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Critical planar Schrödinger–Poisson equations: existence, multiplicity and concentration

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Abstract

In this paper, we are concerned with the study of the following 2-D Schrödinger–Poisson equation with critical exponential growth

$$-\varepsilon^2 \Delta u + V(x)u + \varepsilon^{-\alpha}(I_\alpha * |u|^q)|u|^{q-2}u = f(u),$$

where $\varepsilon > 0$ is a parameter, I_α is the Riesz potential, $0 < \alpha < 2$, $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the critical exponential growth. By variational methods, we first prove the existence of ground state solutions for the above system with the periodic potential. Then we obtain that there exists a positive ground state solution of the above system concentrating at a global minimum of V in the semi-classical limit under some suitable conditions. Meanwhile, the exponential decay of this ground state solution is detected. Finally, we establish the multiplicity of positive solutions by using the Ljusternik–Schnirelmann theory.

Keywords Schrödinger–Poisson system · Ground state solutions · Concentration behavior · Critical exponential growth

Mathematics Subject Classification 35B33 · 35J20 · 35J61

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1 Introduction and main results

In this paper, we consider the existence and concentration properties of solutions to the following 2-D Schrödinger–Poisson problem:

$$\begin{cases} -\varepsilon^2\Delta v + V(x)v + \varepsilon^{-\alpha}(I_\alpha * |v|^q)|v|^{q-2}v = f(v) \ \text{ in } \mathbb{R}^2, \\ v \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, $1 < q < \infty$, while I_α is the Riesz potential with $0 < \alpha < 2$, defined as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{2-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{2-\alpha}{2})}{\pi 2^\alpha \Gamma(\frac{\alpha}{2})}. \tag{1.2}$$

In fact, problem (1.1) is equivalent to the following system

$$\begin{cases} -\varepsilon^2\Delta v + V(x)v + \phi|v|^{q-2}v = f(v) \ \text{ in } \mathbb{R}^2, \\ \varepsilon^\alpha(-\Delta)^{\frac{\alpha}{2}}\phi = |v|^q \ \text{ in } \mathbb{R}^2. \end{cases} \tag{1.3}$$

This is due to the fact that I_α is the Green function of the fractional Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$ (see for instance [40, Section 5.1.1]). This is a Hartree-type model with the fractional Poisson equation. From a physical viewpoint, fractional powers of the Laplace operator play an important role in many situations in which one may need to consider nonlocal interaction and anomalous diffusion, see [18, 33] and references therein.

For the nonperturbed case $\varepsilon = 1$, let $\alpha \to 2^-$, then problem (1.3) reduces to the planar Schrödinger–Poisson system:

$$\begin{cases} -\Delta v + V(x)v + \phi|v|^{q-2}v = f(v) \ \text{ in } \mathbb{R}^2, \\ -\Delta\phi = |v|^q \ \text{ in } \mathbb{R}^2. \end{cases} \tag{1.4}$$

This type of system like (1.4) appearing in the physical literature can been seen as an approximation of the Hartree–Fock model about quantum many-body system of electrons, for example, see [25] for a mathematical presentation of Hartree–Fock approach. Under this context, problem (1.1) is known as a Schrödinger–Poisson type system. We refer to [6, 29, 30, 36] for more physical backgrounds. Furthermore, since $I_\alpha * \psi \to \psi$ as $\alpha \to 0^+$, for any $\psi \in C_0^\infty(\mathbb{R}^2)$, the following local Schrödinger equations can be viewed as the formal limit of (1.4):

$$-\Delta v + V(x)v + |v|^{2q-2}v = f(v) \qquad \text{in } \mathbb{R}^2.$$

Similar equations have been investigated in [1, 10–12, 15, 16, 19, 22–24, 26, 31, 37, 42] and further references therein.

Especially, in the two dimensional case, we exhibit some related results. Chen and Tang in [12] considered the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \gamma\phi u = f(u) \ \text{ in } \mathbb{R}^2, \\ \Delta\phi = u^2 \ \text{ in } \mathbb{R}^2, \end{cases} \tag{1.5}$$

under the axially symmetric potential, they developed some new variational technique to verify the existence of nontrivial solution and ground state solution for system (1.5) with $\gamma = 1$. Liu et al. in [31] focused on the system (1.5) with $V = 1$, and proved the existence of positive solutions by introducing a new variational method. Du and Weth in [19] concerned the Schrödinger–Poisson system as above with $V = 1$, $f(u) = |u|^{p-2}u$, where $p > 2$. Based on variational approach, they proved the existence of ground state and high energy solutions. Cingolani and Jeanjean in [14] dealt with the Schrödinger–Poisson system with prescribed mass as follows

$$\begin{cases} -\Delta u + \gamma \left(\log|\cdot| * |u|^2\right) u = a|u|^{p-2}u \ \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x = c, \end{cases} \tag{1.6}$$

here $c > 0, \gamma, a \in \mathbb{R}$, $p > 2$. They investigated the existence of standing wave solutions for system (1.6) by variational arguments. Shen et al. in [39] studied the Choquard-type Schrödinger equation

$$-\Delta u + V(x)u = \left(|x|^{-\mu} * (Q(x)F(u))\right) Q(x)F(u) \ \ x \in \mathbb{R}^2,$$

here V and Q decay to 0 at infinity. Under the critical exponential growth on nonlinearity, the authors obtained the existence of nontrivial solutions and bound state solutions by establishing a weighted Trudinger–Moser inequality.

Before the formal statements, we show the critical exponential growth condition here. In dimension two, we call f has critical exponential growth if the following assumption holds: (f_1) For any $\alpha_0 > 0$,

$$\lim_{t \to +\infty} \frac{f(t)}{e^{\alpha_0 t^2}} = \begin{cases} 0, & \forall \alpha_0 > 4\pi, \\ +\infty, & \forall \alpha_0 < 4\pi. \end{cases}$$

The above critical exponential growth in dimension two was established under the Trudinger–Moser inequality sense in Sobolev space, and describes the sharp maximal exponential integrability of the functions. Furthermore, the Trudinger–Moser inequality can be used as substitute for the Sobolev inequality, which proposed by J. Moser [35]. Indeed, Cao gave the first version of the Trudinger–Moser inequality in $\mathbb{R}^2$ in [9], one can see as follows:

Lemma 1.1 (Trudinger–Moser inequality [9])

(i) *If* $\alpha_0 > 0$*and* $u \in H^1(\mathbb{R}^2)$*,then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha_0 u^2} - 1\right) \mathrm{d}x < \infty;$$

(ii) *If* $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leqslant 1$, $\|u\|_2 \leqslant G < \infty$, *and* $\alpha_0 < 4\pi$*, then there exists a constant* $C(G, \alpha_0)$*, which depends only on G and* α_0*, such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha_0 u^2} - 1\right) \mathrm{d}x \leqslant C(G, \alpha_0).$$

Now we recall some achievements about the concentration behavior of solutions when the nonlocal term is on the right-hand side of equation which is called Charquard type equation. Moroz and Schaftingen in [34] consider the following nonlocal equation

$$-\varepsilon^2 \Delta u + Vu = \varepsilon^{-\alpha}(I_\alpha * |u|^p)|u|^{p-2}u \ \text{ in } \mathbb{R}^N.$$

Using variational methods and nonlocal penalization technique, the authors obtained that there exist a family of solutions concentrating to the local minimum of V. In [7], Bonheure, Cingolani and Secchi performed a semiclassical analysis for the planar Schrödinger–Poisson system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = E(x)u \ \text{ in } \mathbb{R}^2, \\ -\Delta E = |u|^2 \ \text{ in } \mathbb{R}^2, \end{cases}$$

where $E(x) = \int_{\mathbb{R}^2} \log \frac{1}{|x-y|}|u(y)|^2 \mathrm{d}y$, V is a bounded external potential. By adapting a perturbation method [3, 4] and a nondegeneracy result [8], they proved the existence of solution pairs as $\varepsilon \to 0$. In [2], Alves et al. investigated the following critical nonlocal Schrödinger equation with singularly perturbation:

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-2}\left[\frac{1}{|x|^{\mu}} * F(u)\right] f(u) \quad \text{in } \mathbb{R}^2,$$

where $0 < \mu < 2$. They obtained two results by variational methods. Specifically, the first one they established is the existence of ground state solutions for above equation with periodic potential and $\varepsilon = 1$ under the assumptions as follows:

- (V_1) $V(x) \geqslant V_0 > 0$ for some $V_0 > 0$;
- (V_2) $V(x) \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ is a 1-periodic function;
- $(\widetilde{f}_1)$ (i) $f(t) = 0$ for all $t \leqslant 0$, and $0 \leqslant f(t) \leqslant Ce^{4\pi t^2}$ for all $t \geqslant 0$;
  (ii) there exist $t_1 > 0$, $M_1 > 0$, $\bar{q} \in (0, 1]$ such that $0 < t^{\bar{q}} F(t) \leqslant M_1 f(t)$, for any $|t| \geqslant t_1$, where $F(s) = \int_0^s f(t)\mathrm{d}t$;
- $(\widetilde{f}_2)$ there exist $\bar{p} > (2-\mu)/2$ and $C_{\bar{p}} > 0$ satisfying $f(t) \sim C_{\bar{p}} t^{\bar{p}}$, when $t \to 0$;
- $(\widetilde{f}_3)$ there is $K_1 > 1$ such that $f(t)t > K_1 F(t)$, $\forall t > 0$;
- $(\widetilde{f}_4)$ $\lim_{t\to\infty} \dfrac{tf(t)F(t)}{e^{8\pi t^2}} \geqslant \bar{\beta}$, here $\bar{\beta} > \inf_{\rho_0 > 0} \dfrac{e^{\frac{4-\mu}{4}} V_0 \rho_0^2}{16\pi^2 \rho_0^{4-\mu}} \dfrac{(4-\mu)^2}{(2-\mu)(3-\mu)}$.

The second one they proved is about the existence and concentration behavior of ground state solutions with positive and bounded below potential when $\varepsilon \to 0$, in which they used the hypotheses $(\widetilde{f}_1)$–$(\widetilde{f}_4)$ and the following assumptions:

- (V) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ and $0 < V_0 := \inf_{\mathbb{R}^2} V(x) < V_\infty = \lim_{|x|\to\infty} V(x) < \infty$;
- $(\widetilde{f}_5)$ $t \to f(t)$ is strictly increasing on $(0, \infty)$.

There is a result of Schrödinger–Poisson type involving the Riesz potential in $\mathbb{R}^2$. As a particular case, Mercuri et al. in [32] studied the nonlocal equation of Schrödinger–Poisson–Slater type:

$$-\Delta u + (I_\alpha * |u|^p)|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^2,$$

where $p > 1, q > 1$. They considered the Coulomb-Sobolev function space, and obtained the existence of solutions to the above equation by proving a family of optimal interpolation inequalities. The authors also established some qualitative properties of solutions, such as regularity, positivity, radially symmetry and so on. A natural question is whether there exists the concentration behavior of ground state solutions. A goal of this paper is to gives an affirmative answer in the two dimensional case.

In the paper mentioned above, authors always used the Hardy–Littlewood–Sobolev inequality to handle the convolution term, here we would like to demonstrate as follows.

Lemma 1.2 (Hardy–Littlewood–Sobolev inequality [28]) *Let $\alpha \in (0, 2)$ and $s \in (1, 2/\alpha)$. Then for any $\phi \in L^s(\mathbb{R}^2)$, $I_\alpha * \phi \in L^{\frac{2s}{2-\alpha s}}(\mathbb{R}^2)$, and*

$$\int_{\mathbb{R}^2} |I_\alpha * \phi|^{\frac{2s}{2-\alpha s}} \mathrm{d}x \leqslant C(\alpha, s)\left(\int_{\mathbb{R}^2} |\phi|^s \mathrm{d}x\right)^{\frac{2}{2-\alpha s}}.$$

Remark 1.1 The normalisation constant A_α of (1.2) guarantee following property of I_α holds: $I_{s+t} = I_s * I_t$, where $s, t \in (0, 2)$, $s + t < 2$, see [20]. Then, one can read the Hardy–Littlewood–Sobolev inequality as follows:

$$\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q \mathrm{d}x = \int_{\mathbb{R}^2} \left| I_{\frac{\alpha}{2}} * |u|^q \right|^2 \mathrm{d}x \leqslant C \left(\int_{\mathbb{R}^2} |u|^{\frac{4q}{2+\alpha}} \mathrm{d}x \right)^{\frac{2+\alpha}{2}}.$$

Furthermore, by using the Sobolev embedding inequality, we have

$$\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q \mathrm{d}x \leqslant C(\alpha, q) \left[\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \mathrm{d}x \right]^{2q}. \tag{1.7}$$

In the present paper, we study the existence, multiplicity and concentration behavior of positive solutions for the Schrödinger–Poisson problem (1.1). Meanwhile, we give some properties of positive solutions. Firstly, performing the scaling $u(x) = v(\varepsilon x)$ in (1.1), we obtain the following rescaled problem

$$-\Delta u + V(\varepsilon x)u + (I_\alpha * |u|^q)|u|^{q-2}u = f(u) \quad \text{in } \mathbb{R}^2, \tag{1.8}$$

One can easily get the corresponding functional as follows:

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} V(\varepsilon x)u^2 \mathrm{d}x + \frac{1}{2q} \int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q \mathrm{d}x - \int_{\mathbb{R}^2} F(u) \mathrm{d}x. \tag{1.9}$$

Let $H^1(\mathbb{R}^2)$ denote the Sobolev space endowed with the standard norm

$$\|u\| := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \mathrm{d}x \right)^{\frac{1}{2}}.$$

Denote by E the Sobolev space endowed with the norm

$$\|u\|_E := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V(\varepsilon x)u^2) \mathrm{d}x \right)^{\frac{1}{2}}.$$

It is obvious that $\|u\|_E$ is equivalent to the standard norm under the assumption (V). Also, let

$$\|u\|_{V_0} := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V_0 u^2) \mathrm{d}x \right)^{\frac{1}{2}}, \quad \|u\|_{V_\infty} := \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V_\infty u^2) \mathrm{d}x \right)^{\frac{1}{2}}.$$

It is standard to verify that $I_\varepsilon \in \mathcal{C}^1(E, \mathbb{R})$ and the critical points of I_ε correspond to the weak solutions of problem (1.1).

Inspired by the approach of [2, 32], in this paper we prove the existence and concentration behavior of positive solutions for problem (1.1). Precisely, by using variational method, we obtain the existence of ground state solution of (1.1) with $\varepsilon = 1$ under periodic potential hypotheses (V_1) and (V_2). And, compared with the constant potential problem, we get the existence of ground state solutions for problem (1.1) under bounded potential assumption (V), and then the concentration behavior of the corresponding ground state solution is proved by Moser iteration argument. Besides, the exponential decay of this ground state solution is investigated by maximum principle. Finally, we shows the multiplicity of positive solutions based on the Ljusternik–Schnirelmann theory. Similar to the case of ground state solutions, concentration behavior and exponential decay of positive solutions are obtained.

More specifically, the main results we obtained in this paper are three-fold. The first one we establish the existence of nontrivial solution for (1.8) under the periodic potential hypotheses (V_1) and (V_2). In addition, we suppose (f_1) and the following assumptions about nonlinear term hold:

(f_2) $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$, $f(t) = 0$ for $t \leqslant 0$ and $f(t) \geqslant 0$ for $t > 0$;

(f_3) there exists $\rho > 0$ such that $\beta_0 > \frac{[1 + C(\alpha, q)]e^{\frac{1}{2}V(\rho)\rho^2}}{\pi\rho^2}$ and

$$\lim_{|t|\to\infty} \frac{tf(t)}{e^{4\pi t^2}} \geqslant \beta_0,$$

where $C(\alpha, q) > 0$ is a constant comes from (1.7), $V(\rho) = \sup_{|x|\leqslant\rho} V(x)$.

(f_4) $t \mapsto \frac{f(t)t + 2F(t)}{t^{2q+3}}$ is strictly increasing on $(0, \infty)$.

There is a simple example of function f satisfying (f_1)–(f_4):

$$f(t) = \begin{cases} 0, & t \leqslant 0, \\ t^a e^{4\pi t^2} + t^{a-2}, & t > 0, \end{cases}$$

where $a > 2q + 4$.

Now we can demonstrate the first main result as follows.

Theorem 1.1 *Assume that (V_1), (V_2) and (f_1)–(f_4) hold. Then problem (1.1) with $\varepsilon = 1$ has a nonnegative ground state solution in E.*

The second part of paper is about the existence and concentration behavior of positive ground state solutions for (1.8) under the assumption (V). It is worth mentioning that there are only three papers [2, 7, 34] considering the concentration behavior of solutions for Schrödinger–Poisson problem in $\mathbb{R}^2$, to the best of our knowledge. In fact, since the nonlocal term is fractional, it is difficult to get the upper bound of mountain pass value for semi-classical limit about problem (1.1). Moreover, we concern the problem with more general convolution term and nonlinear term. Here is the second main result.

Theorem 1.2 *Suppose that (V) and (f_1)–(f_4) hold. Then, for $\varepsilon > 0$, problem (1.1) has at least a positive ground state solution $v_\varepsilon(\varepsilon x)$ in $H^1(\mathbb{R}^2)$. Moreover, the following statements hold:*

(i) *let $\eta_\varepsilon \in \mathbb{R}^2$ be a global maximum point of this positive ground state solution $v_\varepsilon(\varepsilon x)$ to (1.1), then*

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

Meanwhile, for all $\varepsilon_n \to 0$, $v_{\varepsilon_n}(\varepsilon_n x + \eta_{\varepsilon_n})$ converges to a positive ground state solution of the following equation

$$\begin{cases} -\Delta u + V_0 u + (I_\alpha * |u|^q)|u|^{q-2}u = f(u), & in\ \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2). \end{cases} \tag{1.10}$$

(ii) *there exists $C > 0, \kappa > 0$ independent of η_ε such that*

$$|v_\varepsilon(x)| \leqslant Ce^{-\frac{\kappa}{\varepsilon}|x-\eta_\varepsilon|}, \quad \forall x \in \mathbb{R}^2.$$

In the third part, we concern the multiplicity and concentration behavior of positive solutions for problem (1.8) under the assumption (V). In order to correlate the number of solutions with the topology of the set of minima of the potential V, we present the sets

$$\Theta := \{x \in \mathbb{R}^2 : V(x) = V_0\}$$

and

$$\Theta_\delta := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Theta) \leqslant \delta\} \text{ for } \delta > 0.$$

Note condition (V) introduced by Rabinowitz in [38] guarantees that the set Θ is compact. In addition, we recall that, if Y is a closed subset of a topological space X, $\mathrm{cat}_X(Y)$ is the Ljusternik–Schnirelmann category, that is, the number of closed and contractible set in X which cover Y. The third main result can be stated as follows.

Theorem 1.3 *Suppose that (V) and (f_1)–(f_4) hold. Then, for any $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (1.1) has at least $\mathrm{cat}_{\Theta_\delta}(\Theta)$ positive solutions for $\varepsilon \in (0, \varepsilon_\delta)$. Moreover, the following statements hold:*

(i) *let $\bar{\eta}_\varepsilon \in \mathbb{R}^2$ be a global maximum point of $\bar{v}_\varepsilon(\varepsilon x)$ which is one of these positive solutions to problem (1.1), then*

$$\lim_{\varepsilon \to 0} V(\bar{\eta}_\varepsilon) = V_0.$$

(ii) *for all $\varepsilon_n \to 0$, $\bar{v}_{\varepsilon_n}(\varepsilon_n x + \bar{\eta}_{\varepsilon_n})$ converges to a positive solution of Eq. (1.10). Moreover, there exists $\bar{C} > 0, \bar{\kappa} > 0$ independent of $\bar{\eta}_\varepsilon$ such that*

$$|\bar{v}_\varepsilon(x)| \leqslant \bar{C} e^{-\frac{\bar{\kappa}}{\varepsilon}|x - \bar{\eta}_\varepsilon|}, \quad \forall x \in \mathbb{R}^2.$$

Compared with [2], we impose mild hypotheses on nonlinear term f, such as, we do not suppose $(\widetilde{f}_1)$(ii) and $(\widetilde{f}_3)$ hold. In addition, Alves et al. in [2] considered the problem with the convolution term on the right-hand side, that is, this type of equation is a Choquard-type problem which has been investigated extensively in recent years. In this paper, we consider the convolution term on the left-hand side, which makes the problem more delicate.

This paper is organized as follows. Section 1 proves Theorem 1.1. Section 1 and Sect. 4 give the proof of Theorem 1.2, exactly, Sect. 1 illustrates the existence of ground state solutions for problem (1.8), Sect. 4 explains the concentration behavior and exponential decay of the ground state solution for problem (1.1). Section 5 shows the proof of Theorem 1.3.

Here we state some notations used in this paper:

- $L^s(\mathbb{R}^N)$ denotes the Lebesgue space equipped with the norm $\|u\|_s = \left(\int_{\mathbb{R}^N} |u|^s \mathrm{d}x\right)^{1/s}$, $2 \leqslant s < +\infty$;
- $B_R(y)$ denotes the open ball centered at y with radius $R > 0$;
- C, $\bar{C}$ and $\widetilde{C}$ denote different positive constants in different places.

2 Proof of Theorem 1.1

In this section, we are dedicated to the proof of Theorem 1.1. First of all, we investigate some technical lemmas which will be used later.

Lemma 2.1 *Assume that (f_1)–(f_3) hold. Then*

(i) *there exists $\rho > 0$ such that $I_\varepsilon|_{S_\rho}(u) > 0$, $\forall u \in S_\rho = \{u \in H^1(\mathbb{R}^2) : \|u\|_E = \rho\}$;*

(ii) *there is* $e \in H^1(\mathbb{R}^2)$ *with* $\|e\|_E > \rho$ *such that* $I_\varepsilon(e) < 0$.

Proof (i) From (f_1) and (f_2), for any $\epsilon > 0$, $p > 2$ and $\beta > 1$, there exists $C_\epsilon > 0$ such that

$$|F(t)| \leqslant \epsilon |t|^2 + C_\epsilon |t|^p \left[e^{4\pi\beta t^2} - 1\right], \quad \forall t \in \mathbb{R}. \tag{2.1}$$

As in [13, Lemma 2.1], let $\xi \in (0, 1)$ and $2\beta\|u\|_E^2 = \xi < 1$, for some $C(\epsilon) > 0$, one has

$$\int_{\mathbb{R}^2} F(u)\mathrm{d}x \leqslant \epsilon\|u\|_2^2 + C(\epsilon)\|u\|_{2p}^p, \quad \text{for } \|u\|_E = \left(\frac{\xi}{2\beta}\right)^{\frac{1}{2}}.$$

Then, using Sobolev embedding inequality, we have

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} V(\varepsilon x)u^2\mathrm{d}x + \frac{1}{2q}\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q\mathrm{d}x - \int_{\mathbb{R}^2} F(u)\mathrm{d}x \\ &\geqslant \frac{1}{2}\|u\|_E^2 - \epsilon\|u\|_2^2 - C(\epsilon)\|u\|_{2p}^p \\ &\geqslant \frac{1}{2}\|u\|_E^2 - \epsilon C\|u\|_E^2 - C(\epsilon)\|u\|_E^p, \end{aligned}$$

where constant $C > 0$. We can choose $\rho = \left(\frac{\xi}{2\beta}\right)^{\frac{1}{2}}$ with $\xi \approx 0^+$ such that (i) holds.

(ii) From (f_3), for any $\bar{\varepsilon} > 0$, there exists $R_{\bar{\varepsilon}} > 0$ such that

$$F(s) \geqslant (\beta_0 - \bar{\varepsilon})e^{4\pi s^2}, \quad \forall s \geqslant R_{\bar{\varepsilon}}. \tag{2.2}$$

By (f_2) and the Taylor's expansion, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} F(tu)\mathrm{d}x &\geqslant \int_{\mathbb{R}^2} (\beta_0 - \bar{\varepsilon})e^{4\pi(tu)^2}\mathrm{d}x \\ &= \int_{\mathbb{R}^2} (\beta_0 - \bar{\varepsilon})\sum_{n=1}^{\infty} \frac{(4\pi t^2u^2)^n}{n!}\mathrm{d}x. \end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned} I_\varepsilon(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^2} V(\varepsilon x)u^2\mathrm{d}x + \frac{t^{2q}}{2q}\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2} F(tu)\mathrm{d}x. \end{aligned}$$

It follows from (f_2), (f_3) and (2.3) that $\lim_{t\to\infty} I_\varepsilon(tu) = -\infty$. Then we can choose $T > 0$ such that $e = Tu \in \{u \in H^1(\mathbb{R}^2) : \|u\|_E > \rho\}$ and $I_{\varepsilon(e)} < 0$. □

Now we verify that the nontrivial weak solution of (1.8) is nonnegative.

Lemma 2.2 *Assume that* (V) *(or* (V_1)*) and* (f_2) *hold. Then any nontrivial critical point of* I_ε *is nonnegative.*

Proof Set $u \in H^1(\mathbb{R}^2)\backslash\{0\}$ is the critical point of I_ε, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. Since $\langle I_\varepsilon'(u), -u^-\rangle = 0$, we know

$$\int_{\mathbb{R}^2} |\nabla u| \cdot |\nabla(-u^-)| \mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon x) u(-u^-) \mathrm{d}x$$
$$+ \int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^{q-1}(-u^-) \mathrm{d}x = \int_{\mathbb{R}^2} f(u)(-u^-) \mathrm{d}x.$$

From (f_2), one has $f(u)(-u^-) = 0$ a.e. in $\mathbb{R}^2$. Moreover,

$$\int_{\mathbb{R}^2} V(\varepsilon x) u(-u^-) \mathrm{d}x = \int_{\mathbb{R}^2} V(\varepsilon x) |u^-|^2 \mathrm{d}x,$$
$$\int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^{q-1}(-u^-) \mathrm{d}x = \int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u^-|^q \mathrm{d}x.$$

From above equations, we conclude that

$$\int_{\mathbb{R}^2} |\nabla(u^-)|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V(\varepsilon x)|u^-|^2 \mathrm{d}x + \int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u^-|^q \mathrm{d}x = 0.$$

Using the fact that $V(\varepsilon x) > 0$, one has $u^- \equiv 0$, which implies that $u \geqslant 0$ a.e. in $\mathbb{R}^2$. □

From Lemma 2.1, there exists a Palais–Smale sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ such that

$$I_\varepsilon(u_n) \to c, \quad I'_\varepsilon(u_n) \to 0,$$

where c is characterized by

$$0 < c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) \tag{2.4}$$

with

$$\Gamma := \{\gamma \in \mathcal{C}^1([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}.$$

Lemma 2.3 *Suppose that (f_1)–(f_3) hold. Then*

$$c < \frac{1}{2}.$$

Proof Define $V(\rho) = \sup_{|x| \leqslant \rho} V(x)$ for any $\rho > 0$, then $V(\rho)$ is a positive continuous function. By (f_3), fixed $\rho > 0$ satisfying

$$\beta_0 > \frac{[1 + C(\alpha, q)] e^{\frac{1}{2} V(\rho) \rho^2}}{\pi \rho^2}, \tag{2.5}$$

where $C(\alpha, q)$ from (1.7). Now we consider the following Moser type functions

$$\bar{w}_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leqslant |x| \leqslant \frac{\rho}{n}, \\ \dfrac{\log(\rho/|x|)}{\sqrt{\log n}}, & \frac{\rho}{n} \leqslant |x| \leqslant \rho, \\ 0, & |x| \geqslant \rho. \end{cases}$$

By direct calculation, one has

$$\begin{aligned} \|\bar{w}_n\|_E^2 &= \int_{B_\rho} |\nabla \bar{w}_n|^2 \mathrm{d}x + \int_{B_\rho} V(x)|\bar{w}_n|^2 \mathrm{d}x \\ &\leqslant \int_{\frac{\rho}{n}}^{\rho} \frac{\mathrm{d}r}{r \log n} + V(\rho) \int_0^{\frac{\rho}{n}} r \log n \mathrm{d}r + V(\rho) \int_{\frac{\rho}{n}}^{\rho} r \frac{\log^2(\rho/r)}{\log n} \mathrm{d}r \\ &= 1 + \delta_n, \end{aligned}$$

where

$$\delta_n = V(\rho)\rho^2\left[\frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2}\right] > 0.$$

Let $w_n = \bar{w}_n/\|\bar{w}_n\|_E$, we can easily get that $\|w_n\|_E = 1$. Now, we just need to prove that there is $n \in \mathbb{N}$ such that $\max_{t\geqslant 0} I_\varepsilon(tw_n) < \frac{1}{2}$. With the reduction to absurdity, we assume that there exists $t_n > 0$ such that for all $n \in \mathbb{N}$, $\max_{t\geqslant 0} I_\varepsilon(tw_n) = I_\varepsilon(t_nw_n) \geqslant \frac{1}{2}$. From (f_3), there exists $R_{\bar{\varepsilon}} > 0$ such that

$$sf(s) \geqslant (\beta_0 - \bar{\varepsilon})e^{4\pi s^2}, \quad \forall s \geqslant R_{\bar{\varepsilon}}. \tag{2.6}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}I_\varepsilon(tw_n)\mid_{t=t_n}= 0$, using Hardy–Littlewood–Sobolev inequality and Sobolev embedding inequality, one has

$$\begin{aligned}
t_n^2 &= \int_{\mathbb{R}^2} f(t_nw_n)t_nw_n\mathrm{d}x - t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |w_n|^q)|w_n|^q\mathrm{d}x \\
&\geqslant \int_{B_{\frac{\rho}{n}}} f(t_nw_n)t_nw_n\mathrm{d}x - Ct_n^{2q}\|w_n\|^{2q}_{\frac{4q}{2+\alpha}} \\
&\geqslant \int_{B_{\frac{\rho}{n}}} (\beta_0 - \bar{\varepsilon})e^{2t_n^2(1+\delta_n)^{-1}\log n}\mathrm{d}x - C(\alpha, q)t_n^{2q} \\
&= \frac{\pi\rho^2(\beta_0 - \bar{\varepsilon})}{n^2}e^{2t_n^2(1+\delta_n)^{-1}\log n} - C(\alpha, q)t_n^{2q} \\
&= \pi\rho^2(\beta_0 - \bar{\varepsilon})e^{\log n[2t_n^2(1+\delta_n)^{-1}-2]} - C(\alpha, q)t_n^{2q},
\end{aligned}$$

thus, by means of taylor's expansion, t_n is bounded and

$$\log n[2t_n^2(1 + \delta_n)^{-1} - 2] \leqslant \tilde{C}, \tag{2.7}$$

also, $\limsup_{n\to\infty} t_n \leqslant 1$.

Define

$$A_n = \left\{y \in B_\rho : t_nw_n(y) > R_{\bar{\varepsilon}}\right\}, \quad B_n = B_\rho\backslash A_n.$$

Applying (2.2), we have

$$\begin{aligned}
t_n^2 &= \int_{\mathbb{R}^2} f(t_nw_n)t_nw_n\mathrm{d}x - t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |w_n|^q)|w_n|^q\mathrm{d}x \\
&= \int_{B_\rho} f(t_nw_n)t_nw_n\mathrm{d}x - t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |w_n|^q)|w_n|^q\mathrm{d}x \\
&= \int_{A_n+B_n} f(t_nw_n)t_nw_n\mathrm{d}x - t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |w_n|^q)|w_n|^q\mathrm{d}x.
\end{aligned}$$

From (f_1) and (f_2), for any $\epsilon > 0$, $p > 2$ and $\beta > 1$, there exists $C_\epsilon > 0$ such that

$$|sf(s)| \leqslant \epsilon|s|^2 + C_\epsilon|s|^p\left[e^{4\pi\beta s^2} - 1\right], \quad \forall s \in \mathbb{R}. \tag{2.8}$$

Then, using Hölder's inequality, we know

$$\int_{\mathbb{R}^2} f(t_nw_n)t_nw_n\mathrm{d}x \leqslant \epsilon\|t_nw_n\|_2^2 + C_\epsilon\|t_nw_n\|_{2p}^p\left[\int_{\mathbb{R}^2}(e^{8\pi\beta t_n^2w_n^2} - 1)\mathrm{d}x\right]^{\frac{1}{2}}. \tag{2.9}$$

From Lemma 1.1 and (2.7), note that $\|\nabla \bar{w}_n\|_2 = 1$, $\bar{w}_n^2 \leqslant 2\pi \log n$, one has

$$\int_{\mathbb{R}^2} (e^{8\pi\beta t_n^2 w_n^2} - 1)\mathrm{d}x \leqslant \int_{B_\rho} e^{8\pi\beta t_n^2 w_n^2}\mathrm{d}x \leqslant \int_{B_\rho} e^{8\pi\beta(1+\frac{C}{\log n})\bar{w}_n^2}\mathrm{d}x \leqslant \int_{B_\rho} Ce^{8\pi\beta\bar{w}_n^2}\mathrm{d}x \leqslant C. \tag{2.10}$$

Due to $t_n w_n \to 0$ a.e. in $\mathbb{R}^2$, and $t_n w_n$ is bounded in B_n, it follows from Lebesgue's dominated convergence theorem that

$$\int_{B_n} f(t_n w_n)t_n w_n \mathrm{d}x = o(1).$$

Consequently, by the Hardy–Littlewood–Sobolev inequality and sobolev embedding inequality, we know

$$\begin{aligned} t_n^2 &= \int_{A_n} f(t_n w_n)t_n w_n \mathrm{d}x - t_n^{2q}\int_{\mathbb{R}^2} (I_\alpha * |w_n|^q)|w_n|^q \mathrm{d}x + o(1) \\ &\geqslant \int_{A_n} f(t_n w_n)t_n w_n \mathrm{d}x - C(\alpha, q)t_n^{2q}. \end{aligned}$$

Analogous to [43, Lemma 2.4], one has

$$\lim_{n\to\infty} t_n^2 \geqslant (\beta_0 - \bar{\varepsilon})\pi\rho^2 e^{-V(\rho)\rho^2/2} - C(\alpha, q)\lim_{n\to\infty} t_n^{2q}.$$

Since $\limsup_{n\to\infty} t_n \leqslant 1$, we obtain

$$1 + C(\alpha, q) \geqslant (\beta_0 - \bar{\varepsilon})\pi\rho^2 e^{-V(\rho)\rho^2/2}.$$

Because of the arbitrariness of $\bar{\varepsilon}$, one has

$$\beta_0 \leqslant \frac{[1 + C(\alpha, q)]e^{\frac{1}{2}V(\rho)\rho^2}}{\pi\rho^2},$$

which contradicts (2.5). The proof is completed. □

As the argument in [13, Lemma 2.3], we can conclude the following lemma. For reader's convenience, we sketch the proof here.

Lemma 2.4 *Assume that (f_1)–(f_4) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^2)$ be a (PS) sequence for I_ε, that is*

$$I_\varepsilon(u_n) \to c, \quad I'_\varepsilon(u_n) \to 0 \ \ in \ \ \left(H^1(\mathbb{R}^2)\right)^*, \ \ as \ \ n \to \infty.$$

Then, up to subsequence, there exists $u \in H^1(\mathbb{R}^2)$ such that, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^2)$,

$$F(u_n) \to F(u) \ \ in \ \ L^1_{loc}(\mathbb{R}^2) \tag{2.11}$$

and u is a weak solution of (1.8).

Proof Since $\{u_n\}$ is a (PS) sequence for I_ε, one has

$$\frac{1}{2}\|u_n\|_E^2 + \frac{1}{2q}\int_{\mathbb{R}^2} (I_\alpha * |u_n|^q)|u_n|^q \mathrm{d}x - \int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \to c, \tag{2.12}$$

$$\left| \|u_n\|_E^2 + \int_{\mathbb{R}^2} (I_\alpha * |u_n|^q)|u_n|^q \mathrm{d}x - \int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \right| \leqslant \epsilon_n \|u_n\|_E, \tag{2.13}$$

where $\epsilon_n \to 0$ as $n \to \infty$. We claim that for any $u_n \in H^1(\mathbb{R}^2)$,

$$f(u_n)u_n - 2qF(u_n) \geqslant 0. \tag{2.14}$$

Indeed, for $t > 0$, set

$$G(t) = t^2 F(tu) - F(u) + \frac{1 - t^{2+2q}}{2+2q}[f(u)u + 2F(u)], \ \forall u \in H^1(\mathbb{R}^2).$$

Then, it follows from (f_4) that

$$\begin{aligned} G'(t) &= 2tF(tu) + t^2 f(tu)u - t^{1+2q}[f(u)u + 2F(u)] \\ &= t^{1+2q}\left\{\frac{f(tu)tu + 2F(tu)}{t^{2q}} - [f(u)u + 2F(u)]\right\} \\ &\begin{cases} \geqslant 0, \ t \geqslant 1, \\ < 0, \ t < 1, \end{cases} \end{aligned}$$

which implies $G(t) \geqslant G(1) = 0$, let $t \to 0$ in $G(t)$, we get the claim. Thus, using this claim, we get

$$\begin{aligned} c + o(1) &= I_\varepsilon(u_n) - \frac{1}{2q}\langle I'_\varepsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2q}\right)\|u_n\|_E^2 + \frac{1}{2q}\int_{\mathbb{R}^2}[f(u_n)u_n - 2qF(u_n)]\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2q}\right)\|u_n\|_E^2. \end{aligned} \tag{2.15}$$

Above inequality means that $\|u_n\|_E$ is bounded. Hence, it follows from (2.12) and (2.13) that

$$\int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x \leqslant C, \ \int_{\mathbb{R}^2} F(u_n)\mathrm{d}x \leqslant C.$$

Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we have $u_n \to u$ in $L^s_{loc}(\mathbb{R}^2)$, $2 \leqslant s < \infty$, $u_n \to u$ a.e. in $\mathbb{R}^2$.

Analogous to [17, Lemma 2.1], one can conclude that

$$\left|\int_\Omega F(u_n)\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x\right| \to 0, \ \ \forall\, \Omega \subset\subset \mathbb{R}^2.$$

Furthermore, the proof that u is a weak solution of (1.8) can be obtained similarly to [13, Lemma 2.3], we omit it here. □

Proof of Theorem 1.1 In terms of Lemmas 2.1, 2.2 and 2.4, it remains only to prove that the weak solution u is nontrivial. Argue by contradiction, we assume that $u \equiv 0$. Since $\{u_n\}$ is bounded, we distinguish two cases: $\{u_n\}$ is vanishing or non-vanishing.

If $\{u_n\}$ is vanishing, applying Lion's concentration compactness lemma (see [41, Lemma 1.21]), for any $s > 2$, we have

$$u_n \to 0 \quad \text{in } L^s(\mathbb{R}^2), \ \text{ as } n \to \infty. \tag{2.16}$$

Repeating the proof of (2.11), one can get that

$$F(u_n) \to 0 \quad \text{in } L^1(\mathbb{R}^2), \text{ as } n \to \infty.$$

Since $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence, it follows from Lemma 2.3 that

$$\lim_{n\to\infty}\left[\frac{1}{2}\|u_n\|_E^2+\frac{1}{2q}\int_{\mathbb{R}^2}(I_\alpha*|u_n|^q)|u_n|^q\mathrm{d}x\right]=c\leqslant\frac{1}{2}.$$

Hence, there is $\bar{K}>0$ and $\delta>0$ small enough, such that

$$\frac{1}{2}\|u_n\|_E^2+\frac{1}{2q}\int_{\mathbb{R}^2}(I_\alpha*|u_n|^q)|u_n|^q\mathrm{d}x\leqslant\frac{1-\delta}{2},\ \forall n>\bar{K}.$$

From line by line [13, Lemma 2.4], we get

$$\left|\int_{\mathbb{R}^2}f(u_n)u_n\mathrm{d}x\right|\to 0\quad\text{as } n\to\infty.$$

Then, $u_n\to 0$ in $H^1(\mathbb{R}^2)$, $I_\varepsilon(u_n)\to 0$, which implies $c=0$. That is impossible.

If $\{u_n\}$ is non-vanishing, then there exist $\delta_1, r_0>0$ and a sequence $\{y_n\}\in\mathbb{Z}^2$ such that

$$\lim_{n\to\infty}\int_{B_{r_0}(y_n)}|u_n|^2\mathrm{d}x\geqslant\delta_1.$$

Set $v_n:=u_n(x-y_n)$, one has

$$\int_{B_{r_0}(0)}|v_n|^2\mathrm{d}x\geqslant\delta_1. \tag{2.17}$$

Since I_ε and I'_ε are invariant by $\mathbb{Z}^2$ translations, then $\{v_n\}$ is also a (PS) sequence. Therefore, we can obtain that $v_n\rightharpoonup v$ in $H^1(\mathbb{R}^2)$. By (2.17) and $v_n\to v$ in $L^2_{loc}(\mathbb{R}^2)$, we have $v\neq 0$. Thus, v is a nontrivial critical point of I_ε, the proof of Theorem 1.1 is now complete. □

3 Existence of ground state solutions

Now, we consider the problem (1.1). Define the Nehari manifold associated to I_ε as follows:

$$\mathcal{N}_\varepsilon=\left\{u\in H^1(\mathbb{R}^2): u\neq 0, \langle I'_\varepsilon(u),u\rangle=0\right\}.$$

Lemma 3.1 *Assume that (f_1)–(f_3) hold. Then there exist $C>0$ and $\alpha_1>0$ independent of ε, such that*

$$\|u\|_E\geqslant\alpha_1,\quad u\in\mathcal{N}_\varepsilon.$$

Proof By (2.8)–(2.10), we obtain

$$\int_{\mathbb{R}^2}f(u)u\mathrm{d}x\leqslant\epsilon\|u\|_2^2+C\|u\|_{2p}^p,\qquad p>2 \tag{3.1}$$

for some positive constant C. Since $u\in\mathcal{N}_\varepsilon$,

$$\|u\|_E^2+\int_{\mathbb{R}^2}(I_\alpha*|u|^q)|u|^q\mathrm{d}x=\int_{\mathbb{R}^2}f(u)u\mathrm{d}x,$$

then, using Sobolev embedding inequality, we have

$$\|u\|_E^2\leqslant\int_{\mathbb{R}^2}f(u)u\mathrm{d}x\leqslant\epsilon\|u\|_2^2+C\|u\|_{2p}^p\leqslant\epsilon C\|u\|_E^2+C\|u\|_E^p. \tag{3.2}$$

Thus, $\|u\|_E$ is bounded from below. The proof is thus complete. □

In view of Lemma 2.1, the functional I_ε satisfies the Mountain Pass geometry. Thus, there is a sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$ satisfying

$$I_\varepsilon(u_n) \to c_\varepsilon, \quad I'_\varepsilon(u_n) \to 0,$$

where

$$c_\varepsilon := \inf_{u \in H^1(\mathbb{R}^2)\backslash\{0\}} \max_{t \geqslant 0} I_\varepsilon(tu), \tag{3.3}$$

and $0 < C < c_\varepsilon$. From (f_4), for all $u \in H^1(\mathbb{R}^2)\backslash\{0\}$, there exists a unique $t = t(u)$ such that

$$I_\varepsilon(t(u)u) = \max_{s \geqslant 0} I_\varepsilon(su), \quad t(u)u \in \mathcal{N}_\varepsilon.$$

Then, by standard argument as in [41, Theorem 4.2], one has

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

Lemma 3.2 *Assume that* (V) *and* (f_1)–(f_4) *hold. Then*

$$\lim_{\varepsilon \to 0} c_\varepsilon = c_{V_0} < c_{V_\infty},$$

where c_{V_∞} *is defined as* (2.4).

Proof It follows from Theorem 1.1 that there is a ground state solution $w \in H^1(\mathbb{R}^2)$, namely

$$\|w\|_{V_0}^2 + \int_{\mathbb{R}^2} (I_\alpha * |w|^q)|w|^q \mathrm{d}x = \int_{\mathbb{R}^2} f(w)w\mathrm{d}x. \tag{3.4}$$

For all $\delta > 0$, setting $w_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ satisfying

$$w_\delta \in \mathcal{N}_{V_0}, \quad w_\delta \to w \text{ in } H^1(\mathbb{R}^2), \quad I_{V_0}(w_\delta) < c_{V_0} + \delta.$$

Fix $\eta \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0, 1])$ such that $\eta = 1$ on $B_1(0)$ and $\eta = 0$ on $\mathbb{R}^2\backslash B_2(0)$. Now define $v_n(x) = \eta(\varepsilon_n x)w_\delta(x)$ with $\varepsilon_n \to 0$, we have

$$v_n \to w_\delta \quad \text{in } H^1(\mathbb{R}^2), \text{ as } n \to \infty.$$

Since the definition of $\mathcal{N}_\varepsilon$, there is a unique t_n satisfying $t_n v_n \in \mathcal{N}_{\varepsilon_n}$. Then,

$$c_{\varepsilon_n} \leqslant I_{\varepsilon_n}(t_n v_n) = \frac{t_n^2}{2}\|v_n\|_E^2 + \frac{t_n^{2q}}{2q}\int_{\mathbb{R}^2} (I_\alpha * |v_n|^q)|v_n|^q \mathrm{d}x - \int_{\mathbb{R}^2} F(t_n v_n)\mathrm{d}x.$$

Note that $\langle I'_{\varepsilon_n}(t_n v_n), t_n v_n\rangle = 0$, using (f_2)–(f_3) and Taylor's expansion as (2.3), we have

$$\begin{aligned} t_n^2\|v_n\|_E^2 + t_n^{2q}\int_{\mathbb{R}^2} (I_\alpha * |v_n|^q)|v_n|^q \mathrm{d}x &= \int_{\mathbb{R}^2} f(t_n v_n)t_n v_n \mathrm{d}x \\ &\geqslant \int_{t_n v_n \geqslant R_{\bar{\varepsilon}}} (\beta_0 - \bar{\varepsilon})e^{4\pi t_n^2 v_n^2}\mathrm{d}x \end{aligned}$$

then $\{t_n\}$ is bounded. Up to a subsequence, we suppose that $t_n \to t_0 \geqslant 0$. Since $c_{\varepsilon_n} > C > 0$, we know $t_0 > 0$. Let $n \to \infty$ in above inequality, one has

$$t_0^2\|w_\delta\|_{V_0}^2 + t_0^{2q}\int_{\mathbb{R}^2} (I_\alpha * |w_\delta|^q)|w_\delta|^q \mathrm{d}x = \int_{\mathbb{R}^2} f(t_0 w_\delta)t_0 w_\delta \mathrm{d}x. \tag{3.5}$$

Thus, combining (3.4) with (3.5), we have as $n \to \infty$

$$t_0^{2-2q}\|w\|_{V_0}^2 - \|w\|_{V_0}^2 = \int_{\mathbb{R}^2} \frac{f(t_0 w)t_0 w}{t_0^{2q}}\mathrm{d}x - \int_{\mathbb{R}^2} f(w)w\mathrm{d}x,$$

from (f_4), when $t_0 > 1$, the right-hand side of above equality is more than zero, while the left side is less than zero, which is impossible. We consider $t_0 < 1$ by a similar way. Then we conclude that $t_0 = 1$. Hence, analogous to [2, Lemma 3.3], we obtain $\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0}$. □

In what follows, one can get that I_ε satisfying $(\mathrm{PS})_{c_\varepsilon}$ condition for $\varepsilon \in [0, \varepsilon_0)$.

Lemma 3.3 *Assume that (V) and (f_1)–(f_4) hold. For any $\varepsilon \in [0, \varepsilon_0)$, let $\{u_n\}$ be a $(PS)_{c_\varepsilon}$ sequence and $u_n \rightharpoonup u_\varepsilon$, then $u_n \to u_\varepsilon$ in $H^1(\mathbb{R}^2)$.*

Proof The proof of this lemma is analogous to [13, Lemma 3.3], for convenience to the reader, we sketch the proof briefly. Similar to (2.15) and Lemma 3.1, for functional I_ε with $V(\varepsilon x) = V_\infty$, there are $\bar{C}, \tilde{C} > 0$, we have

$$\bar{C} < \|u_n\|_{V_\infty} < \tilde{C}. \tag{3.6}$$

Then we can obtain that $u_\varepsilon \neq 0$ by Lion's concentration compactness lemma. For this purpose, we give the proof as follows.

Fix $t_n > 0$ satisfying $t_n u_n \in \mathcal{N}_{V_\infty}$, now we prove that $\{t_n\}$ is bounded. Set $v_n = u_n(x+y_n)$, passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$. Moreover, since $u_n \geqslant 0$ for $n \in \mathbb{N}$, we get that there exists $C > 0$ and $\Omega \subset \mathbb{R}^2$ with positive measure, such that, for any $x \in \Omega$, $v(x) > C$. With the reduction to absurdity, we suppose

$$\liminf_{n\to\infty} \frac{f(t_n v_n)t_n v_n}{t_n^{2q}} = +\infty, \quad a.e. \text{ in } H^1(\mathbb{R}^2). \tag{3.7}$$

Since $\langle I'(u_n), u_n\rangle = 0$, one has

$$t_n^2\|u_n\|_{V_\infty}^2 + t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)|u_n|^q\mathrm{d}x = \int_{\mathbb{R}^2} f(t_n u_n)t_n u_n\mathrm{d}x,$$

Consequently, we have

$$t_n^2\|u_n\|_{V_\infty}^2 + t_n^{2q}\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)|u_n|^q\mathrm{d}x = \int_{\Omega} f(t_n v_n)t_n v_n\mathrm{d}x.$$

Using (3.7) and Fatou's lemma, we get

$$\liminf_{n\to\infty}\left[t_n^{2-2q}\|u_n\|_{V_\infty}^2 + \int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)|u_n|^q\mathrm{d}x\right] = \infty,$$

which contradicts (3.6). Then we may assume that

$$\lim_{n\to\infty} t_n = t_0 > 0.$$

We divide the following proof into three steps, that is, $t_0 \leqslant 1$, $t_0 = 1$ and $t_0 < 1$, we conclude that $u_\varepsilon \neq 0$ by make a contradiction in above three steps. Hence, using Fatou's lemma, one can get that $u_n \to u_\varepsilon$ in $H^1(\mathbb{R}^2)$. Thus the proof is complete. □

Lemma 3.4 *Assume that (V) and (f_2) hold. Let u be the nonnegative solution of problem (1.8), then $u > 0$ for all $x \in \mathbb{R}^2$.*

Proof Set $W(x) = V(\varepsilon x) + (I_\alpha * |u|^q)|u|^{q-2} - \frac{f(u)}{u}$, then

$$-\Delta u + W(x)u = 0 \ \text{ in } \mathbb{R}^2.$$

As (3.6), we have $\|u\|_E \leqslant C$, then by Sobolev embedding theorem, $\|u\|_s \leqslant C\|u\|_E$. Consequently, from (V), (f_2) and $u \in L^s(\mathbb{R}^2)$, $s \in [2, \infty)$, we have $W \in L^s(\mathbb{R}^2)$. Using strong maximum principle [21, Theorem 8.19], one has $u > 0$ for all $x \in \mathbb{R}^2$. □

From Lemma 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.1 *Assume that (V) and (f_1)–(f_4) hold. Then for $\varepsilon > 0$ small enough, c_ε is achieved and problem (1.8) has a positive ground state solution.*

4 Proof of Theorem 1.2

To obtain the concentration behavior from ground state solution of problem (1.8), we first recall the following lemma.

Lemma 4.1 *Assume that (V) and (f_1)–(f_4) hold. Let $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to c_{V_0}$. Then, there is a sequence $\{y_n\} \subset \mathbb{R}^2$ such that $v_n = u_n(x + y_n)$ has a convergent subsequence in $H^1(\mathbb{R}^2)$. Moreover, up to a subsequence, $y_n \to y \in \Theta$.*

Proof Analogous to [2, Proposition 4.2], we can prove Lemma 4.1, so we omit the proof here. □

Let $\varepsilon_n \to 0$ as $n \to \infty$, and let u_n be the ground state solution of (1.8). Using Lemma 3.2, we have

$$I_{\varepsilon_n}(u_n) \to c_{V_0}.$$

Hence, there is $\{\bar{y}_n\} \subset \mathbb{R}^2$ such that $v_n = u_n(x + \bar{y}_n)$ is a solution of the following equation

$$-\Delta v_n + V(\varepsilon_n x + \varepsilon_n \bar{y}_n)v_n + (I_\alpha * |v_n|^q)|v_n|^{q-2}v_n = f(v_n), \quad \text{in } \mathbb{R}^2. \tag{4.1}$$

Furthermore, $\{v_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^2)$ and $y_n \to y \in \Theta$. Consequently, there exists $g \in H^1(\mathbb{R}^2)$ satisfying

$$|v_n(x)| \leqslant g(x), \ \text{ a.e. in } \mathbb{R}^2, \ \forall n \in \mathbb{N}. \tag{4.2}$$

Lemma 4.2 *Assume that (V) and (f_1)–(f_4) hold. Then there is a constant $C > 0$ and $\bar{\delta} > 0$ such that $\bar{\delta} \leqslant \|v_n\|_{L^\infty(\mathbb{R}^2)} \leqslant C$, for any $n \in \mathbb{N}$. Moreover,*

$$\lim_{|x|\to\infty} v_n(x) = 0 \ \ uniformly \ in \ n \in \mathbb{N}.$$

Proof For any $R > 0$, $0 < r \leqslant R/2$, set $\eta \in \mathcal{C}^\infty(\mathbb{R}^2)$, $0 \leqslant \eta \leqslant 1$ with $\eta(x) = 1$ if $|x| \geqslant R$ and $\eta(x) = 0$ if $|x| \leqslant R - r$ and $|\nabla\eta| \leqslant 2/r$. For any $L > 0$, fix

$$v_{L,n} = \begin{cases} v_n(x), & v_n(x) \leqslant L, \\ L, & v_n(x) > L, \end{cases}$$

and

$$z_{L,n} = \eta^2 v_{L,n}^{2(\gamma-1)} v_n, \ \ w_{L,n} = \eta v_n v_{L,n}^{\gamma-1},$$

where $\gamma > 1$ to be determined later. Setting $z_{L,n}$ as a test function, i.e. $\langle I'_\varepsilon(u_n), z_{L,n}\rangle$, we know that

$$\begin{aligned}&\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^2} V(\varepsilon x)|v_n|^2\eta^2 v_{L,n}^{2(\gamma-1)}\,\mathrm{d}x+\int_{\mathbb{R}^2}(I_\alpha * |v_n|^q)\eta^2|v_n|^q v_{L,n}^{2(\gamma-1)}\mathrm{d}x\\ &=\int_{\mathbb{R}^2} f(v_n)\eta^2 v_n v_{L,n}^{2(\gamma-1)}\,\mathrm{d}x-2(\gamma-1)\int_{\mathbb{R}^2} v_n v_{L,n}^{2\gamma-3}\eta^2\nabla v_n\nabla v_{L,n}\,\mathrm{d}x\\ &\quad -2\int_{\mathbb{R}^2}\eta v_{L,n}^{2(\gamma-1)} v_n\nabla v_n\nabla\eta\,\mathrm{d}x.\end{aligned}\tag{4.3}$$

Let $\sigma=\sqrt{s},\ h>\frac{2\sigma}{\sigma-1}>2$ and $\gamma=\frac{h(\sigma-1)}{2\sigma}$, for any $\delta>0, \beta>1$, there exists $C(\delta,h,\beta)$ such that

$$f(u)u\leqslant \delta u^2+C(\delta,h,\beta)|u|^h\left[e^{4\pi\beta|u|^2}-1\right],\quad \forall u\in\mathbb{R}.$$

For δ small enough, from (4.3) and Young's inequality, for any $\tau>0$, one has

$$\begin{aligned}&\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^2} V_0|v_n|^2\eta^2 v_{L,n}^{2(\gamma-1)}\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^2} v_n^h\eta^2 v_{L,n}^{2(\gamma-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x+2\int_{\mathbb{R}^2}\eta v_{L,n}^{2(\gamma-1)}v_n\nabla v_n\nabla\eta\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^2} v_n^h\eta^2 v_{L,n}^{2(\gamma-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x+2C_\tau\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)}|\nabla\eta|^2\mathrm{d}x\\ &\quad +2\tau\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\mathrm{d}x.\end{aligned}$$

Choosing $\tau=\frac{1}{4}$, we get

$$\begin{aligned}&\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\,\mathrm{d}x+2\int_{\mathbb{R}^2} V_0|v_n|^2\eta^2 v_{L,n}^{2(\gamma-1)}\,\mathrm{d}x\\ &\leqslant 2C\int_{\mathbb{R}^2} v_n^h\eta^2 v_{L,n}^{2(\gamma-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x+C\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)}|\nabla\eta|^2\mathrm{d}x.\end{aligned}\tag{4.4}$$

In addition, using Sobolev embedding inequality, we know

$$\|w_{L,n}\|_h^2\leqslant C\gamma^2\left[\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)}|\nabla\eta|^2\mathrm{d}x+\int_{\mathbb{R}^2}\eta^2 v_{L,n}^{2(\gamma-1)}|\nabla v_n|^2\mathrm{d}x\right].\tag{4.5}$$

Combining (4.4) with (4.5), we obtain

$$\begin{aligned}\|w_{L,n}\|_h^2&\leqslant C\gamma^2\left[\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\gamma-1)}|\nabla\eta|^2\mathrm{d}x+\int_{\mathbb{R}^2} v_n^h\eta^2 v_{L,n}^{2(\gamma-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x\right]\\ &\leqslant C\gamma^2\left[\int_{R\geqslant|x|\geqslant R-r} v_n^2 v_{L,n}^{2(\gamma-1)}\mathrm{d}x+\int_{|x|\geqslant R-r} v_n^h v_{L,n}^{2(\gamma-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x\right].\end{aligned}\tag{4.6}$$

Hence, by the Hölder inequality, we have

$$
\begin{aligned}
\|w_{L,n}\|_h^2 &\leqslant C\gamma^2\left[\int_{R\geqslant|x|\geqslant R-r} v_n^{2\gamma}\mathrm{d}x + \int_{|x|\geqslant R-r} v_n^{h-2}v_n^{2\gamma}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x\right]\\
&\leqslant C\gamma^2\left\{\left(\int_{R\geqslant|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}\left(\int_{R\geqslant|x|\geqslant R-r} 1\mathrm{d}x\right)^{\frac{1}{t}}\right.\\
&\quad\left.+\left(\int_{|x|\geqslant R-r} v_n^{(h-2)t}\left[e^{4\pi\beta|v_n|^2}-1\right]^t\mathrm{d}x\right)^{\frac{1}{t}}\left(\int_{|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}\right\}\\
&\leqslant C\gamma^2\left\{\left(\int_{R\geqslant|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}\left(\int_{R\geqslant|x|\geqslant R-r} 1\mathrm{d}x\right)^{\frac{1}{t}}\right.\\
&\quad+\left(\int_{|x|\geqslant R-r} v_n^{th}\mathrm{d}x\right)^{\frac{h-2}{ht}}\left(\int_{|x|\geqslant R-r}\left[e^{4\pi\beta|v_n|^2}-1\right]^{\frac{th}{2}}\mathrm{d}x\right)^{\frac{2}{th}}\\
&\quad\left.\left(\int_{|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}\right\}.
\end{aligned}
\tag{4.7}
$$

Furthermore, from (4.2) and Trudinger–Moser inequality (see Lemma 1.1), we know that

$$
\int_{\mathbb{R}^2}\left[e^{4\pi\beta|v_n|^2}-1\right]^s\mathrm{d}x \leqslant \int_{\mathbb{R}^2}\left[e^{4\pi\beta|g|^2}-1\right]^s\mathrm{d}x = C<\infty,\quad \forall n\in\mathbb{N},\ s>1. \tag{4.8}
$$

On the other hand, set $\gamma = h/2 > 1$, it follows from (4.6), (4.8) that,

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^2}(\eta v_n v_{L,n}^{\frac{h}{2}-1})^h\mathrm{d}x\right)^{\frac{2}{h}}\\
&\leqslant C\gamma^2\left(\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\frac{h}{2}-1)}|\nabla\eta|^2\mathrm{d}x + \int_{|x|\geqslant\frac{R}{2}} v_n^h\eta^2 v_{L,n}^{2(\frac{h}{2}-1)}\left[e^{4\pi\beta|v_n|^2}-1\right]\mathrm{d}x\right)\\
&\leqslant C\gamma^2\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\frac{h}{2}-1)}|\nabla\eta|^2\mathrm{d}x + C\gamma^2\left(\int_{|x|\geqslant\frac{R}{2}} v_n^h\left[e^{4\pi\beta|v_n|^2}-1\right]^{\frac{h}{h-2}}\mathrm{d}x\right)^{\frac{h-2}{h}}\\
&\quad\cdot\left(\int_{|x|\geqslant\frac{R}{2}}\eta^h v_n^h v_{L,n}^{h(\frac{h}{2}-1)}\mathrm{d}x\right)^{\frac{1}{h}}\left(\int_{|x|\geqslant\frac{R}{2}}\eta^h v_n^h v_{L,n}^{h(\frac{h}{2}-1)}\mathrm{d}x\right)^{\frac{1}{h}}\\
&\leqslant C\gamma^2\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\frac{h}{2}-1)}|\nabla\eta|^2\mathrm{d}x + C\gamma^2\left(\int_{|x|\geqslant\frac{R}{2}} v_n^h\mathrm{d}x\right)^{\frac{h-2}{h}}\left(\int_{\mathbb{R}^2}\eta^h v_n^h v_{L,n}^{h(\frac{h}{2}-1)}\mathrm{d}x\right)^{\frac{2}{h}},
\end{aligned}
$$

Since $v_n\to v$ in $H^1(\mathbb{R}^2)$, for R sufficiently large, let $\epsilon_1\leqslant\frac{1}{2C\gamma^2}$, we have

$$
\int_{|x|\geqslant\frac{R}{2}} v_n^h\mathrm{d}x \leqslant \epsilon_1 \text{ uniformly in } n.
$$

Then, by the definition of $w_{L,n}$, we have

$$\left(\int_{|x|\geqslant R}(v_n v_{L,n}^{(\frac{h}{2}-1)})^h \mathrm{d}x\right)^{\frac{2}{h}} \leqslant C\gamma^2\int_{\mathbb{R}^2} v_n^2 v_{L,n}^{2(\frac{h}{2}-1)}|\nabla\eta|^2\mathrm{d}x$$
$$\leqslant C\gamma^2\int_{\mathbb{R}^2} v_n^h\mathrm{d}x \leqslant C.$$

Applying Fatou's lemma, when $L\to\infty$, we get

$$\int_{|x|\geqslant R} v_n^{\frac{h^2}{2}}\mathrm{d}x < \infty,$$

which means $v_n \in L^{\frac{h^2}{2}}(|x|\geqslant R)$.

Together (4.7) with (4.8), applying Hölder's inequality, set $\gamma = t = h/2 > 1$, one has

$$\|w_{L,n}\|_h^2 \leqslant C\gamma^2\left(\int_{|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}.$$

Thus,

$$\|v_{L,n}\|_{h\gamma(|x|\geqslant R)}^{2\gamma} \leqslant \left(\int_{|x|\geqslant R-r} v_{L,n}^{h\gamma}\mathrm{d}x\right)^{\frac{2}{h}} \leqslant \left(\int_{\mathbb{R}^2}\eta^h v_n^h v_{L,n}^{h(\gamma-1)}\mathrm{d}x\right)^{\frac{2}{h}}$$
$$= \|w_{L,n}\|_h^2 \leqslant C\gamma^2\left(\int_{|x|\geqslant R-r} v_n^{\frac{2\gamma t}{t-1}}\mathrm{d}x\right)^{\frac{t-1}{t}}$$
$$= C\gamma^2\|v_n\|_{\frac{2\gamma t}{t-1}(|x|\geqslant R-r)}^{2\gamma}.$$

By Fatou's lemma, we derive that

$$\|v_n\|_{h\gamma(|x|\geqslant R)}^{2\gamma} \leqslant C\gamma^2\|v_n\|_{\frac{2\gamma t}{t-1}(|x|\geqslant R-r)}^{2\gamma}.$$

From line by line of [27, Lemma 4.5], let $\chi = \frac{h(t-1)}{2t}$, $s = \frac{2t}{t-1}$, we conclude that

$$\|v_n\|_{\chi^{m+1}s(|x|\geqslant R)} \leqslant C^{\sum_{i=1}^m \chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\|v_n\|_{h(|x|\geqslant R-r)},$$

which means

$$\|v_n\|_{L^\infty(|x|\geqslant R)} \leqslant C\|v_n\|_{h(|x|\geqslant R-r)}. \tag{4.9}$$

For $\bar{x}\in B_R$, using the same argument as above, let $\eta\in\mathcal{C}^\infty(\mathbb{R}^2)$, $0\leqslant\eta\leqslant 1$ with $\eta(x)=1$ if $|x-\bar{x}|\geqslant\bar{R}$ and $\eta(x)=0$ if $|x-\bar{x}|>2\bar{R}$ and $|\nabla\eta|\leqslant 2/\bar{R}$, one can get that

$$\|v_n\|_{L^\infty(|x-\bar{x}|\geqslant\bar{R})} \leqslant C\|v_n\|_{h(|x-\bar{x}|\leqslant 2\bar{R})}. \tag{4.10}$$

By (4.9) and (4.10), it follows from a standard covering argument that

$$\|v_n\|_{L^\infty(\mathbb{R}^2)} < C.$$

In addition, using $v_n\to v$ in $H^1(\mathbb{R}^2)$ and (4.9), for fixed $\delta>0$, there is $R>0$ satisfying $\|v_n\|_{L^\infty(|x|\geqslant R)}<\delta$. Hence,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in } n\in\mathbb{N}.$$

Thus, following the similar arguments of [2, Lemma 4.4], one can get that

$$\bar{\delta} \leqslant \|v_n\|_{L^\infty(\mathbb{R}^2)}.$$

This completes the proof. □

Proof of Theorem 1.2 From Corollary 3.1, we obtain that problem (1.8) has a positive ground state solution.

For item (i), set b_n is the maximum of v_n. Note that b_n is a bounded sequence, that is, there is $R > 0$ such that $b_n \in B_R(0)$. Then, we can get the global maximum of u_{ε_n}, we denoted it by $z_n = b_n + \bar{y}_n$ and $\varepsilon_n z_n = \varepsilon_n b_n + \varepsilon_n \bar{y}_n = \varepsilon_n b_n + y_n$. Since $\{b_n\}$ is bounded, one has

$$\lim_{n\to\infty} z_n = y$$

and

$$\lim_{n\to\infty} V(\varepsilon_n z_n) = V_0.$$

Fix u_ε be the ground state solution of (1.8), then $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a ground state solution of (1.1). Hence, the maxima points of w_ε and u_ε, denoted by η_ε and ς_ε respectively, such that $\eta_\varepsilon = \varepsilon\varsigma_\varepsilon$ and

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

Thus, from (4.1), we can get that (i) holds.

Now we prove the statement (ii). Indeed, note that u_ε is the ground state solution of (1.8), for convenience, we denoted it by u, let ς_ε be the maxima point of u, we have $|u(\varsigma_\varepsilon)| = \max_{x\in\mathbb{R}^2} |u(x)|$. Since ς_ε is bounded, it follows from Lemma 4.2 that

$$u(x) \to 0 \quad \text{as } |x - \varsigma_\varepsilon| \to \infty.$$

Using assumption (f_1), there is $\bar{R} > 0$ such that

$$f(u) \leqslant \bar{C}_\varepsilon |u|, \quad \forall |x - \varsigma_\varepsilon| \geqslant \bar{R}.$$

Combining the above inequality with the boundedness of $|u|$ in $H^1(\mathbb{R}^2)$, one has

$$\begin{aligned}\Delta|u| = \frac{u \cdot \Delta u}{|u|} &= \frac{V(\varepsilon x)u^2 + (I_\alpha * |u|^q)|u|^q - f(u)u}{|u|} \\ &\geqslant (V_0 - \bar{C}_\varepsilon)|u| := \varrho|u|.\end{aligned}$$

Fix $\tilde{w} = |u(x)| - Ce^{-\sqrt{\varrho}(|x-\varsigma_\varepsilon|-\bar{R})}$, where C is given by Lemma 4.2. Thus, we have

$$\Delta\tilde{w} \geqslant \varrho\tilde{w}, \quad |x - \varsigma_\varepsilon| \geqslant \bar{R}.$$

Applying the maximum principle, we obtain

$$\tilde{w} \leqslant 0, \quad \forall |x - \varsigma_\varepsilon| \geqslant \bar{R},$$

namely,

$$|u(x)| \leqslant Ce^{-\sqrt{\varrho}(|x-\varsigma_\varepsilon|-\bar{R})}, \quad \forall |x - \varsigma_\varepsilon| \geqslant \bar{R},$$

which means that (ii) holds. □

5 Proof of Theorem 1.3

In this section, we denote by w the ground state solution of problem (1.8) with $V \equiv V_0$, let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^+, [0,1])$ be a smooth non-increasing cut-off function such that $\varphi(t) = 1$ on $[0, \frac{1}{2}]$ and $\varphi(t) = 0$ on $[1, \infty)$. For any $y \in \Theta$, we define the function

$$\Psi_{\varepsilon,y}(x) := \varphi(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

and $t_\varepsilon > 0$ verifying

$$I_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}) = \max_{t \geqslant 0} I_\varepsilon(t\Psi_{\varepsilon,y}).$$

Furthermore, we define $\Phi_\varepsilon : \Theta \to \mathcal{N}_\varepsilon$ by

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

By construction, one can easily get that $\Phi_\varepsilon(y)$ has compact support for any $y \in \Theta$.

Lemma 5.1 *Assume that (V) and (f_1)–(f_4) hold. Then, uniformly for $y \in \Theta$, the following limit*

$$\lim_{\varepsilon \to 0} I_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}.$$

holds.

Proof Arguing by contradiction, we suppose that the lemma is false. Thus, there exist δ_0, $\{y_n\} \subset \Theta$ and $\varepsilon_n \to 0$ such that

$$\left| I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0} \right| \geqslant \delta_0. \tag{5.1}$$

Now, we claim that $\lim_{n\to\infty} t_{\varepsilon_n} = 1$. Indeed, it follows from Lemma 3.1 and the definition of t_{ε_n} that

$$\begin{aligned}\alpha_1^2 &= \int_{\mathbb{R}^2} \left[t_{\varepsilon_n}^2 |\nabla \Psi_{\varepsilon_n,y_n}|^2 + V(\varepsilon_n x)(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n})^2\right] \mathrm{d}x + t_{\varepsilon_n}^{2q} \int_{\mathbb{R}^2} (I_\alpha * |\Psi_{\varepsilon_n,y_n}|^q)|\Psi_{\varepsilon_n,y_n}|^q \mathrm{d}x \\ &= \int_{\mathbb{R}^2} f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n} \mathrm{d}x.\end{aligned}$$

Using (3.1), one can get that $t_{\varepsilon_n} \geqslant \bar{t} > 0$ for some $\bar{t}$. If $t_{\varepsilon_n} \to \infty$, in view of the boundedness of Ψ_{ε_n,y_n} and (f_4), one has

$$\begin{aligned}&t_{\varepsilon_n}^{2-2q} \|\Psi_{\varepsilon_n,y_n}\|_E^2 + \int_{\mathbb{R}^2} (I_\alpha * |\Psi_{\varepsilon_n,y_n}|^q)|\Psi_{\varepsilon_n,y_n}|^q \mathrm{d}x \\ &\quad= \int_{\mathbb{R}^2} \frac{f(t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}) t_{\varepsilon_n}\Psi_{\varepsilon_n,y_n}}{t_{\varepsilon_n}^{2q}} \mathrm{d}x > \int_{B_{\frac{1}{2}}(0)} \frac{f(t_{\varepsilon_n}\varphi(|\varepsilon_n z|) w(z))}{t_{\varepsilon_n}^{2q-1}} \varphi(|\varepsilon_n z|) w(z) \mathrm{d}z \\ &\quad= \int_{B_{\frac{1}{2}}(0)} \frac{f(t_{\varepsilon_n} w)}{t_{\varepsilon_n}^{2q-1}} w \mathrm{d}x \geqslant \int_{B_{\frac{1}{2}}(0)} \frac{f(t_{\varepsilon_n}\bar{w})}{t_{\varepsilon_n}^{2q-1}} \bar{w} \mathrm{d}x \to \infty,\end{aligned}$$

where $\bar{w} = \inf_{x \in B_{\frac{1}{2}}(0)} w(x)$. Since $t_{\varepsilon_n} \to \infty$ as $\varepsilon_n \to 0$, the left side of above inequality tends to $\int_{\mathbb{R}^2} (I_\alpha * |w|^q)|w|^q \mathrm{d}x$, that is a contradiction. Thus, $t_{\varepsilon_n} \leqslant C$. We may assume that $t_{\varepsilon_n} \to \bar{t}_0$. Repeating the proof of Lemma 3.2 line by line, we can verify that $\bar{t}_0 = 1$.

Note that

$$I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2}\int_{\mathbb{R}^2}\left(\left|\nabla(\varphi(|\varepsilon_n z|)w)\right|^2 + V(\varepsilon_n z + y_n)\left|\varphi(|\varepsilon_n z|)w\right|^2\right)\mathrm{d}x$$
$$+\frac{t_{\varepsilon_n}^{2q}}{2q}\int_{\mathbb{R}^2}\left(I_\alpha * \left|\varphi(|\varepsilon_n z|)w\right|^q\right)\left|\varphi(|\varepsilon_n z|)w\right|^q\mathrm{d}x - \int_{\mathbb{R}^2}F(t_{\varepsilon_n}\varphi(|\varepsilon_n z|)w\mathrm{d}x.$$

Let $n \to \infty$ in above equation, one has $\lim_{n\to\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = c_{V_0}$, which contradicts (5.1). This ends the proof. □

For any $\delta > 0$, set $\rho = \rho(\delta) > 0$ satisfying $\Theta_\delta \subset B_\rho(0)$. We define $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ as follows: $\psi(x) = x$ for $|x| \leqslant \rho$ and $\psi(x) = \rho x/|x|$ for $|x| \geqslant \rho$. Now, we consider the map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^2$, and

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^2}\psi(\varepsilon x)u^{2q}\mathrm{d}x}{\int_{\mathbb{R}^2}u^{2q}\mathrm{d}x}.$$

By using $\Theta \subset B_\rho(0)$ and the Lebesgue's theorem, one has

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \text{uniformly in } y \in \Theta.$$

Lemma 5.2 *Assume that* (V) *and* (f_1)*–*(f_4) *hold. Then, for any* $\delta > 0$*,*

$$\lim_{\varepsilon\to 0}\sup_{u\in\widetilde{\mathcal{N}_\varepsilon}}\mathrm{dist}(\beta_\varepsilon(u), \Theta_\delta) = 0,$$

where $\widetilde{\mathcal{N}_\varepsilon} := \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leqslant c_{V_0} + a(\varepsilon)\}$, $a : \mathbb{R}^+ \to \mathbb{R}^+$ *is a positive function such that* $a(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof Fix $\{\varepsilon_n\} \subset \mathbb{R}^+$ satisfying $\varepsilon_n \to 0$. In fact, there is $\{u_n\} \subset \widetilde{\mathcal{N}_{\varepsilon_n}}$ such that

$$\mathrm{dist}(\beta_{\varepsilon_n}(u_n), \Theta_\delta) = \sup_{u\in\widetilde{\mathcal{N}_{\varepsilon_n}}}\mathrm{dist}(\beta_{\varepsilon_n}(u), \Theta_\delta) + o(1).$$

Hence, we only need to prove that there is a sequence $\{y_n\} \subset \Theta_\delta$ such that

$$\left|\beta_{\varepsilon_n}(u_n) - y_n\right| = o(1). \tag{5.2}$$

By using $\{u_n\} \subset \widetilde{\mathcal{N}_{\varepsilon_n}} \subset \mathcal{N}_{\varepsilon_n}$, we obtain

$$c_{V_0} \leqslant c_{\varepsilon_n} \leqslant I_{\varepsilon_n}(u_n) \leqslant c_{V_0} + a(\varepsilon_n).$$

Then we have $I_{\varepsilon_n}(u_n) \to c_{V_0}$. Applying Lemma 4.1, one can get that there is a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^2$ such that $y_n = \varepsilon_n\tilde{y}_n \in \Theta_\delta$, for n large enough. Thus,

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^2}(\psi(\varepsilon_n z + y_n) - y_n))u_n^{2q}(z + \tilde{y}_n)\mathrm{d}x}{\int_{\mathbb{R}^2}u^{2q}(z + \tilde{y}_n)\mathrm{d}x}.$$

It follows from $\varepsilon_n z + y_n \to y \in \Theta_\delta$ that $\beta_{\varepsilon_n}(u_n) = y_n + o(1)$. Therefore, $\{y_n\}$ verifies (5.2). This completes the proof. □

Lemma 5.3 *Assume that* (V) *and* (f_1)*–*(f_4) *hold. Let* $\{u_n\}$ *be the* $(PS)_{c_\varepsilon}$ *sequence for* I_ε *in* $\mathcal{N}_\varepsilon$*. Then,* $\{u_n\}$ *possesses a convergent subsequence in* $H^1(\mathbb{R}^2)$.

Proof By hypotheses, we have

$$I_\varepsilon(u_n) \to c_\varepsilon, \quad \|I'_\varepsilon(u_n)\|_{(H^1)^*} = o(1).$$

Consequently, there is $\{\lambda_n\} \subset \mathbb{R}$ satisfying

$$I'_\varepsilon(u_n) = \lambda_n J'_\varepsilon(u_n) + o(1), \tag{5.3}$$

where $J_\varepsilon : H^1 \to \mathbb{R}$ is given by

$$J_\varepsilon(u) = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2\right) \mathrm{d}x + \int_{\mathbb{R}^2} (I_\alpha * |u|^q)|u|^q \mathrm{d}x - \int_{\mathbb{R}^2} f(u)u\mathrm{d}x.$$

Thus, from $(f4)$ and Hölder's inequality, one has

$$\begin{aligned}
\langle J'_\varepsilon(u_n), u_n\rangle &= 2\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right) \mathrm{d}x + 2q\int_{\mathbb{R}^2} (I_\alpha * |u_n|^q)|u_n|^q \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x - \int_{\mathbb{R}^2} f'(u_n)u_n^2\mathrm{d}x \\
&\leqslant -\left[\int_{\mathbb{R}^2} f'(u_n)u_n^2\mathrm{d}x - (2q-1)\int_{\mathbb{R}^2} f(u_n)u_n\mathrm{d}x\right] \\
&\leqslant -\left(\int_{\mathbb{R}^2} [f'(u_n)u_n - (2q-1)f(u_n)]^2\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u_n|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&< 0.
\end{aligned}$$

We assume that $\langle J'_\varepsilon(u_n), u_n\rangle \to l \leqslant 0$. Now we claim that $l \neq 0$. Indeed, if $l = 0$, the above inequality shows that $u_n \to 0$ in $L^2(\mathbb{R}^2)$, then $u_n \to 0$ in $H^1(\mathbb{R}^2)$, which contradicts to Lemma 3.1. Hence, $l \neq 0$, and so, $\lambda_n = o_n(1)$. By using (5.3), one has $I'_\varepsilon(u_n) = o(1)$. Then, $\{u_n\}$ is a $(PS)_{c_\varepsilon}$ sequence in $H^1(\mathbb{R}^2)$. From Lemma 3.3, we obtain that $\{u_n\}$ possesses a convergent subsequence in $H^1(\mathbb{R}^2)$. □

By the similar arguments explored in above lemma, we get the following result.

Corollary 5.1 *Assume that (V) and (f_1)–(f_4) hold. Then, the critical points of I_ε on $\mathcal{N}_\varepsilon$ are critical points of I_ε in $H^1(\mathbb{R}^2)$.*

Proof of Theorem 1.3 Applying Lemmas 5.1–5.2, we obtain that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding map $\iota : \Theta \to \Theta_\delta$. Consequently, from Lemma 4.3 in [5], we conclude that

$$\mathrm{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \geqslant \mathrm{cat}_{\Theta_\delta}(\Theta).$$

Because I_ε satisfies the $(PS)_{c_\varepsilon}$ condition for $c \in (c_{V_0}, c_{V_0} + a(\varepsilon))$, according to the Ljusternik–Schnirelmann theory [41], I_ε possesses at least $\mathrm{cat}_{\Theta_\delta}(\Theta)$ critical points in $\mathcal{N}_\varepsilon$. Then, by Corollary 5.1, I_ε possesses at least $\mathrm{cat}_{\Theta_\delta}(\Theta)$ critical points in $H^1(\mathbb{R}^2)$. Repeating the proof of Theorem 1.2, we can also conclude that the positive solutions satisfy concentration behavior and exponential decay. Hence we complete the proof. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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