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Abstract. We consider a parametric Robin problem driven by the p-Laplacian and with a Carathéodory reaction. Our hypotheses on the reaction incorporate a special case p-logistic equations with a superdiffusive reaction. Using variational methods coupled with suitable truncation, perturbation and comparison techniques, we prove a bifurcation near infinity result.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Robin problem

$$\begin{cases} -\Delta_p u(z) = \lambda f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega, \end{cases} \tag{P_λ}$$

where $1 < p < \infty$. Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \text{ for all } u \in W^{1,p}(\Omega).$$

We are looking for solutions in the Sobolev space $W^{1,p}(\Omega)$. The weak distributional formulation of (P_λ) is

$$\int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz + \int_{\partial\Omega} \beta(z)|u|^{p-2} u h d\sigma$$
$$= \lambda \int_\Omega f(z, u) h dz \quad \text{for all } h \in W^{1,p}(\Omega)$$

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(or, equivalently, for all $h \in C^{\infty}(\overline{\Omega})$; recall that since $\partial\Omega$ is a C^2-manifold, the space $C^{\infty}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, see for example Gasinski and Papageorgiou [9, p. 189]). In the above equation, $d\sigma$ denotes the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$ (the surface measure on $\partial\Omega$).

The Robin boundary condition is interpreted using the nonlinear Green's identity (see Casas and Fernandez [5], Kenmochi [12], and Gasinski and Papageorgiou [9, p. 211]) as is the case in Lieberman [13]. So, according to that identity, there exists a unique element

$$\frac{\partial u}{\partial n_p} \in W^{-1/p',p'}(\partial\Omega) \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1\right),$$

which by extension we denote by

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{\mathbb{R}^N}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ such that if $\Delta_p u \in L^{p'}(\Omega)$, then

$$\int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega (\Delta_p u) h dz = \left\langle \frac{\partial u}{\partial n_p}, \gamma_0(h) \right\rangle \quad \text{for all } h \in W^{1,p}(\Omega).$$

Here γ_0 is the trace map and we have denoted by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $\left(W^{-1/p',p'}(\partial\Omega), W^{1/p',p}(\partial\Omega)\right)$.

The reaction $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \longmapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous). On $f(z,\cdot)$ we impose conditions near $+\infty$ and near 0^+. Our conditions are general and incorporate as a special case, the so-called *superdiffusive reaction* of the *p-logistic equation*. Finally $\lambda > 0$ is a parameter.

Our goal is to study the existence, nonexistence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. More precisely, we prove a bifurcation-type result for large values of the parameter $\lambda > 0$ (bifurcation near $+\infty$). So, we establish the existence of a critical parameter value $\lambda_* > 0$ such that for every $\lambda > \lambda_*$ problem (P_λ) admits at least two positive solutions, when $\lambda = \lambda_*$ problem (P_λ) has at least one positive solution and finally for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solutions.

Such bifurcation type results, were proved by Brock et al. [3], Filippakis et al. [8], Gasinski and Papageorgiou [10], Rădulescu and Repovs [17], Takeuchi [19,20] (semilinear or nonlinear Dirichlet problems) and by Cardinali et al. [4], Papageorgiou and Rădulescu [15] (for nonlinear Neumann problems). All the aforementioned results impose more restrictive conditions on the reaction $f(z,\cdot)$. Moreover, our work here complements the recent one by Papageorgiou and Rădulescu [16], where the authors prove for Robin problems a bifurcation theorem for small values of $\lambda > 0$ (bifurcation near zero). However, it should be pointed out that in [16] the differential operator is considerably more general and nonhomogeneous. It is an interesting open problem whether our work here can be extended to equations driven by such operators.

An important role in our analysis is played by the regularity theory of Lieberman [13], who established regularity up to the boundary (global regularity) for solutions of equations driven by a broad class of nonhomogeneous differential operators, which includes as a special case the p-Laplacian. The results of Lieberman [13] extend local regularity results of DiBenedetto [7] and Tolksdorf [21].

Our approach is variational and it is based on the critical point theory combined with suitable truncation, perturbation and comparison techniques.

2. Mathematical background

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), it the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This compactness-type condition, is in general weaker than the usual Palais–Smale condition ("PS-condition" for short). The two conditions coincide if φ is bounded below (see, for example, Denkowski et al. [6, p. 174]). The C-condition leads to a deformation theorem, from which one can derive the minimax theory of the critical values of φ. Prominent in that theory, is the so-called "mountain pass theorem" of Ambrosetti and Rabinowitz [2], which here we state in a slightly more general form (see, for example, Denkowski et al. [6, p. 179]).

Theorem 1. *Assume that $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$, $\rho > 0$, $||u_1 - u_0|| > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = \eta_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$. Then $c \geqslant \eta_\rho$ and c is a critical value of φ.

The analysis of problem (P_λ) will involve the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. We will also make use of the fact that $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In what follows by $||\cdot||$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$. So, we have

$$||u|| = \left(||u||_p^p + ||Du||_p^p\right)^{1/p} \text{ for all } u \in W^{1,p}(\Omega)$$

On $\partial\Omega$ we use the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define the Lebesgue spaces $L^p(\partial\Omega)$, $1 \leqslant p \leqslant \infty$. We know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ known as the "trace map", which satisfies $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. In fact γ_0 is compact and we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \ \left(\text{with } \frac{1}{p} + \frac{1}{p'} = 1\right)$$
$$\ker\gamma_0 = W_0^{1,p}(\Omega)$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map. All the restrictions of functions on $\partial\Omega$, are understood in the sense of traces.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in the $x \in \mathbb{R}$ variable, that is

$$|f_0(z,x)| \leqslant a_0(z)(1 + |x|^{r-1}) \text{ for a.a } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p \end{cases}$. We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)|u(z)|^p d\sigma - \int_\Omega F_0(z,u(z))dz \text{ for all } u \in W^{1,p}(\Omega).$$

The next proposition can be found in Papageorgiou and Rădulescu [16] and it is essentially a consequence of the nonlinear regularity theory of Lieberman [13].

Proposition 2. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \textit{ with } ||h||_{C^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and it is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_) + h) \textit{ for all } h \in W^{1,p}(\Omega) \textit{ with } ||h|| \leqslant \rho_1.$$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega |Du|^{p-2}(Du, Dy)_{\mathbb{R}^N} dz \text{ for all } u, y \in W^{1,p}(\Omega). \tag{1}$$

The next result about the map A is well-known (see, for example, Gasinski and Papageorgiou [9, p. 745]).

Proposition 3. *The map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (1) is demicontinuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is*

$$\text{“}\textit{if } u_n \xrightarrow{w} u \textit{ in } W^{1,p}(\Omega) \textit{ and } \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leqslant 0,$$
$$\textit{then } u_n \to u \textit{ in } W^{1,p}(\Omega).\text{”}$$

Finally, let us fix our notation. In what follows, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. If $x \in \mathbb{R}^N$, then we set $x^{\pm} = \max\{\pm x, 0\}$. So, if $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-$$

If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently, the mapping $z \longmapsto N_h(u)(z)$ is measurable.

3. Bifurcation near infinity

The hypotheses in the reaction $f(z, x)$ are the following:

H : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leqslant a(z)\,(1+x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^{\infty}(\Omega)_+,\ p \leqslant r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $F(z, x) \to -\infty$ uniformly for a.a. $z \in \Omega$ as $x \to +\infty$ and there exists $\tilde{u} \in L^1(\Omega)$ such that $\int_\Omega F(z, \tilde{u}(z))dz > 0$;

(iii) there exist $\tau > p$ and $M > 0$ such that $\tau F(z, x) \leqslant f(z, x)x$ for a.a. $z \in \Omega$, all $x \geqslant M$;

(iv) $-\hat{c} \leqslant \liminf\limits_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} \leqslant \limsup\limits_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} \leqslant 0$ uniformly for a.a. $z \in \Omega$;

(v) there exists $\mu > p$ such that for all $\eta > 0$, we can find $\tilde{\xi}_\eta > 0$ for which we have

$$y - x \geqslant \eta \Rightarrow \frac{f(z, x)}{x^{\mu-1}} - \frac{f(z, y)}{y^{\mu-1}} \geqslant \tilde{\xi}_\eta \quad \text{for a.a.} \quad z \in \Omega.$$

Remark 1. Since we are interested on positive solutions and all the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$.

Example 1. The following functions satisfy hypotheses H. For the sake of simplicity, we drop the z-dependence:

$$f_1(x) = x^{q-1} - x^{r-1} \text{ for all } x \geqslant 0,\ \text{ with } p < q < r < p^*,$$
$$f_2(x) = x^{\tau-1}\left(\frac{1}{\tau} - \ln x\right) \text{ for all } x \geqslant 0,\ \text{ with } p < \tau < p^*$$

The function $f_1(x)$ corresponds to the superdiffusive reaction of the p-logistic equation.

Our condition on the boundary weight $\beta(\cdot)$ is the following:

$H(\beta)$: $\beta \in C^{1,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1)$, $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$ and $\beta \neq 0$.

Under this hypothesis, we know that the eigenvalue problem

$$-\Delta_p u(z) = \lambda(u(z))^{p-2}u(z) \text{ in } \Omega, \ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, \tag{2}$$

has a first eigenvalue $\hat{\lambda}_1 > 0$ which is isolated, simple and admits the following variational characterization

$$\hat{\lambda}_1 = \inf\left[\frac{||Du||_p^p + \int_{\partial\Omega}\beta(z)|u|^p d\sigma}{||u||_p^p} : \ u \in W^{1,p}(\Omega), \ u \neq 0\right]. \tag{3}$$

This infimum is realized on the corresponding one dimensional eigenspace whose elements are in $C^1(\overline{\Omega})$ and have constant sign. For details, we refer to Papageorgiou and Rădulescu [14].

We introduce the following truncation-perturbation of the term $\lambda f(z,\cdot)$:

$$\hat{f}_\lambda(z,x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ \lambda f(z,x) + x^{p-1} & \text{if } 0 < x. \end{cases}$$

This is a Carathéodory function. We set $\hat{F}_\lambda(z,x) = \int_0^x \hat{f}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)u^+(z)^p d\sigma - \int_\Omega \hat{F}_\lambda(z,u(z))dz$$

for all $u \in W^{1,p}(\Omega)$.

Proposition 4. *If hypotheses $H, H(\beta)$ hold and $\lambda > 0$, then the functional $\hat{\varphi}_\lambda$ is bounded below.*

Proof. Hypotheses $H(i), (ii)$, imply that we can find $c_1 > 0$ such that

$$F(z,x) \leqslant c_1 \quad \text{for a.a.} \quad z \in \Omega, \quad \text{all } x \geqslant 0. \tag{4}$$

Then for all $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned} \hat{\varphi}_\lambda(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}_\lambda(z,u)dz \\ &\geqslant \frac{1}{p}||Du||_p^p - c_1|\Omega|_N \text{ (see (3), (4) and hypothesis } H(\beta)), \\ &\Rightarrow \hat{\varphi}_\lambda \text{ is bounded below.} \end{aligned}$$

The proof is complete. □

Proposition 5. *If hypotheses $H, H(\beta)$ hold and $\lambda > 0$, then the functional $\hat{\varphi}_\lambda$ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subset W^{1,p}(\Omega)$ be a sequence such that

$$|\hat{\varphi}_\lambda(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \geqslant 1 \tag{5}$$
$$(1+||u_n||)\hat{\varphi}_\lambda'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{6}$$

From (6), we have

$$\left|\left\langle \hat{\varphi}_\lambda'(u_n), h\right\rangle\right| \leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+,$$
$$\Rightarrow \left|\langle A(u_n), h\rangle + \int_\Omega |u_n|^{p-2}u_n h dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p-1} h d\sigma - \int_\Omega \hat{f}_\lambda(z, u_n) h dz\right|$$
$$\leqslant \frac{\epsilon_n ||h||}{1+||u_n||}$$
$$\text{for all } h \in W^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+. \tag{7}$$

In (7) we choose $h = -u_n^- \in W^{1,p}(\Omega)$ and obtain

$$||u_n^-||^p \leqslant \epsilon_n \text{ for all } n \geqslant 1 \text{ (see (3))}, \Rightarrow u_n^- \to 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty. \tag{8}$$

Next in (7) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-||Du_n^+||_p^p - \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma + \int_\Omega \lambda f(z, u_n^+) u_n^+ dz \leqslant \epsilon_n$$
$$\text{for all } n \geqslant 1 \text{ (see (3))}. \tag{9}$$

On the other hand from (5) and (8), we have

$$||Du_n^+||_p^p + \int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma - \lambda \int_\Omega pF(z, u_n^+) dz \leqslant M_2 \tag{10}$$
$$\text{for some } M_2 > 0, \text{ all } n \geqslant 1.$$

Adding (9) and (10), we obtain

$$\lambda \int_\Omega \left[f(z, u_n^+)u_n^+ - pF(z, u_n^+)\right] dz \leqslant M_3 \text{ for some } M_3 > 0, \text{ all } n \geqslant 1,$$
$$\Rightarrow \lambda(\tau - p) \int_\Omega F(z, u_n^+) dz + \lambda \int_\Omega \left[f(z, u_n^+)u_n^+ - \tau F(z, u_n^+)\right] dz$$
$$\leqslant M_3 \text{ for all } n \geqslant 1,$$
$$\Rightarrow \lambda(\tau - p) \int_\Omega F(z, u_n^+) dz \leqslant M_4 \text{ for some } M_4 > 0, \text{ all } n \geqslant 1 \tag{11}$$
$$\text{(see hypotheses } H(i), (iii)).$$

From (5) and (8), we have

$$\frac{1}{p}||Du_n^+||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma - \lambda \int_\Omega F(z, u_n^+) dz \leqslant M_5 \tag{12}$$
$$\text{for some } M_5 > 0, \text{ all } n \geqslant 1 \text{ (see (3))},$$
$$\Rightarrow ||Du_n^+||_p^p \leqslant M_6 \text{ for some } M_6 > 0, \text{ all } n \geqslant 1 \tag{13}$$
$$\text{(see hypothesis } H(\beta), \text{ (11) and recall } \tau > p \text{ (see } H(iii))).$$

Let $V = \{u \in W^{1,p}(\Omega) : \int_\Omega u(z)dz = 0\}$. We have

$$W^{1,p}(\Omega) = \mathbb{R} \oplus V.$$

Then every $u_n \in W^{1,p}(\Omega)$ $n \geqslant 1$, can be written in a unique way as

$$u_n = \bar{u}_n + \hat{u}_n \text{ with } \bar{u}_n \in \mathbb{R},\ \hat{u}_n \in V \text{ for all } n \geqslant 1.$$

From (8) and (13), we see that

$$\{Du_n\}_{n\geqslant 1} \subseteq L^p(\Omega, \mathbb{R}^N) \text{ is bounded, } \Rightarrow \{D\hat{u}_n\}_{n\geqslant 1} \subseteq L^p(\Omega, \mathbb{R}^N) \text{ is bounded.}$$

By virtue of the Poincaré–Wirtinger inequality (see, for example, Gasinski and Papageorgiou [9, p. 84]), we have that

$$\{\hat{u}_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{14}$$

Suppose that $|\bar{u}_n| \to \infty$ as $n \to \infty$. We know that there exists $c_2 > 0$ such that

$$\begin{aligned} &||\bar{u}_n|| \leqslant c_2||u_n|| \text{ (see Goldberg [11, p.48])} \\ &\Rightarrow ||u_n|| \to +\infty \text{ as } n \to \infty. \end{aligned}$$

Also, we have

$$\begin{aligned} u_n^- &= (\bar{u}_n + \hat{u}_n)^- = -\min\{\bar{u}_n + \hat{u}_n, 0\} \\ &\geqslant \bar{u}_n^- + \hat{u}_n^- \text{ for all } n \geqslant 1 \text{ (see Schaefer [18, p. 53).} \end{aligned}$$

From (8) we see that $\bar{u}_n^- \to 0$ and so $\bar{u}_n^+ \to +\infty$ as $n \to \infty$. Therefore $u_n(z) \to +\infty$ for a.a. $z \in \Omega$, hence $u_n^+(z) \to \infty$ for a.a. $z \in \Omega$.

Let $y_n = \frac{u_n}{||u_n||}$ $n \geqslant 1$. Then $||y_n|| = 1$ for all $n \geqslant 1$ and so by passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{15}$$

We have

$$y_n = \bar{y}_n + \hat{y}_n \text{ with } \bar{y}_n = \frac{\bar{u}_n}{||u_n||},\ \hat{y}_n = \frac{\hat{u}_n}{||u_n||} \text{ for all } n \geqslant 1.$$

From (14) and (15), we infer that

$$y_n \to y = \bar{y} \in \mathbb{R} \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ and } ||y|| = 1.$$

Since $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$ as $n \to \infty$, from hypothesis $H_2(ii)$ and Fatou's lemma, we have

$$\int_\Omega F(z, u_n^+)dz \to -\infty \text{ as } n \to \infty. \tag{16}$$

On the other hand from (12), we have

$$-\lambda \int_\Omega F(z, u_n^+)dz \leqslant M_5 \text{ for all } n \geqslant 1. \tag{17}$$

Comparing (16) and (17), we reach a contradiction (recall that $\lambda > 0$). So, we have that

$$\begin{aligned}&\{u_n^+\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded,}\\ &\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (8)).}\end{aligned}$$

Therefore, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \quad (18)$$

In (7) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (18). Then we have

$$\begin{aligned}&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,\\ &\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 3),}\\ &\Rightarrow \hat{\varphi}_\lambda \text{ satisfies the } C-\text{condition.}\end{aligned}$$

This completes the proof. □

Remark 2. Propositions 4 and 5 imply that for all $\lambda > 0$, the functional $\hat{\varphi}_\lambda$ satisfies the PS-condition too (see [6, p. 174]).

We introduce the following sets

$$\begin{aligned}\mathcal{L} &= \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\}\\ S(\lambda) &= \text{set of positive solutions of problem } (P_\lambda).\end{aligned}$$

Proposition 6. *If hypotheses H and $H(\beta)$ hold, then for every $\lambda > 0$, $S(\lambda) \subseteq$ int C_+ and $\lambda_* = \inf \mathcal{L} > 0$.*

Proof. We may assume that $\lambda \in \mathcal{L}$ (otherwise $S(\lambda) = \varnothing$). Let $u \in S(\lambda)$. Then we have

$$\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z)|u|^{p-2}uh d\sigma = \int_\sigma \lambda f(z,u)h dz \text{ for all } h \in W^{1,p}(\Omega). \quad (19)$$

Let $\langle\cdot,\cdot\rangle_0$ denote the duality brackets for the pair $(W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*, W_0^{1,p}(\Omega))$. From the representation theorem for the elements of the dual space $W^{-1,p'}(\Omega)$ (see, for example, Gasinski and Papageorgiou [9, p. 212]), we have

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \in W^{-1,p'}(\Omega).$$

Integration by parts (Green's identity), shows that

$$\langle A(u), h\rangle = \left\langle -\Delta_p u, h\right\rangle_0 \text{ for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

Using this equation in (19) and recalling that, if $h \in W_0^{1,p}(\Omega)$, then $h|_{\partial\Omega} = 0$ in the sense of trace. So, we have

$$\langle -\Delta_p u, h \rangle_0 = \int_\Omega \lambda f(z,u) h dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

Note that hypothesis $H(i)$ implies that $N_f(u) \in L^{r'}(\Omega) \subseteq L^{p'}(\Omega)$ (recall that $p \leqslant r$, hence $r' \leqslant p'$; $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{r} + \frac{1}{r'} = 1$). So, we have

$$\begin{aligned}
&\int_\Omega \lambda f(z,u) h dz = \langle \lambda N_f(u), h \rangle_0 \quad \text{for all } h \in W_0^{1,p}(\Omega),\\
&\Rightarrow \langle -\Delta_p u - \lambda N_f(u), h \rangle_0 = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega),\\
&\Rightarrow -\Delta_p u(z) = \lambda f(z, u(z)) \quad \text{for a.a. } z \in \Omega. \qquad (20)
\end{aligned}$$

Then using the nonlinear Green's identity mentioned in the Introduction (see also Casas and Fernandez [5], Gasinski and Papageorgiou [9, p. 210], and Kenmochi [12]), we have

$$\langle A(u), h \rangle + \int_\Omega (\Delta_p u) h dz = \left\langle \frac{\partial u}{\partial n_p}, h \right\rangle_{\partial\Omega} \quad \text{for all } h \in W^{1,p}(\Omega). \qquad (21)$$

Here by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair $(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p'},p}(\partial\Omega))$. Returning to (19) and using (20) and (21), we obtain

$$\left\langle \frac{\partial u}{\partial n_p} + \beta(z) u^{p-1}, h \right\rangle_{\partial\Omega} = 0 \text{ for all } h \in W^{1,p}(\Omega). \qquad (22)$$

Let γ_0 denote the trace map on $W^{1,p}(\Omega)$. We know that

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega).$$

So, from (22) it follows that

$$\frac{\partial u}{\partial n_p} + \beta(z) u^{p-1} = 0 \text{ in } W^{-1/p',p'}(\partial\Omega). \qquad (23)$$

In fact the nonlinear regularity theory will help us to interpret this boundary condition in a pointwise sense. More precisely, from (20), (23) and Winkert [22], we know that $u \in L^\infty(\Omega)$. Thus we can apply Theorem 2 of Lieberman [13] and have that $u \in C_+ \backslash \{0\}$. So, it follows that relation (23) holds for all $z \in \partial\Omega$.

Hypotheses $H(i)$, (iv) imply that given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z,x) + \xi_\rho x^{p-1} \geqslant 0 \text{ for a.a } z \in \Omega, \text{ all } x \in [0,\rho]. \qquad (24)$$

Let $\rho = ||u||_\infty$ and let $\xi_\rho > 0$ as in (24). Then

$$\begin{aligned}
&-\Delta_p u(z) + \lambda \xi_\rho u(z)^{p-1} = \lambda f(z, u(z)) + \lambda \xi_\rho u(z)^{p-1}\\
&\quad \geqslant 0 \text{ for a.a. } z \in \Omega \text{ (see (20), (24))}\\
&\quad \Rightarrow \Delta_p u(z) \leqslant \lambda \xi_\rho u(z)^{p-1} \text{ for a.a. } z \in \Omega,\\
&\quad \Rightarrow u \in \operatorname{int} C_+ \text{ (see, for example, Gasinski and Papageorgiou [9, p. 738]).}
\end{aligned}$$

Therefore, we conclude that $S(\lambda) \subseteq \operatorname{int} C_+$.

Note that hypotheses $H(i)$, (ii), (iv) imply that

$$f(z,x) \leqslant c_3 x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0 \text{ with } c_3 > 0. \tag{25}$$

Let $\lambda \in \left(0, \frac{\hat{\lambda}_1}{c_2}\right)$. Recall that $\hat{\lambda}_1 > 0$ is the principal eigenvalue of the negative Robin p-Laplacian (see (2)). We have

$$\lambda f(z,x) < \hat{\lambda}_1 x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x > 0 \text{ (see (25))}. \tag{26}$$

Suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ and we have

$$\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma = \lambda \int_\Omega f(z, u_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega).$$

Choosing $h = u_\lambda \in \operatorname{int} C_+$, we have

$$||Du_\lambda||_p^p + \int_{\partial\Omega} \beta(z) u_\lambda^p d\sigma < \hat{\lambda}_1 ||u_\lambda||_p^p \text{ (see (26) and recall } u_\lambda \in \operatorname{int} C_+),$$

which contradicts (2). Therefore $\lambda_* \geqslant \frac{\hat{\lambda}_1}{c_2} > 0$. $\square$

Next we show the nonemptiness and a structural property of the admissible set $\mathcal{L}$.

Proposition 7. *If hypotheses H and $H(\beta)$ hold, then $\mathcal{L} \neq \emptyset$ and $\lambda \in \mathcal{L}$, $\eta > \lambda$ imply $\eta \in \mathcal{L}$.*

Proof. From Propositions 4 and 5 and Theorem 2.1.14 of Denkowski et al. [6, p. 184], we know that there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(u_0) = \inf\left[\hat{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\right]. \tag{27}$$

The integral functional $u \longmapsto I_F(u) = \int_\Omega F(z, u(z)) dz$ is continuous on $L^1(\Omega)$ and, by hypothesis $H(ii)$, $I_F(\tilde{u}) > 0$. Exploiting the density of $W^{1,p}(\Omega)$ in $L^1(\Omega)$, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$I_F(\tilde{u}_0) = \int_\Omega F(z, \tilde{u}_0(z)) dz > 0. \tag{28}$$

Recalling that $F(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$, we see that we can replace $\tilde{u}_0$ by $\tilde{u}_0^+ \in W^{1,p}(\Omega)$. So, without any loss of generality, we may assume that $\tilde{u}_0 \geqslant 0$.

We have

$$\hat{\varphi}_\lambda(\tilde{u}_0) = \frac{1}{p}||D\tilde{u}_0||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z) \tilde{u}_0^p d\sigma - \lambda \int_\Omega F(z, \tilde{u}_0) dz.$$

Because of (28), we see that we can find $\tilde{\lambda} > 0$ such that

$$\begin{aligned} &\hat{\varphi}_\lambda(\tilde{u}_0) < 0 \text{ for all } \lambda > \tilde{\lambda}, \\ &\Rightarrow \hat{\varphi}_\lambda(u_0) < 0 = \hat{\varphi}_\lambda(0) \text{ (see (27)), hence } u_0 \neq 0. \end{aligned}$$

From (27) we have

$$\hat{\varphi}_{\lambda}'(u_0) = 0,$$

$$\Rightarrow \langle A(u_0), h\rangle + \int_{\Omega} |u_0|^{p-2} u_0 h dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1} h d\sigma$$
$$= \int_{\Omega} \hat{f}_{\lambda}(z, u_0) h dz \text{ for all } h \in W^{1,p}(\Omega). \quad (29)$$

In (29) we choose $h = -u_0^- \in W^{1,p}(\Omega)$ and obtain

$$||Du_0^-||_p^p + ||u_0^-||_p^p = 0 \text{ (see (3)), hence } u_0 \geqslant 0,\ u_0 \neq 0.$$

So, (29) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z) u_0^{p-1} h d\sigma = \lambda \int_{\Omega} f(z, u_0) h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (3)).}$$

From this equality, as in the proof of Proposition 6, using the nonlinear Green's identity, we obtain

$$-\Delta_p u_0(z) = \lambda f(z, u_0(z)) \text{ for a.a. } z \in \Omega,\ \frac{\partial u_0}{\partial n_p} + \beta(z) u_0^{p-1} = 0 \text{ on } \partial\Omega,$$
$$\Rightarrow u_0 \in S(\lambda) \text{ and so } (\tilde{\lambda}, \infty) \subseteq \mathcal{L},\ \text{ hence } \mathcal{L} \neq \emptyset.$$

Next let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Choose $\vartheta \in (0, 1)$ such that $\lambda = \vartheta^{\tau-p}\eta$ (here $\tau > p$ is as in hypothesis $H(v)$). Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 6).

We have

$$\begin{aligned} -\Delta_p(\vartheta u_\lambda)(z) &= \vartheta^{p-1}\lambda f(z, u_\lambda(z)) \text{ (since } u_\lambda \in S(\lambda)) \\ &= \vartheta^{\tau-1}\eta f(z, u_\lambda(z)) \text{ (since } \lambda = \vartheta^{\tau-p}\eta) \\ &\leqslant \eta f(z, \vartheta u_\lambda(z)) \text{ for a.a. } z \in \Omega \\ &\quad \text{(since } \vartheta \in (0, 1),\ \text{ see hypothesis } H(v)). \end{aligned} \quad (30)$$

Also, we have

$$\frac{\partial(\vartheta u_\lambda)}{\partial n_p} + \beta(z)(\vartheta u_\lambda)^{p-1} = 0 \quad \text{on } \partial\Omega. \quad (31)$$

Let $\underline{u} = \vartheta u_\lambda \in \operatorname{int} C_+$ and consider the following truncation-perturbation of the reaction in problem (P_η) :

$$\hat{g}_\eta(z, x) = \begin{cases} \eta f(z, \underline{u}(z)) + \underline{u}(z)^{p-1} & \text{if } x \leqslant \underline{u}(z) \\ \eta f(z, x) + x^{p-1} & \text{if } \underline{u}(z) < x. \end{cases} \quad (32)$$

This is a Carathéodory function. We set $\hat{G}_\eta(z, x) = \int_0^x \hat{g}_\eta(z, s) ds$ and consider the C^1−functional $\hat{\psi}_\eta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\eta(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^+(z)^p d\sigma - \int_{\Omega} \hat{G}_\eta(z, u(z)) dz.$$

From (32) it is clear that for $u \geqslant \underline{u}$

$$\hat{\psi}_\eta = \hat{\varphi}_\eta + \xi^*_\eta \text{ for some } \xi^*_\eta \in \mathbb{R}.$$

So, it follows that

- $\hat{\psi}_\eta$ is bounded below (see Proposition 4).
- $\hat{\psi}_\eta$ satisfies the C-condition (see Proposition 5).

Therefore we can find $u_\eta \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\hat{\psi}_\eta(u_\eta) = \inf\left[\hat{\psi}_\eta(u) : u \in W^{1,p}(\Omega)\right] \text{ (see [6, p. 184])},\\
&\Rightarrow \hat{\psi}'_\eta(u_\eta) = 0,\\
&\Rightarrow \langle A(u_\eta), h\rangle + \int_\Omega |u_\eta|^{p-2}u_\eta h dz + \int_{\partial\Omega} \beta(z)(u^+_\eta)^{p-1} h d\sigma = \int_\Omega \hat{g}_\eta(z, u_\eta) h dz\\
&\text{for all } h \in W^{1,p}(\Omega).
\end{aligned} \tag{33}$$

Choosing $h = (\underline{u} - u_\eta)^+ \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
&\langle A(u_\eta), (\underline{u} - u_\eta)^+\rangle + \int_\Omega |u_\eta|^{p-2}u_\eta(\underline{u} - u_\eta)^+ dz\\
&\quad + \int_{\partial\Omega} \beta(z)(u^+_\eta)^{p-1}(\underline{u} - u_\eta)^+ d\sigma = \int_\Omega \hat{g}_\eta(z, u_\eta)(\underline{u} - u_\eta)^+ dz\\
&= \int_\Omega \left[\eta f(z, \underline{u}) + \underline{u}^{p-1}\right](\underline{u} - u_\eta)^+ dz \text{ (see (32))}\\
&\geqslant \langle A(\underline{u}), (\underline{u} - u_\eta)^+\rangle + \int_\Omega \underline{u}^{p-1}(\underline{u} - u_\eta)^+ dz + \int_{\partial\Omega} \beta(z)\underline{u}^{p-1}(\underline{u} - u_\eta)^+ d\sigma\\
&\text{(see (30), (31) and use the nonlinear Green's identity)}\\
&\Rightarrow \langle A(\underline{u}) - A(u_\eta), (\underline{u} - u_\eta)^+\rangle\\
&\quad + \int_\Omega (\underline{u}^{p-1} - |u_\eta|^{p-2}u_\eta)(\underline{u} - u_\eta)^+ dz\\
&\quad + \int_{\partial\Omega} \beta(z)(\underline{u}^{p-1} - (u^+_\eta)^{p-1})(\underline{u} - u_\eta)^+ d\sigma \leqslant 0.
\end{aligned} \tag{34}$$

Note that

$$\begin{aligned}
&\underline{u}(z) > u_\eta(z) \text{ for all } z \in \{\underline{u} > u_\eta\},\\
&\quad \Rightarrow \underline{u}(z) \geqslant u^+_\eta(z) \text{ for all } z \in \{\underline{u} > u_\eta\} \text{ (recall } \underline{u} = \vartheta u_\lambda \in \operatorname{int} C_+)\\
&\quad \Rightarrow \int_{\partial\Omega} \beta(z)(\underline{u}^{p-1} - (u^+_\eta)^{p-1})(\underline{u} - u_\eta)^+ d\sigma \geqslant 0 \text{ (see hypothesis } H(\beta)).
\end{aligned}$$

Using this last inequality in (34), we obtain

$$\begin{aligned}
&\langle A(\underline{u}) - A(u_\eta), (\underline{u} - u_\eta)^+\rangle + \int_\Omega (\underline{u}^{p-1} - |u_\eta|^{p-2}u_\eta)(\underline{u} - u_\eta)^+ dz \leqslant 0,\\
&\quad \Rightarrow |\{\underline{u} > u_\eta\}|_N = 0, \text{ hence } \underline{u} \leqslant u_\eta.
\end{aligned}$$

So, relation (33) becomes

$$\langle A(u_\eta), h\rangle + \int_{\partial\Omega} \beta(z) u_\eta^{p-1} h d\sigma = \int_\Omega \eta f(z, u_\eta) h dz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (32))}$$
$$\Rightarrow u_\eta \in S(\eta) \subseteq \operatorname{int} C_+ \text{ (see Proposition 6 its proof)},$$
$$\Rightarrow \eta \in \mathcal{L} \text{ and so } [\lambda, +\infty) \subseteq \mathcal{L}.$$

This completes the proof. □

Remark 3. According to Proposition 7, the admissible set $\mathcal{L}$ is an upper half line.

Next we prove a multiplicity result for the positive solutions of problem (P_λ). To do this we need to strengthen the conditions on $f(z, \cdot)$.

The new hypotheses on the reaction function $f(z, x)$, are the following:

$H^{'}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H^{'}(i) \to (v)$ are the same as the corresponding hypotheses $H(i) \to (v)$ and
(vi) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \longmapsto f(z, x) + \xi_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 4. If $p = 2$ and $f(z, \cdot)$ is differentiable with $f_x^{'}(z, \cdot)$ being $L^\infty(\Omega)$-bounded on bounded sets in $\mathbb{R}$, then hypothesis $H^{'}(vi)$ is satisfied. Also, the two examples given after hypotheses H (the functions $f_1(x)$ and $f_2(x)$), satisfy the new hypothesis $H^{'}(vi)$.

Proposition 8. *If hypotheses H, $H(\beta)$ hold and $\lambda > \lambda_*$, then problem (P_λ) has at least two positive solutions*

$$u_0, \hat{u} \in \mathit{int}\, C_+, u_0 \neq \hat{u}.$$

Proof. Let $\nu \in (\lambda_*, \lambda)$. By virtue of Proposition 7, $\nu \in \mathcal{L}$ and so we can find $u_\nu \in S(\nu) \subseteq \operatorname{int} C_+$. We choose $\vartheta \in (0, 1)$ such that $\nu = \vartheta^{\tau - p}\lambda$ (see hypothesis $H^{'}(iii)$). We set $\underline{u} = \vartheta u_\nu \in \operatorname{int} C_+$ and define

$$\hat{g}_\lambda(z, x) = \begin{cases} \lambda f(z, \underline{u}(z)) + \underline{u}(z)^{p-1} & \text{if } x \leqslant \underline{u}(z) \\ \lambda f(z, x) + x^{p-1} & \text{if } \underline{u}(z) < x. \end{cases} \tag{35}$$

This is a Carathéodory function. We set $\hat{G}_\lambda(z, x) = \int_0^x \hat{g}_\lambda(z, s) ds$ and consider the C^1-functional $\hat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z) u^+(z)^p dz$$
$$- \int_\Omega \hat{G}_\lambda(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Reasoning as in the second half of the proof of Proposition 7, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\psi}_\lambda(u_0) = \inf[\hat{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)]. \tag{36}$$

From (36) it follows that

$$u_0 \in S(\lambda) \subseteq \operatorname{int} C_+ \text{ and } \underline{u} \leqslant u_0 \text{ (see the proof of Proposition 7).}$$

Let $\rho = ||u_0||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H^{'}(vi)$. Let $\delta > 0$ and set $\underline{u}^\delta = \underline{u} + \delta \in \operatorname{int} C_+$. We have

$$\begin{aligned}
&-\Delta_p \underline{u}^\delta(z) + \lambda\xi_\rho \underline{u}^\delta(z)^{p-1}\\
&\leqslant -\Delta_p \underline{u}(z) + \lambda\xi_\rho \underline{u}(z)^{p-1} + \gamma(\delta) \text{ with } \gamma(\delta) \to 0^+ \text{ as } \delta \to 0^+\\
&= \vartheta^{p-1}(-\Delta_p u_\nu(z)) + \lambda\xi_\rho \underline{u}(z)^{p-1} + \gamma(\delta)\\
&= \vartheta^{p-1}\nu f(z, u_\nu(z)) + \lambda\xi_\rho \underline{u}(z)^{p-1} + \gamma(\delta) \text{ (since } u_\nu \in S(\nu))\\
&= \vartheta^{\tau-1}\lambda f(z, u_\nu(z)) + \lambda\xi_\rho \underline{u}(z)^{p-1} + \gamma(\delta) \text{ (since } \nu = \vartheta^{\tau-p}\lambda).
\end{aligned} \tag{37}$$

Let $m_\nu = \min\limits_{\overline{\Omega}} u_\nu > 0$ (recall that $u_\nu \in \operatorname{int} C_+$). By virtue of hypothesis $H^{'}(v)$ for $\eta = (1-\vartheta)m_\nu$, we can find $\tilde{\xi}_\eta > 0$ such that

$$\begin{aligned}
&\frac{f(z, \underline{u}(z))}{\underline{u}(z)^{\tau-1}} - \frac{f(z, u_\nu(z))}{u_\nu(z)^{\tau-1}} \geqslant \tilde{\xi}_\eta \text{ (recall } \underline{u} = \vartheta u_\nu, \vartheta \in (0,1))\\
&\Rightarrow \frac{f(z, \underline{u}(z))}{\vartheta^{\tau-1}} - f(z, u_\nu(z)) \geqslant \tilde{\xi}_\eta m_\nu^{\tau-1},\\
&\Rightarrow f(z, \underline{u}(z)) - \vartheta^{\tau-1} f(z, u_\nu(z)) \geqslant \vartheta^{\tau-1}\tilde{\xi}_\eta m_\nu^{\tau-1} > 0.
\end{aligned} \tag{38}$$

Using (38) in (37), we obtain

$$\begin{aligned}
&-\Delta_p \underline{u}^\delta(z) + \lambda\xi_\rho \underline{u}^\delta(z)^{p-1}\\
&\leqslant \lambda f(z, \underline{u}(z)) - \vartheta^{\tau-1}\tilde{\xi}_\eta m_\nu^{\tau-1} + \lambda\xi_\rho \underline{u}(z)^{p-1} + \gamma(\delta)\\
&\leqslant \lambda f(z, u_0(z)) + \lambda\xi_\eta u_0(z)^{p-1} - \vartheta^{\tau-1}\tilde{\xi}_\eta m_\nu^{\tau-1} + \gamma(\delta)\\
&\quad \text{(see hypothesis } H^{'}(vi) \text{ and recall } \underline{u} \leqslant u_0)\\
&\leqslant -\Delta_p u_0(z) + \lambda\xi_\rho u_0(z)^{p-1} \text{ for a.a. } z \in \Omega, \text{ for } \delta > 0 \text{ small}\\
&\quad \text{(recall } \gamma(\delta) \to 0^+ \text{ as } \delta \to 0^+).
\end{aligned} \tag{39}$$

Note that

$$\frac{\partial \underline{u}}{\partial n_\eta} + \beta(z)\underline{u}^{p-1} = 0 \text{ and } \frac{\partial u_0}{\partial n_p} + \beta(z)u_0^{p-1} = 0 \text{ on } \partial\Omega. \tag{40}$$

From (39), (40) and using the nonlinear Green's identity, as before we obtain

$$\begin{aligned}
&u_0 - \underline{u}^\delta \geqslant 0 \text{ for } \delta > 0 \text{ small},\\
&\Rightarrow u_0 - \underline{u} \in \operatorname{int} C_+.
\end{aligned} \tag{41}$$

Let $\left[\underline{u}\right) = \left\{u \in W^{1,p}(\Omega) : \underline{u}(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\right\}$. From (35) it is clear that on $[\underline{u})$

$$\hat{\psi}_\lambda = \hat{\varphi}_\lambda + \xi_\lambda^* \text{ with } \xi_\lambda^* \in \mathbb{R}.$$

Then (41) implies that u_0 is a local $C^1(\overline{\Omega})$-minimizer of $\hat{\psi}_\lambda$. Invoking Proposition 2, we infer that u_0 is a local $W^{1,p}(\Omega)$-minimizer of $\hat{\psi}_\lambda$.

By virtue of hypothesis $H^{'}(iv)$, given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p} x^p \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{42}$$

Then for $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta$, we have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{p}||u^-||_p^p + \frac{1}{p}\int_{\partial\Omega}\beta(z)(u^+)^p d\sigma - \int_\Omega \lambda F(z,u^+)dz \\
&\geqslant \frac{1}{p}\left[||Du^+||_p^p + \int_{\partial\Omega}\beta(z)(u^+)^p d\sigma\right] + \frac{1}{p}||u^-||_p^p - \frac{\epsilon}{p}\lambda||u^+||_p^p \text{ (see (42))} \\
&\geqslant \frac{\hat{\lambda}_1 - \lambda\epsilon}{p}||u^+||_p^p + \frac{1}{p}||u^-||_p^p \text{ (see (21))} \\
&\geqslant c_4||u||_p^p \text{ for some } c_4 > 0 \text{ (choosing } \epsilon \in (0, \hat{\lambda}_1/\lambda)).
\end{aligned}$$

So, we have

$$\begin{aligned}
&\hat{\varphi}_\lambda(0) = 0 < \hat{\varphi}_\lambda(u) \text{ for all } u \in C^1(\overline{\Omega}),\ 0 < ||u||_{C^1(\overline{\Omega})} \leqslant \delta, \\
&\Rightarrow u = 0 \text{ is a (strict) local } C^1(\overline{\Omega}) - \text{minimizer of } \hat{\varphi}_\lambda, \\
&\Rightarrow u = 0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \hat{\varphi}_\lambda \text{ (see Proposition 2).}
\end{aligned}$$

Without any loss of generality, we may assume that $\hat{\varphi}_\lambda(0) = 0 \leqslant \hat{\varphi}_\lambda(u_0)$ (the analysis is similar if the opposite inequality holds). We may assume that the critical set of $\hat{\varphi}_\lambda$ is finite (otherwise, we already have infinitely many positive solutions for problem (P_λ)). Since $u_0 \in \operatorname{int} C_+$ is a local minimizer of $\hat{\varphi}_\lambda$, we can find $\rho \in (0,1)$ small such that

$$0 = \hat{\varphi}_\lambda(0) \leqslant \hat{\varphi}_\lambda(u_0) < \inf[\hat{\varphi_\lambda}(u) : ||u - u_0|| = \rho] = m_\rho,\ ||u_0|| > \rho \tag{43}$$

(see Aizicovici, Papageorgiou and Staicu [1] (proof of Proposition 29). Recall that $\hat{\varphi}_\lambda$ satisfies the C-condition (see Proposition 5). This fact and (43) permit the use of Theorem 1 (the mountain pass theorem). So, se can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda'(\hat{u}) = 0 \text{ and } m_\rho \leqslant \hat{\varphi}_\lambda(\hat{u}). \tag{44}$$

From (43) and (44) it follows that $\hat{u} \notin \{0, u_0\}$ and $\hat{u} \in S(\lambda) \subseteq \operatorname{int} C_+$. □

Next we examine what happens in the critical case $\lambda = \lambda_*$.

Proposition 9. *If hypotheses H and $H(\beta)$ hold, then $\lambda_* \in \mathcal{L}$.*

Proof. Le $\{\lambda_n\}_{n\geqslant 1} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow \lambda_*$ as $n \to \infty$ and let $u_n \in S(\lambda_n)$ for all $n \geqslant 1$.

We have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \lambda_n \int_\Omega f(z, u_n) h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \geqslant 1. \tag{45}$$

Hypotheses $H(i),\ (ii),\ (iv)$, imply that given $\epsilon > 0$, we can find $c_5 = c_5(\epsilon) > 0$ such that

$$f(z, x) \leqslant \epsilon x^{p-1} + c_5 \text{ for a.a. } z \in \Omega,\ \text{all } x \geqslant 0. \tag{46}$$

In (45) we choose $h = u_n \in W^{1,p}(\Omega)$ and use (46). Then

$$\begin{aligned} ||Du_n||_p^p + \int_{\partial\Omega} \beta(z) u_n^p d\sigma &\leqslant \lambda_n \epsilon ||u_n||_p^p + \lambda_n c_5 |\Omega|_N \\ &\leqslant \lambda_1 \epsilon ||u_n||_p^p + \lambda_1 c_5 |\Omega|_N \qquad (47) \\ &\qquad \text{(since } \lambda_n \leqslant \lambda_1 \text{ for all } n \geqslant 1), \\ \Rightarrow (\hat{\lambda}_1 - \lambda_1 \epsilon) ||u_n||_p^p &\leqslant \lambda_1 c_5 |\Omega|_N. \end{aligned}$$

Choosing $\epsilon \in \left(0, \frac{\hat{\lambda}_1}{\lambda_1}\right)$, we see that $\{u_n\}_{n\geqslant 1} \subseteq L^p(\Omega)$ is bounded. Using this in (47) and since $\beta \geqslant 0$ (see hypothesis $H(\beta)$), we infer that

$$\begin{aligned} &\{Du_n\}_{n\geqslant 1} \subseteq L^p(\Omega, \mathbb{R}^N) \text{ is bounded,} \\ &\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{48}$$

In (45) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (48). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0, \\ &\Rightarrow u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 3).} \end{aligned} \tag{49}$$

So, if in (45) we pass to the limit as $n \to \infty$ and use (49), then

$$\begin{aligned} \langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h d\sigma &= \lambda_* \int_\Omega f(z, u_*) h dz \text{ for all } h \in W^{1,p}(\Omega), \\ &\Rightarrow u_* \text{ is a solution of problem } (P_{\lambda_*}). \end{aligned}$$

We need to show that $u_* \neq 0$ in order to conclude that $u_* \in S(\lambda_*)$, hence that $\lambda_* \in \mathcal{L}$. Arguing by contradiction, suppose $u_* = 0$. Then

$$u_n \to 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see (49)).}$$

Let $y_n = \frac{u_n}{||u_n||}$ $n \geqslant 1$. Then $||y_n|| = 1$, $y_n \geqslant 0$, for all $n \geqslant 1$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \quad (50)$$

From (45), we have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n^{p-1} h d\sigma = \int_\Omega \frac{N_f(u_n)}{||u_n||^{p-1}} h dz \text{ for all } h \in W^{1,p}(\Omega),$$
$$\text{all } n \geqslant 1. \quad (51)$$

Hypotheses $H(i)$, (iv) imply that

$$|f(z,x)| \leqslant c_6(x^{p-1} + x^{r-1}) \text{ for a.a. } z \in \Omega \text{ with } c_6 > 0$$
$$\Rightarrow \left\{ \frac{N_f(u_n)}{||u_n||^{p-1}} \right\}_{n \geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

Passing to a subsequence if necessary and using hypothesis $H(iv)$ we obtain

$$\frac{N_f(u_n)}{||u_n||^{p-1}} \xrightarrow{w} \xi y^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } -\hat{c} \leqslant \xi(z) \leqslant 0 \text{ for a.a. } z \in \Omega. \quad (52)$$

In (51), we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (52). Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$
$$\Rightarrow y_n \to y \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty, \text{ hence } ||y|| = 1, \ y \geqslant 0. \quad (53)$$

If in (51) we pass to the limit as $n \to \infty$ and use (52) and (53), we obtain

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y^{p-1} h d\sigma = \lambda_* \int_\Omega \xi(z) y^{p-1} h dz \text{ for all } h \in W^{1,p}(\Omega).$$

Choose $h = y \in W^{1,p}(\Omega)$. Then using (2), we have

$$0 < \hat{\lambda}_1 ||y||_p^p \leqslant ||Dy||_p^p + \int_{\partial\Omega} \beta(z) y^p d\sigma = \lambda_* \int_\Omega \xi(z) y^p dz \leqslant 0 \text{ (see (52))},$$

a contradiction. This proves that $u_* \neq 0$ and so $u_* \in S(\lambda_*)$, that is, $\lambda_* \in \mathcal{L}$. □

So, we can summarize the situation for problem (P_λ), by stating the following bifurcation near infinity result.

Theorem 10. *If hypotheses H' and $H(\beta)$ hold, then there exists $\lambda_* > 0$ such that*
(a) for all $\lambda > \lambda_*$ problem (P_λ) admits at least two positive solutions

$$u_0, \hat{u} \in int\, C_+, \ u_0 \neq \hat{u};$$

(b) for $\lambda = \lambda_*$ problem (P_λ) admits at least one positive solution

$$u_* \in int\, C_+;$$

(c) for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has no positive solutions.

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