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Blow-up boundary solutions of semilinear elliptic problems

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1. Introduction and the main results

We consider the following semilinear elliptic equation:

$$\begin{aligned} &\Delta u = p(x) f(u) \quad \text{in } \Omega,\\ &u \geq 0,\ u \not\equiv 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume throughout this paper that p is a non-negative function such that $p \in C^{0,\alpha}(\bar{\Omega})$ if Ω is bounded, and $p \in C^{0,\alpha}_{\text{loc}}(\Omega)$, otherwise. The nonlinearity f is assumed to fulfill

(f1) $\quad f \in C^1[0,\infty),\ f' \geq 0,\ f(0) = 0$ and $f > 0 \quad$ on $(0,\infty)$

and the Keller–Osserman condition (see [7,13])

(f2) $\quad \displaystyle\int_1^{\infty} [2F(t)]^{-1/2}\,\mathrm{d}t < \infty \quad$ where $F(t) = \displaystyle\int_0^t f(s)\,\mathrm{d}s.$

The main purpose of the paper is to find properties of *large solutions* of (1), that is solutions u satisfying $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbf{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbf{R}^N$). In the latter case the solution is called to be an *entire large* solution.

Problems of this type have been originally studied by Loewner and Nirenberg in their celebrated paper [11]. Their work deals with partial differential equations having

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a "partial conformal invariance" and is motivated by a concrete problem arising in Riemannian Geometry. More precisely, in [11] Loewner and Nirenberg proved the remarkable result that (1) has a maximal solution, provided that $\Omega \neq \mathbf{R}^N$, $p \equiv \text{Const.} > 0$ in Ω and $f(u) = u^{(N+2)/(N-2)}$.

In [1,12] problem (1) is considered in the special case when Ω is bounded and $p > 0$ in $\bar{\Omega}$. More precisely, in [1] Bandle and Marcus described the precise asymptotic behavior of large solutions near the boundary and established the uniqueness of such solutions, while in [12] Marcus obtained existence results for large solutions.

The first result we obtain in this paper is an existence theorem for large solutions when Ω is bounded.

Theorem 1. *Suppose Ω is bounded and p satisfies*

(p1) *for every $x_0 \in \Omega$ with $p(x_0) = 0$, there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega_0} \subset \Omega$ and $p > 0$ on $\partial\Omega_0$.*

Then problem (1) *has a positive large solution.*

This result generalizes Theorem 3.1 in Marcus [12] and Lemma 2.6 in [4] since condition (p1) is weaker than the assumption that $p > 0$ on $\partial\Omega$, as required in [4, Lemma 2.6] and in [12, Theorem 3.1]. Indeed, the continuity of p, the compactness of $\partial\Omega$ and the fact that $p > 0$ on $\partial\Omega$ imply the existence of some $\delta > 0$ such that $p > 0$ in

$$\Omega_\delta := \{x \in \bar{\Omega};\ \operatorname{dist}(x, \partial\Omega) \leq \delta\}.$$

Therefore, all the zeros of p are included in $\Omega_0 = \bar{\Omega} \setminus \Omega_\delta \subset\subset \Omega$. Hence $p > 0$ on $\partial\Omega_0$, so (p1) is fulfilled.

We now consider problem (1) when $\Omega = \mathbf{R}^N$, and first observe that any entire large solution of (1) is positive. Indeed, assume there exists $x_0 \in \mathbf{R}^N$ such that $u(x_0) = 0$. Since u is an entire large solution, we can choose $R > |x_0|$ such that $u > 0$ on $\partial B(0, R)$. Thus, by Theorem 5 in the appendix, the problem

$$\begin{aligned} &\Delta\zeta = p(x) f(\zeta) \quad \text{in } B(0, R), \\ &\zeta = u \quad \text{on } \partial B(0, R), \\ &\zeta \geq 0 \quad \text{in } B(0, R) \end{aligned}$$

has a unique solution, which is positive. By uniqueness, of course, $\zeta = u$, which is the required contradiction. This shows that u cannot vanish in $\mathbf{R}^N$.

The next purpose of the paper is to prove the existence of an entire maximal solution for (1), under more general hypotheses than in [4]. They investigate the structure of all positive solutions of (1) in the special case when $f(u) = u^\gamma$, $\gamma > 1$, and they also establish existence of the maximal classical solution U of (1), under the hypotheses that this equation possesses at least a positive entire solution and there is a sequence of smooth bounded domains $(\Omega_n)_{n \geq 1}$ such that, for any $n \geq 1$,

$$\overline{\Omega_n} \subseteq \Omega_{n+1}, \quad \mathbf{R}^N = \bigcup_{n=1}^{\infty} \Omega_n, \quad p > 0 \text{ on } \partial\Omega_n. \tag{2}$$

Cheng and Ni [4] also proved that the maximal solution U is the unique entire large solution of problem (1), under the additional restriction that for some $l > 2$ there exist two positive constants C_1, C_2 such that

$$C_1 p(x) \le |x|^{-l} \le C_2 p(x) \quad \text{for large } |x|. \tag{3}$$

Our result in the case $\Omega = \mathbf{R}^N$ is the following.

Theorem 2. *Assume that $\Omega = \mathbf{R}^N$ and that problem* (1) *has at least a solution. Suppose that p satisfies the condition*

(p1)′ *There exists a sequence of smooth bounded domains $(\Omega_n)_{n\ge 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\mathbf{R}^N = \bigcup_{n=1}^{\infty} \Omega_n$, and* (p1) *holds in Ω_n, for any $n \ge 1$.*

Then there exists a maximal classical solution U of (1).

If p verifies the additional condition

(p2) $\int_0^\infty r\Phi(r)\,\mathrm{d}r < \infty$ *where* $\Phi(r) = \max\{p(x)\colon |x| = r\}$,

then U is an entire large solution.

In view of the remark above that condition (p1) on Ω is weaker than the requirement that $p > 0$ on $\partial\Omega$, it follows that condition (p1)′ is weaker than the assumption (2) required in [4], and also assumption (p2) is weaker than condition (3) imposed in [4].

We now observe that if $p(x) > 0$ for $|x|$ sufficiently large, then (p1)′ is automatically satisfied. Therefore, it is natural to ask us if there exists $p \ge 0$ which satisfies (p2) and (p1)′, with p vanishing in every neighborhood of infinity. The answer is positive by the following example. Take

$$p(r) = 0 \quad \text{for } r = |x| \in [n - 1/3, n + 1/3],\ n \ge 1,$$

$$p(r) > 0 \quad \text{in } \boldsymbol{R}_+ \setminus \bigcup_{n=1}^{\infty} [n - 1/3, n + 1/3],$$

$$p \in C^1[0,\infty) \quad \text{and} \quad \max_{r\in[n,n+1]} p(r) = \frac{2}{n^2(2n+1)}.$$

Of course, (p1)′ is fulfilled, for $\Omega_n = B(0, n + 1/2)$. On the other hand, condition (p2) is also satisfied since

$$\int_1^\infty r\Phi(r)\,\mathrm{d}r = \sum_{n=1}^{\infty} \int_n^{n+1} r p(r)\,\mathrm{d}r \le \sum_{n=1}^{\infty} \int_n^{n+1} \frac{2}{n^2(2n+1)} r\,\mathrm{d}r = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We now consider the case in which $\Omega \ne \boldsymbol{R}^N$ and Ω is unbounded; we say that a large solution u of (1) is *regular* if u tends to zero at infinity. In [12, Theorem 3.1] Marcus proved for this case the existence of regular large solutions to problem (1) by assuming that there exist $\gamma > 1$ and $\beta > 0$ such that

$$\liminf_{t\to 0} f(t)t^{-\gamma} > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} p(x)|x|^{\beta} > 0.$$

The large solution constructed in Marcus [12] is the smallest large solution of problem (1). In the next result we show that problem (1) admits a maximal classical solution U and that U blows-up at infinity if $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$.

Theorem 3. *Suppose that $\Omega \neq \mathbf{R}^N$ is unbounded and that problem* (1) *has at least a solution. Assume that p satisfies condition* (p1)$'$ *in Ω. Then there exists a maximal classical solution U of problem* (1).

If $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$ and p satisfies the additional condition (p2), *with $\Phi(r) = 0$ for $r \in [0,R]$, then the maximal solution U is a large solution that blows-up at infinity.*

In conclusion, by Theorem 3 and the recalled result of Marcus, in the case $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$, problem (1) admits large solutions tending to zero or to infinity as $|x| \to \infty$ (regular or normal large solutions).

In Section 2, we prove Theorem 1, while in Section 3 we prove Theorems 2 and 3. In Section 4, we prove the following necessary condition for the existence of entire large solutions to Eq. (1) if p satisfies (p2), and for which f is not assumed to satisfy (f2), and p is not required to be so regular as before. More precisely, we prove

Theorem 4. *Assume that $p \in C(\mathbf{R}^N)$ is a non-negative and non-trivial function which satisfies* (p2). *Let f be a function satisfying assumption* (f1). *Then condition*

$$\int_1^\infty \frac{dt}{f(t)} < \infty \tag{4}$$

is necessary for the existence of entire large solutions to (1).

For further results in this direction we refer to [3] *and* [10].

2. Existence results for bounded domains

Lemma 1. *Assume that conditions* (f1) *and* (f2) *are fulfilled. Then*

$$\int_1^\infty \frac{dt}{f(t)} < \infty.$$

Proof. Fix $R > 0$ and denote $B = B(0,R)$. By Theorem 5 in the appendix the boundary value problem

$$\begin{aligned} &\Delta u_n = f(u_n) \quad \text{in } B, \\ &u_n = n \quad \text{on } \partial B, \\ &u_n \geq 0,\ u_n \not\equiv 0 \quad \text{in } B \end{aligned} \tag{5}$$

has a unique positive solution. Since f is non-decreasing, it follows by the maximum principle that $u_n(x)$ increases with n, for any fixed $x \in B$.

We first claim that (u_n) is uniformly bounded in every compact subdomain of B. Indeed, let $K \subset B$ be any compact set and $d := \operatorname{dist}(K, \partial B)$. Then

$$0 < d \leq \operatorname{dist}(x, \partial B) \quad \forall x \in K. \tag{6}$$

By Proposition 1 of Bandle–Marcus [1], there exists a continuous, non-increasing function $\mu: \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$u_n(x) \le \mu(\mathrm{dist}(x, \partial B)) \quad \forall x \in K.$$

The claim now follows from (6). Thus, for every $x \in B$ we can define $u(x) := \lim_{n\to\infty} u_n(x)$.

We next show that u is a classical large solution of

$$\Delta u = f(u) \quad \text{in } B. \tag{7}$$

Fix $x_0 \in B$ and let $r > 0$ be such that $\overline{B(x_0,r)} \subset B$. Let $\Psi \in C^\infty(B)$ be such that $\Psi \equiv 1$ in $\overline{B(x_0,r/2)}$ and $\Psi \equiv 0$ in $B \setminus B(x_0,r)$. We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n,$$

where $p_n = u_n\Delta\Psi + \Psi\Delta u_n$. Since (u_n) is uniformly bounded on $\overline{B(x_0,r)}$ and f is non-decreasing on $[0,\infty)$, it follows that $\|p_n\|_\infty \le C$, where C is a constant independent of n. From now on, using the same argument given in the proof of Lemma 3 of [9], we find that (u_n) converges in $C^{2,\alpha}(\overline{B(x_0,r_1)})$, for some $r_1 > 0$. Since $x_0 \in B$ is arbitrary, this shows that $u \in C^2(B)$ and u is a positive solution of (7). Moreover, by the Gidas–Ni–Nirenberg theorem in [6], u is radially symmetric in B, namely $u(x) = u(r)$, $r = |x|$, and u satisfies in the r variable the equation

$$u''(r) + \frac{N-1}{r}u'(r) = f(u(r)), \quad 0 < r < R.$$

This equation can be rewritten as follows:

$$(r^{N-1}u'(r))' = r^{N-1}f(u(r)), \quad 0 < r < R. \tag{8}$$

Integrating (8) from 0 to r we obtain

$$u'(r) = r^{1-N}\int_0^r s^{N-1}f(u(s))\,\mathrm{d}s, \quad 0 < r < R.$$

Hence u is a non-decreasing function and

$$u'(r) \le r^{1-N}f(u(r))\int_0^r s^{N-1}\,\mathrm{d}s = \frac{r}{N}f(u(r)), \quad 0 < r < R. \tag{9}$$

Similarly, u_n is non-decreasing on $(0,R)$, for any $n \ge 1$.

In order to show that u is a large solution of (7), it remains to prove that $u(r) \to \infty$ as $r \nearrow R$. Assume the contrary. Then there exists $C > 0$ such that $u(r) < C$ for all $0 \le r < R$. Let $N_1 \ge 2C$ be fixed. The monotonicity of u_{N_1} and the fact that $u_{N_1}(r) \to N_1$ as $r \nearrow R$ imply the existence of some $r_1 \in (0,R)$ such that $C \le u_{N_1}(r)$, for $r \in [r_1,R)$. Hence

$$C \le u_{N_1}(r) \le u_{N_1+1}(r) \le \cdots \le u_n(r) \le u_{n+1}(r) \le \cdots \quad \forall n \ge N_1, \ \forall r \in [r_1,R).$$

Passing to the limit as $n \to \infty$, we obtain $u(r) \ge C$ for all $r \in [r_1,R)$, which is a contradiction.

Integrating (9) on $(0,r)$ and taking $r \nearrow R$ we find

$$\int_{u(0)}^{\infty} \frac{1}{f(t)}\,\mathrm{d}t \le \frac{R^2}{2N}.$$

The conclusion of Lemma 1 is therefore proved. □

Proof of Theorem 1. By Theorem 5 in the appendix, the boundary value problem

$$\begin{aligned} &\Delta v_n = p(x) f(v_n) \quad \text{in } \Omega,\\ &v_n = n \quad \text{on } \partial\Omega,\\ &v_n \ge 0,\ v_n \not\equiv 0 \quad \text{in } \Omega \end{aligned} \tag{10}$$

has a unique positive solution, for any $n \ge 1$.

We now claim that

(a) for all $x_0 \in \Omega$ there exist an open set $\mathcal{O} \subset\subset \Omega$ containing x_0 and $M_0 = M_0(x_0) > 0$ such that $v_n \le M_0$ in $\mathcal{O}$, for any $n \ge 1$;

(b) $\lim_{x\to\partial\Omega} v(x) = \infty$, where $(x) = \lim_{n\to\infty} v_n(x)$.

We first remark that the sequence (v_n) is non-decreasing. Indeed, by Theorem 5 in the appendix, the boundary value problem

$$\begin{aligned} &\Delta\zeta = \|p\|_\infty f(\zeta) \quad \text{in } \Omega,\\ &\zeta = 1 \quad \text{on } \partial\Omega,\\ &\zeta > 0 \quad \text{in } \Omega \end{aligned}$$

has a unique solution. Then, by the maximum principle,

$$0 < \zeta \le v_1 \le \cdots \le v_n \le \cdots \quad \text{in } \Omega. \tag{11}$$

We also observe that (a) and (b) are sufficient to conclude the proof. In fact, assertion (a) shows that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Standard elliptic regularity arguments (see the proof of Lemma 3 in [9]) show that v is a solution of problem (1). Then, by (11) and (b), it follows that v is the desired solution.

To prove (a) we distinguish two cases:

Case $p(x_0) > 0$: By the continuity of p, there exists a ball $B = B(x_0,r) \subset\subset \Omega$ such that

$$m_0 := \min\{p(x);\ x \in \bar{B}\} > 0.$$

Let w be a positive solution of problem

$$\begin{aligned} &\Delta w = m_0 f(w) \quad \text{in } B,\\ &w(x) \to \infty \quad \text{as } x \to \partial B. \end{aligned} \tag{12}$$

The existence of w follows by [7, Theorem III], due to Keller. By the maximum principle it follows that $v_n \le w$ in B. Furthermore, w is bounded in $\overline{B(x_0,r/2)}$. Setting $M_0 = \sup_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0,r/2)$, we obtain (a).

Case $p(x_0) = 0$: Our hypothesis (p1) and the boundedness of Ω imply the existence of a domain $\mathcal{O} \subset\subset \Omega$ which contains x_0 such that $p > 0$ on $\partial\mathcal{O}$. The above case shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in Ω and a constant $M_x > 0$ such that $v_n \le M_x$ on $B(x, r_x/2)$, for any $n \ge 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \ldots, k_0$. Setting $M_0 = \max\{M_{x_1}, \ldots, M_{x_{k_0}}\}$ we have $v_n \le M_0$ on $\partial\mathcal{O}$, for any $n \ge 1$. Applying the maximum principle we obtain $v_n \le M_0$ in $\mathcal{O}$ and (a) follows.

Let us now consider the problem

$$\begin{aligned} &-\Delta z = p(x) \quad \text{in } \Omega,\\ &z = 0 \quad \text{on } \partial\Omega,\\ &z \ge 0,\ z \not\equiv 0 \quad \text{in } \Omega. \end{aligned} \tag{13}$$

Applying Theorem 1 in Brezis–Oswald [2] we deduce that (13) has a unique solution which is positive in Ω, by the maximum principle.

We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le z(x) \quad \text{for any } x \in \Omega. \tag{14}$$

By Lemma 1, the left-hand side of (14) is well defined in Ω. Fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, there is $n_1 = n_1(\varepsilon)$ such that

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le \varepsilon(1+R^2)^{-1/2} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \forall x \in \partial\Omega\ \forall n \ge n_1, \tag{15}$$

where $R > 0$ is chosen so that $\bar{\Omega} \subset B(0, R)$.

In order to prove (14), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \le z(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \forall x \in \Omega\ \forall n \ge n_1. \tag{16}$$

Indeed, putting $n \to \infty$ in (16) we deduce (14), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (16) fails. Then

$$\max_{x \in \bar{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right\} > 0.$$

Using (15) we see that the point where the maximum is achieved must lie in Ω. At this point, say x_0, we have

$$\begin{aligned} 0 &\ge \Delta\left(\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon(1+|x|^2)^{-1/2} \right)_{|x=x_0}\\ &= \left(-\frac{1}{f(v_n)}\Delta v_n - \left(\frac{1}{f}\right)'(v_n)\cdot|\nabla v_n|^2 - \Delta z(x) - \varepsilon\Delta(1+|x|^2)^{-1/2} \right)_{|x=x_0} \end{aligned}$$

$$= \left(-p(x) - \left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + p(x) - \varepsilon\Delta(1+|x|^2)^{-1/2} \right)_{|x=x_0}$$

$$= \left(-\left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\,\varepsilon(1+|x|^2)^{-5/2} \right)_{|x=x_0} > 0.$$

This contradiction shows that inequality (16) holds and the proof of Theorem 1 is complete. □

3. Existence results for unbounded domains

In this section we are interested mainly in the question of finding and describing the behavior on the boundary and at infinity of the maximal solution to problem (1), where Ω is now an unbounded domain, possibly $\mathbf{R}^N$. For the significance of such a study we refer to Dynkin [5] where it is showed that there exist certain relations between hitting probabilities for superdiffusions and maximal solutions of (1) with $f(u) = u^\gamma$, $1 < \gamma \le 2$.

It is clear that a unique normal large solution is necessarily a maximal solution. In view of this remark the problem of maximal solution seems to be connected with the uniqueness of large solutions. But this is not the best way to be followed because we lose the control if the uniqueness of large solutions fails. The advantage offered by our results is that we find a direct method which establishes an interesting connection between the maximal solution and *any* sequence of large solutions taken on bounded domains of the type given in condition $(p1)'$ in Ω.

Proof of Theorem 2. By Theorem 1, the boundary value problem

$$\begin{aligned} &\Delta v_n = p(x) f(v_n) \quad \text{in } \Omega_n, \\ &v_n(x) \to \infty \quad \text{as } x \to \partial\Omega_n, \\ &v_n > 0 \quad \text{in } \Omega_n \end{aligned} \tag{17}$$

has solution. Since $\overline{\Omega_n} \subset \Omega_{n+1}$ we can apply, for each $n \ge 1$, the maximum principle (in the same manner as in the uniqueness proof of Theorem 5 in the appendix) in order to find that $v_n \ge v_{n+1}$ in Ω_n. Since $\mathbf{R}^N = \bigcup_{n=1}^{\infty} \Omega_n$ and $\overline{\Omega_n} \subset \Omega_{n+1}$ it follows that for every $x_0 \in \mathbf{R}^N$ there exists $n_0 = n_0(x_0)$ such that $x_0 \in \Omega_n$ for all $n \ge n_0$. In view of the monotonicity of the sequence $(v_n(x_0))_{n \ge n_0}$ we can define $U(x_0) = \lim_{n\to\infty} v_n(x_0)$. By applying the standard bootstrap argument (see [8, Theorem 1]) we find that $U \in C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^N)$ and $\Delta U = p(x) f(U)$ in Ω.

We now prove that U is the maximal solution of problem (1). Indeed, let u be an arbitrary solution of (1). Applying again the maximum principle we obtain that $v_n \ge u$ in Ω_n for all $n \ge 1$. By the definition of U, it is clear that $U \ge u$ in $\mathbf{R}^N$.

We point out that U is independent of the choice of the sequence of domains Ω_n and the number of solutions of problem (17). This follows easily by the uniqueness of the maximal solution.

We suppose, in addition, that p satisfies (p2) and we shall prove that U blows-up at infinity. For this aim, it is sufficient to find a positive function $w \in C(\mathbf{R}^N)$ such that $U \geq w$ in $\mathbf{R}^N$ and $w(x) \to \infty$ as $|x| \to \infty$. We first observe that (p2) implies

$$K = \int_0^\infty r^{1-N} \left(\int_0^r \sigma^{N-1} \Phi(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}r < \infty. \tag{18}$$

Note that (18) is a simple consequence of the fact that for all $R > 0$ we have

$$\begin{aligned}\int_0^R r^{1-N} \left(\int_0^r \sigma^{N-1} \Phi(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}r &= \frac{1}{2-N} \int_0^R \frac{\mathrm{d}}{\mathrm{d}r}(r^{2-N}) \left(\int_0^r \sigma^{N-1} \Phi(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}r \\ &= \frac{1}{2-N} R^{2-N} \int_0^R \sigma^{N-1} \Phi(\sigma) \, \mathrm{d}\sigma - \frac{1}{2-N} \\ &\quad \int_0^R r\Phi(r) \, \mathrm{d}r \leq \frac{1}{N-2} \int_0^\infty r\Phi(r) \, \mathrm{d}r < \infty.\end{aligned}$$

Using (18) and the maximum principle we obtain that the problem

$$-\Delta z = \Phi(r), \quad r = |x| < \infty,$$

$$z(x) \to 0, \quad \text{as } |x| \to \infty$$

has a unique positive radial solution which is given by

$$z(r) = K - \int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau) \, \mathrm{d}\tau \right) \mathrm{d}\sigma \qquad \forall r \geq 0.$$

Let w be the positive function defined implicitely by

$$z(x) = \int_{w(x)}^\infty \frac{\mathrm{d}t}{f(t)} \quad \forall x \in \mathbf{R}^N. \tag{19}$$

Assumption (f1) and L'Hospital rule yield

$$\lim_{t \searrow 0} \frac{f(t)}{t} = \lim_{t \searrow 0} f'(t) = f'(0) \in [0, \infty),$$

which implies the existence of some $\delta > 0$ such that

$$\frac{f(t)}{t} < f'(0) + 1 \quad \text{for all } 0 < t < \delta.$$

Thus for every $s \in (0, \delta)$ we have

$$\int_s^\delta \frac{\mathrm{d}t}{f(t)} > \frac{1}{f'(0)+1} \int_s^\delta \frac{\mathrm{d}t}{t} = \frac{1}{f'(0)+1} (\ln \delta - \ln s).$$

It follows that $\lim_{s \searrow 0} \int_s^\delta \mathrm{d}t/f(t) = \infty$, which gives the possibility to define w as in (19).

We claim that $w \le v_n$ in Ω_n for all $n \ge 1$. Obviously this inequality is true on $\partial\Omega_n$. Using the same arguments as in the proof of the inequality (26) in the appendix (with Ω replaced by Ω_n) we obtain that for any $\varepsilon > 0$ and $n \ge 1$ we have

$$w(x) \le v_n(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \text{in } \Omega_n$$

and the claim follows. Consequently, $U \ge w$ in $\mathbf{R}^N$ and, by (19), $w(x) \to \infty$ as $|x| \to \infty$. This completes the proof. □

Proof of Theorem 3. We argue in a similar manner as in the proof of Theorem 2, but with some changes due to the fact that $\Omega \ne \mathbf{R}^N$.

Let $(\Omega_n)_{n\ge 1}$ be the sequence of bounded smooth domains given by condition $(p1)'$. For $n \ge 1$ fixed, let v_n be a positive solution of problem (17) and recall that $v_n \ge v_{n+1}$ in Ω_n. Set $U(x) = \lim_{n\to\infty} v_n(x)$, for every $x \in \Omega$. With the same arguments as in Theorem 2, we find that U is a classical solution to (1) and that U is the maximal solution. Hence the first part of Theorem 3 is proved.

For the second part, in which $\Omega = \mathbf{R}^N \setminus \overline{B(0,R)}$, we suppose that (p2) is fulfilled, with $\Phi(r) = 0$ for $r \in [0,R]$. In order to prove that U is a normal large solution it is enough to show the existence of a positive function $w \in C(\mathbf{R}^N \setminus \overline{B(0,R)})$ such that $U \ge w$ in $\mathbf{R}^N \setminus \overline{B(0,R)}$, and $w(x) \to \infty$ as $|x| \to \infty$ and as $|x| \searrow R$. This will be done as in the proof of Theorem 2, with the function z given now as the unique positive radial solution of the problem

$$-\Delta z = \Phi(r) \quad \text{if } |x| = r > R,$$

$$z(x) \to 0 \quad \text{as } |x| \to \infty,$$

$$z(x) \to 0 \quad \text{as } |x| \searrow R.$$

The uniqueness of z follows by the maximum principle. Moreover,

$$z(r) = \left(\frac{1}{R^{N-2}} - \frac{1}{r^{N-2}}\right) \int_R^\infty \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau\right) \mathrm{d}\sigma - \frac{1}{R^{N-2}} \int_R^r \sigma^{1-N} \times \left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,\mathrm{d}\tau\right) \mathrm{d}\sigma.$$

This completes the proof. □

4. Proof of Theorem 4

Let u be an entire large solution of problem (1). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left(\int_a^{u(x)} \frac{\mathrm{d}t}{f(t)}\right) \mathrm{d}S = \frac{1}{\omega_N} \int_{|\xi|=1} \left(\int_a^{u(r\xi)} \frac{\mathrm{d}t}{f(t)}\right) \mathrm{d}S,$$

where ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$ and a is chosen such that $a \in (0, u_0)$, where $u_0 = \inf_{\mathbf{R}^N} u > 0$. By the divergence theorem we have

$$\overline{u}'(r) = \frac{1}{\omega_N} \int_{|\xi|=1} \frac{1}{f(u(r\xi))} \nabla u(r\xi) \cdot \xi \, \mathrm{d}S = \frac{1}{\omega_N r^N} \int_{|y|=r} \frac{1}{f(u(y))} \nabla u(y) \cdot y \, \mathrm{d}S$$

$$= \frac{1}{\omega_N r^N} \int_{|y|=r} \nabla \left(\int_a^{u(y)} \frac{\mathrm{d}t}{f(t)} \right) \cdot y \, \mathrm{d}S = \frac{1}{\omega_N r^{N-1}} \int_{|y|=r} \frac{\partial}{\partial \nu} \left(\int_a^{u(y)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S$$

$$= \frac{1}{\omega_N r^{N-1}} \int_{B(0,r)} \Delta \left(\int_a^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}x.$$

Since u is a positive classical solution it follows that

$$|\overline{u}'(r)| \le Cr \to 0 \quad \text{as } r \to 0.$$

On the other hand,

$$\omega_N (R^{N-1} \overline{u}'(R) - r^{N-1} \overline{u}'(r)) = \int_D \Delta \left(\int_a^{u(x)} \frac{1}{f(t)} \, \mathrm{d}t \right) \mathrm{d}x$$

$$= \int_r^R \left(\int_{|x|=z} \Delta \left(\int_a^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S \right) \mathrm{d}z,$$

where $D = \{x \in \mathbf{R}^N \colon r < |x| < R\}$. Dividing by $R - r$ and letting $R \to r$ we find

$$\omega_N (r^{N-1} \overline{u}'(r))' = \int_{|x|=r} \Delta \left(\int_a^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S = \int_{|x|=r} \operatorname{div} \left(\frac{1}{f(u(x))} \nabla u(x) \right) \mathrm{d}S$$

$$= \int_{|x|=r} \left[\left(\frac{1}{f} \right)' (u(x)) \cdot |\nabla u(x)|^2 + \frac{1}{f(u(x))} \Delta u(x) \right] \mathrm{d}S$$

$$\le \int_{|x|=r} \frac{p(x) f(u(x))}{f(u(x))} \, \mathrm{d}S \le \omega_N r^{N-1} \Phi(r).$$

Integrating the above inequality yields

$$\overline{u}(r) \le \overline{u}(0) + \int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau) \, \mathrm{d}\tau \right) \mathrm{d}\sigma \quad \forall r \ge 0. \tag{20}$$

Since (p2) implies (18) we have

$$\overline{u}(r) \le \overline{u}(0) + K \quad \forall r \ge 0.$$

Thus $\bar{u}$ is bounded and assuming that (4) is not fulfilled it follows that u cannot be a large solution. □

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Appendix

The following result is mentioned without proof in Marcus [12] and it was applied several times in this paper. For the sake of completeness we present in this section a simple proof of this theorem.

Theorem 5. *Let Ω be a bounded domain. Assume that $p \in C^{0,\alpha}(\bar{\Omega})$ is a non-negative function, f satisfies $(f1)$ and $g : \partial\Omega \to (0,\infty)$ is continuous. Then the boundary value problem*

$$\begin{aligned} &\Delta u = p(x) f(u) \quad \text{in } \Omega, \\ &u = g \quad \text{on } \partial\Omega, \\ &u \geq 0,\ u \not\equiv 0 \quad \text{in } \Omega \end{aligned} \tag{A.1}$$

has a unique classical solution, which is positive.

Proof of Theorem 5. We first observe that the function $u^+(x) = n$ is a super-solution of problem (A.1), provided that n is sufficiently large. To find a positive subsolution, we look for an arbitrary positive solution to the following auxiliary problem:

$$\Delta v = \Phi(r) \quad \text{in } A(\underline{r}, \bar{r}) = \{x \in \mathbf{R}^N;\ \underline{r} < |x| < \bar{r}\}, \tag{A.2}$$

where

$$\underline{r} = \inf\{\tau > 0;\ \partial B(0,\tau) \cap \bar{\Omega} \neq \emptyset\}, \quad \bar{r} = \sup\{\tau > 0;\ \partial B(0,\tau) \cap \bar{\Omega} \neq \emptyset\}$$

$$\Phi(r) = \max_{|x|=r} p(x) \quad \text{for any } r \in [\underline{r}, \bar{r}].$$

The function

$$v(r) = 1 + \int_{\underline{r}}^{r} \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau)\, d\tau \right) d\sigma, \quad \underline{r} \leq r \leq \bar{r}$$

verifies Eq. (A.2). The assumptions on f and g imply

$$g_0 := \min_{\partial\Omega} g > 0 \quad \text{and} \quad \lim_{z \searrow 0} \int_z^{g_0} \frac{dt}{f(t)} = \infty.$$

This will be used to justify the existence of a positive number c such that

$$\max_{\partial\Omega} v = \int_c^{g_0} \frac{dt}{f(t)}. \tag{A.3}$$

Next, we define the function u_- such that

$$v(x) = \int_c^{u_-(x)} \frac{\mathrm{d}t}{f(t)} \quad \forall x \in \Omega. \tag{A.4}$$

It turns out that u_- is a positive subsolution of problem (A.1). Indeed, it is clear that

$$u_- \in C^2(\Omega) \cap C(\bar{\Omega}) \quad \text{and} \quad u_- \geq c \quad \text{in } \Omega.$$

On the other hand, from (A.2), (A.4) and (f1) it follows that

$$\begin{aligned} p(x) \leq \Delta v(x) &= \frac{1}{f(u_-(x))} \Delta u_-(x) + \left(\frac{1}{f}\right)'(u_-(x)) \cdot |\nabla u_-(x)|^2 \\ &\leq \frac{1}{f(u_-(x))} \Delta u_-(x) \quad \text{in } \Omega, \end{aligned}$$

which yields

$$\Delta u_-(x) \geq p(x) f(u_-(x)) \quad \text{in } \Omega.$$

On the other hand, taking into account (A.3) and (A.4) we find

$$u_-(x) \leq g(x) \quad \forall x \in \partial\Omega.$$

So, we have proved that u_- is a positive subsolution to problem (A.1). Therefore, this problem has at least a positive solution u. Furthermore, taking into account the regularity of p and f, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

Let us now assume that u_1 and u_2 are arbitrary solutions of (A.1). In order to prove the uniqueness, it is enough to show that $u_1 \geq u_2$ in Ω. Denote

$$\omega := \{x \in \Omega;\ u_1(x) < u_2(x)\}$$

and suppose that $\omega \neq \emptyset$. Then the function $\tilde{u} = u_1 - u_2$ satisfies

$$\begin{aligned} &\Delta \tilde{u} = p(x)(f(u_1) - f(u_2)) \quad \text{in } \omega, \\ &\tilde{u} = 0 \quad \text{on } \partial\omega. \end{aligned} \tag{A.5}$$

Since f is non-decreasing and $p \geq 0$, it follows by (A.5) that $\tilde{u}$ is a super-harmonic function in ω which vanishes on $\partial\omega$. Thus, by the maximum principle, either $\tilde{u} \equiv 0$ or $\tilde{u} > 0$ in ω, which yield a contradiction. Thus $u_1 \geq u_2$ in Ω.

We give in what follows an alternative proof for the uniqueness. Let u_1, u_2 be two arbitrary solutions of problem (A.1). As above, it is enough to show that $u_1 \geq u_2$ in Ω. Fix $\varepsilon > 0$. We claim that

$$u_2(x) \leq u_1(x) + \varepsilon(1 + |x|^2)^{-1/2} \quad \text{for any } x \in \Omega. \tag{A.6}$$

Suppose the contrary. Since (A.6) is obviously fulfilled on $\partial\Omega$, we deduce that

$$\max_{x \in \bar{\Omega}} \{u_2(x) - u_1(x) - \varepsilon(1 + |x|^2)^{-1/2}\}$$

is achieved in Ω. At that point we have

$$0 \geq \Delta(u_2(x) - u_1(x) - \varepsilon(1 + |x|^2)^{-1/2}) = p(x)(f(u_2(x)) - f(u_1(x))) \\ -\varepsilon\Delta(1 + |x|^2)^{-1/2} \\ = p(x)(f(u_2(x)) - f(u_1(x))) + \varepsilon(N - 3)(1 + |x|^2)^{-3/2} + 3\,\varepsilon(1 + |x|^2)^{-5/2} > 0,$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (A.6) implies $u_2 \leq u_1$ in Ω. □

We point out that the hypothesis that f is differentiable in the origin is essential in order to find a *positive* solution to problem (A.1). Indeed, consider $\Omega = B_1$, and $f(u) = u^{(\beta-2)/\beta}$, where $\beta > 2$. Choose $p \equiv 1$ and $g \equiv C$ on ∂B_1, where $C = (\beta^2 + (N-2)\beta)^{-\beta/2}$. For this choice of Ω, p, f and g, the function $u(r) = Cr^\beta$, $0 \leq r \leq 1$, is the unique solution of problem (A.1), but $u(0) = 0$.

Under the hypotheses on f made in the statement of Theorem 5, except f is of class C^1 at the origin (but $f \in C^{0,\alpha}$ in $u = 0$), problem (A.1) has a unique solution which may vanish in Ω. For this purpose it is sufficient to choose as a sub-solution in the above proof the function $u_- = 0$.

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