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $(y,t) \in \partial\Omega \times \mathbb{R}$

($\mathbf{h_4}$) $f(x,t) = h(x,t)\mathrm{e}^{bt^2}$, $g(y,t) = k(x,t)\mathrm{e}^{\theta t^2}$;
($\mathbf{h_5}$) $\lim_{t\to\infty} \sup_{x\in\overline{\Omega}} h(x,t)\mathrm{e}^{-\epsilon t^2} = 0$, $\lim_{t\to\infty} \sup_{y\in\partial\Omega} k(y,t)\mathrm{e}^{-\epsilon t^2} = 0$;
($\mathbf{h_6}$) $\lim_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x,t)\mathrm{e}^{\epsilon t^2} = \infty$, $\lim_{t\to\infty} \inf_{y\in\partial\Omega} k(y,t)\mathrm{e}^{\epsilon t^2} = \infty$,

where }}$\prime$" denotes the derivative with respect to t and $(F(x,t), G(y,t)) = \int_0^t (f(x,s), g(y,s))ds$. The class of functions which satisfies assumptions $(h_1) - (h_6)$ is quite big. For example $(f(t), g(t)) = t^{m+1}(\mathrm{e}^{bt^2 + c_1 t^\alpha}, \mathrm{e}^{\theta t^2 + c_2 t^\beta})$, where $m, c_1, c_2 \geq 0$ and $0 \leq \alpha, \beta < 2$.

Remark 2.1. From (h_2) we can deduce that for any $t > 0$, $2F(x,t) < f(x,t)t$ and $2G(y,t) < g(y,t)t$.

For any $u \in H^1(\Omega)$, by Sobolev embedding we get that $u \in L^q(\Omega)$ for all $q \in [1, \infty)$. As a result, we get $F(x,u) \in L^q(\Omega)$ for all $q \geq 1$. Furthermore, from the classical Sobolev trace embedding of Trudinger–Moser, $H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$ holds for all $q \in [1, \infty)$ and thus $G(x,u) \in L^q(\partial\Omega)$ for all $q \geq 1$.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality). *Let $t, r > 1$ and $0 < \mu < N$ with $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C(t, r, \mu, N)$ independent of f, h such that*

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^{\mu}}\, dxdy \le C(t, r, \mu, N)\|f\|_{L^t(\mathbb{R}^N)}\|h\|_{L^r(\mathbb{R}^N)}.$$

In particular, the above proposition holds true for $N = 2$. By taking $t = r = \frac{4}{4-\mu_1}$, where $\mu_1 \in (0, 2)$ and using Sobolev embedding we see that

$$\iint_{\Omega\times\Omega} \frac{f(x)f(y)}{|x-y|^{\mu_1}}\, dxdy \le C(\mu_1)\|f\|^2_{L^{\frac{4}{4-\mu_1}}(\Omega)}. \tag{2.1}$$

Moreover, it is easy to see that Proposition 2.1 remains valid when we are considering boundary conditions $\partial\Omega \subset \mathbb{R}$. For $\mu_2 \in (0, 1)$, $f \in L^t(\partial\Omega)$ and $h \in L^r(\partial\Omega)$, the following holds:

$$\iint_{\partial\Omega\times\partial\Omega} \frac{f(x)h(y)}{|x-y|^{\mu_2}}\, d\sigma d\nu \le C(t, r, \mu_2)\|f\|_{L^t(\partial\Omega)}\|h\|_{L^r(\partial\Omega)}.$$

By taking $t = r = \frac{2}{2-\mu_2}$ and using the Sobolev trace embedding, we get inequality similar to Equation (2.1). Hence, the functional $J(u)$ is well-defined.

Definition 2.1 (**Moser functions**). Let $y \in \Omega$ and $R \le d(y, \partial\Omega)$, where d denotes the distance from y to $\partial\Omega$. For $0 < l < R$, define

$$m_l(x, y) = \frac{1}{\sqrt{2\pi}} \begin{cases} \left(\log(\frac{R}{l})\right)^{1/2} & \text{if } 0 \le |x-y| \le l \\ \dfrac{\log(\frac{R}{|x-y|})}{\left(\log(\frac{R}{l})\right)^{1/2}} & \text{if } l \le |x-y| \le R \\ 0 & \text{if } R \le |x-y|. \end{cases}$$

With the help of the Moser functions, we can obtain the following lemma.

Lemma 2.1 [4, Lemma 3.3]. *Let $\partial\Omega$ is $C^{1,r}$ manifold. For every $x_0 \in \partial\Omega$, we can find a $L > 0$ such that for each $0 < l < L$ there exists a function $w_l \in H^1(\Omega)$ satisfying*

(i) $w_l \ge 0$, *supp* $w_l \subset B(x_0, L) \cap \overline{\Omega}$,
(ii) $\|w_l\| = 1$,
(iii) for all $x \in B(x_0, l) \cap \overline{\Omega}$, w_l *is constant and*

$$w_l^2(x) = \frac{1}{\pi}\log\frac{L}{l} + O(1) \text{ as } l \to 0,$$

where $B(x_0, L) = \{x \in \mathbb{R}^2 : |x - x_0| < L\}$.

From [4, Lemmas 3.1, 4.2, and 3.2], we have the following result.

Lemma 2.2. *Let* $u \in H^1(\Omega)$

(i) $e^{u^2} \in L^p(\Omega)$ *and* $e^{u^2} \in L^p(\partial\Omega)$, *for any* $p < \infty$
(ii) $\sup\left\{c^2 : \sup_{\|u\|\le 1} \int_\Omega f(x, cu)u(x)\, dx < \infty\right\} = \sup\left\{c^2 : \sup_{\|u\|\le 1} \int_\Omega e^{bc^2u^2(x)}\, dx < \infty\right\} = \frac{2\pi}{b}$

$$\textit{(iii)}\ \sup\left\{c^2 : \sup_{\|u\|\le 1}\int_{\partial\Omega} g(x,cu)u(x)\,dx<\infty\right\}=\sup\left\{c^2 : \sup_{\|u\|\le 1}\int_{\partial\Omega} e^{\theta c^2u^2(x)}\,dx<\infty\right\}=\frac{\pi}{\theta}.$$

For the proof of our Theorem 1.1, we need the result similar to Lemma 2.2.

Lemma 2.3. *For any $u\in H^1(\Omega)$*

$$\textit{(i)}\ \sup\left\{c^2 : \sup_{\|u\|\le 1}\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy<\infty\right\}=\frac{(4-\mu_1)\pi}{2b};$$

$$\textit{(ii)}\ \sup\left\{c^2 : \sup_{\|u\|\le 1}\iint_{\partial\Omega\times\partial\Omega}\frac{e^{\theta c^2u^2(y)}e^{\theta c^2u^2(x)}}{|x-y|^{\mu_2}}\,d\sigma d\nu<\infty\right\}=\frac{(2-\mu_2)\pi}{2\theta}.$$

Proof. **(i)** Employing Lemma 2.2(i) and Hardy–Littlewood–Sobolev inequality, we have

$$\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy\le C(\mu_1)\left(\int_{\Omega} e^{\frac{4bc^2u^2(x)}{4-\mu_1}}\,dx\right)^{\frac{4-\mu_1}{2}}.$$

From Lemma 2.2(ii), we get that if $c^2\le\frac{(4-\mu_1)\pi}{2b}$, then $\sup_{\|u\|\le 1}\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy<\infty$, which implies that

$$\sup\left\{c^2 : \sup_{\|u\|\le 1}\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy<\infty\right\}\ge\frac{(4-\mu_1)\pi}{2b}.$$

Assume that

$$\sup\left\{c^2 : \sup_{\|u\|\le 1}\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy<\infty\right\}>\frac{(4-\mu_1)\pi}{2b}.$$

Let $\epsilon>0$ be such that for $c^2=(1+\epsilon)\frac{(4-\mu_1)\pi}{2b}$, we have a constant C_1 such that

$$\sup_{\|u\|\le 1}\iint_{\Omega\times\Omega}\frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy=C_1.$$

In addition, from [7, Equation (2.11)], it is easy to get that, for some positive constant $C(\mu_1)$

$$\iint_{[B(x_0,l)]^2}\frac{dxdy}{|x-y|^{\mu_1}}\ge\frac{C(\mu_1)}{l^{4-\mu_1}}.$$

Recall function w_l, defined in Lemma 2.1, we get

$$\begin{aligned} C_1 &\ge \iint_{[B(x_0,l)\cap\Omega]^2}\frac{e^{bc^2w_l^2(y)}e^{bc^2w_l^2(x)}}{|x-y|^{\mu_1}}\,dxdy\\ &= e^{(1+\epsilon)(4-\mu_1)\pi w_l^2(x_0)}\iint_{[B(x_0,l)\cap\Omega]^2}\frac{1}{|x-y|^{\mu_1}}\,dxdy\\ &\ge C(\mu_1)l^{-(4-\mu_1)\epsilon}\to\infty \text{ as } l\to 0, \end{aligned}$$

which is a contradiction. Hence,

$$\sup\left\{c^2 : \sup_{\|u\|\le 1} \iint\limits_{\Omega\times\Omega} \frac{e^{bc^2u^2(y)}e^{bc^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy < \infty\right\} = \frac{(4-\mu_1)\pi}{2b}.$$

(ii) From Lemma 2.2(i) and Hardy–Littlewood–Sobolev inequality, we have

$$\iint\limits_{\partial\Omega\times\partial\Omega} \frac{e^{\theta c^2u^2(y)}e^{\theta c^2u^2(x)}}{|x-y|^{\mu_2}}\,d\sigma d\nu \le C(\mu_2)\left(\int\limits_{\partial\Omega} e^{\frac{2\theta c^2u^2(x)}{2-\mu_2}}\,d\sigma\right)^{2-\mu_2}.$$

Now proceeding with the same analysis as in (i), and utilizing Lemma 2.2(iii) and the following inequality:

$$\iint\limits_{[B(x_0,l)\cap\partial\Omega]^2} \frac{d\sigma d\nu}{|x-y|^{\mu_2}} \ge \frac{C(\mu_2)}{l^{2-\mu_2}},$$

we can obtain the desired equality. □

Let us define

$$2I(u) = \iint\limits_{\Omega\times\Omega} \frac{F(y,u)[f(x,u)u(x)-F(x,u)]}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)[g(x,u)u(x)-G(x,u)]}{|x-y|^{\mu_2}}\,d\sigma d\nu. \tag{2.2}$$

Lemma 2.4. *Let (f,g) be a function of critical growth on $(\Omega,\partial\Omega)$. Then, we have*

(i) If $u\in H^1(\Omega)$, then $f(x,u)\in L^p(\Omega)$ and $g(x,u)\in L^p(\partial\Omega)$ for all $p\ge 0$;
(ii) For $u\in H^1(\Omega)$

$$\sup\left\{c^2 : \sup_{\|u\|\le 1} \iint\limits_{\Omega\times\Omega} \frac{F(y,cu)f(x,cu)u(x)}{|x-y|^{\mu_1}}\,dxdy < \infty\right\} = \frac{(4-\mu_1)\pi}{2b}$$

$$\sup\left\{c^2 : \sup_{\|u\|\le 1} \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,cu)g(x,cu)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu < \infty\right\} = \frac{(2-\mu_2)\pi}{2\theta};$$

(iii) Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences, converging weakly in $H^1(\Omega)$ and for almost every x in Ω to u and v, respectively. Further, assume that

$$\sup_n \|u_n\|^2 < \frac{(4-\mu_1)\pi}{2b}.$$

Then for every $l\ge 0$,

$$\lim_{n\to\infty} \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy = \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)v^l(x)}{u(x)|x-y|^{\mu_1}}\,dxdy.$$

(iv) Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences, converging weakly in $H^1(\Omega)$ and for almost every x in $\partial\Omega$ to u and v, respectively. Further, assume that

$$\sup_n \|u_n\|^2 < \frac{(2-\mu_2)\pi}{2\theta}.$$

Then for every $l \geq 0$,

$$\lim_{n\to\infty} \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_2}}\, d\sigma d\nu = \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)v^l(x)}{u(x)|x-y|^{\mu_2}}\, d\sigma d\nu;$$

(v) *Let* $\{u_n\}$ *be a sequence in* $H^1(\Omega)$ *converging weakly and for almost every* x *in* Ω *to* u *such that*

$$\sup_n \iint_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\, dxdy < \infty$$

then

$$\lim_{n\to\infty} \|F(u_n)\|_{0,\Omega} = \|F(u)\|_{0,\Omega};$$

(vi) *Let* $\{u_n\}$ *be a sequence in* $H^1(\Omega)$ *converging weakly and for almost every* y *in* $\partial\Omega$ *to* u *such that*

$$\sup_n \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu < \infty$$

then

$$\lim_{n\to\infty} \|G(u_n)\|_{0,\partial\Omega} = \|G(u)\|_{0,\partial\Omega};$$

(vii) *Let* $I(u)$ *be as defined in Equation (2.2). For all* $u \in H^1(\Omega)$, $I(u) \geq 0$ *and* $I(u) = 0 \iff u \equiv 0$. *Further, for all* $u \in H^1(\Omega)$

$$\iint_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\, dxdy + \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu \leq 4I(u).$$

Proof. **(i)** The proof follows directly from the assumption (h_5) and Lemma 2.2(i).

(ii) From Remark 2.1 and assumptions (h_4) and (h_5), we infer that there exists a constant $C_1(\epsilon) > 0$ such that

$$2F(x,t) < f(x,t)t \leq C_1(\epsilon)e^{b(1+\epsilon)t^2}. \tag{2.3}$$

Employing the Hardy–Littlewood–Sobolev inequality and Equation (2.3), we obtain that

$$\begin{aligned} \iint_{\Omega\times\Omega} \frac{F(y,cu)f(x,cu)u(x)}{|x-y|^{\mu_1}}\, dxdy &\leq C(\mu_1)\|F(y,cu)\|_{L^{\frac{4}{4-\mu_1}}} \|f(x,cu)u\|_{L^{\frac{4}{4-\mu_1}}} \\ &\leq C(\mu_1,\epsilon)\left[\int_\Omega e^{\frac{4b(1+\epsilon)c^2u^2}{4-\mu_1}}\, dx\right]^{\frac{4-\mu_1}{2}}. \end{aligned} \tag{2.4}$$

From Lemmas 2.2 (ii) and (2.4), we get that if $c > 0$ such that

$$(1+\epsilon)c^2 < \frac{(4-\mu_1)\pi}{2b},$$

then

$$\sup_{\|u\|\leq 1} \iint_{\Omega\times\Omega} \frac{F(y,cu)f(x,cu)u(x)}{|x-y|^{\mu_1}}\, dxdy < \infty.$$

On the other hand, from the assumption (h_6), for $|t|$ large there exist positive constants $C_2(\epsilon), C_3(\epsilon)$ such that

$$f(x,t) \geq C_2(\epsilon)e^{b(1-\epsilon)t^2}, \tag{2.5}$$

and

$$F(x,t) \geq C_3(\epsilon)e^{b(1-\epsilon)t^2}. \tag{2.6}$$

Hence, if $c > 0$ such that

$$\sup_{\|u\|\leq 1} \iint_{\Omega\times\Omega} \frac{F(y,cu)f(x,cu)u(x)}{|x-y|^{\mu_1}}\,dxdy < \infty,$$

then it implies from Equations (2.5) and (2.6), that for every $\epsilon > 0$

$$\sup_{\|u\|\leq 1} \iint_{\Omega\times\Omega} \frac{e^{b(1-\epsilon)c^2u^2(y)}e^{b(1-\epsilon)c^2u^2(x)}}{|x-y|^{\mu_1}}\,dxdy < \infty.$$

Therefore, from Lemma 2.3, we have

$$(1-\epsilon)c^2 \leq \frac{(4-\mu_1)\pi}{2b}.$$

Thus, we get the desired equality

$$\sup\left\{c^2 : \sup_{\|u\|\leq 1} \iint_{\Omega\times\Omega} \frac{F(y,cu)f(x,cu)u(x)}{|x-y|^{\mu_1}}\,dxdy < \infty\right\} = \frac{(4-\mu_1)\pi}{2b}.$$

Proceeding in similar manner, we can also deduce

$$\sup\left\{c^2 : \sup_{\|u\|\leq 1} \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,cu)g(x,cu)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu < \infty\right\} = \frac{(2-\mu_2)\pi}{2\theta}.$$

(iii) Given that $\|u_n\|^2 < \frac{(4-\mu_1)\pi}{2b}$, we deduce from Lemma 2.2(ii) using Remark 2.1

$$\sup_n \|F(x,u_n)\|_{L^{\frac{4}{4-\mu_1}}} < \infty, \tag{2.7}$$

moreover, we can choose a $p > 1$ such that

$$\sup_n \|f(x,u_n)\|_{L^{\frac{4p}{4-\mu_1}}} < \infty. \tag{2.8}$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and $\{v_n\}$ be a bounded sequence, then for every $l \geq 0$

$$\sup_n \|v_n^l\|_{L^{\frac{4q}{4-\mu_1}}} < \infty. \tag{2.9}$$

For any $N > 0$

$$\begin{aligned}\iint_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy &= \iint_{|u_n(x)|\leq N, |u_n(y)|\leq N} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy \\ &+ \iint_{|u_n(x)|\leq N, |u_n(y)|> N} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy + \iint_{|u_n(x)|> N, y\in\Omega} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy\end{aligned} \tag{2.10}$$

First, let us consider the case when $\{(x,y) : |u_n(x)| > N,\ y \in \Omega\}$. By employing Hardy–Littlewood–Sobolev inequality, Hölder's inequality, Equations (2.7)–(2.9), we get

$$\begin{aligned}\iint\limits_{|u_n(x)|>N, y\in\Omega} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy &\le \frac{C(\mu_1)}{N}\sup_n \|F(y,u_n)\|_{L^{\frac{4}{4-\mu_1}}} \|f(x,u_n)v_n^l\|_{L^{\frac{4}{4-\mu_1}}} \\ &\le \frac{C(\mu_1)}{N}\sup_n \|f(x,u_n)\|_{L^{\frac{4p}{4-\mu_1}}} \sup_n \|v_n^l\|_{L^{\frac{4q}{4-\mu_1}}} \\ &\le \frac{C(\mu_1)}{N}.\end{aligned} \tag{2.11}$$

Next, we consider the case when $\{(x,y) : |u_n(x)| \le N,\ |u_n(y)| > N\}$. Using Remark 2.1, Hardy–Littlewood–Sobolev inequality, Hölder's inequality, Equations (2.8) and (2.9), we get

$$\begin{aligned}&\iint\limits_{|u_n(x)|\le N,|u_n(y)|>N} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy \\ &\quad< \frac{1}{2N}\iint\limits_{|u_n(x)|\le N,|u_n(y)|>N} \frac{f(y,u_n)u_n^2(y)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy \\ &\quad\le \frac{C(\mu_1)}{N}\sup_n \|f(y,u_n)u_n^2\|_{L^{\frac{4}{4-\mu_1}}} \sup_{|u_n(x)|\le N} \left\|\frac{f(x,u_n)v_n^l}{u_n}\right\|_{L^{\frac{4}{4-\mu_1}}} \\ &\quad\le \frac{C(\mu_1)}{N}.\end{aligned} \tag{2.12}$$

Hence, putting together Equations (2.11) and (2.12) in Equation (2.10), and applying the dominated convergence theorem by taking $n \to \infty$ and $N \to \infty$, we obtain the desired equality

$$\lim_{n\to\infty}\iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)v_n^l(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy = \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)v^l(x)}{u(x)|x-y|^{\mu_1}}\,dxdy.$$

(iv) The desired equality is obtained by following similar reasoning as in (iii).
(v) For any $N > 0$

$$\begin{aligned}\iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy &= \iint\limits_{|u_n(x)|\le N,|u_n(y)|\le N} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy \\ &+ 2\iint\limits_{|u_n(x)|>N,\,|u_n(y)|\le N} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{|u_n(x)|>N,|u_n(y)|>N} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy\end{aligned} \tag{2.13}$$

As $\sup_n \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy < \infty$, from (ii) we have $\|u_n\|^2 \le \frac{(4-\mu_1)\pi}{2b}$. This consequently leads to Equation (2.7). Now, let us consider the case when $\{(x,y) : |u_n(x)| > N,\ |u_n(y)| \le N\}$. From the assumption (h_3), Hardy–Littlewood–

Sobolev inequality and Equation (2.7), we have

$$
\begin{aligned}
&2 \iint\limits_{|u_n(x)|>N,\, |u_n(y)|\le N} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy \\
&\le \frac{2M}{N} \iint\limits_{|u_n(x)|>N,\, |u_n(y)|\le N} \frac{F(y,u_n)u_n(x) + F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy \\
&\le \frac{2M}{N} \sup_n \|F(y,u_n)\|_{L^{\frac{4}{4-\mu_1}}} \sup_n \|u_n\|_{L^{\frac{4}{4-\mu_1}}} + O\left(\frac{1}{N}\right) \\
&\le \frac{C(\mu_1)}{N}.
\end{aligned}
\tag{2.14}
$$

By following similar argument for the case $\{(x,y) : |u_n(x)| > N,\ |u_n(y)| > N\}$, we get

$$
\iint\limits_{|u_n(x)|>N, |u_n(y)|>N} \frac{F(y,u_n)F(x,u_n)}{|x-y|^{\mu_1}}\,dxdy = O\left(\frac{1}{N}\right). \tag{2.15}
$$

Hence, putting together Equations (2.14) and (2.15) in Equation (2.13), and applying the dominated convergence theorem by taking $n \to \infty$ and $N \to \infty$, we obtain the desired equality

$$
\lim_{n\to\infty} \|F(u_n)\|_{0,\Omega} = \|F(u)\|_{0,\Omega}
$$

(vi) The desired equality is obtained by following similar reasoning as in (v). **(vii)** Using the assumption $(h_1) - (h_2)$ and Remark 2.1, for $t > 0$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{F(y,tu)(f(x,tu)tu - F(x,tu))}{|x-y|^{\mu_1}}\right) \\
&= \frac{f(y,tu)u(y)[f(x,tu)tu(x) - F(x,tu)] + F(y,tu)f'(x,tu)tu^2(x)}{|x-y|^{\mu_1}} \\
&> \frac{f(y,tu)u(y)F(x,tu) + F(y,tu)f(x,tu)u(x)}{|x-y|^{\mu_1}} \ge 0.
\end{aligned}
$$

This implies for all $t > 0$, $\frac{F(y,tu)(f(x,tu)tu - F(x,tu))}{|x-y|^{\mu_1}}$ is an increasing function. Also, at $t = 0$, it is zero. Hence, for $t > 0$

$$
\frac{F(y,tu)(f(x,tu)tu - F(x,tu))}{|x-y|^{\mu_1}} > 0,
$$

and similarly

$$
\frac{G(y,tu)(g(x,tu)tu - G(x,tu))}{|x-y|^{\mu_2}} > 0.
$$

Therefore, $I(u) \ge 0$ for all $u \in H^1(\Omega)$ and $I(u) = 0 \iff u \equiv 0$. Further, using Remark 2.1 we have

$$
\begin{aligned}
2I(u) &= \iint\limits_{\Omega\times\Omega} \frac{F(y,u)[f(x,u)u(x) - F(x,u)]}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)[g(x,u)u(x) - G(x,u)]}{|x-y|^{\mu_2}}\,d\sigma d\nu \\
&\ge \frac{1}{2} \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\,dxdy + \frac{1}{2} \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu.
\end{aligned}
$$

Hence, we obtain the desired inequality. □

To obtain the compactness result, we require the following theorem by Lions [26] and [4, Lemma 3.5].

Lemma 2.5. *Let $\{u_n\}$ be a bounded sequence in $H^1(\Omega)$ with $\|u_n\| = 1$ converging weakly to a nonzero $u \in H^1(\Omega)$. Then for every $p < \frac{1}{1-\|u\|^2}$,*

$$\sup_n \int_\Omega e^{2\pi p u_n^2}\,dx < \infty, \quad \textit{and} \quad \sup_n \int_{\partial\Omega} e^{\pi p u_n^2}\,dx < \infty.$$

3 | PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. To obtain the existence of a positive solution, we deploy the Nehari constraint method and prove that $a(\Omega)^2$ defined in Equation (1.7) is below the Palais–Smale condition. Subsequently, with the Lions' compactness lemma, we establish that the minimizer of $a(\Omega)^2$ is indeed the critical point of our functional J.

Lemma 3.1. *Assume that*

(i) $\overline{\lim_{t\to\infty}} \inf_{x\in\overline{\Omega}} h(x,t)t = \overline{\lim_{t\to\infty}} \inf_{y\in\partial\Omega} k(y,t)t = \infty$
(ii) $f'(x,0) = g'(y,0) = 0$*, for all* $(x,y) \in \Omega \times \partial\Omega$.

Then, $0 < a(\Omega)^2 < \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}$*, where* $a(\Omega)^2$ *is as defined in Equation (1.7).*

Proof. We divide the proof into the following steps:

Step 1: We prove that $a(\Omega)^2 > 0$.

Let us suppose that $a(\Omega) = 0$. Then, there exists a sequence $\{u_n\} \in \partial B(\Omega)$ (as defined in Equation (1.6)) such that

$$I(u_n) = J(u_n) \to 0 \ \text{ as } n \to \infty, \tag{3.1}$$

From Lemma 2.4(vii) and Equation (3.1), we obtain that

$$\sup_n \iint_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy < \infty, \tag{3.2}$$

$$\sup_n \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu < \infty, \tag{3.3}$$

which implies that $\sup\limits_n \|u_n\|^2 < \infty$. Thus, up to a subsequence there exists a function $u \in H^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. By Fatou's lemma and Equation (3.1)

$$0 \le I(u) \le \underline{\lim}_{n\to\infty} I(u_n) = 0.$$

Hence, from Lemma 2.4(vii), we get that $u \equiv 0$. Given that we have Equations (3.2) and (3.3), we can consequently infer, using Lemma 2.4 (v) and (vi) that

$$\lim_{n\to\infty} \|u_n\|^2 = \lim_{n\to\infty} \left\{2J(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega}\right\} = 0. \tag{3.4}$$

Let $v_n = \frac{u_n}{\|u_n\|}$ and up to a subsequence there exists a function $v \in H^1(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ and for almost all $x \in \Omega$ and $y \in \partial\Omega$.

With the help of Equation (3.4), we utilize Lemma 2.4(iii)and (iv) and since $u_n \in \partial B(\Omega)$, we get

$$1 = \iint_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)v_n^2(x)}{u_n(x)|x-y|^{\mu_1}}\,dxdy + \iint_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)v_n^2(x)}{u_n(x)|x-y|^{\mu_2}}\,d\sigma d\nu \to 0, \quad \text{as } n \to \infty,$$

which gives us a contradiction. Hence, $a(\Omega) > 0$.

Step 2: We prove that $a(\Omega)^2 \leq \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}$.
For any $u \in H^1(\Omega)\backslash\{0\}$ and $\gamma > 0$, define

$$\Psi(\gamma) := \frac{1}{\gamma}\left[\iint\limits_{\Omega\times\Omega} \frac{F(y,\gamma u)f(x,\gamma u)u(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,\gamma u)g(x,\gamma u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\right].$$

As $F(y,t), G(y,t), \frac{f(x,t)}{t}$ and $\frac{g(x,t)}{t}$ are all increasing functions with respect to t and remain positive for $t > 0$, we observe that

$$\lim_{\gamma\to\infty} \Psi(\gamma) = \infty.$$

Further, by utilizing the hypothesis (ii), we deduce that $\lim_{\gamma\to 0}\Psi(\gamma) = 0$. Thus, for any $u \in H^1(\Omega)\backslash\{0\}$, there exists $\gamma > 0$ such that

$$\begin{aligned} \|u\|^2 &= \Psi(\gamma) \\ &= \frac{1}{\gamma^2}\left[\iint\limits_{\Omega\times\Omega} \frac{F(y,\gamma u)f(x,\gamma u)\gamma u(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,\gamma u)g(x,\gamma u)\gamma u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\right], \end{aligned} \tag{3.5}$$

that is, $\gamma u \in \partial B(\Omega)$. Moreover, through a contradictory argument, we establish that

$$\text{if } \|u\|^2 \leq \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu, \quad \text{then } \gamma \leq 1. \tag{3.6}$$

Now, let $w \in H^1(\Omega)$ with $\|w\| = 1$. Choose $\gamma > 0$ corresponding to w such that Equations (3.5) and (3.6) hold. Since $\gamma w \in \partial B(\Omega)$, we have

$$0 < \frac{a(\Omega)^2}{2} \leq J(\gamma w) \leq \|\gamma w\|^2 = \gamma^2.$$

Hence,

$$\begin{aligned} &\iint\limits_{\Omega\times\Omega} \frac{F(y,a(\Omega)w)f(x,a(\Omega)w)w^2(x)}{a(\Omega)w(x)|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,a(\Omega)w)g(x,a(\Omega)w)w^2(x)}{a(\Omega)w(x)|x-y|^{\mu_2}}\,d\sigma d\nu \\ &\leq \iint\limits_{\Omega\times\Omega} \frac{F(y,\gamma w)f(x,\gamma w)w^2(x)}{\gamma w(x)|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,\gamma w)g(x,\gamma w)w^2(x)}{\gamma w(x)|x-y|^{\mu_2}}\,d\sigma d\nu \\ &= \|w\|^2 = 1. \end{aligned}$$

This implies, for any $u \in H^1(\Omega)$,

$$\sup_{\|u\|\leq 1} \iint\limits_{\Omega\times\Omega} \frac{F(y,a(\Omega)u)f(x,a(\Omega)u)u(x)}{|x-y|^{\mu_1}}\,dxdy \leq a(\Omega), \tag{3.7}$$

and

$$\sup_{\|u\|\leq 1} \iint\limits_{\Omega\times\Omega} \frac{G(y,a(\Omega)u)g(x,a(\Omega)u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \leq a(\Omega). \tag{3.8}$$

Thus, combining Equations (3.7) and (3.8) with Lemma 2.4(ii), we obtain

$$a(\Omega)^2 \leq \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}.$$

Step 3: We will prove that $a(\Omega)^2 < \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}$.

Let us suppose

$$a(\Omega)^2 = \frac{(4-\mu_1)\pi}{2b}.$$

Using Equation (3.7) and recalling the function w_l defined in Lemma 2.1, we obtain $\delta > 0$ such that

$$\begin{aligned} a(\Omega) &\geq \iint\limits_{[B(x_0,l)\cap\Omega]^2} \frac{F(y, a(\Omega)w_l) f(x, a(\Omega)w_l)w_l(x)}{|x-y|^{\mu_1}}\, dxdy \\ &\geq C \iint\limits_{[B(x_0,l)\cap\Omega]^2} \inf_x h(x, a(\Omega)w_l)w_l(x)e^{2ba(\Omega)^2 w_l^2(x)}\, dx \\ &= C \inf_x h(x, a(\Omega)w_l(x_0))w_l(x_0)e^{(4-\mu_1)\pi w_l^2(x_0)} \iint\limits_{[B(x_0,l)\cap\Omega]^2} \frac{dxdy}{|x-y|^{\mu_1}} \\ &\geq C\delta \inf_x h(x, a(\Omega)w_l(x_0))w_l(x_0) \to \infty \text{ as } l \to 0, \end{aligned}$$

which is a contradiction. Hence, $a(\Omega)^2 < \frac{(4-\mu_1)\pi}{2b}$. Similarly, by using Equation (3.8) and the same function w_l we can show that $a(\Omega)^2 < \frac{(2-\mu_2)\pi}{2\theta}$. Indeed, let us suppose $a(\Omega)^2 = \frac{(2-\mu_2)\pi}{2\theta}$,

$$\begin{aligned} a(\Omega) &\geq C \iint\limits_{[B(x_0,l)\cap\partial\Omega]^2} \inf_x k(x, a(\Omega)w_l)w_l(x)\mathrm{e}^{2\theta a(\Omega)^2 w_l^2(x)}\, d\sigma \\ &= C \inf_x k(x, a(\Omega)w_l(x_0))w_l(x_0)\mathrm{e}^{(2-\mu_2)\pi w_l^2(x_0)} \iint\limits_{[B(x_0,l)\cap\partial\Omega]^2} \frac{d\sigma d\nu}{|x-y|^{\mu_2}} \to \infty \text{ as } l \to 0. \end{aligned}$$

Hence, $a(\Omega)^2 < \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}$. □

Lemma 3.2. *Let $c \in \left[0, \min\left\{\frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta}\right\}\right]$ and $\{u_n\}$ be a bounded sequence in $H^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and for almost all $x \in \Omega$ and for almost all $y \in \partial\Omega$.*

(i) $\lim_{n\to\infty} J(u_n) = c$

(ii) $\|u\|^2 \geq \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu$

(iii) $\sup_n \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu \right\} < \infty.$

Then

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu \right\} \\ &= \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu. \end{aligned}$$

Proof. Since $u_n \rightharpoonup u$, with $u \not\equiv 0$, we use (ii) and Lemma 2.4 (vii), to establish $J(u) \geq I(u) > 0$. Furthermore, applying Fatou's lemma, $J(u) \leq \varliminf_{n\to\infty} J(u_n) = c$. This leads to the existence of an $\epsilon > 0$ such that

$$(1+\epsilon)(c - J(u)) < \min\left\{\frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta}\right\} := \frac{\delta}{4} \tag{3.9}$$

Let $K = \|F(u)\|_{0,\Omega} + \|G(u)\|_{0,\partial\Omega}$. Then from (iii) and Lemma 2.4(v) and (vi), we have

$$\begin{aligned} \lim_{n\to\infty} \|u_n\|^2 &= \lim_{n\to\infty} \left\{2J(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega}\right\} \\ &= 2c + K. \end{aligned} \tag{3.10}$$

Employing Equations (3.9) and (3.10) such that for large values of n

$$\frac{2(1+\epsilon)}{\delta}\|u_n\|^2 < \frac{2c+K}{2(c-J(u))} = \frac{1}{\left(1 - \frac{\|u\|^2}{2c+K}\right)}.$$

Now, choose p such that

$$\frac{2(1+\epsilon)}{\delta}\|u_n\|^2 \leq p < \frac{1}{\left(1 - \frac{\|u\|^2}{2c+K}\right)}. \tag{3.11}$$

Applying Lemma 2.5 to the sequence $\frac{u_n}{\|u_n\|}$ and using Equation (3.11), we have

$$\sup_n \int_\Omega e^{\frac{2b\delta}{4-\mu_1} p \left(\frac{u_n}{\|u_n\|}\right)^2} dx < \infty, \quad \text{and} \quad \sup_n \int_{\partial\Omega} e^{\frac{\theta\delta}{2-\mu_2} p\left(\frac{u_n}{\|u_n\|}\right)^2} d\sigma < \infty. \tag{3.12}$$

Hence, from Equations (3.11) and (3.12), we have

$$\sup_n \int_\Omega e^{(1+\epsilon)\frac{4b}{4-\mu_1}u_n^2} dx < \infty, \quad \text{and} \quad \sup_n \int_{\partial\Omega} e^{(1+\epsilon)\frac{2\theta}{2-\mu_2}u_n^2} d\sigma < \infty. \tag{3.13}$$

Let

$$M_1 := \sup_{(x,t)\in\Omega\times\mathbb{R}} h(x,t)te^{\frac{-\epsilon b t^2}{2}} + \sup_{(x,t)\in\partial\Omega\times\mathbb{R}} k(x,t)te^{\frac{-\epsilon\theta t^2}{2}},$$

and for any $N > 0$, we have

$$\begin{aligned} \iint_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy &= \iint_{|u_n(x)|\leq N, |u_n(y)|\leq N} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy \\ &+ \iint_{|u_n(x)|\leq N, |u_n(y)|>N} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint_{|u_n(x)|>N, y\in\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy \end{aligned} \tag{3.14}$$

First, let us consider the case when $\{(x, y) : |u_n(x)| > N,\ y \in \Omega\}$. By employing Hardy–Littlewood–Sobolev inequality, Equations (3.13) and (2.3)

$$\begin{aligned}
&\iint\limits_{|u_n(x)|>N, y\in\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy \\
&\le C\mathrm{e}^{\frac{-\epsilon b N^2}{2}} \iint\limits_{|u_n(x)|>N, y\in\Omega} \frac{\mathrm{e}^{b(1+\epsilon)u_n^2(y)} h(x,u_n)u_n(x)\mathrm{e}^{\frac{-\epsilon b u_n^2(x)}{2}}\mathrm{e}^{b(1+\epsilon)u_n^2(x)}}{|x-y|^{\mu_1}}\,dxdy \\
&\le CM_1\mathrm{e}^{\frac{-\epsilon b N^2}{2}} \left(\int\limits_{\Omega} \mathrm{e}^{(1+\epsilon)\frac{4b}{4-\mu_1}u_n^2(x)}\,dx\right)^{\frac{4-\mu_1}{2}} \le C\mathrm{e}^{\frac{-\epsilon b N^2}{2}}.
\end{aligned} \tag{3.15}$$

Next, we consider the case when $\{(x, y) : |u_n(x)| \le N,\ |u_n(y)| > N\}$. From Remark 2.1 and Equation (2.3), we see that the result follows by same analysis

$$\iint\limits_{|u_n(x)|\le N, |u_n(y)|>N} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy \le C\mathrm{e}^{\frac{-\epsilon b N^2}{2}}. \tag{3.16}$$

Hence putting together Equations (3.15) and (3.16) in Equation (3.14), and applying the dominated convergence theorem by taking $n \to \infty$ and $N \to \infty$, we obtain the desired equality

$$\lim_{n\to\infty} \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy = \iint\limits_{\Omega\times\Omega} \frac{F(y,u)f(x,u)u(x)}{|x-y|^{\mu_1}}\,dxdy.$$

By similar analysis, we get

$$\lim_{n\to\infty} \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu = \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u)g(x,u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu.$$

This proves the lemma. □

Lemma 3.3. *Let $u_0 \in \partial B(\Omega)$ such that $J'(u_0) \neq 0$. Then*

$$J(u_0) > \inf\{J(u) : u \in \partial B(\Omega)\} = \frac{a(\Omega)^2}{2}.$$

Proof. For $\alpha, t \in \mathbb{R}$, define

$$m_t(\alpha) = \alpha u_0 - tJ'(u_0).$$

Then

$$\lim_{t\to 0,\alpha\to 1} \frac{d}{dt} J(m_t(\alpha)) = -\|J'(u_0)\|^2 < 0.$$

We can choose $\epsilon, \delta > 0$ such that for all $\alpha \in [1-\epsilon, 1+\epsilon]$ and $0 < t < \delta$

$$J(m_t(\alpha)) < J(m_0(\alpha)) = J(\alpha u_0). \tag{3.17}$$

Let

$$\rho_t(\alpha) = \|m_t(\alpha)\|^2 - \iint\limits_{\Omega\times\Omega} \frac{F(y, m_t(\alpha))f(x, m_t(\alpha))m_t(\alpha)(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y, m_t(\alpha))g(x, m_t(\alpha))m_t(\alpha)(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu.$$

Since $u \in \partial B(\Omega)$, by decreasing ϵ and δ if necessary, we have for $0 < t < \delta$,

$$\rho_t(\alpha) = \begin{cases} >0 & \text{if } \alpha = 1-\epsilon; \\ <0 & \text{if } \alpha = 1+\epsilon. \end{cases}$$

Thus, there exists α_t such that $\rho_t(\alpha_t) = 0$ and hence $m_t(\alpha_t) \in \partial B(\Omega)$. Using Equation (3.17) and analyzing $\rho_0(\alpha) = \alpha\left(\frac{d}{d\alpha}J(\alpha u_0)\right)$, we get

$$\inf\{J(u) : u \in \partial B(\Omega)\} \le J(m_t(\alpha_t)) < J(\alpha_t u_0) \le \sup_{\alpha\in\mathbb{R}} J(\alpha u_0) = J(u_0).$$

□

Lemma 3.4. *Let $c \in \left(-\infty, \min\left\{\frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta}\right\}\right)$ and $\{u_n\}$ be a sequence in $H^1(\Omega)$ such that $\lim_{n\to\infty} J(u_n) = c$, $\lim_{n\to\infty} J'(u_n) = 0$. Then, $\{u_n\}$ has a convergent subsequence.*

Proof. We claim that

$$\sup_n \left\{ \|u_n\| + \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \right\} < \infty.$$

Indeed, from Equations (1.4) and (1.5), we get

$$\frac{\|u_n\|^2}{2} - \frac{\|F(u_n)\|_{0,\Omega}}{2} - \frac{\|G(u_n)\|_{0,\partial\Omega}}{2} \to c. \tag{3.18}$$

$$\|u_n\|^2 - \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy - \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \le \epsilon_n\|u_n\|, \tag{3.19}$$

where $\epsilon_n \to 0$, as $n \to \infty$. Now, from Remark 2.1, it is easy to see that

$$C + \frac{\epsilon_n\|u_n\|}{4} \ge J(u_n) - \frac{\langle J'(u_n), u_n\rangle}{4} \ge \frac{\|u_n\|^2}{4},$$

which implies that $\|u_n\|$ is bounded. As a consequence, we have from Equations (3.18) and (3.19)

$$\sup_n \left\{\|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega}\right\} < \infty,$$

$$\sup_n \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \right\} < \infty.$$

Moreover, there exists a function $u_0 \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$, weakly in $H^1(\mathbb{R}^N)$, and for almost all $x \in \Omega$ and $y \in \partial\Omega$. Hence, we have established the claim. Next, let us divide the proof into the following cases:

Case I: $c \leq 0$
Employing Lemma 2.4(vii) and Fatou's lemma, we obtain

$$0 \leq I(u_0) \leq \underline{\lim}_{n\to\infty} I(u_n) = \underline{\lim}_{n\to\infty} \left[J(u_n) - \frac{1}{2}\left\langle J'(u_n), u_n \right\rangle \right] = c \leq 0.$$

Hence, when $c < 0$, there exists no Palais–Smale sequence, leading to the conclusion that $c = 0$, $I(u_0) = 0$, and consequently, $u_0 \equiv 0$. Further, using Lemma 2.4(v) and (vi), we get

$$\lim_{n\to\infty} \|u_n\|^2 = \lim_{n\to\infty} \left\{ 2J(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega} \right\} = 2c = 0.$$

This implies $u_n \to 0$, strongly in $H^1(\Omega)$.
Case II: $c \in \left[0, \min\left\{ \frac{(4-\mu_1)\pi}{4b}, \frac{(2-\mu_2)\pi}{4\theta} \right\}\right)$
We claim that $u_0 \not\equiv 0$ and $u_0 \in \partial B(\Omega)$. Let if possible, $u_0 \equiv 0$. Using Lemma 2.4(v) and (vi), we get

$$\begin{aligned} \lim_{n\to\infty} \|u_n\|^2 &= \lim_{n\to\infty} \left\{ 2J(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega} \right\} \\ &= 2c < \min\left\{ \frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta} \right\}. \end{aligned}$$

Hence, from Lemma 2.4 iii) and (iv), we have

$$\lim_{n\to\infty} \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \right\} = 0.$$

Thus, $\lim\limits_{n\to\infty} I(u_n) = 0$ and

$$0 < c = \lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} \left[I(u_n) + \frac{1}{2}\left\langle J'(u_n), u_n \right\rangle \right] = 0,$$

which is a contradiction. Hence, $u_0 \not\equiv 0$. For all $h \in C^\infty(\overline{\Omega})$,

$$0 = \lim_{n\to\infty} \left\langle J'(u_n), h \right\rangle.$$

By density this property also extends to any $u_0 \in H^1(\Omega)$,

$$\|u_0\|^2 = \iint\limits_{\Omega\times\Omega} \frac{F(y,u_0)f(x,u_0)u_0(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_0)g(x,u_0)u_0(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu,$$

which implies that $u_0 \in \partial B(\Omega)$ and this proves the claim. Given that (u_n, u_0) fulfills all the hypothesis of Lemma 3.2, we can conclude that

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu \right\} \\ &= \iint\limits_{\Omega\times\Omega} \frac{F(y,u_0)f(x,u_0)u_0(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_0)g(x,u_0)u_0(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu. \end{aligned}$$

Now to prove $u_n \to u_0$ strongly in $H^1(\Omega)$, it suffices to show that $\|u_n\| \to \|u_0\|$. By Fatou's lemma and the fact that $u_0 \in \partial B(\Omega)$, we get

$$\begin{aligned}
\|u_0\|^2 &\leq \underline{\lim}_{n\to\infty} \|u_n\|^2 \\
&= \underline{\lim}_{n\to\infty} \left\{2I(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega} + \left\langle J'(u_n), u_n \right\rangle\right\} \\
&= \underline{\lim}_{n\to\infty} \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu \right. \\
&\quad \left. + \left\langle J'(u_n), u_n \right\rangle \right\} \\
&= \|u_0\|^2.
\end{aligned}$$

This implies $u_n \to u_0$, strongly in $H^1(\Omega)$. □

From Lemma 3.4, we have obtained the Palais–Smale condition. Since the critical points of the functional J correspond to the solutions of (P). Thus, considering Lemmas 3.1 and 3.3, it suffices to prove that $J(u_0) = \frac{a(\Omega)^2}{2}$.

Lemma 3.5. *Assume that*

(i) $\overline{\lim_{t\to\infty}} \inf_{x\in\overline{\Omega}} h(x,t)t = \overline{\lim_{t\to\infty}} \inf_{y\in\partial\Omega} k(y,t)t = \infty$
(ii) $f'(x,0) = g'(y,0) = 0$, *for all* $(x,y) \in \Omega \times \partial\Omega$,

then there exists $u_0 \in \partial B(\Omega)$ *such that*

$$J(u_0) = \frac{a(\Omega)^2}{2}.$$

Proof. Let $\{u_n\}$ be any minimizing sequence in $\partial B(\Omega)$ such that

$$I(u_n) = J(u_n) \to \frac{a(\Omega)^2}{2} \quad \text{as } n \to \infty, \tag{3.20}$$

From Lemma 2.4(vii) and Equation (3.20), we obtain that

$$\sup_n \left\{ \iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu \right\} < \infty, \tag{3.21}$$

which implies that $\sup_n \|u_n\|^2 < \infty$. Thus, up to a subsequence there exists a function $u_0 \in H^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H^1(\Omega)$. From Equation (3.21), we can consequently infer, using Lemma 2.4(v) and (vi) that

$$\lim_{n\to\infty} \|F(u_n)\|_{0,\Omega} + \lim_{n\to\infty} \|G(u_n)\|_{0,\partial\Omega} = \|F(u_0)\|_{0,\Omega} + \|G(u_0)\|_{0,\partial\Omega}. \tag{3.22}$$

We claim that $u_0 \not\equiv 0$ and

$$\|u_0\|^2 \leq \iint\limits_{\Omega\times\Omega} \frac{F(y,u_0)f(x,u_0)u_0(x)}{|x-y|^{\mu_1}}\, dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_0)g(x,u_0)u_0(x)}{|x-y|^{\mu_2}}\, d\sigma d\nu.$$

Let us suppose, $u_0 \equiv 0$. From Equations (3.20) and (3.22)

$$\lim_{n\to\infty} \|u_n\|^2 = \lim_{n\to\infty} \left\{2J(u_n) + \|F(u_n)\|_{0,\Omega} + \|G(u_n)\|_{0,\partial\Omega}\right\}$$
$$= a(\Omega)^2 < \min\left\{\frac{(4-\mu_1)\pi}{2b}, \frac{(2-\mu_2)\pi}{2\theta}\right\}.$$

Consequently, from Lemma 2.4(iii) and (iv), we have

$$\lim_{n\to\infty}\left\{\iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\right\} = 0,$$

which further implies that

$$0 < \frac{a(\Omega)^2}{2} = \lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} I(u_n) = 0,$$

which is a contradiction. Hence, $u \not\equiv 0$. Now suppose that

$$\|u_0\|^2 > \iint\limits_{\Omega\times\Omega} \frac{F(y,u_0)f(x,u_0)u_0(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_0)g(x,u_0)u_0(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu.$$

Since all the hypotheses of Lemma 3.2 hold, we have

$$\begin{aligned}\|u_0\|^2 &\le \varliminf_{n\to\infty} \|u_n\|^2 \\ &= \varliminf_{n\to\infty}\left\{\iint\limits_{\Omega\times\Omega} \frac{F(y,u_n)f(x,u_n)u_n(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_n)g(x,u_n)u_n(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\right. \\ &= \iint\limits_{\Omega\times\Omega} \frac{F(y,u_0)f(x,u_0)u_0(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,u_0)g(x,u_0)u_0(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\end{aligned}$$

which is a contradiction. This proves the claim. Similar to Lemma 3.1, we construct $\Psi(\gamma)$

$$\Psi(\gamma) := \frac{1}{\gamma}\left[\iint\limits_{\Omega\times\Omega} \frac{F(y,\gamma u)f(x,\gamma u)u(x)}{|x-y|^{\mu_1}}\,dxdy + \iint\limits_{\partial\Omega\times\partial\Omega} \frac{G(y,\gamma u)g(x,\gamma u)u(x)}{|x-y|^{\mu_2}}\,d\sigma d\nu\right].$$

Thus for $u_0 \not\equiv 0$, there exists $\gamma > 0$ such that $\Psi(\gamma) = \|u_0\|^2$, consequently implying that $\gamma u_0 \in \partial B(\Omega)$. Moreover, from the previous claim and Equation (3.6), we deduce that $0 < \gamma \le 1$. Hence,

$$\frac{a(\Omega)^2}{2} \le J(\gamma u_0) = I(\gamma u_0) \le I(u_0) \le \varliminf_{n\to\infty} I(u_n) = \varliminf_{n\to\infty} J(u_n) = \frac{a(\Omega)^2}{2},$$

which implies $\gamma = 1$. Thus, $u_0 \in \partial B(\Omega)$, and $J(u_0) = \frac{a(\Omega)^2}{2}$. This concludes the proof of the lemma. □

Proof of Theorem 1.1: From Lemmas 3.3 and 3.5, we conclude that u_0 is a critical point of the functional J and since $J(u) = J(|u|)$, this implies $u_0 \ge 0$. Hence, u_0 is a nonnegative solution of (P). From regularity theory and the strong maximum principle, it follows that $u_0 \in H^2(\Omega) \cap C(\overline{\Omega})$ and $u_0 > 0$ in Ω. This concludes the theorem's proof.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this paper as no datasets were generated or analyzed during this study.

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