DOUBLE PHASE EIGENVALUE PROBLEMS WITH AN INDEFINITE PERTURBATION

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Abstract. We consider a perturbed double phase eigenvalue problem. Using the Nehari method, we show that for all small values of the parameter $\lambda > 0$, the problem has at least three nontrivial bounded solutions all with sign information (positive, negative and nodal).

Key Words. Double phase operator, generalized Orlicz spaces, indefinite perturbation, Nehari manifold, nonlinear eigenvalue problem, constant sign and nodal solutions.

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This paper is dedicated with esteem to Professor Enzo Mitidieri on the occasion of his 70th anniversary

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following parametric double phase problem

$$\left\{ \begin{array}{l} -\Delta_p^a(z) - \Delta_q u(z) = \lambda a(z)|u(z)|^{p-2}u(z) + \beta(z)|u(z)|^{r-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \lambda > 0, 1 < q < p < r. \end{array} \right\} \tag{1}$$

Here, Δ_p^a denotes the weighted p-Laplacian with weight $a \in C^{0,1}(\bar{\Omega}), a \geq 0, a \neq 0$, defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

When $a \equiv 1$, we recover the standard p-Laplacian. Equation (1) is driven by a weighted p-Laplacian and a standard q-Laplacian with $q < p$. So, the differential operator of (1) is not homogeneous and is related to the so-called double phase integral functional

$$u \to \int_\Omega [a(z)|Du|^p + |Du|^q] \mathrm{d}z.$$

Let $\eta(z,t)$ denote the density of this integral functional, that is

$$\eta(z,t) = a(z)t^p + t^q, z \in \Omega, t \geq 0.$$

We do not assume that the weight $a(\cdot)$ is bounded away from zero (that is, we do not assume that $0 < \min_{\bar{\Omega}} a$). So, this density exhibits unbalanced growth in t, that is,

$$t^q \leq \eta(z,t) \leq c_0(1 + t^p) \text{ for some } c_0 > 0, \text{ all } t \geq 0.$$

This is a new class of functionals, which were first investigated by Marcellini [12, 13] and Zhikov [22, 23], in the context of problems of the calculus of variations and of nonlinear elasticity. The unbalanced growth of the density $\eta(z, \cdot)$ leads to a setting that uses generalized Orlicz spaces. For boundary value problems driven by the double phase operator, there is no global regularity theory as for balanced growth problems (see Lieberman [9]). There are only local regularity results. A comprehensive account of the known regularity theory

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for double phase problems can be found in the papers of Marcellini [14, 15] and of Mingione & Rădulescu [16]. Double phase operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable, differ considerably with the point in space, with the modulating coefficient $a(\cdot)$ dictating the geometry of a composite made of two different materials.

Problem (1) can be viewed as a perturbed version of an eigenvalue problem for the double phase operator, with the perturbation being $\beta(z)|u|^{r-2}u$. The novel feature of (1), is that the coefficient $\beta(\cdot)$ is sign-changing, that is, the perturbation term is indefinite. Recently there have been existence and multiplicity results for parametric double phase problems. We mention the works of Colasuonno & Squassina [2], Gasinski & Papageorgiou [4], Joe, Kim, Kim & Oh [7], Leonardi & Papageorgiou [8], Liu & Papageorgiou [10, 11], Papageorgiou, Vetro & Vetro [20, 21]. None of the aforementioned works studies the case of an indefinite perturbation.

2. Mathematical background

As we already mentioned in the introduction, the appropriate space framework for double phase problems is provided by generalized Orlicz spaces. A detailed account of the theory of these spaces can be found in the book of Harjulehto & Hästo [6].

Let $L^0(\Omega)$ denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a Lebesgue–null set. Recall that

$$\eta(z,t) = a(z)t^p + t^q, z \in \Omega, t \geq 0,$$

with $a \in C^{0,1}(\bar{\Omega})\backslash\{0\}, a \geq 0, 1 < q < p < N$. Then the generalized Orlicz space $L^\eta(\Omega)$, is defined by

$$L^\eta(\Omega) = \{u \in L^0(\Omega) : \rho_\eta(u) < \infty\},$$

where

$$\rho_\eta(u) = \int_\Omega \eta(z,|u|)\mathrm{d}z = \int_\Omega [a(z)|u|^p + |u|^q]\mathrm{d}z$$

is the modular function corresponding to η. We equip $L^\eta(\Omega)$ with the so-called Luxemburg norm $\|\cdot\|\eta$ defined by

$$\|u\|_\eta = \inf\left\{\lambda > 0 : \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

With this norm $L^\eta(\Omega)$ becomes a Banach space which is separable and reflexive (in fact, uniformly convex). The corresponding generalized Orlicz–Sobolev space $W^{1,\eta}(\Omega)$, is defined by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\}.$$

This space is endowed with the following norm

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta,$$

with $\|Du\|_\eta = \||Du|\|_\eta$. Also we set $W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}$. We know that the Poincaré inequality holds on $W_0^{1,\eta}(\Omega)$, namely we can find $\hat{c} = \hat{c}(\Omega) > 0$ such that

$$\|u\|_\eta \leq \hat{c}\|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega)$$

(see Crespo Blanco, Gasinski, Harjulehto & Winkert [3]). So, on $W_0^{1,\eta}(\Omega)$, we can consider the equivalent norm

$$\|u\| = \|Du\|_\eta \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

The spaces $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are separable and reflexive Banach spaces (in fact, uniformly convex).

Let $\mathcal{A}_p$ be the p-Muckenhoupt class (see Harjulehto & Hästo [6, p.106]). Our hypotheses on the data of (1) are the following:

(H): $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p, a(z) > 0$ for all $z \in \Omega, 1 < q < p < N, \beta \in L^\infty(\Omega), p < r < q^* = \frac{Nq}{N-q}$ and $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 1. The last condition on the exponents p, q, implies that they can not be far apart and that $p < q^* = \frac{Nq}{N-q}$. So the condition on r makes sense and in addition we have compact embeddings of some relevant spaces (see Proposition 2 below).

There is a close relation between the norm $\|\cdot\|_\eta$ and the modular function $\rho_\eta(\cdot)$.

Proposition 1. *If hypotheses (H) hold, then*

(a) $\|u\|_\eta = \vartheta \Leftrightarrow \rho_\eta\left(\frac{u}{\vartheta}\right) = 1$;
(b) $\|u\|_\eta < 1\,(\textit{resp.} = 1, > 1) \Leftrightarrow \rho_\eta(u) < 1\,(\textit{resp.} = 1, > 1)$;
(c) $\|u\|_\eta < 1 \Rightarrow \|u\|_\eta^p \le \rho_\eta(u) \le \|u\|_\eta^q$;
(d) $\|u\|_\eta > 1 \Rightarrow \|u\|_\eta^q \le \rho_\eta(u) \le \|u\|_\eta^p$;
(e) $\|u\|_\eta \to 0\,(\textit{resp.} \to +\infty) \Leftrightarrow \rho_\eta(u) \to 0\,(\textit{resp.} \to +\infty)$.

There are some useful embeddings between the spaces introduced above.

Proposition 2. *If hypotheses (H) hold, then*

(a) $L^\eta(\Omega) \hookrightarrow L^\tau(\Omega)$ *and* $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,\tau}(\Omega)$ *continuously and densely for all* $\tau \in [1, q]$;
(b) $W_0^{1,\eta}(\Omega) \hookrightarrow L^\tau(\Omega)$ *continuously and densely for all* $\tau \in [1, q^*]$ *and* $W_0^{1,\eta}(\Omega) \hookrightarrow L^\tau(\Omega)$ *compactly for all* $\tau \in [1, q^*)$;
(c) $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously and densely.*

Let $\eta_0(z,t) = a(z)t^p$ for all $z \in \Omega$, all $t \ge 0$. As we did for $\eta(\cdot,\cdot)$, we introduce the spaces $L^{\eta_0}(\Omega)$ and $W_0^{1,\eta_0}(\Omega)$. These are separable and reflexive Banach spaces (see Harjulehto & Hästo [6, pp.52,66]). Additionally, we have the following properties:
(i) $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,\eta_0}(\Omega)$ continuously and densely;
(ii) $W_0^{1,\eta_0}(\Omega) \hookrightarrow L^{\eta_0}(\Omega)$ compactly (see Papageorgiou, Rădulescu & Zhang [19]).
We consider the following eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) = \hat{\lambda} a(z)|u(z)|^{p-2}u(z) \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0. \end{array} \right\} \tag{2}$$

Exploiting the compact embedding of $W_0^{1,\eta_0}(\Omega)$ into $L^{\eta_0}(\Omega)$, Papageorgiou, Pudelko & Rădulescu [17], proved that problem (2) has a smallest eigenvalue $\hat{\lambda}_1^a > 0$ which is isolated and simple. Moreover, we have the following variational characterization of $\hat{\lambda}_1^a > 0$

$$\begin{aligned} \hat{\lambda}_1^a &= \inf\left\{\frac{\rho_{\eta_0}(Du)}{\rho_{\eta_0}(u)} : u \in W_0^{1,\eta_0}(\Omega), u \ne 0\right\} \\ &= \inf\{\rho_{\eta_0}(Du) : \rho_{\eta_0}(u) = 1\} \ \text{(by } p\text{-homogeneity).} \end{aligned} \tag{3}$$

For a measurable function $u : \Omega \to [0, \infty)$, we write $0 \prec u$, if for all $K \subseteq \Omega$ compact, $0 < c_K \le u(z)$ for a.a. $z \in K$. We write $u \prec 0$ if $0 \prec -u$. Now let $\hat{u}_1$ be the positive eigenfunction corresponding to $\hat{\lambda}_1^a$ with $\|\hat{u}_1\|_{\eta_0} = 1$. We know that $0 \prec \hat{u}_1$.
We introduce the following quantity

$$\lambda^* = \inf\left\{\frac{\rho_{\eta_0}(Du)}{\rho_{\eta_0}(u)} : u \in W_0^{1,\eta_0}(\Omega), u \ne 0, \int_\Omega \beta(z)|u|^r \mathrm{d}z > 0\right\}. \tag{4}$$

Proposition 3. *If hypotheses (H) hold and* $\int_\Omega \beta(z)\hat{u}_1^r \mathrm{d}z > 0$, *then* $\lambda^* = \hat{\lambda}_1^a$.

Proof. From (3) and (4), we see that $\hat{\lambda}_1^a \leq \lambda^*$. On the other hand, the hypotheses imply that

$$\lambda^* \leq \frac{\rho_{\eta_0}(D\hat{u}_1)}{\rho_{\eta_0}(\hat{u}_1)} = \hat{\lambda}_1^a,$$
$$\Rightarrow \lambda^* = \hat{\lambda}_1^a.$$

The proof is now complete. □

In the sequel we will also use the following strengthened version of hypotheses (H).

(H'): Hypotheses (H) hold and in addition $\int_\Omega \beta(z)\hat{u}_1^r \mathrm{d}z > 0$.

For every $\lambda > 0$, let $\varphi_\lambda : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{p}\rho_{\eta_0}(u) - \frac{1}{r}\int_\Omega \beta(z)|u|^r \mathrm{d}z \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(W_0^{1,\eta}(\Omega))$.

We also consider the positive and negative truncations of $\varphi_\lambda(\cdot)$, namely the C^1 functionals $\varphi_\lambda^\pm : W_0^{1,\eta} \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\rho_{\eta_0}(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{p}\rho_{\eta_0}(u^\pm) - \frac{1}{r}\int_\Omega \beta(z)(u^\pm)^r \mathrm{d}z \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Using the Nehari method we will show that for all $\lambda \in (0, \hat{\lambda}_1^a)$ problem (1) has at least three nontrivial solutions, one positive, one negative and the third nodal (sign-changing).

So, for the functionals φ_λ, $\varphi_\lambda^\pm$ introduced above, we define the corresponding Nehari manifolds. We have

$$N_\lambda = \{u \in W_0^{1,\eta}(\Omega) : \langle \varphi_\lambda'(u), u\rangle, u \neq 0\},$$
$$N_\lambda^\pm = \{u \in W_0^{1,\eta}(\Omega) : \langle (\varphi_\lambda^\pm)'|u|, u\rangle, u \neq 0\}.$$

Note that

$$u \in N_\lambda \Leftrightarrow u^+ \in N_\lambda^+, u^- \in N_\lambda^- \text{ and } u \in N_\lambda^+ \Leftrightarrow -u \in N_\lambda^-.$$

The properties of these sets are linked to the behavior of the fibering function $t \mapsto \varphi_\lambda(tu)$ for given $u \in W_0^{1,\eta}(\Omega)$, all $t \geq 0$.

3. Multiplicity theorem

We start by proving some important properties of the Nehari manifolds $N_\lambda, N_\lambda^\pm$.

Proposition 4. *If hypotheses (H) hold, $u \in W_0^{1,\eta}(\Omega)$, $u \neq 0$, $\int_\Omega \beta(z)|u|^r \mathrm{d}z > 0$, and $\lambda \in (0, \hat{\lambda}_1^a)$, then there exists unique $t_u > 0$ such that $t_u u \in N_\lambda$.*

Proof. We consider the fibering map $t \mapsto \varphi_\lambda(tu), t \geq 0$. We have

$$\varphi_\lambda(tu) = \frac{t^p}{p}\rho_{\eta_0}(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{\lambda t^p}{p}\rho_{\eta_0}(u) - \frac{t^r}{r}\int_\Omega \beta(z)|u|^r \mathrm{d}z,$$
$$\Rightarrow \frac{\varphi_\lambda(tu)}{t^q} \geq \frac{t^{p-q}}{p}(1 - \frac{\lambda}{\hat{\lambda}_1^a})\rho_\eta(Du) + \frac{1}{q}\|Du\|_q^q - \frac{t^{r-q}}{r}\int_\Omega \beta(z)|u|^r \mathrm{d}z \text{ (see (3))}$$

Since $1 < q < p < r$ and $0 < \lambda < \hat{\lambda}_1^a$, we see that

$$\liminf_{t \to 0^*} \frac{\varphi_\lambda(tu)}{t^q} \geq \frac{1}{q}\|Du\|_q^q > 0. \tag{5}$$

On the other hand dividing with t^r, we have

$$\begin{aligned}&\frac{\varphi_\lambda(tu)}{t^r} \leq \frac{1}{p}\frac{1}{t^{r-p}}\rho_{\eta_0}(Du) + \frac{1}{t^{r-q}}\|Du\|_q^q - \frac{1}{r}\int_\Omega \beta(z)|u|^r \mathrm{d}z,\\ \Rightarrow&\limsup_{t\to+\infty}\frac{\varphi_\lambda(tu)}{t^r} \leq -\frac{1}{r}\int_\Omega \beta(z)|u|^r\mathrm{d}z < 0 \text{ (by hypothesis).}\end{aligned} \tag{6}$$

From (5) we see that

$$\varphi_\lambda(tu) > 0, \text{ for } t > 0 \text{ small,} \tag{7}$$

while from (5) we have

$$\varphi_\lambda(tu) < 0, \text{ for } t > 0 \text{ large,} \tag{8}$$

The fibering function is continuous. So, from (7) and (8) we infer that these exists $t_u > 0$ such that

$$\varphi_\lambda(t_u u) = \max_{t\geq 0}\varphi_\lambda(tu).$$

Then we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda(tu)|_{t=t_u} = \langle\varphi_\lambda'(t_u u), u\rangle = 0 \text{ (by the chain rule)},\\ \Rightarrow&\langle\varphi_\lambda'(t_u u), t_u u\rangle = 0,\\ \Rightarrow&t_u u \in N_\lambda.\end{aligned}$$

We show that $t_u > 0$ is unique. Suppose for some $t > 0$, we have $tu \in N_\lambda$, then

$$\begin{aligned}&t^p[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + t^q\|Du\|_q^q = t^r\int_\Omega \beta(z)|u|^r\mathrm{d}z,\\ \Leftrightarrow&\frac{1}{t^{r-p}}[\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u)] + \frac{1}{t^{r-q}}\|Du\|_q^q = \int_\Omega \beta(z)|u|^r\mathrm{d}z,\end{aligned} \tag{9}$$

Since $\lambda \in (0, \hat{\lambda}_1^a)$, we have $\rho_{\eta_0}(Du) - \lambda\rho_{\eta_0}(u) > 0$ and so in (9) we see that the left-hand side is a strictly decreasing function of $t > 0$, which goes to $+\infty$ as $t \to 0^+$ and to zero as $t \to +\infty$. Since $\int_\Omega \beta(z)|u|^r\mathrm{d}z > 0$ (by hypothesis), we conclude that $t_u > 0$ is unique. □

An immediate consequence of the above proposition, is the following result concerning the Nehari manifold N_λ.

Corollary 1. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then $N_\lambda \neq \varnothing$.*

In a similar fashion, we prove analogous results for the Nehari manifolds $N_\lambda^\pm$.

Proposition 5. *If hypotheses (H) hold, $u \in W_0^{1,\eta}(\Omega)$, $\int_\Omega \beta(z)(u^\pm)^r\mathrm{d}z > 0$, and $\lambda \in (0, \hat{\lambda}_1^a)$, then there exist unique $t_u^\pm > 0$ such that*

$$t_u^\pm u \in N_\lambda^\pm \text{ and so } N_\lambda^\pm \neq 0.$$

Remark 2. Therefore if hypotheses (H') hold, then $N_\lambda^\pm \neq \varnothing$ for all $\lambda \in (0, \hat{\lambda}_1^a)$.

Next, we show that the functionals φ_λ and $\varphi_\lambda^\pm$ are bounded away from zero when evaluated on the corresponding Nehari manifolds.

Proposition 6. *If hypotheses (H) hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then*

(a) for all $u \in N_\lambda, \varphi_\lambda(u) \geq c_1(\lambda)\min\{\|u\|^p, \|u\|^q\}$ with $c_1(\lambda) > 0$;
(b) for all $u \in N_\lambda^\pm, \varphi_\lambda^\pm(u) \geq c_2(\lambda)\min\{\|u\|^p, \|u\|^q\}$ with $c_2(\lambda) > 0$.

Proof. (a) If $u \in N_\lambda$, then

$$\rho_{\eta_0}(Du) + \|Du\|_q^q - \lambda \rho_{\eta_0}(u) = \int_\Omega \beta(z)|u|^r \mathrm{d}z, \tag{10}$$

Then we have

$$\begin{aligned}
\varphi_\lambda(u) &\geq \frac{1}{p}[\rho_{\eta_0}(Du) + \|Du\|_q^q - \lambda\rho_{\eta_0}(u)] - \frac{1}{r}\int_\Omega \beta(z)|u|^r \mathrm{d}z \ \ (\text{since } \ q < p) \\
&= \left[\frac{1}{p} - \frac{1}{r}\right](\rho_{\eta_0}(Du) + \|Du\|_q^q - \lambda\rho_{\eta_0}(u)) \ \ (\text{see } \ (10)) \\
&\geq \frac{r-p}{rp}\left((1 - \frac{\lambda}{\hat{\lambda}_1^a})\rho_{\eta_0}(Du) + \|Du\|_q^q\right) \ \ (\text{see } \ (3)) \\
&\geq \frac{r-p}{rp} c_1(\lambda)\rho_{\eta_0}(Du) \ \ \text{for some } \ c_1(\lambda) > 0 \ \ (\text{since } \ \lambda \in (0, \hat{\lambda}_1^a)) \\
&\geq \frac{r-p}{rp} c_1(\lambda)\min\{\|u\|^p, \|u\|^q\} \ \ (\text{see Proposition 1}).
\end{aligned}$$

(b) The proof of this part is similar to that of (a). ☐

From this proposition, we infer the following result.

Corollary 2. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then $\varphi_\lambda|_{N_\lambda}$ and $\varphi_\lambda^\pm|_{N_\lambda^\pm}$ are all coercive.*

Remark 3. This corollary highlights the significance of the Nehari manifold. Note that since $r > p > q$, we can not have coercivity of $\varphi_\lambda(\cdot)$ on all of $W_0^{1,\eta}(\Omega)$. However $\varphi_\lambda|_{N_\lambda}$ is coercive (similarly for $\varphi_\lambda^\pm$).

Next we show that the elements of the Nehari manifolds are bounded below away from zero.

Proposition 7. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then we can find $c(\lambda) > 0$ such that*

$$0 < c(\lambda) \leq \|u\| \ \ \textit{for all} \ \ u \in N_\lambda, u \in N_\lambda^\pm.$$

Proof. If $u \in N_\lambda$, then

$$\begin{aligned}
\rho_{\eta_0}(Du) + \|Du\|_q^q &= \lambda\rho_{\eta_0}(u) + \int_\Omega \beta(z)|u|^r \mathrm{d}z \\
&\leq \frac{\lambda}{\hat{\lambda}_1^a}\rho_{\eta_0}(Du) + \int_\Omega \beta(z)|u|^r \mathrm{d}z \ \ (\text{see } \ (3)),
\end{aligned}$$

$$\Rightarrow \left(1 - \frac{\lambda}{\hat{\lambda}_1^a}\right)\rho_{\eta_0}(Du) + \|Du\|_q^q \leq \int_\Omega \beta(z)|u|^r \mathrm{d}z, \tag{11}$$

$$\Rightarrow c_3(\lambda)\rho_\eta(Du) \leq \|u\|^r (\text{for some } \ c_3(\lambda) > 0)$$

$$(\text{recall that } \ W_0^{1,\eta}(\Omega) \hookrightarrow L^r(\Omega) \ \text{ continuously, see Proposition } \ 2).$$

If $\|u\| \leq 1$, then from (11) and Proposition 1 we have

$$0 < c_3(\lambda) \leq \|u\|^{r-p}.$$

If $\|u\| > 1$, then from (11) and Proposition 1 we have

$$0 < c_3(\lambda) \leq \|u\|^{r-q}.$$

We conclude that there exists $\hat{c} > 0$ such that

$$0 < \hat{c}(\lambda) \leq \|u\| \text{ for all } u \in N_\lambda.$$

Similarly for the Nehari manifolds $N_\lambda^\pm$. □

Proposition 8. *If hypotheses (H) hold and $\lambda \in (0, \hat{\lambda}_1^a)$, and $u \in N_\lambda$ (resp. $u \in N_\lambda^\pm$), then $\int_\Omega \beta(z)|u|^r \mathrm{d}z > 0$, (resp. $\int_\Omega \beta(z)(u^\pm)^r \mathrm{d}z > 0$).*

Proof. We know that

$$\left(1 - \frac{\lambda}{\hat{\lambda}_1^a}\right)\rho_a(Du) + \|Du\|_q^q \leq \int_\Omega \beta(z)|u|^r \mathrm{d}z$$
$$\Rightarrow 0 < c_4(\lambda)\rho_\eta(Du) \leq \int_\Omega \beta(z)|u|^r \mathrm{d}z.$$

Similarly we show that

$$0 < \int_\Omega \beta(z)(u^\pm)^r \mathrm{d}z \text{ for all } u \in N_\lambda^\pm.$$

This completes the proof. □

To generate a nodal solution, we will need the following subset of the Nehari manifold N_λ

$$N_\lambda^0 = \{u \in W_0^{1,\eta}(\Omega) : u^\pm \in N_\lambda\}.$$

Evidently $N_\lambda^0 \subseteq N_\lambda$ and next we show that $N_\lambda^0 \neq \emptyset$ when $0 < \lambda < \lambda_1^a$.

Proposition 9. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then $N_\lambda^0 \neq \varnothing$.*

Proof. On account of hypotheses (H'), we know that $N_\lambda, N_\lambda^\pm \neq \varnothing$ (see Corollary 1 and Remark 2). Let $u \in N_\lambda^+$ and $v \in N_\lambda^-$. From Proposition 8, we know that

$$\int_\Omega \beta(z)(u^+)^r \mathrm{d}z > 0, \int_\Omega \beta(z)(v^-)^r \mathrm{d}z > 0.$$

Since $u^+, v^- \in W_0^{1,\eta}(\Omega) \setminus 0$, by Proposition 4, we can find unique $t_1, t_2 > 0$ such that

$$t_1 u^+ \in N_\lambda,\ t_2 v^- \in N_\lambda. \tag{12}$$

We set $y = t_1 u^+ - t_2 v^-$. Evidently $y^+ = t_1 u^+, y^- = t_2 v^-$ and so from (12) we conclude that $y \in N_\lambda^0$ and so $N_\lambda^0 \neq \varnothing$. □

Now we consider the following constrained minimization problems

$$m_\lambda^+ = \inf_{N_\lambda^+} \varphi_\lambda^+,\ m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda^-, \text{ and } m_\lambda^0 = \inf_{N_\lambda^0} \varphi_\lambda. \tag{13}$$

To solve these minimization problems, we will need the following lemma.

Lemma 1. *If hypotheses (H) hold and $\lambda \in (0, \hat{\lambda}_1^a)$ and $u \in N_\lambda^\pm$, then $\varphi_\lambda^\pm(tu) \leq \varphi_\lambda^\pm(u)$ for all $t \geq 0$.*

Proof. We do the proof for the pair $(\varphi_\lambda^+, N_\lambda^+)$, the proof for the pair $(\varphi_{\lambda'}^-, N_{\lambda'}^-)$ being similar.

Reasoning as in the proof of Proposition 4, we show that

$$\varphi_\lambda^+(tu) > 0 \text{ for all } t \in (0,1) \text{ small,}$$
$$\varphi_\lambda^+(tu) < 0 \text{ for } t > 1 \text{ large.}$$

Therefore as in that proof, we produce a unique $\hat{t}_u > 0$ such that

$$\begin{aligned} &\varphi_\lambda^+(\hat{t}_u u) = \max_{t\geq 0} \varphi_\lambda^+(tu), \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda^+(tu)|_{t=\hat{t}_u} = 0, \\ &\Rightarrow \langle (\varphi_\lambda^+)'(\hat{t}_u u), \hat{t}_u u\rangle = 0, \\ &\Rightarrow \hat{t}_u u \in N_\lambda^+. \end{aligned} \tag{14}$$

But by hypothesis, $u \in N_\lambda^+$. Therefore $\hat{t}_u = 1$ and so from (14) it follows that

$$\varphi_\lambda^+(tu) \leq \varphi_\lambda^+(u) \text{ for all } t \geq 0.$$

Similarly for the pair $(\varphi_\lambda^-, N_\lambda^-)$. □

Using this lemma, we can solve the minimization problems in (14).

Proposition 10. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then*

- *(a) there exists $\hat{u}_\lambda \in N_\lambda^+$ such that $\varphi_\lambda^+(\hat{u}_\lambda) = m_\lambda^+$;*
- *(b) there exists $\hat{v}_\lambda \in N_\lambda^-$ such that $\varphi_\lambda^-(\hat{v}_\lambda) = m_\lambda^-$;*
- *(c) there exists $\hat{\varphi}_\lambda \in N_\lambda^0$ such that $\varphi_\lambda(\hat{y}_\lambda) = m_\lambda^0$.*

Proof. (a) Let $\{u_n\}_{n\in\mathbb{N}} \subseteq N_\lambda^+$ be such that

$$\varphi_\lambda^+(u_n) \downarrow m_\lambda^+.$$

From Corollary 2 we know that $\varphi_\lambda^+|_{N_\lambda^+}$ is coercive. Therefore $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded. So, we may assume that

$$u_n \to u_\lambda \text{ weakly in } W_0^{1,\eta}(\Omega), u_n \to u_\lambda \text{ in } L^r(\Omega) \text{ (see Proposition 2)}.$$

If $u_\lambda = 0$, then

$$u_n \to 0 \text{ weakly in } W_0^{1,\eta}(\Omega), u_n^+ \to 0 \text{ in } L^r(\Omega). \tag{15}$$

Since $u_n \in N_\lambda^+$, we have

$$\begin{aligned} &\rho_\eta(Du_n) = \lambda\rho_{\eta_0}(u_n^+) + \int_\Omega \beta(z)(u_n^+)^r \mathrm{d}z, \\ &\Rightarrow \rho_\eta(Du_n) \to 0 \text{ (see (15))}, \\ &\Rightarrow u_n \to 0 \text{ (in } W_0^{1,\eta}(\Omega) \text{ (see Proposition 1)} \\ &\Rightarrow \varphi_\lambda^+(u_n) \to \varphi_\lambda^+(0) = 0 = m_\lambda^+. \end{aligned}$$

But from Propositions 6 and 7, we see that $m_\lambda^+ > 0$, a contradiction. Therefore $u_\lambda \neq 0$. Note that

$$\begin{aligned} &\int_\Omega \beta(z)(u_n^+)^r \mathrm{d}z > 0, \text{ for all } n \in \mathbb{N} \text{ (see Proposition 8)}, \\ \Rightarrow &\int_\Omega \beta(z)(u_\lambda^+)^r \mathrm{d}z \geq 0 \text{ (see (15))}. \end{aligned}$$

Suppose $\int_\Omega \beta(z)(u_\lambda^+)\mathrm{d}z = 0$. Then

$$\lim_{n\to\infty} \int_\Omega \beta(z)(u_n^+)^r \mathrm{d}z = 0. \tag{16}$$

Since $u_n \in N_\lambda^+, n \in \mathbb{N}$, we have

$$\begin{aligned}&\rho_\eta(Du_n) = \lambda\rho_{\eta_0}(u_n^+) + \int_\Omega \beta(z)(u_n^+)^r \mathrm{d}z,\\ \Rightarrow &\rho_{\eta_0}(Du_\lambda^+) < \lambda\rho_{\eta_0}(u_\lambda^+) \text{ (see (16) and note that } \|Du_\lambda\|_q > 0),\end{aligned}$$

which contradicts (3). Therefore

$$0 < \int_\Omega \beta(z)(u_\lambda^+)^r \mathrm{d}z.$$

According to Proposition 5, there exists unique $t_{u_\lambda}^+ = t_\lambda^+ > 0$, such that $t_\lambda^+ u_\lambda = \acute{u}_\lambda \in N_\lambda^+$. So we have

$$\begin{aligned}m_\lambda^+ &= \lim_{n\to\infty} \varphi_\lambda^+(u_n)\\ &\geq \liminf_{n\to\infty} \varphi_\lambda^+(t_\lambda^+ u_n) \text{ (see Lemma 1)}\\ &\geq \varphi_\lambda^+(\hat{u}_\lambda) \text{ (since } \varphi_\lambda^+ \text{ is sequentially weakly lower semicontinuous)}\\ &\geq m_\lambda^+ \text{ (since } \hat{u}_\lambda \in N_\lambda^+),\\ \Rightarrow &\varphi_\lambda^+(\hat{u}_\lambda) = m_\lambda^+ \text{ with } \hat{u}_\lambda \in N_\lambda^+.\end{aligned}$$

(b) Arguing as in (a), we produce $\hat{v}_\lambda \in N_\lambda^-$ such that

$$\varphi_\lambda^-(\hat{v}_\lambda) = m_\lambda^-.$$

(c) Let $\{y_n\}_{n\in\mathbb{N}} \subseteq N_\lambda^0$ such that

$$\varphi_\lambda(y_n) \downarrow m_\lambda^0.$$

From Corollary 2, we know that $\varphi_\lambda|_{N_\lambda}$ is coercive. Since $N_\lambda^0 \subseteq N_\lambda$, it follows that $\{y_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$ is bounded. Therefore

$$\{y_n^+\}_{n\in\mathbb{N}}, \{y_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega)$$

are both bounded.

We may assume that

$$y_n^+ \to w_1,\ y_n^- \to w_2 \text{ weakly in } W_0^{1,\eta}(\Omega), w_1, w_2 \geq 0, \{w_1 > 0\} \cap \{-w_2 < 0\} = \varnothing. \tag{17}$$

We have

$$\begin{aligned}&\varphi_\lambda(y_n) = \varphi_\lambda(y_n^+) + \varphi_\lambda(y_n^-) \geq m_\lambda^+ + m_\lambda^-,\\ \Rightarrow &m_\lambda^0 \geq m_\lambda^+ + m_\lambda^- > 0. \text{(see Propositions 6 and 7)}.\end{aligned}$$

Since $y_n \in N_\lambda^0$, by definition we have $y_n^+, y_n^- \in N_\lambda$ for all $n \in \mathbb{N}$. Therefore the following equalities hold:

$$\rho_\eta(Dy_n^+) = \lambda\rho_{\eta_0}(y_n^+) + \int_\Omega \beta(z)(y_n^+)^r \mathrm{d}z, \tag{18}$$

$$\rho_\eta(Dy_n^-) = \lambda\rho_{\eta_0}(y_n^-) + \int_\Omega \beta(z)(y_n^-)^r \mathrm{d}z \text{ for all } n \in \mathbb{N}. \tag{19}$$

Suppose that $w_1 = 0$ (see (17)). Then from (17) and (18), we see that

$$\begin{aligned}&\rho_\eta(Dy_n^+) \to 0,\\ \Rightarrow &y_n^+ \to 0 \text{ in } W_0^{1,\eta}(\Omega) \text{ (see Proposition 1)}.\end{aligned} \tag{20}$$

But $0 < m_\lambda^+ \leq \varphi_\lambda^+(y_n^+)$ for all $n \in \mathbb{N}$ and $\varphi_\lambda^+(y_n^+) \to \varphi_\lambda^+(0) = 0$ (see (20)), a contradiction. Therefore $w_1 \neq 0$. Similarly using this time (17) and (19), we show that $w_2 \neq 0$. As before using (17) and (18), (19), we show that

$$0 < \int_\Omega \beta(z) w_1^r \mathrm{d}z, 0 < \int_\Omega \beta(z) w_2^r \mathrm{d}z.$$

On account of Proposition 4, we can find unique $t_1, t_2 > 0$ such that

$$t_1 w_1 \in N_\lambda, t_2 w_2 \in N_\lambda.$$

We set $\hat{y}_\lambda = t_1 w_1 - t_2 w_2 \in W_0^{1,\eta}(\Omega), w \neq 0$ and observe that

$$\begin{aligned} &\hat{y}_\lambda^+ = t_1 w_1 \in N_\lambda, \hat{y}_\lambda^- = t_2 w_2 \in N_\lambda, \\ \Rightarrow &\hat{y}_\lambda \in N_\lambda^0. \end{aligned}$$

Note that

$$\begin{aligned} m_\lambda^0 &= \lim_{n\to\infty} \varphi_\lambda(y_n) \\ &= \lim_{n\to\infty} [\varphi_\lambda(y_n^+) + \varphi_\lambda(y_n^-)] \\ &= \lim_{n\to\infty} [\varphi_\lambda^+(y_n^+) + \varphi_\lambda^-(y_n^-)] \\ &\geq \liminf_{n\to\infty} [\varphi_\lambda^+(t_1 y_n^+) + \varphi_\lambda^-(t_2 y_n^-)] \\ &= \liminf_{n\to\infty} [\varphi_\lambda(t_1 y_n^+) + \varphi_\lambda(t_2 y_n^-)] \\ &\geq \varphi_\lambda(t_1 w_1) + \varphi_\lambda(t_2 w_2) \\ &= \varphi_\lambda(\hat{y}_\lambda) \\ &\geq m_\lambda^0 \text{ (since } \hat{y}_\lambda \in N_\lambda^0), \\ &\Rightarrow \varphi_\lambda(\hat{y}_\lambda) = m_\lambda^0 \text{ with } \hat{y}_\lambda \in N_\lambda^0. \end{aligned}$$

The proof is now complete. □

Next, we show that the minimizers produced in Proposition 10, are critical points of $\varphi_\lambda^\pm$ and φ_λ respectively, that is, $N_\lambda^\pm$ and N_λ^0 are natural constraints for the functionals $\varphi_\lambda^\pm$ and φ_λ respectively (see Papageorgiou, Rădulescu & Repovs [18, p.425]).

In what follows, we denote by $K_{\varphi_\lambda^\pm}$ and K_{φ_λ} the critical sets of $\varphi_\lambda^\pm$ and φ_λ respectively, that is,

$$K_{\varphi_\lambda^\pm} = \{u \in W_0^{1,\eta}(\Omega) : (\varphi_\lambda^\pm)'(u) = 0\},$$

$$K_{\varphi_\lambda} = \{u \in W_0^{1,\eta}(\Omega) : \varphi_\lambda'(u) = 0\}.$$

Proposition 11. *If hypotheses (H') hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then*

$$\hat{u}_\lambda \in K_{\varphi_\lambda^+}, \hat{v}_\lambda \in K_{\varphi_\lambda^-} \text{ and } \hat{y}_\lambda \in K_{\varphi_\lambda}.$$

Proof. Let $\psi_\lambda^+ : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ be defined by

$$\psi_\lambda^+(u) = \rho_\eta(Du) - \lambda \rho_{\eta_0}(u^+) - \int_\Omega \beta(z)(u^+)^r \mathrm{d}z.$$

Evidently, $\psi_\lambda^+ \in C^1(W_0^{1,\eta}(\Omega))$. We see that

$$m_\lambda^+ = \inf\{\varphi_\lambda^+(u) : u \in W_0^{1,\eta}(\Omega) \setminus \{0\}, \psi_\lambda^+(u) = 0\}.$$

The Lagrange multiplier rule (see Theorem 5.5.9 of Papageorgiou, Rădulescu, Repovs [18, p.422]) implies that we can find $(\xi, \mu) \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\xi(\varphi_\lambda^+)'(\hat{u}_\lambda) + \mu(\psi_\lambda^+)'(\hat{u}_\lambda) = 0 \text{ in } W_0^{1,\eta}(\Omega). \tag{21}$$

If $\mu = 0$, then $\xi \neq 0$ and we have

$$\begin{aligned} &\xi(\varphi_\lambda^+)'(\hat{u}_\lambda) = 0, \\ \Rightarrow &(\varphi_\lambda^+)'(\hat{u}_\lambda) = 0 \\ \Rightarrow &\hat{u}_\lambda \in K_{\varphi_\lambda^+}. \end{aligned}$$

So, we need to show that $\mu = 0$. To this end on (21), we act with $\hat{u}_\lambda \in N_\lambda^+$. Then since $\langle(\varphi_\lambda^+)'(\hat{u}_\lambda), \hat{u}_\lambda\rangle = 0$ (recall that $\hat{u}_\lambda \in N_\lambda^+$), we obtain

$$\begin{aligned} &\mu\langle(\psi_\lambda^+)'(\hat{u}_\lambda), \hat{u}_\lambda\rangle = 0, \\ \Rightarrow &\mu[p\rho_{\eta_0}(D\hat{u}_\lambda) + q\|D\hat{u}_\lambda\|_q^q - \lambda p\rho_{\eta_0}(u_\lambda^+) - \int_\Omega r\beta(z)(u_\lambda^+)^r \mathrm{d}z] = 0, \\ \Rightarrow &\mu p[\rho_\eta(D\hat{u}_\lambda) - \lambda\rho_{\eta_0}(\hat{u}_\lambda^+) - \int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z] \\ &\quad + \mu[-(p-q)\|D\hat{u}_\lambda\|_q^q - (r-p)\int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z] \\ = &0. \end{aligned} \tag{22}$$

If $\mu \neq 0$, then we may assume that $\mu > 0$ (the reasoning is similar if $\mu < 0$). Then from Proposition 8 and since $p < r$, we see that

$$\mu[-(p-q)\|D\hat{u}_\lambda\|_q^q - (r-p)\int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z] < 0. \tag{23}$$

So from (22) and (23), we infer that

$$\mu p[\rho_\eta(D\hat{u}_\lambda) - \lambda\rho_{\eta_0}(\hat{u}_\lambda^+) - \int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z] > 0. \tag{24}$$

But recall that $\hat{u}_\lambda \in N_\lambda^+$. Hence

$$\begin{aligned} &\rho_{\eta_0}(D\hat{u}_\lambda) + \|D\hat{u}_\lambda\|_q^q = \lambda\rho_{\eta_0}(\hat{u}_\lambda^+) + \int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z, \\ \Rightarrow &\rho_\eta(D\hat{u}_\lambda) - \lambda\rho_{\eta_0}(\hat{u}_\lambda^+) - \int_\Omega \beta(z)(\hat{u}_\lambda^+)^r \mathrm{d}z = 0. \end{aligned} \tag{25}$$

Comparing (24) and (25), we have a contradiction. This proves that $\mu = 0$, hence $\hat{u}_\lambda \in K_{\varphi_\lambda^+}$.

Similarly we show that $\hat{v}_\lambda \in K_{\varphi_\lambda^-}$, using this time the C^1-constraint function $\psi_\lambda^- : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^-(u) = \rho_\eta(Du) - \lambda\rho_{\eta_0}(u^-) - \int_\Omega \beta(z)(u^-)^r \mathrm{d}z \text{ for all } u \in W_0^{1,\eta}(\Omega).$$

Finally, we show that $\hat{y}_\lambda \in K_{\varphi_\lambda}$. To this end, note that

$$m_\lambda = \inf\{\varphi_\lambda(u) : u^\pm \in W_0^{1,\eta}(\Omega) \setminus \{0\}, \psi_\lambda^+(u^+) = 0, \psi_\lambda^-(u^-) = 0\}.$$

Consider the maps $\tau_\pm : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)$ defined by

$$\tau_\pm(u) = u^\pm.$$

These are Lipschitz mappings and we have

$$\psi_\lambda^\pm(u^\pm) = (\psi_\lambda^\pm \circ \tau_\pm)(u).$$

Then by the nonsmooth Lagrange multiplier rule of Clarke (see Theorem 10.47 in [1, p. 221]), we can find $(\xi, \mu_1, \mu_2) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$0 \in \xi\varphi_\lambda'(\hat{y}_\lambda) + \mu_1(\psi_\lambda^+)'(\hat{y}_\lambda^+)\partial\tau_+(\hat{y}_\lambda) + \mu_2(\psi_\lambda^-)'(\hat{y}_\lambda^-)\partial\tau_-(\hat{y}_\lambda). \tag{26}$$

Here by $\partial\vartheta(\cdot)$ we denote the Clarke subdifferential of a locally Lipschitz function $\vartheta(\cdot)$ and we have used the nonsmooth chain rule for this subdifferential (see Clarke [1, pp. 196, 202]).

Suppose that $(\mu_1, \mu_2) \in \mathbb{R}^2 \setminus \{0\}$. Hence one of the components is nonzero. To fix things, suppose $\mu_1 > 0$ (the reasoning is similar if $\mu_1 < 0$). On (26) we act with $\hat{y}_\lambda^+$. Since $\langle\varphi_\lambda'(\hat{y}_\lambda), \hat{y}_\lambda^+\rangle = \langle\varphi_\lambda'(\hat{y}_\lambda^+), \hat{y}_\lambda^+\rangle = 0$ (recall that $\hat{y}_\lambda^+ \in N_\lambda$), we obtain

$$\begin{aligned}
&\mu_1\langle(\psi_\lambda')'(\hat{y}_\lambda^+), \hat{y}_\lambda^+\rangle = 0 \ \text{(note that } \{\hat{y}_\lambda^+ > 0\} \cap \{\hat{y}_\lambda^- > 0\} = \varnothing), \\
\Rightarrow &\mu_1 p[\rho_\eta(D\hat{y}_\lambda^+) - \lambda\rho_{\eta_0}(\hat{y}_\lambda^+) - \int_\Omega \beta(z)(\hat{y}_\lambda^+)^r \mathrm{d}z] \\
&+ \mu_1[-(p-q)\|D\hat{y}_\lambda^+\|_q^q - (r-p)\int_\Omega \beta(z)(\hat{y}_\lambda^+)^r \mathrm{d}z] = 0, \\
\Rightarrow &\mu_1 p[\rho_\eta(D\hat{y}_\lambda^+) - \lambda\rho_{\eta_0}(\hat{y}_\lambda^+) - \int_\Omega \beta(z)(\hat{y}_\lambda^+)^r \mathrm{d}z] > 0 \\
&\qquad \text{(as before, using Proposition 8 and since } p < r).
\end{aligned} \tag{27}$$

But since $\hat{y}_\lambda^+ \in N_\lambda$, we have

$$\rho_\eta(D\hat{y}_\lambda^+) - \lambda\rho_{\eta_0}(\hat{y}_\lambda^+) - \int_\Omega \beta(z)(\hat{y}_\lambda^+)^r \mathrm{d}z = 0. \tag{28}$$

Comparing (27) and (28), we have a contradiction. Therefore $\mu_1 = 0$. Similarly we show that $\mu_2 = 0$. Therefore $\mu_1 = \mu_2 = 0$ and so $\xi \neq 0$. We have

$$\begin{aligned}
&\xi\varphi_\lambda'(\hat{y}_\lambda) = 0 \ \text{ in } W_0^{1,\eta}(\Omega) \ \text{(see (26))}, \\
\Rightarrow &\varphi_\lambda'(\hat{y}_\lambda) = 0 \ \text{ and so } \hat{y}_\lambda \in K_{\varphi_\lambda}.
\end{aligned}$$

The proof is now complete. □

Now we can state and prove the multiplicity theorem for problem (1).

Theorem 1. *If hypotheses (H) hold and $\lambda \in (0, \hat{\lambda}_1^a)$, then problem* (1) *has at least three nontrivial solutions*

$$\begin{aligned}
&\hat{u}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), 0 \prec \hat{u}_\lambda, \\
&\hat{v}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), \hat{v}_\lambda \prec 0, \\
&\hat{y}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)\, \textit{nodal}.
\end{aligned}$$

Proof. From Proposition 10 we know that there exists $\hat{u}_\lambda \in N_\lambda^+$ such that

$$m_\lambda^+ = \varphi_\lambda^+(\hat{u}_\lambda).$$

Moreover, from Proposition 11, we know that

$$\langle(\varphi_\lambda^+)'(\hat{u}_\lambda), h\rangle = 0 \ \text{ for all } \ h \in W_0^{1,\eta}(\Omega). \tag{29}$$

In (29) we use the test function $h = -\hat{u}_\lambda^- \in W_0^{1,\eta}(\Omega)$. We have

$$\rho_\eta(D\hat{u}_\lambda^-) = 0,$$
$$\Rightarrow \hat{u}_\lambda \geq 0, \hat{u}_\lambda \neq 0 \text{ (recall that } \hat{u}_\lambda \in N_\lambda^+).$$

From Gasinski & Winkert [5, Theorem 3.1], we have

$$\hat{u}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega).$$

Let $\rho = \|\hat{u}_\lambda\|_\infty$. We can find $\hat{\xi}_\rho > 0$ such that

$$\lambda a(z)x^{p-1} - \beta(z)x^{r-1} + \hat{\xi}_\rho x^{p-1} \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

So, we have

$$-\Delta_p^a \hat{u}_\lambda - \Delta_q \hat{u}_\lambda + \hat{\xi}_\rho \hat{u}_\lambda^{p-1} \geq 0 \text{ in } \Omega.$$

Then invoking Proposition 2.4 of Papageorgiou, Vetro & Vetro [20], we infer that $0 \prec \hat{u}_\lambda$. Similarly we show that $\hat{v}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty$ is a negative solution of problem (1) such that $\hat{v}_\lambda \prec 0$.

Finally, let $\hat{y}_\lambda \in N_\lambda$ be such that

$$m_\lambda^0 = \varphi_\lambda(\hat{y}_\lambda) \text{ (see Proposition 10)}$$

Then from Proposition 11 we know that $\hat{y}_\lambda \in K_{\varphi_\lambda} \subseteq W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ and so $\hat{y}_\lambda$ is a nontrivial solution of (1). Since $\hat{y}_\lambda \in N_\lambda^0$, we have $\hat{y}_\lambda^\pm \neq 0$ and so $\hat{y}_\lambda$ is a nodal solution of problem (1). This completes the proof. □

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