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Constraint minimizers of mass critical fractional Kirchhoff equations: concentration and uniqueness

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Abstract

This paper focuses on the constraint minimization problem associated with the fractional Kirchhoff equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+|x|^2u=\mu u+\beta u^{\frac{8s}{N}+1} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=1, \end{cases}$$

where $s\in(N/4,1), N=2,3$, $a\geqslant 0, b>0$ are constants, $\mu\in\mathbb{R}$ is the corresponding Lagrange multiplier and $(-\Delta)^s$ is the fractional Laplacian operator, $8s/N+1$ is the corresponding mass critical exponent. The purpose of this paper is threefold: to establish the existence and non-existence of the L^2-constraint minimizers to the degenerate fractional Kirchhoff problem, that is $a=0$, to

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prove some classical concentration behaviors of constraint minimizers and to reveal the local uniqueness of constraint minimizers of above problem under double nonlocal effect. In particular, we will give some energy estimates, decay estimates and uniform regularity to find that the maximal point of constraint minimizer concentrates on the bottom point of the homogeneous potential. Furthermore, we introduce several new techniques based on the combination of the localization method of $(-\Delta)^s$ and by establishing the nonlocal Pohozăev identity, which allow us to get over some new challenges due to the nonlocal property of $(-\Delta)^s$ and the fact that $\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x(-\Delta)^s u$ does not vanish as $a\searrow 0$. We believe that these techniques will have some potential applications in various related problems.

Keywords: constraint minimization, concentration behavior, local uniqueness, Pohozăev identity

Mathematics Subject Classification numbers: 35J20, 35J62, 35Q55

1. Introduction

1.1. Background and relevant progress

In this paper, we are concerned with the mass critical fractional Kirchhoff equations

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\right)(-\Delta)^s u+|x|^2u=\mu u+\beta u^{\frac{8s}{N}+1} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2\mathrm{d}x=1, \end{cases} \tag{1}$$

where $s\in(N/4,1), N=2,3$, $a\geqslant 0, b>0$ are constants, $\mu\in\mathbb{R}$ is the corresponding Lagrange multiplier, $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u(x):=C_sP.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y=C_s\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

for $x\in\mathbb{R}^N$, where $P.V.$ is the principal value, and C_s is a normalization constant.

The first feature of our problem is the appearance of the fractional Laplace operator. Such type of operators, which has special properties and the connection with the Fourier transform, arise in a quite natural way in many different applications, such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws and water waves. A series of studies have been done on the fractional Laplacian operator, we refer readers to [1, 7, 34, 35, 47] and references therein for physical background.

When $s=1$, the classical Kirchhoff type equation is analogous to the stationary case of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt}-M\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u=h(x,u)\quad \text{in } \mathbb{R}^N,$$

where u denotes the displacement, and $h(x,u)$ the external force, such equations were proposed by Kirchhoff [31] in 1883 to describe the transversal oscillations of a stretched string, furthermore, taking into account the subsequent change in string length caused by oscillations. The most typical feature of such an equation is the appearance of the term $\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u$, which makes Kirchhoff's model a nonlocal one and more challenging to deal with. Scholars have done a lot of research on Kirchhoff type equations on account of its extensive application in physics and biology, since the nonlocal effect also finds its applications in biological systems, we refer readers to [2, 3, 17, 40, 41, 48, 49] and references therein for different existence results.

The solvability of the Kirchoff type equation (1) has been well-studied in general dimensions by various authors and its main feature is the normalization condition

$$\int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x = \|u\|_2^2 = 1,$$

the solution of such an equation under which is commonly called normalized solution accordingly. From a physical point of view, the normalization condition can also be regarded as a mass conservation, which makes such problems more practical and physically significant, for example, it can represent the total number of atoms or the power source in nonlinear optics. Therefore, the research of the correlation properties about the normalized solution has become an important topic in the research of nonlinear equations.

Among the investigations of the above problems, due to the appearance of normalization condition and Lagrange multiplier μ, the common idea is to use the constraint minimization method on a constrained set, more specially on some submanifolds. For the classical case, the energy functional of the following form contains a lot of classical models

$$\bar{E}_{a,b,p,q}(u) := \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)|u|^2\right) \mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{2p}{q+2}\int_{\mathbb{R}^N} |u|^{q+2} \mathrm{d}x \tag{2}$$

and its constraint minimization problem

$$\bar{e}(a,b,p,q,c) := \inf_{\left\{\|u\|_2^2 = c^2\right\}} \bar{E}_{a,b,p,q}(u) \tag{3}$$

has been studied in many different contexts. For instance, as $N = 2$, $a = 1$, $b = 0$ and $q = 2$, (2) turns to Gross–Pitaevskii energy functional which describes Bose–Einstein condensates of a dilute gas with attractive interactions, we refer to Guo *et al* [21–23] and references therein for more details. For the classical Laplacian, it is well known that the L^2-critical exponent $2 + 4/N$ plays an important role in the Schrödinger equation, when $b = 0$ and $2 < q < 2 + 4/N$, the energy functional $\bar{E}_{a,b,p,q}$ is bounded below on the constrained set. It is worth mentioning here that the corresponding classical Schrödinger equations also receive much attention, see, for example [9, 25, 39, 42, 53, 54] and references therein for more details.

Back to the classical Kirchhoff equation, that is a, and b are arbitrary but fixed, and its corresponding L^2-critical exponent becomes $2 + 8/N$. Ye [55] first demonstrated the existence and non-existence of normalized solutions to a kind of Kirchhoff equation when $s = 1$, then a series of subsequent study has been done on the existence of normalized solutions to the nonlinear Kirchhoff equation, for L^2-critical problem with $s = 1$, we refer to [56, 57]. Luo and Wang [43] obtained the multiplicity existence of normalized solutions to a kind of Kirchoff equation when $N = 3$. Zeng *et al* [58] recently used a global branch approach for handling the nonlinearities in a unified way, and obtained the existence of normalized solutions to the Kirchhoff equation with general nonlinearities.

As for nonlinear equations involving fractional Laplacian, Cingolani *et al* [11] studied the following fractional problem

$$\begin{cases} (-\Delta)^s u + \mu u = g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = m, \\ u \in H_r^s(\mathbb{R}^N), \end{cases}$$

where $N \geqslant 2$, $s \in (0,1)$, $m > 0$, μ is an unknown Lagrange multiplier and $g \in C(\mathbb{R},\mathbb{R})$ satisfies Berestycki-Lions type conditions. Using a Lagrangian formulation of the above problem, they proved the existence of a weak solution with prescribed mass when g has L^2-subcritical growth. Luo and Zhang [42] proved some existence and nonexistence results about the normalized solutions of the fractional nonlinear Schrödinger equations with combined nonlinearities

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

where $0 < s < 1$, $N \geqslant 2$, $\mu \in \mathbb{R}$ and $2 < q < p < 2_s^* = \frac{2N}{N-2s}$. After that, Zhen and Zhang [59] considered the case of $p = 2_s^*$. For more results on this direction, see for example, [25, 33, 38, 39, 45, 52]. It is worth mentioning here that the Sobolev critical index makes it difficult to verify the existence of gauge solutions, even for unconstrained problems. We refer to He and Zou [26] for the details about the existence and concentration of solutions to the fractional critical Schrödinger equation.

Back to our concerned fractional Kirchhoff equation, there are a few literature studies on this issue. Huang and Zhang [30] provided a thorough classification for the existence of solutions to the fractional Kirchhoff functional on the L^2-normalized manifold when f is pure power nonlinearity, taking into account p. Chen and Huang [8] considered the existence and nonexistence of solutions to the fractional Kirchhoff equation with an external potential V and doubly critical exponents: critical Sobolev exponent and the fractional Gagliardo–Nirenberg–Sobolev critical exponent. By decomposing the Pohozaev set and constructing a fiber map, Liu *et al* [39] established the existence and properties of normalized ground states to fractional Kirchhoff equation with combined nonlinearities. Kong and Chen [32] studied the existence of normalized ground states for nonlinear fractional Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities in $\mathbb{R}^3$, and analyzed the asymptotic behavior of the obtained normalized solutions. We refer to [8, 30, 32, 39] and the reference therein for more details. For unconstrained problems, here we refer [14, 27, 28] to the readers for the details of ground state solution of fractional Kirchhoff equations with critical growth and multiplicity of concentrating solutions to the nonlinear fractional Kirchhoff equation.

1.2. Main goal and difficulties

In this paper, we are concerned with the concentration and uniqueness of constraint minimizers of the mass critical fractional Kirchhoff equations (1). It is worth mentioning that the study of the concentration behavior of solutions is an excellent approach to further study the properties of solutions and their development trend in the near critical case.

In the model we mentioned before, that is $N = 2$, $a = 1$, $b = 0$ and $q = 2$ in (2), Guo *et al* [21–23] explored the limiting behaviour of minimizers for the Gross–Pitaevskii energy functional (3) by some techniques consisting of blow-up estimate and elliptic regularity theory, then they also proved the local uniqueness of constraint minimizer by establishing Pohozaev identities and by contradiction. When $b \neq 0$ and $N = 2$, the energy functional (2) becomes a classical Kirchhoff type energy functional. Guo and Zhou [18, 19] considered the limiting behavior of constraint minimizers as $b \to 0^+$, in particular, they first illustrated the existence and nonexistence of constraint minimizers for different parameters p, and then they considered concentration behaviors of the minimizers of (3) as $b \to 0^+$ and also the uniqueness of the minimizers of (3) for b sufficiently close to 0 were shown. For the concentration phenomenon and local uniqueness of constraint minimizers of (3) as $a \searrow 0$, recently, Hu and Tang [29] had overcame the difficulties in estimates and blow-up analysis caused by the nonlocal term in the process of establishing decay estimate and setting Pohozăev identities, they improved the concentration phenomenon in the sense of that they provided the accurate estimating exponent instead of $o(1)$, and then they finally established the corresponding results. Motivated by [29], Guo *et al* [20] studied the normalized solutions for a kind of Kirchhoff type equation on a suitable weighted Sobolev space, and they investigated the limit behaviors of the normalized solutions for this equation as $(a,b) \to (0^+,0^+)$, furthermore, they also discussed the uniqueness of the normalized solution when a,b close to 0. For more concentration phenomenon and local uniqueness results of the classical elliptic equation, we also refer to [37, 44] and the references therein.

In the present paper, we are likewise interested in the concentration phenomenon and local uniqueness of the solution under the effect of the fractional Laplace operator and the nonlocal Kirchhoff term. Motivated by the above works, *a natural question is one may ask whether we could use the insights and methods therein to continue the subsequent research, to establish the concentration phenomenon and local uniqueness of constraint minimizers of fractional Kirchhoff equation (1) as* $a \searrow 0$?

Note that until now there have been no results in this field. Two intuitive difficulties arise, on the one hand, some well-known regularity theories for classical elliptic problems are invalid due to the nonlocal properties of fractional Laplace operators. On the other hand, the nonlocal Kirchhoff term will not vanish as $a \searrow 0$ and it will cause great difficulty concerning the uniqueness since it could influence the rate of decay of solutions sequence.

In the following, we shall give the relevant results to equation (1). The energy functional of (1) can be defined by

$$E_{a,\beta}(u) := \int_{\mathbb{R}^N} \left(a|(-\Delta)^{\frac{s}{2}} u|^2 + |x|^2 u^2 \right) \mathrm{d}x + \frac{b}{2} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \right)^2 - \frac{N\beta}{N+4s} \int_{\mathbb{R}^N} |u|^{\frac{8s+2N}{N}} \mathrm{d}x \quad u \in H,$$

where the work space H is defined by

$$H := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2 u^2 < \infty \right\}.$$

From the variational method, one could turn the constraint solution of (1) into the research of the following minimization problem

$$e(a,\beta) := \inf_{\{u \in H, \|u\|_2^2 = 1\}} E_{a,\beta}(u).$$

In order to consider the asymptotic properties of constraint minimizers, we first need to figure out their existence under the degenerate condition, that is $a = 0$. Similar to the classical case, fractional Gagliardo–Nirenberg–Sobolev inequality plays a crucial role in verifying the existence of a constraint minimizer. Du *et al* [12] established the following fractional

Gagliardo–Nirenberg–Sobolev inequality

$$\int_{\mathbb{R}^N} |u(x)|^{\frac{8s+2N}{N}} \mathrm{d}x \leqslant \frac{N+4s}{N\|\varphi\|_2^{\frac{8s}{N}}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u(x)|^2 \mathrm{d}x \right)^2 \left(\int_{\mathbb{R}^N} |u(x)|^2 \mathrm{d}x \right)^{\frac{4s-N}{N}}, \tag{4}$$

where $\varphi(x)$ is the unique radial positive ground state solution of the following equation

$$2(-\Delta)^s u + \frac{4s-N}{N} u - u^{\frac{8s}{N}+1} = 0 \quad \text{in } \mathbb{R}^N, \tag{5}$$

with

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |\varphi(x)|^2 \mathrm{d}x = \frac{N}{N+4s} \int_{\mathbb{R}^N} |\varphi(x)|^{\frac{8s+2N}{N}} \mathrm{d}x \tag{6}$$

and

$$\frac{C_1}{1+|x|^{N+2s}} \leqslant \varphi(x) \leqslant \frac{C_2}{1+|x|^{N+2s}} \quad \text{for } x \in \mathbb{R}^N. \tag{7}$$

Our first result is the existence and non-existence of degenerate fractional Kirchhoff equation.

Theorem 1. *Let $\varphi(x) > 0$ be the unique positive solution of (5), then we have*

(i) if $0 < \beta < \beta^* := \frac{b}{2}\|\varphi\|_2^{\frac{8s}{N}}$, there exists at least one minimizer to $e(0,\beta)$;
(ii) if $\beta \geqslant \beta^*$, $e(0,\beta)$ has no minimizer.
Moreover, $e(0,\beta^*) = 0$ and $e(0,\beta) = -\infty$ for all $\beta > \beta^*$.

Next, we consider the concentration phenomenon of constraint minimizer as $a \searrow 0$. Our second result is as follows.

Theorem 2. *Let u_k be a nonnegative minimizer of $e(a_k, \beta^*)$ with $a_k \to 0$ as $k \to \infty$. Then there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$ such that each u_k has a unique global maximum point $\bar{z}_k$ with*

$$\lim_{k\to\infty} \bar{z}_k = 0 \tag{8}$$

and

$$\lim_{k\to\infty} \bar{\varepsilon}_k^{\frac{N}{2}} u_k(\bar{\varepsilon}_k x + \bar{z}_k) = \left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}} \varphi(x) \quad \text{in } L^\infty(\mathbb{R}^N), \tag{9}$$

where $\bar{\varepsilon}_k$ is defined by

$$\bar{\varepsilon}_k := \left[\frac{s a_k \|\varphi\|_2^2}{\int_{\mathbb{R}^N} |x|^2 \varphi^2(x) \mathrm{d}x} \right]^{\frac{1}{2+2s}}. \tag{10}$$

Moreover, we have

$$\lim_{k\to\infty} \frac{\bar{z}_k}{\bar{\varepsilon}_k} = 0. \tag{11}$$

Remark 1. Note that in the process of establishing the concentration phenomenon of constraint minimizers, some classical local elliptic theories such as De Giorgi-Nash-Morser theory, regularity theory and comparison principle are no longer applicable. Therefore, some uniform regularity and decay estimates for the solutions to the fractional Kirchhoff equation are established by taking full advantage of the Bessel kernel and employing the Morse iteration. In addition, for the sake of subsequent uniqueness results, as in theorem 2, we need to verify the convergence of minimizers in $L^{\infty}(\mathbb{R}^N)$ instead of $H^s(\mathbb{R}^N)$, where the most convergence results of minimizers to fractional nonlinear equations were established in the latter space.

Based on the above concentration phenomenon of constraint minimizer, we give the following result about the uniqueness of the constraint minimizer.

Theorem 3. *Let u_k be a nonnegative minimizer of $e(a_k,\beta^*)$ with $a_k \to 0$ as $k\to\infty$, then the minimizer u_k is unique as $k\to\infty$.*

Comments on the theorems 2 and 3:

1. The essential idea to prove theorem 3 is to derive contradiction by establishing nonlocal Pohozǎev identities. First, select two suitable $\hat{u}_1$ and $\hat{u}_2$ which are scaling functions of two different constraint minimizers u_1 and u_2. By studying the properties of fractional Laplacian, we can decompose $\bar{\eta} := (\hat{u}_1-\hat{u}_2)/\|\hat{u}_1-\hat{u}_2\|_{\infty}$ as the linear combination of some functions consisting of the unique radial positive ground state solution to (5). Inspired by [5], then we consider the localization method introduced by Caffarelli and Silvestre. More precisely, for a function $u\in H^s(\mathbb{R}^N)$, set

$$\tilde{u}(x,t) = \mathcal{P}_s[u] = \int_{\mathbb{R}^N} \mathcal{P}_s(x-z,t)\,u(z)\,\mathrm{d}z, \quad (x,t)\in\mathbb{R}^{N+1}_{+} := \left\{(x,t): x\in\mathbb{R}^N, t>0\right\}, \tag{12}$$

where

$$\mathcal{P}_s(x,t) = \beta_s \frac{t^{2s}}{\left(|x|^2+t^2\right)^{\frac{N+2s}{2}}} \tag{13}$$

with a constant β_s such that $\int_{\mathbb{R}^N}\mathcal{P}_s(x,1)\mathrm{d}x = 1$. Moreover, $\tilde{u}$ satisfies

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla\tilde{u}\right) = 0 & x\in\mathbb{R}^{N+1}_{+}, \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t\tilde{u}(x,t) = \omega_s(-\Delta)^s u(x) & x\in\mathbb{R}^N, \end{cases} \tag{14}$$

in the distribution sense, where $\omega_s = 2^{1-s}\Gamma(1-s)/\Gamma(s)$. Without loss of generality, we may assume $\omega_s = 1$. Taking advantage of decay estimates and blow up analysis, we obtain a contradiction which indicates the local uniqueness as a closes to zero sufficiently by establishing two nonlocal Pohozǎev identities.

2. Compared to the classical case, constraint minimizers in our problem only satisfy polynomial decay (it can be exponential decay in the classical case) which requires more precision in our estimates. In the process of our research, we find that the scaling functions used in the classical case (such as the ones in [23, 29]) can not meet our requirement of estimates due to the polynomial decay and the non-vanishing of the nonlocal fractional Kirchhoff term. The testing function defined in this paper, which is defined by

$$\hat{u}_{ik}(x) := \left(\frac{2\beta^*}{b}\right)^{\frac{N}{8s}} \bar{\varepsilon}_k^{\frac{N}{2}} u_{ik}(\bar{\varepsilon}_k^{\tau} x) \quad i=1,2,$$

achieves the best decay speed, please see equation (96) for more details, which improves the accuracy of our estimates. As a new attempt, we believe it will be useful in the relevant research of fractional nonlinear equations.

3. Some technique difficulties appear, for instance, the appearance of $(-\Delta)^s$ makes the classical elliptic regularity invalid, we have to set the decay estimate of $|\nabla u|$ (where u stands for the constraint minimizer). Moreover, the property of the linearized operator $\Gamma := (-\Delta)^s + \frac{4s-N}{2N} - \frac{8s+N}{2N}\varphi^{\frac{8s}{N}}$ is also an important topic of our research.

4. Although the corresponding harmonic extension problem can help us to establish Pohozăev identities, we have to deal with several new integral terms that never appeared in the classical local elliptic problems. As a result, the nonlocal Pohozăev identity we establish may be the first result of the fractional Kirchhoff equation and we believe it will be useful in the relevant research of fractional nonlinear equations.

This paper is organized as follows. In section 2, we study the existence and nonexistence of the constraint minimizers and prove the theorem 1. In section 3, we show some general concentration behaviors and finally prove theorem 2. Theorem 3 will be established in section 4.

2. Existence and non-existence of minimizers

In this section, we discuss the existence and non-existence of the minimizer to $e(0,\beta)$. Inspired by [12, 29], we consider this problem in three cases and complete the proof of theorem 1. We first give a compactness result, see [10] for the details.

Lemma 1. *The embedding $H \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in [2, 2_s^*)$.*

Next, we give the proof of theorem 1.

Proof of theorem 1: For $\beta \in (0,\beta^*)$. Following (4), for any $u \in H$ with $\|u\|_2 = 1$, it has

$$\begin{aligned} E_{0,\beta}(u) &= \int_{\mathbb{R}^N} |x|^2 u^2 \mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2 - \frac{N\beta}{N+4s}\int_{\mathbb{R}^N} |u|^{\frac{8s+2N}{N}} \mathrm{d}x \\ &\geqslant \left(\frac{b}{2} - \frac{\beta}{\|\varphi\|_2^{\frac{8s}{N}}}\right)\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\right)^2, \end{aligned} \tag{15}$$

which implies $E_{0,\beta}(u)$ is bounded from below in this case. Letting $\{u_n\} \in H$ be a minimizing sequence for $e(0,\beta)$ for $\beta \in (0,\beta^*)$, it is easy to see from (15) that sequences $\left\{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \mathrm{d}x\right\}$ and $\left\{\int_{\mathbb{R}^N} |x|^2 u_n^2 \mathrm{d}x\right\}$ are bounded uniformly in n. By the weak lower semicontinuity of the norm in H and the lemma 1, we can conclude that $e(0,\beta)$ has at least one minimizer in this case.

For $\beta > \beta^*$, we take advantage of the cut-off function to prove the non-existence of minimizer of $e(0,\beta)$ in this case. Let $\omega \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function with $0 \leqslant \omega \leqslant 1$, $\omega(x) = 1$ when $|x| \leqslant 1$, $\omega(x) = 0$ when $|x| > 2$ and $|\nabla\omega| \leqslant 2$, and define the following trail function

$$U_\tau(x) := \frac{A_\tau \tau^{\frac{N}{2}}}{\|\varphi\|_2}\omega\left(\frac{x-x_0}{\tau^2}\right)\varphi(\tau|x-x_0|), \tag{16}$$

where $A_\tau > 0$ is chosen such that $\|U_\tau(x)\|_2^2 = 1$. From the polynomial decay of φ, it holds

$$\frac{1}{A_\tau^2} = 1 + \frac{1}{\|\varphi\|_2^2}\int_{\mathbb{R}^N \setminus B_{\tau^3}(0)} \left[\omega^2\left(\frac{x}{\tau^3}\right) - 1\right]\varphi^2 \mathrm{d}x = 1 + O\left(\tau^{-3N+2s}\right) \quad \text{as } \tau \to \infty. \tag{17}$$

Following (6), [12, lemma 3.2] and (17), one has

$$\begin{aligned}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U_\tau(x)|^2 \mathrm{d}x &= \frac{A_\tau^2 \tau^{2s}}{\|\varphi\|_2^2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\omega\left(\frac{x}{\tau^3}\right)\varphi(x) - \omega\left(\frac{y}{\tau^3}\right)\varphi(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\leqslant \frac{A_\tau^2 \tau^{2s}}{\|\varphi\|_2^2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\varphi(x)|^2\mathrm{d}x + O\left(\tau^{-12s}\right)\right) \\ &= \tau^{2s} + O\left(\tau^{-10s}\right) \quad \text{as } \tau\to\infty,\end{aligned} \tag{18}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |U_\tau(x)|^{\frac{8s+2N}{N}}\mathrm{d}x &= \frac{A_\tau^{\frac{8s+2N}{N}}\tau^{4s}}{\|\varphi\|_2^{\frac{8s+2N}{N}}} \int_{\mathbb{R}^N} \omega^{\frac{8s+2N}{N}}\left(\frac{x}{\tau^3}\right)\varphi^{\frac{8s+2N}{N}}\mathrm{d}x \\ &= \frac{(N+4s)\tau^{4s}}{N\|\varphi\|_2^{\frac{8s}{N}}} + O\left(\tau^{-3N-36s-\frac{48s^2}{N}}\right) \quad \text{as } \tau\to\infty,\end{aligned} \tag{19}$$

and

$$\int_{\mathbb{R}^N} |x|^2|U_\tau(x)|^2\mathrm{d}x = \frac{A_\tau^2}{\|\varphi\|_2^2}\int_{B_{2\tau^3}(0)} \left|\frac{x}{\tau}+x_0\right|^2 \omega^2\left(\frac{x}{\tau^3}\right)\varphi^2\mathrm{d}x \to |x_0|^2 \quad \text{as } \tau\to\infty. \tag{20}$$

Therefore, we derive from (18)–(20) that for $\beta > \beta^*$,

$$e(0,\beta) \leqslant E_{0,\beta}(u) \leqslant \left(\frac{b}{2} - \frac{\beta}{\|\varphi\|_2^{\frac{8s}{N}}}\right)\tau^{4s} + |x_0|^2 + O\left(\tau^{-8s}\right) + o(1) \to -\infty \quad \text{as } \tau\to\infty, \tag{21}$$

which implies that $e(0,\beta)$ has no minimizer for $\beta > \beta^*$.

For $\beta = \beta^*$, we may assume that there exists a positive constraint minimizer v, then taking $x_0 = 0$ in (16), we can deduce from (21) that $e(0,\beta) \leqslant 0$. Moreover, (15) shows that

$$E_{0,\beta^*}(v) \geqslant \int_{\mathbb{R}^N} |x|^2 v^2 \mathrm{d}x \geqslant 0.$$

Then we obtain $e(0,\beta^*) = 0$ and $v(x) = 0$ in $\mathbb{R}^N \setminus \{0\}$ which is a contradiction to the fact $\int_{\mathbb{R}^N} |v|^2\mathrm{d}x = 1$.

3. Concentration phenomenon

In this section, we fix $\beta = \beta^*$ and focus on some concentration phenomenon of the minimizer u_a to $e(a,\beta^*)$. Similar to the proof of theorem 1, we can prove that for any $a > 0$, there exists a nonnegative minimizer u_a of $e(a,\beta^*)$. On the other hand, there is no minimizer to $e(0,\beta^*)$ as $a = 0$. Therefore, a nature question is what would happen if $a \searrow 0$? We shall establish theorem 2 on the concentration phenomenon of u_a as $a \searrow 0$. As a bedding, the following lemma describes the limiting behavior of energy as $a \searrow 0$.

Lemma 2. *Let u_a be a nonnegative minimizer of $e(a,\beta^*)$, then*

$$\begin{cases} e(a,\beta^*) \to e(0,\beta^*) = 0 \quad \text{as } a\searrow 0, \\ \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_a|^2\mathrm{d}x \to +\infty \quad \text{as } a\searrow 0, \\ a\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_a|^2\mathrm{d}x \to 0 \quad \text{as } a\searrow 0, \\ \int_{\mathbb{R}^N} |x|^2|u_a|^2\mathrm{d}x \to 0 \quad \text{as } a\searrow 0. \end{cases}$$

Proof. In view of theorem 1 and (4), it holds

$$e(a,\beta^*) = E_{a,\beta^*}(u_a) \geqslant \int_{\mathbb{R}^N} \left(a|(-\Delta)^{\frac{s}{2}} u_a|^2 + |x|^2 u_a^2\right) \mathrm{d}x > e(0,\beta^*) = 0. \quad (22)$$

Defining $U_\tau(x)$ as (16), it follows from (17)–(20) that

$$0 < e(a,\beta^*) \leqslant E_{a,\beta^*}(U_\tau) \leqslant a\tau^{2s} + |x_0|^2 + Ca\tau^{-10s} + C\tau^{-8s} + o(1) \quad \text{as } \tau \to \infty. \quad (23)$$

Taking $x_0 = (0,0,\ldots 0)$ and letting $\tau = a^{-\frac{1}{4s}}$, then (23) shows that

$$0 < e(a,\beta^*) \to 0 \quad \text{as } a \searrow 0, \quad (24)$$

from which and (22) we have

$$a\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_a|^2 \mathrm{d}x \to 0, \quad \int_{\mathbb{R}^N} |x|^2 |u_a|^2 \mathrm{d}x \to 0 \quad \text{as } a \searrow 0. \quad (25)$$

We next prove that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_a|^2 \mathrm{d}x \to +\infty \quad \text{as } a \searrow 0. \quad (26)$$

On the contrary, we now assume that (26) is false, then (25) shows that there exists a sequence $\{a_k\}$ with $a_k \to 0$ as $k \to \infty$ such that $\{u_k\}$ is bounded in H, where $u_k := u_{a_k}$. Thus lemma 1 implies that there exists $u_0 \in H$ such that $u_k \rightharpoonup u_0$ weakly in H and $u_k \to u_0$ strongly in $L^p(\mathbb{R}^N)$ as $k \to \infty$ for $p \in [2, 2_s^*)$. Then,

$$e(0,\beta^*) \leqslant \liminf_{k\to\infty} E_{a_k,\beta^*}(u_k) = \lim_{k\to\infty} e(a_k,\beta^*) = 0 = e(0,\beta^*).$$

This shows that u_0 is a minimizer of $e(0,\beta^*)$, which is a contradiction. □

We next investigate asymptotic behaviors of the minimizers ulteriorly by scaling it and give some properties of the trail function.

Lemma 3. *Let u_k be a nonnegative minimizer of $e(a_k,\beta^*)$ with $a_k \to 0$ as $k \to \infty$, then there exists a sequence $\{y_k\}$, $K_0 > 0$ and $\delta > 0$ such that the normalized function*

$$w_k(x) = \varepsilon_k^{\frac{N}{2}} u_k(\varepsilon_k x + \varepsilon_k y_k) \quad \textit{where} \quad \varepsilon_k := \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_k|^2 \mathrm{d}x\right)^{-\frac{1}{2s}} \quad (27)$$

satisfies

$$\liminf_{k\to\infty} \int_{B_{K_0}(0)} |w_k|^2 \mathrm{d}x \geqslant \delta > 0. \quad (28)$$

Moreover, for any sequence $\{a_k\}$ with $a_k \to 0$ as $k \to \infty$, there exists a subsequence, still denoted by $\{a_k\}$, such that $z_k := \varepsilon_k y_k \to 0$ as $k \to \infty$. In addition, for any $\rho > 0$ small enough

$$u_k(x) = \varepsilon_k^{-\frac{N}{2}} w_k\left(\frac{x - z_k}{\varepsilon_k}\right) \to 0 \quad \textit{as } k \to \infty \textit{ for } x \in B_\rho^c(0), \quad (29)$$

and

$$w_k(x) \to w_0(x) := \left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}} \varphi(|x-y_0|) \quad \textit{as } k \to \infty \textit{ for some } y_0 \in \mathbb{R}^N \textit{ in } H^s(\mathbb{R}^N). \tag{30}$$

Proof. We divide this proof into six steps.

Step 1: Define $\tilde{w}_k(x) := \varepsilon_k^{\frac{N}{2}} u_k(\varepsilon_k x)$, we can derive from lemma 2 that

$$\begin{aligned}\int_{\mathbb{R}^N} |\tilde{w}_k|^{\frac{8s+2N}{N}} \mathrm{d}x =& \varepsilon_k^{4s} \int_{\mathbb{R}^N} |u_k|^{\frac{8s+2N}{N}} \mathrm{d}x \\ =& \frac{\varepsilon_k^{4s}(N+4s)}{N\beta^*} \left[\int_{\mathbb{R}^N} \left(a_k|(-\Delta)^{\frac{s}{2}} u_k|^2 + |x|^2 u_k^2\right) \mathrm{d}x \right. \\ & \left. + \frac{b}{2}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_k|^2 \mathrm{d}x\right)^2 - e(a_k, \beta^*)\right] \\ \to& \frac{b(N+4s)}{2N\beta^*} \quad \text{as } k \to \infty.\end{aligned} \tag{31}$$

We now claim that there exists a sequence $\{y_k\} \subset \mathbb{R}^N$ and $K_0, \delta > 0$ such that

$$\liminf_{\varepsilon_k \to 0} \int_{B_{K_0}(y_k)} |\tilde{w}_k|^2 \mathrm{d}x \geqslant \delta > 0. \tag{32}$$

Suppose by contradiction that for any $K > 0$, there exists a subsequence of $\{\tilde{w}_k\}$, still denoted by $\{\tilde{w}_k\}$, satisfying

$$\lim_{k\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_K(y)} |\tilde{w}_k|^2 \mathrm{d}x = 0.$$

By the fact that $s \in (N/4, 1)$, we deduce from the vanishing lemma that $\tilde{w}_k \to 0$ in $L^{\frac{8s+2N}{N}}(\mathbb{R}^N)$ as $k \to \infty$, which contradicts (31). Then (32) shows that (28) holds.

Step 2: Applying lemma 2 and (27), we have

$$\int_{\mathbb{R}^N} |x|^2 u_k^2 \mathrm{d}x = \int_{\mathbb{R}^N} |\varepsilon_k x + \varepsilon_k y_k|^2 w_k^2 \mathrm{d}x \to 0 \quad \text{as } k \to \infty,$$

which implies that

$$0 = \liminf_{k\to\infty} \int_{\mathbb{R}^N} |\varepsilon_k x + \varepsilon_k y_k|^2 w_k^2 \mathrm{d}x \geqslant \liminf_{k\to\infty} \int_{B_{K_0}(0)} |\varepsilon_k x + \varepsilon_k y_k|^2 w_k^2 \mathrm{d}x.$$

Since $|x|^2 \to \infty$ as $|x| \to \infty$, then (28) shows that $\{\varepsilon_k y_k\}$ is bounded in $\mathbb{R}^N$. Up to a subsequence if necessary, there exists a $x_0 \in \mathbb{R}^N$ such that $z_k := \varepsilon_k y_k \to x_0$ as $k \to \infty$. By Fatou lemma, it then follows from (28) that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^N}|\varepsilon_k x+\varepsilon_k y_k|^2 w_k^2\mathrm{d}x\geqslant|x_0|^2\int_{B_{K_0(0)}}\lim_{k\to\infty}w_k^2\mathrm{d}x\geqslant\frac{|x_0|^2\delta}{2},$$

which shows that $|x_0|=0$.

Step 3: Since u_k is a nonnegative minimizer of $e(a_k,\beta^*)$, then it satisfies the following fractional Kirchhoff equation

$$\left(a_k+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\right)(-\Delta)^s u_k+|x|^2u_k=\mu_k u_k+\beta^* u_k^{\frac{8s}{N}+1}\quad\text{in }\mathbb{R}^N,\tag{33}$$

where $\mu_k\in\mathbb{R}$ is the Lagrange multiplier. It is easy to check that

$$\mu_k=\int_{\mathbb{R}^N}\left(a_k|(-\Delta)^{\frac{s}{2}}u_k|^2+|x|^2u_k^2\right)\mathrm{d}x+b\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\right)^2-\beta^*\int_{\mathbb{R}^N}u_k^{\frac{8s+2N}{N}}\mathrm{d}x\tag{34}$$

and

$$\begin{aligned}e(a_k,\beta^*)=&\int_{\mathbb{R}^N}\left(a_k|(-\Delta)^{\frac{s}{2}}u_k|^2+|x|^2u_k^2\right)\mathrm{d}x+\frac{b}{2}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\right)^2\\&-\frac{N\beta^*}{N+4s}\int_{\mathbb{R}^N}u_k^{\frac{8s+2N}{N}}\mathrm{d}x.\end{aligned}\tag{35}$$

Combing (27) and (35), it yields from lemma 2 that

$$\begin{aligned}&\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}w_k|^2\mathrm{d}x=\int_{\mathbb{R}^N}|w_k|^2=1,\\&\int_{\mathbb{R}^N}w_k^{\frac{8s+2N}{N}}\mathrm{d}x=\varepsilon_k^{4s}\int_{\mathbb{R}^N}u_k^{\frac{8s+2N}{N}}\mathrm{d}x\to\frac{b(N+4s)}{2N\beta^*}\quad\text{as }k\to\infty.\end{aligned}\tag{36}$$

Following (34) and (36), one has

$$\mu_k\varepsilon_k^{4s}\to\frac{(N-4s)b}{2N}\quad\text{as }k\to\infty.\tag{37}$$

Moreover, one can check that $w_k(x)$ satisfies the following equation

$$\left(a_k\varepsilon_k^{2s}+b\right)(-\Delta)^s w_k+\varepsilon_k^{4s}|\varepsilon_k x+\varepsilon_k y_k|^2w_k=\mu_k\varepsilon_k^{4s}w_k+\beta^* w_k^{\frac{8s}{N}+1}\quad\text{in }\mathbb{R}^N.\tag{38}$$

Applying (36), we conclude that $\{w_k\}$ is bounded in $H^s(\mathbb{R}^N)$. Passing to subsequence, there exists $w_0\in H^s(\mathbb{R}^N)$ such that

$$w_k\rightharpoonup w_0\geqslant 0\quad\text{in }H^s\left(\mathbb{R}^N\right)\text{ as }k\to\infty.\tag{39}$$

Since w_k satisfies (38), using (37) and passing to weak limit, one has

$$b(-\Delta)^s w_0=\frac{(N-4s)b}{2N}w_0+\beta^* w_0^{\frac{8s}{N}+1}\quad\text{in }\mathbb{R}^N\tag{40}$$

in weak sense. Furthermore, (28) implies that $w_0 \not\equiv 0$. Similar to the proof of [50, proposition 4.4], we have that $w_0 \in C^{2,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Then, following [13, lemma 3.2], we have

$$(-\Delta)^s w_0(x) = -\frac{C_s}{2}\int_{\mathbb{R}^N} \frac{w_0(x+y)+w_0(x-y)-2w_0(x)}{|y|^{N+2s}}\mathrm{d}y \quad \text{in } \mathbb{R}^N.$$

Assume that there exists $x_0 \in \mathbb{R}^N$ such that $w_0(x_0) = 0$, it then follows from $w_0(x) \geqslant 0$ and $w_0 \not\equiv 0$ that

$$(-\Delta)^s w_0(x_0) = -\frac{C_s}{2}\int_{\mathbb{R}^N} \frac{w_0(x_0+y)+w_0(x_0-y)}{|y|^{N+2s}}\mathrm{d}y < 0.$$

However, from (40) we know that $(-\Delta)^s w_0(x_0) = 0$, which is a contradiction. Hence, $w(x) > 0$ for any $x \in \mathbb{R}^N$. Comparing (5) and (40), the uniqueness of positive radial solution of (5) implies that

$$w_0(x) = \left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}} \varphi(|x-y_0|), \quad \text{for some } y_0 \in \mathbb{R}^N. \tag{41}$$

Moreover, we derive from (4) and (41) that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w_0|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |w_0|^2 = 1. \tag{42}$$

In view of (36), (39) and (42), one has

$$w_k \to w_0 := \left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}} \varphi(|x-y_0|) \quad \text{for some } y_0 \in \mathbb{R}^N.$$

Step 4: Now we show that $\|w_k\|_\infty < \infty$ uniformly for large k. In fact, from (37), (38) and (42), it has

$$(-\Delta)^s w_k \leqslant \frac{2\beta^*}{b} w_k^{\frac{8s}{N}+1}. \tag{43}$$

Define

$$h(t) :\equiv h_{T,\gamma}(t) = \begin{cases} 0 & t \leqslant 0, \\ t^\gamma & 0 < t \leqslant T, \\ \gamma T^{\gamma-1}(t-T) + T^\gamma & t > T, \end{cases}$$

where $\gamma > 1$ and $T > 0$. It is easy to check that $h(t)$ is convex and Lipschitz continuous, so

$$(-\Delta)^s h(w_k) \leqslant h'(w_k)(-\Delta)^s w_k$$

in the weak sense. Clearly, $h \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and it is positive, increasing and convex.

Define

$$\tilde{h}(t) := \int_0^t (h'(r))^2 \mathrm{d}r,$$

which is positive, increasing, convex and belongs to $C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, then we have $\tilde{h}'(t)=(h'(t))^2$ and $\tilde{h}(t)-\tilde{h}(r)=\tilde{h}'(t)(t-r)$ for $t,r\in\mathbb{R}$. Also, by the Jensen inequality, the fact $0\leqslant h(t)\leqslant|t|^\gamma$ and $0\leqslant th'(t)\leqslant\gamma h(t)$ for $r,t\in\mathbb{R}$, we deduce $0\leqslant\tilde{h}(t)\leqslant\|h'\|_\infty|t|^\gamma$ and $|h(t)-h(r)|^2\leqslant(\tilde{h}(t)-\tilde{h}(r))(t-r)$. Thus, we conclude $\tilde{h}(w_k)\in H^s(\mathbb{R}^N)$ since $\tilde{h}$ is Lipschitz continuous and $\tilde{h}(0)=0$.

Assume that $1<\gamma\leqslant 2_s^*/2$, it has $\tilde{h}(w_k)\leqslant\|h'\|_\infty|w_k|^\gamma\in L^2(\mathbb{R}^N)$. By using Sobolev inequality, (43) and the properties of $\tilde{h}(t)$, we can see that

$$\begin{aligned}
\|h(w_p)\|_{2_s^*}^2 &\leqslant C\|(-\Delta)^{\frac{s}{2}}h(w_k)\|_2^2\\
&\leqslant C\int_{\mathbb{R}^{2N}}\frac{\left(\tilde{h}(w_k(x))-\tilde{h}(w_k(y))\right)(w_k(x)-w_k(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&= C\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}w_k(-\Delta)^{\frac{s}{2}}\tilde{h}(w_k)\,\mathrm{d}x\\
&\leqslant C\int_{\mathbb{R}^N}\left(w_k+w_k^{2_s^*-1}\right)\tilde{h}(w_k)\,\mathrm{d}x\\
&\leqslant C\int_{\mathbb{R}^N}\left(w_k+w_k^{2_s^*-1}\right)w_k\tilde{h}'(w_k)\,\mathrm{d}x\\
&\leqslant C\int_{\mathbb{R}^N}\left(w_k+w_k^{2_s^*-1}\right)w_k(h'(w_k))^2\,\mathrm{d}x\\
&\leqslant \gamma^2C\int_{\mathbb{R}^N}\left(1+w_k^{2_s^*-2}\right)(h(w_k))^2\,\mathrm{d}x\\
&= \gamma^2C\int_{\mathbb{R}^N}(h(w_k))^2\mathrm{d}x+\gamma^2C\int_{\mathbb{R}^N}w_k^{2_s^*-2}(h(w_k))^2\mathrm{d}x.
\end{aligned}\tag{44}$$

Now let $R>0$ be fixed, by the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(h(w_k))^2w_k^{2_s^*-2}\mathrm{d}x\\
&\quad=\int_{\{w_k\leqslant R\}}(h(w_k))^2w_k^{2_s^*-2}\mathrm{d}x+\int_{\{w_k>R\}}(h(w_k))^2w_k^{2_s^*-2}\mathrm{d}x\\
&\quad\leqslant R^{2_s^*-2}\|h(w_k)\|_2^2+\left(\int_{\{w_k>R\}}w_k^{2_s^*}\,\mathrm{d}x\right)^{\frac{2_s^*-2}{2_s^*}}\|h(w_k)\|_{2_s^*}^2.
\end{aligned}\tag{45}$$

Note that $\{w_k\}$ converges strongly in $H^s(\mathbb{R}^N)$, then one has $\{w_k\}$ converges strongly in $L^{2_s^*}(\mathbb{R}^N)$, so we can choose $R>0$ sufficiently large such that

$$\left(\int_{\{w_k>R\}}w_k^{2_s^*}\,\mathrm{d}x\right)^{\frac{2_s^*-2}{2_s^*}}\leqslant\frac{1}{2\gamma^2C}.$$

It follows from (44), (45) and the fact $h(t)\leqslant|t|^\gamma$ that

$$\|h(w_k)\|_{2_s^*}^2\leqslant 2\gamma^2C\left(1+R^{2_s^*-2}\right)\|h(w_k)\|_2^2\leqslant 2\gamma^2C\left(1+R^{2_s^*-2}\right)\|w_k\|_{2\gamma}^{2\gamma}.$$

From the Fatou lemma and the fact $\lim_{T\to+\infty} h_{T,\gamma}(t) = t^\gamma$, we deduce that

$$\|w_k\|_{2_s^*\gamma}^{2\gamma} = \left(\int_{\mathbb{R}^N} \liminf_{T\to+\infty} h_{T,\gamma}^{2_s^*}(w_k)\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \leqslant \liminf_{T\to+\infty}\left(\int_{\mathbb{R}^N} h_{T,\gamma}^{2_s^*}(w_k)\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}$$
$$\leqslant 2\gamma^2 C\left(1+R^{2_s^*-2}\right)\|w_k\|_{2\gamma}^{2\gamma},$$

which implies that $w_k \in L^{2_s^*\gamma}(\mathbb{R}^N)$ since $1<\gamma\leqslant \frac{2_s^*}{2}$. Iterating this argument with $\gamma_0 := \frac{2_s^*}{2}$, $\gamma_i := \frac{2_s^*}{2}\gamma_{i-1}$ and let $\gamma_i \to +\infty$, we can obtain that $w_k \in L^r(\mathbb{R}^N)$ where $r\in[2,+\infty)$. By Fatou lemma, (44) and the fact $h(t)\leqslant |t|^\gamma$, we have

$$\|w_k\|_{2_s^*\gamma}^{2\gamma} \leqslant \gamma^2 C\int_{\mathbb{R}^N}\left(w_k^{2\gamma} + w_k^{2_s^*-2+2\gamma}\right)\mathrm{d}x. \tag{46}$$

By Hölder inequality and Young inequality, we get that

$$\int_{\mathbb{R}^N} w_k^{2_s^*-2+2\gamma}\mathrm{d}x \leqslant \|w_k^{2_s^*-2}\|_{\frac{N}{s}}\|w_k^\gamma\|_2\|w_k^\gamma\|_{2_s^*} \leqslant \|w_k^{2_s^*-2}\|_{\frac{N}{s}}\left(\frac{1}{2\epsilon}\|w_k\|_{2\gamma}^{2\gamma} + \frac{\epsilon}{2}\|w_k\|_{2_s^*\gamma}^{2\gamma}\right). \tag{47}$$

Set

$$L := \|w_k^{2_s^*-2}\|_{\frac{2}{N}},$$

then the fact $w_k\in L^r(\mathbb{R}^N)$ shows that $L<+\infty$, where $r\in[2,+\infty)$. Let $\epsilon = 1/(L\gamma^2 C)$, it follows from (46) and (47) that

$$\|w_k\|_{2_s^*\gamma}^{2\gamma} \leqslant 2\gamma^2 C\left(1+\frac{1}{2L^2\gamma^2 C}\right)\|w_k\|_{2\gamma}^{2\gamma} \leqslant C\gamma^4\|w_k\|_{2\gamma}^{2\gamma}. \tag{48}$$

Letting $2\gamma_i = 2_s^*\gamma_{i-1}$ in (48), we have

$$\|w_k\|_{2_s^*\gamma_i} \leqslant \left(C\gamma_i^4\right)^{\frac{1}{2\gamma_i}}\|w_k\|_{2_s^*\gamma_{i-1}}$$

which implies that

$$\|w_k\|_{2_s^*\gamma_i} \leqslant \prod_{j=0}^{i}\left(C\gamma_j^4\right)^{\frac{1}{2\gamma_j}}\|w_k\|_{2_s^*\gamma_0}. \tag{49}$$

Letting $i\to+\infty$ in (49), we conclude that

$$\|w_k\|_\infty \leqslant e^{\sum_{j=0}^{\infty}\frac{\log\left(C\left(\frac{2_s^*}{2}\right)^{4j}\gamma_0^4\right)}{2\left(\frac{2_s^*}{2}\right)^j\gamma_0}}\|w_k\|_{2_s^*\gamma_0} \leqslant C.$$

Step 5: Next we prove the fact $w_k\to 0$ as $|x|\to\infty$ uniformly for large k. We rewrite problem (38) as follows

$$(-\Delta)^s w_k + w_k = h_k(x) \quad \text{in } \mathbb{R}^N,$$

where

$$h_k := w_k + \left(\varepsilon_k^{2s} a_k + b\right)^{-1} \left(\mu_k \varepsilon_k^{4s} w_k + \beta^* w_k^{\frac{8s}{N}+1} - \varepsilon_k^{4s} |\varepsilon_k x + \varepsilon_k y_k|^2 w_k\right).$$

It is easy to check that $h_k \in L^\infty\left(\mathbb{R}^N\right)$ for large k, thus by the interpolation on the L^p-spaces and the convergence of $\{w_k\}$ in $H^s\left(\mathbb{R}^N\right)$, there exists $h \in L^r\left(\mathbb{R}^N\right)$ such that $h_k \to h$ in $L^r\left(\mathbb{R}^N\right)$ as $k \to \infty$ for $r \geqslant 2$. Using [16], we have

$$w_k = \int_{\mathbb{R}^N} \mathcal{K}(x-y) h_k(x) \, \mathrm{d}y,$$

where $\mathcal{K}$ is a Bessel potential and it satisfies

$(\mathcal{K}_1)$ $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \backslash \{0\}$.

$(\mathcal{K}_2)$ There exists $C > 0$ such that $\mathcal{K}(x) \leqslant \frac{C}{|x|^{N+2s}}$ for $x \in \mathbb{R}^N \backslash \{0\}$.

$(\mathcal{K}_3)$ $\mathcal{K} \in L^r(\mathbb{R}^N)$ for $r \in [1, N/(N-2s))$.

Now argue as in the proof of [51, lemma 6.4], we conclude that

$$\lim_{|x| \to \infty} w_k(x) \to 0 \quad \text{as } k \to \infty. \tag{50}$$

Step 6: We now show (29) holds. In fact, lemma 2 shows that there exists $M > 0$ such that

$$a_k \varepsilon_k^{2s} + b \leqslant M \quad \text{uniformly for large } k. \tag{51}$$

Following [16, lemma 4.3], there exists a function $\mathcal{X}$ such that

$$0 < \mathcal{X} < \frac{1}{1 + |x|^{N+2s}}$$

and

$$(-\Delta)^s \mathcal{X} + \frac{(4s-N) b}{4NM} \mathcal{X} = 0 \quad \text{in } \mathbb{R}^N \backslash B_{R_1}(0) \text{ for some } R_1 > 0.$$

Following (37), (38), (50) and (51), we conclude for large k that

$$\begin{aligned} (-\Delta)^s w_k + \frac{(4s-N) b}{4NM} w_k &\leqslant (-\Delta)^s w_k + \frac{(4s-N) b w_k}{4N\left(a_k \varepsilon_k^{2s} + b\right)} \\ &= \frac{-\varepsilon_k^{4s} |\varepsilon_k x + \varepsilon_k y_k|^2 w_k + \mu_k \varepsilon_k^{4s} w_k + \beta^* w_k^{\frac{8s}{N}+1} + \frac{(4s-N)b}{4N} w_k}{a_k \varepsilon_k^{2s} + b} \\ &\leqslant 0 \end{aligned}$$

for $|x| \geqslant R_2$ with R_2 large enough. Similar to [50, lemma 5.6], we conclude that

$$w_k(x) \leqslant \frac{C}{1 + |x|^{N+2s}} \quad \text{uniformly for large } k. \tag{52}$$

For any $x \in B_\rho^c(0)$, we have

$$\frac{|x - z_k|}{\varepsilon_k} \geqslant \frac{|x|}{2\varepsilon_k} \geqslant \frac{\rho}{2\varepsilon_k} \to +\infty \quad \text{as } k \to \infty. \tag{53}$$

From (52) and (53), it follows that

$$u_k(x)=\varepsilon_k^{-\frac{N}{2}}w_k\left(\frac{x-z_k}{\varepsilon_k}\right)\leqslant\varepsilon_k^{-\frac{N}{2}}\frac{C}{1+|\frac{x-z_k}{\varepsilon_k}|^{N+2s}}$$
$$\leqslant\varepsilon_k^{-\frac{N}{2}}\frac{C}{1+|\frac{\rho}{2\varepsilon_k}|^{N+2s}}\to 0\quad\text{as } k\to\infty \text{ for } x\in B_\rho^c(0),$$

thus the proof of lemma 3 is complete. □

Next, we recall the [36, propostion 3.2], whose regularity property is crucial for our proof.

Lemma 4. *Assume that $g\in C^r(B_R(0)), r>0$ and u is a nonnegative solution of*

$$(-\Delta)^s u=g(x)\quad in\, B_R(0).$$

If $2s+r\leqslant 1$, then $u\in C^{0,2s+r}\left(B_{\frac{R}{2}}(0)\right)$ with

$$\|u\|_{C^{0,2s+r}\left(B_{\frac{R}{2}}(0)\right)}\leqslant C\left[\|u\|_{L^\infty\left(B_{\frac{3R}{4}}(0)\right)}+\|g\|_{C^r\left(B_{\frac{3R}{4}}(0)\right)}\right].$$

If $2s+r>1$, then $u\in C^{1,2s+r-1}\left(B_{\frac{R}{2}}(0)\right)$ with

$$\|u\|_{C^{1,2s+r-1}\left(B_{\frac{R}{2}}(0)\right)}\leqslant C\left[\|u\|_{L^\infty\left(B_{\frac{3R}{4}}(0)\right)}+\|g\|_{C^r\left(B_{\frac{3R}{4}}(0)\right)}\right].$$

Proposition 1. *Let u_k be a nonnegative minimizer of $e(a_k,\beta^*)$ with $a_k\to 0$ as $k\to\infty$, then there exists a sequence $\{u_k\}$, still denoted by $\{u_k\}$, such that each u_k has a unique global maximum point $\bar{z}_k$ satisfying*

$$\lim_{k\to\infty}\bar{z}_k=0 \tag{54}$$

and

$$\lim_{k\to\infty}\varepsilon_k^{\frac{N}{2}}u_k(\varepsilon_k x+\bar{z}_k)=\left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}}\varphi(x)\quad in\ L^\infty\left(\mathbb{R}^N\right), \tag{55}$$

where $\varphi(x)$ is the unique solution of (5), and $\varepsilon_k\to 0$ as $k\to\infty$ is defined by

$$\varepsilon_k:=\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_k|^2\mathrm{d}x\right)^{-\frac{1}{2s}}. \tag{56}$$

Proof. Let $\bar{z}_k$ be a global maximum point of u_k, it then follows from (33) and (37) that

$$u_k(\bar{z}_k)\geqslant\left(\frac{-\mu_k}{\beta^*}\right)^{\frac{N}{8s}}\geqslant C\varepsilon_k^{-\frac{N}{2}}, \tag{57}$$

from which and (29) we derive that $\bar{z}_k\to 0$ as $k\to\infty$. Set

$$\bar{w}_k(x)=\varepsilon_k^{\frac{N}{2}}u_k(\varepsilon_k x+\bar{z}_k). \tag{58}$$

One can check that $\bar{w}_k(x)$ satisfies the following equation

$$\left(a_k\varepsilon_k^{2s}+b\right)(-\Delta)^s\bar{w}_k+\varepsilon_k^{4s}|\varepsilon_k x+\bar{z}_k|^2\bar{w}_k=\mu_k\varepsilon_k^{4s}\bar{w}_k+\beta^*\bar{w}_k^{\frac{8s}{N}+1}\quad \text{in } \mathbb{R}^N. \tag{59}$$

We now claim that $\bar{w}_k$ satisfies (28) for some positive constants R_0 and δ. Indeed, (52) and (57) yield that there exists $R_1>0$, independent of k, such that $\left|\frac{\bar{z}_k-z_k}{\varepsilon_k}\right|<\frac{R_1}{2}$. Since w_k satisfies (38), we then deduce from (58) that

$$\lim_{k\to\infty}\int_{B_{R_0+R_1}(0)}|\bar{w}_k|^2\mathrm{d}x=\lim_{k\to\infty}\int_{B_{R_0+R_1}\left(\frac{\bar{z}_k-z_k}{\varepsilon_k}\right)}|w_k|^2\mathrm{d}x\geqslant\int_{B_{R_0+\frac{R_1}{2}}(0)}|w_k|^2\mathrm{d}x\geqslant\delta>0, \tag{60}$$

which proves the claim. Similar to lemma 3, one can further derive that there exists a subsequence, still denoted by $\{\bar{w}_k\}$ with

$$\mu_k\varepsilon_k^{4s}\to\frac{(N-4s)\,b}{2N}\quad\text{and}\quad\bar{w}_k\rightharpoonup\bar{w}_0\geqslant 0\quad\text{in } H^s\left(\mathbb{R}^N\right)\text{ as } k\to\infty, \tag{61}$$

where $\bar{w}_0$ satisfies (40). Moreover, (60) implies that $\bar{w}_0\not\equiv 0$. Thus, following Step 3 in lemma 3, it has $\bar{w}_0>0$. Further, we obtain that

$$\bar{w}_0(x)=\left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}}\varphi(|x|), \tag{62}$$

since the origin is the unique global maximum point of φ.

We next prove the uniqueness of $\bar{z}_k$ as $k\to\infty$. Rewriting (59) as follows

$$(-\Delta)^s\bar{w}_k=F_k(x)\quad\text{in } \mathbb{R}^N, \tag{63}$$

where

$$F_k(x):=\left(a_k\varepsilon_k^{2s}+b\right)^{-1}\left[-\varepsilon_k^{4s}|\varepsilon_k x+\bar{z}_k|^2\bar{w}_k+\mu_k\varepsilon_k^{4s}\bar{w}_k+\beta^*\bar{w}_k^{\frac{8s}{N}+1}\right].$$

By the fact $\|\bar{w}_k\|_\infty<C$ and $\bar{z}_k\to 0$ as $k\to\infty$, we then deduce from (61) that $\{F_k\}$ is bounded uniformly in $L^\infty(\mathbb{R}^N)$ as $k\to\infty$. Using [47, Proposition 2.9], we have

$$\|\bar{w}_k\|_{C^{1,\alpha}(\mathbb{R}^N)}\leqslant C\left(\|\bar{w}_k\|_{L^\infty(\mathbb{R}^N)}+\|F_k\|_{L^\infty(\mathbb{R}^N)}\right)\leqslant C\quad\text{for } \alpha\in(0,1)\,\text{as}\,k\to\infty.$$

Furthermore, since $\varepsilon_k^{4s}|\varepsilon_k x+\bar{z}_k|^2$ is locally Lipschitz continuous in $\mathbb{R}^N$, in view of $\bar{z}_k\to 0$ as $k\to\infty$. Applying lemma 4 a finite number of times to (63), there exists some positive $\gamma\in(0,1)$ such that

$$\|\bar{w}_k\|_{C^{2,\gamma}\left(\overline{B_{R/2}}\right)}\leqslant C\left[\|\bar{w}_k\|_{L^\infty\left(B_{3R/4}\right)}+\|F_k\|_{C^\gamma\left(B_{3R/4}\right)}\right]\leqslant C\quad\text{as } k\to\infty,$$

which shows that $\{\bar{w}_k\}$ is bounded uniformly in $C^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ as $k\to\infty$ for some $\gamma\in(0,1)$. So there exists $\tilde{w}_0\in C^2_{\mathrm{loc}}(\mathbb{R}^N)$ such that $\bar{w}_k\to\tilde{w}_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $k\to\infty$. Further, by using (61) we have that $\bar{w}_0=\tilde{w}_0$. Thus,

$$\bar{w}_k\to\bar{w}_0\quad\text{in } C^2_{\mathrm{loc}}\left(\mathbb{R}^N\right)\text{ as } k\to\infty. \tag{64}$$

Because the origin is the unique global maximum point of $\varphi(x)$, then (64) shows that all global maximum points of $\bar{w}_k$ stay in a small ball $B_\rho(0)$ as $k\to\infty$ for some $\rho>0$. Since $\varphi''(0)<0$, we know $\varphi''(r)<0$ for $r\in[0,\rho)$. By [46, lemma 4.2], we obtain that $\bar{w}_k$ has no critical point other than origin and $\bar{z}_k$ is the unique global maximum point of $\bar{w}_k$ as $k\to\infty$.

We now prove (55) holds. It then follows from (7) and (52) that for any $\epsilon>0$, there exists a constant $R_\epsilon>0$ independent k, such that

$$|\bar{w}_k(x)|,\quad \left|\left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}}\varphi(x)\right|<\frac{\epsilon}{4}\quad \text{for any } |x|>R_\epsilon \text{ as } k\to\infty,$$

which yields that

$$\sup_{|x|>R_\epsilon}|\bar{w}_k(x)-\bar{w}_0(x)|\leqslant\sup_{|x|>R_\epsilon}\left(|\bar{w}_k(x)|+\left|\left(\frac{b}{2\beta^*}\right)^{\frac{N}{8s}}\varphi(x)\right|\right)<\frac{\epsilon}{2}\quad\text{as } k\to\infty,$$

from which and (64) we can obtain (55). □

Following the proof of Proposition 1, we now address theorem 2 on the local properties of concentration points. Before proving theorem 2, we first establish several Claims.

Claim 1: Define $\lambda_0:=\int_{\mathbb{R}^N}|x|^2\varphi^2(x)\mathrm{d}x$, then we have

$$\limsup_{a\to0}\frac{e(a,\beta^*)}{a^{\frac{1}{1+s}}}\leqslant(1+s)\left(\frac{\lambda_0}{s\|\varphi\|_2^2}\right)^{\frac{s}{1+s}}.$$

Indeed, let $U_\tau(x)$ be the trail function defined by (16). Taking $x_0=0$, we deduce from (18) and (19) that

$$\int_{\mathbb{R}^N}|x|^2|U_\tau(x)|^2\mathrm{d}x=\frac{\lambda_0}{\tau^2\|\varphi\|_2^2}(1+o(1))\quad\text{as } \tau\to\infty, \tag{65}$$

and

$$e(a,\beta^*)=E_{a,\beta^*}(U_\tau)\leqslant a\tau^{2s}+aC\tau^{-10s}+\frac{\lambda_0}{\tau^2\|\varphi\|_2^2}(1+o(1))+C\tau^{-8s}\quad\text{as } \tau\to\infty. \tag{66}$$

Let $\tau=\left(\frac{\lambda_0}{as\|\varphi\|_2^2}\right)^{\frac{1}{2+2s}}$, then $\tau\to\infty$ as $a\searrow0$. We thus obtain from (65) and (66) that

$$e(a,\beta^*)\leqslant a\left(\frac{\lambda_0}{as\|\varphi\|_2^2}\right)^{\frac{s}{1+s}}+\frac{\lambda_0}{\|\varphi\|_2^2}\left(\frac{\lambda_0}{as\|\varphi\|_2^2}\right)^{\frac{-1}{1+s}}(1+o(1))+C\tau^{-8s}+o(1)\quad\text{as } a\searrow0,$$

which shows that

$$\limsup_{a\to0}\frac{e(a,\beta^*)}{a^{\frac{1}{1+s}}}\leqslant(1+s)\left(\frac{\lambda_0}{s\|\varphi\|_2^2}\right)^{\frac{s}{1+s}}.$$

Claim 2: Let u_k be a nonnegative minimizer of $e(a_k,\beta^*)$ as in proposition 1 with $a_k\to0$ as $k\to\infty$, then the unique global maximum point $\bar{z}_k$ of u_k satisfies $\bar{z}_k\to0$ as $k\to\infty$ and $\left\{\frac{\bar{z}_k}{\varepsilon_k}\right\}$ is bounded.

In fact, if $\left|\frac{\bar{z}_k}{\varepsilon_k}\right| \to \infty$ as $k \to \infty$, then for any $M > 0$ large enough, we have

$$\liminf_{k\to\infty} \frac{1}{\varepsilon_k^2} \int_{\mathbb{R}^N} |x|^2 u_k^2 \mathrm{d}x = \liminf_{k\to\infty} \int_{\mathbb{R}^N} \left| x + \frac{\bar{z}_k}{\varepsilon_k} \right|^2 \bar{w}_k^2 \mathrm{d}x \geqslant M. \tag{67}$$

Following (4), we deduce by calculation that there exists a constant $C > 0$ such that

$$e(a_k, \beta^*) = E_{a_k,\beta^*}(u_k) \geqslant a_k \varepsilon_k^{-2s} + M\varepsilon_k^2 \geqslant CM^{\frac{s}{1+s}} a_k^{\frac{1}{1+s}},$$

which is a contradiction in view of **Claim 1** by choosing $M > 0$ large enough.

Proof of theorem 2: Actually, **Claim 2** shows that there exists $z_0 \in \mathbb{R}^N$ such that

$$\frac{\bar{z}_k}{\varepsilon_k} \to z_0 \quad \text{as } k \to \infty. \tag{68}$$

Since $\varphi(x)$ is a radial decreasing function and decays polynomially as $|x| \to \infty$, we then conclude that

$$\begin{aligned} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^2} \int_{\mathbb{R}^N} |x|^2 u_k^2 \mathrm{d}x &= \frac{1}{\|\varphi\|_2^2} \int_{\mathbb{R}^N} |x + z_0|^2 \varphi^2(x) \, \mathrm{d}x \\ &\geqslant \frac{1}{\|\varphi\|_2^2} \int_{\mathbb{R}^N} |x|^2 \varphi^2(x) \, \mathrm{d}x = \frac{\lambda_0}{\|\varphi\|_2^2}. \end{aligned} \tag{69}$$

Together with the Young inequality, we derive from (66) that

$$\begin{aligned} e(a_k, \beta^*) = E_{a_k,\beta^*}(u_k) &\geqslant a_k \varepsilon_k^{-2s} + \frac{\lambda_0 \varepsilon_k^2}{\|\varphi\|_2^2} (1 + o(1)) \\ &\geqslant (1+s) a_k^{\frac{1}{1+s}} (1 + o(1)) \left(\frac{s\|\varphi\|_2^2}{\lambda_0} \right)^{\frac{-s}{1+s}}. \end{aligned}$$

Hence

$$\liminf_{k\to\infty} \frac{e(a_k, \beta^*)}{a_k^{\frac{1}{1+s}}} \geqslant (1+s) \left(\frac{\lambda_0}{s\|\varphi\|_2^2} \right)^{\frac{s}{1+s}}, \tag{70}$$

where the equality holds if and only if

$$\lim_{k\to\infty} \frac{\varepsilon_k}{\bar{\varepsilon}_k} = 1 \quad \text{where } \bar{\varepsilon}_k := \left(\frac{s a_k \|\varphi\|_2^2}{\lambda_0} \right)^{\frac{1}{2+2s}}. \tag{71}$$

In view of (70) and **Claim 1**, it follows that

$$\liminf_{k\to\infty} \frac{e(a_k, \beta^*)}{a_k^{\frac{1}{1+s}}} = (1+s) \left(\frac{\lambda_0}{s\|\varphi\|_2^2} \right)^{\frac{s}{1+s}}.$$

Therefore, (55) and (71) show that (9) holds. Next, we prove (11), indeed, from the definition of $e(a_k, \beta^*)$, we know that

$$\int_{\mathbb{R}^N} |x|^2 u_k^2(x) \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} |x|^2 u_k^2(x + \bar{z}_k) \, \mathrm{d}x. \tag{72}$$

By the decay estimate of $\bar{w}_k$ and φ, we deduce from (71) that

$$\int_{\mathbb{R}^N} |x|^2 u_k^2 (x+\bar{z}_k)\,\mathrm{d}x = \frac{\varepsilon_k^2}{\|\varphi\|_2^2}(1+o(1))\int_{\mathbb{R}^N}|x|^2\varphi^2(x)\,\mathrm{d}x \quad \text{as } k\to\infty. \tag{73}$$

Note that

$$\int_{\mathbb{R}^N} |x|^2 u_k^2(x)\,\mathrm{d}x \geqslant \int_{B_{\frac{1}{\sqrt{\varepsilon_k}}}(0)} |\varepsilon_k x+\bar{z}_k|^2 \bar{w}_k^2(x)\,\mathrm{d}x = \frac{(1+o(1))\,\varepsilon_k^2}{\|\varphi\|_2^2}\int_{B_{\frac{1}{\sqrt{\varepsilon_k}}}(0)} \left|x+\frac{\bar{z}_k}{\varepsilon_k}\right|^2 \varphi^2(x)\,\mathrm{d}x. \tag{74}$$

On the contrary, we assume that there exists a constant $\rho>0$ and a subsequence of $\left\{\frac{\bar{z}_k}{\varepsilon_k}\right\}$ with $\frac{|\bar{z}_k|}{\varepsilon_k}\geqslant\rho>0$ as $k\to\infty$. Then (72)–(74) show that

$$\frac{\varepsilon_k^2}{\|\varphi\|_2^2}\int_{B_{\frac{1}{\sqrt{\varepsilon_k}}}(0)}\left|x+\frac{\bar{z}_k}{\varepsilon_k}\right|^2\varphi^2(x)\mathrm{d}x \leqslant \frac{(1+o(1))\varepsilon_k^2}{\|\varphi\|_2^2}\int_{\mathbb{R}^N}|x|^2\varphi^2(x)\mathrm{d}x \quad \text{as } k\to\infty.$$

Applying the Fatou lemma, we have

$$\int_{\mathbb{R}^N}|x|^2\varphi^2(x)\,\mathrm{d}x < \int_{\mathbb{R}^N}\liminf_{k\to\infty}\left|x+\frac{\bar{z}_k}{\varepsilon_k}\right|^2\varphi^2(x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N}|x|^2\varphi^2(x)\,\mathrm{d}x$$

which is a contradiction. Thus $\frac{\bar{z}_k}{\varepsilon_k}\to 0$ as $k\to\infty$, from which and (71) we can obtain (11). This completes the proof.

4. Local uniqueness

This section is devoted to studying the local uniqueness of nonnegative minimizers when a is small enough. For this purpose, let $\{a_k\}$ be a sequence satisfies $a_k\to 0$ as $k\to\infty$. Suppose by contradiction, let u_{1k} and u_{2k} be two different normalized minimizers to $e(a_k,\beta^*)$ with $\|u_{ik}\|_2^2=1$ for $i=1,2$. Let $\bar{z}_{1k}$ and $\bar{z}_{2k}$ be the unique global maximal point of u_{1k} and u_{2k} as $k\to\infty$, respectively. Obviously, we rewrite (33) as

$$\left(a_k+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_{ik}|^2\mathrm{d}x\right)(-\Delta)^s u_{ik}+|x|^2u_{ik}=\mu_{ik}u_{ik}+\beta^* u_{ik}^{\frac{8s}{N}+1} \quad \text{in } \mathbb{R}^N, i=1,2, \tag{75}$$

where $\mu_{ik}\in\mathbb{R}$ is the corresponding Lagrange multiplier. Define

$$\bar{u}_{ik}(x) := \left(\frac{2\beta^*}{b}\right)^{\frac{N}{8s}}\bar{\varepsilon}_k^{\frac{N}{2}}u_{ik}(\bar{\varepsilon}_k x+\bar{z}_{1k}) \quad i=1,2, \tag{76}$$

and

$$\bar{\eta}_k(x) := \frac{\bar{u}_{1k}(x)-\bar{u}_{2k}(x)}{\|\bar{u}_{1k}-\bar{u}_{2k}\|_\infty}. \tag{77}$$

One can check that $\bar{u}_{ik}$ and $\bar{\eta}_k$ satisfy

$$\begin{aligned}&\left[a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2\mathrm{d}x\right](-\Delta)^s\bar{u}_{ik}+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_kx+\bar{z}_{1k}|^2\bar{u}_{ik}\\&=\mu_{ik}\bar{\varepsilon}_k^{4s}\bar{u}_{ik}+\frac{b}{2}\bar{u}_{ik}^{\frac{8s}{N}+1}\quad\text{in }\mathbb{R}^N,i=1,2,\end{aligned}\tag{78}$$

and

$$\begin{aligned}&\left[a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{2k}|^2\mathrm{d}x\right](-\Delta)^s\bar{\eta}_k+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_kx+\bar{z}_{1k}|^2\bar{\eta}_k\\&+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}(\bar{u}_{1k}+\bar{u}_{2k})(-\Delta)^{\frac{s}{2}}\bar{\eta}_k\mathrm{d}x\,(-\Delta)^{\frac{s}{2}}\bar{u}_{1k}\\&=\mu_{1k}\bar{\varepsilon}_k^{4s}\bar{\eta}_k+\frac{\bar{\varepsilon}_k^{4s}\bar{u}_{2k}(\mu_{1k}-\mu_{2k})}{\|\bar{u}_{1k}-\bar{u}_{2k}\|_\infty}+\frac{(8s+N)b}{2N}\left[A\bar{u}_{1k}+(1-A)\bar{u}_{2k}\right]^{\frac{8s}{N}}\bar{\eta}_k,\end{aligned}\tag{79}$$

where $A\in(0,1)$. The following result shows the polynomial decay of $\bar{u}_{ik}$ and $|\nabla\bar{u}_{ik}|$.

Lemma 5. *Assume that $u_{ik}(i=1,2)$ are two minimizers of $e(a_k,\beta^*)$ with $a_k\to0$ as $k\to\infty$ and $\bar{u}_{ik}$ is defined by (76), then there exist $R>0$ large enough and $2s>\rho>0$ small enough such that*

$$\bar{u}_{ik}(x)\leqslant\frac{C}{1+|x|^{N+2s}}\quad\textit{as } k\to\infty,\tag{80}$$

and

$$|\nabla\bar{u}_{ik}(x)|\leqslant\frac{C}{1+|x|^{N+2s}}\quad\textit{for } |x|>R \textit{ as } k\to\infty.\tag{81}$$

Proof. Clearly (80) follows from the decay estimate of u_{ik} in section 3 and we now show that (81) is true. In fact, $\bar{u}_{ik}\in C^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha\in(0,1)$ as $k\to\infty$. In particular, $|D\bar{u}_{ik}|\in L^\infty(\mathbb{R}^N)$ as $k\to\infty$. Differentiating (78), we get

$$\begin{aligned}&\left[a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2\mathrm{d}x\right](-\Delta)^s\frac{\partial\bar{u}_{ik}}{\partial x_j}+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_kx+\bar{z}_{1k}|^2\frac{\partial\bar{u}_{ik}}{\partial x_j}\\&=\mu_{ik}\bar{\varepsilon}_k^{4s}\frac{\partial\bar{u}_{ik}}{\partial x_j}+\frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}}\frac{\partial\bar{u}_{ik}}{\partial x_j}-2\bar{\varepsilon}_k^{4s+1}\left(\bar{\varepsilon}_kx_j+\bar{z}_{1k}^j\right)\bar{u}_{ik}\end{aligned}\tag{82}$$

where $i=1,2$ and $j=1,2,\cdots,N$. By (61) and (71), we have

$$\mu_{ik}\bar{\varepsilon}_k^{4s}\to\frac{(N-4s)b}{2N}\quad\text{as } k\to\infty\text{ and } i=1,2.\tag{83}$$

Using the fact that $\bar{z}_{1k}\to0$ as $k\to\infty$, we derive from (80) and (83) that there exists $\vartheta>0$ large such that

$$\inf_{|x|\geqslant\vartheta}\left[\frac{-\mu_{ik}\bar{\varepsilon}_k^{4s}-\frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}}+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_kx+\bar{z}_{1k}|^2}{a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2\mathrm{d}x}\right]:=\rho_0>0\quad\text{as } k\to\infty\text{ and } i=1,2.\tag{84}$$

Let $v \in H^s(\mathbb{R}^N)$ be a positive radial solution of

$$(-\Delta)^s v + \rho_0 v = v^{\frac{8s}{N}+1} \quad \text{in } \mathbb{R}^N. \tag{85}$$

From [15], we know that $v \in C(\mathbb{R}^N)$ and there exist constants $C_2 > C_1 > 0$ such that

$$\frac{C_1}{1+|x|^{N+2s}} \leqslant v(x) \leqslant \frac{C_2}{1+|x|^{N+2s}} \quad x \in \mathbb{R}^N. \tag{86}$$

For large k and $\vartheta_0 > \vartheta$, set

$$\bar{v} := \left(1 + \frac{\|D\check{u}_{ik}\|_\infty}{\inf_{|x|\leqslant\vartheta_0} v}\right) v.$$

In view of (82) and (85), it follows that

$$\begin{aligned}
&(-\Delta)^s \left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right) + \left[\frac{-\mu_{ik}\bar{\varepsilon}_k^{4s} - \frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}} + \bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k x + \bar{z}_{1k}|^2}{a_k\bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2 \mathrm{d}x}\right]\frac{\partial \bar{u}_{ik}}{\partial x_j} - \rho_0\bar{v} \\
&\quad = \frac{-2\bar{\varepsilon}_k^{4s+1}\left(\bar{\varepsilon}_k x_j + \bar{z}_{1k}^j\right)\bar{u}_{ik}}{a_k\bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2 \mathrm{d}x} - \left(1 + \frac{\|D\check{u}_{ik}\|_\infty}{\inf_{|x|\leqslant\vartheta_0} v}\right) v^{\frac{8s}{N}+1} \quad i = 1,2 \text{ and } j = 1,2,\cdots,N.
\end{aligned} \tag{87}$$

Testing (87) with $\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)(-\Delta)^{\frac{s}{2}}\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+ \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}\left\{\left[\frac{-\mu_{ik}\bar{\varepsilon}_k^{4s} - \frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}} + \bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k x + \bar{z}_{1k}|^2}{a_k\bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2 \mathrm{d}x}\right]\frac{\partial \bar{u}_{ik}}{\partial x_j} - \rho_0\bar{v}\right\}\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+ \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N}\left[\frac{-2\bar{\varepsilon}_k^{4s+1}\left(\bar{\varepsilon}_k x_j + \bar{z}_{1k}^j\right)\bar{u}_{ik}}{a_k\bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2 \mathrm{d}x} - \left(1 + \frac{\|D\check{u}_{ik}\|_\infty}{\inf_{|x|\leqslant\vartheta_0} v}\right) v^{\frac{8s}{N}+1}\right]\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+ \mathrm{d}x,
\end{aligned} \tag{88}$$

where $i = 1,2$ and $j = 1,2,\cdots,N$. According to the definition of $\bar{v}$, we can see that $\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+ = 0$ in $B_{\vartheta_0}(0)$ as $k \to \infty$. Moreover, we have that as $k \to \infty$

$$\int_{\mathbb{R}^N}\left[\frac{-2\bar{\varepsilon}_k^{4s+1}\left(\bar{\varepsilon}_k x_j + \bar{z}_{1k}^j\right)\bar{u}_{ik}}{a_k\bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2 \mathrm{d}x} - \left(1 + \frac{\|D\check{u}_{ik}\|_\infty}{\inf_{|x|\leqslant\vartheta_0} v}\right) v^{\frac{8s}{N}+1}\right]\left(\frac{\partial \bar{u}_{ik}}{\partial x_j} - \bar{v}\right)^+ \mathrm{d}x \leqslant 0, \tag{89}$$

where $i = 1,2$ and $j = 1,2,\cdots,N$. By the fact that $\vartheta_0 > \vartheta$, we obtain from (84) that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left\{\left[\frac{-\mu_{ik}\bar{\varepsilon}_k^{4s}-\frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}}+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k x+\bar{z}_{1k}|^2}{a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2\mathrm{d}x}\right]\frac{\partial\bar{u}_{ik}}{\partial x_j}-\rho_0\bar{v}\right\}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\mathrm{d}x\\
&=\int_{B^c_{\vartheta_0}(0)}\left\{\left[\frac{-\mu_{ik}\bar{\varepsilon}_k^{4s}-\frac{b}{2}\frac{8s+N}{N}\bar{u}_{ik}^{\frac{8s}{N}}+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k x+\bar{z}_{1k}|^2}{a_k\bar{\varepsilon}_k^{2s}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\bar{u}_{ik}|^2\mathrm{d}x}\right]\frac{\partial\bar{u}_{ik}}{\partial x_j}-\rho_0\bar{v}\right\}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\mathrm{d}x\\
&\geqslant\int_{B^c_{\vartheta_0}(0)}\left(\rho_0\frac{\partial\bar{u}_{ik}}{\partial x_j}-\rho_0\bar{v}\right)\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\mathrm{d}x\\
&=\rho_0\int_{\mathbb{R}^N}\left|\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\right|^2\mathrm{d}x\quad\text{as } k\to\infty,\ i=1,2\text{ and } j=1,2,\cdots,N.
\end{aligned}\tag{90}$$

Let $\omega_i := \frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}$, where $i = 1,2$ and $j = 1,2,\cdots,N$. Some calculations show that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(\omega_i(x)-\omega_i(y))\left(\omega_i^+(x)-\omega_i^+(y)\right)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&=\int_{\{\omega_i(x)\geqslant 0\}}\int_{\{\omega_i(y)<0\}}\frac{(\omega_i(x)-\omega_i(y))\,\omega_i(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\{\omega_i(x)<0\}}\int_{\{\omega_i(y)\geqslant 0\}}\frac{(\omega_i(y)-\omega_i(x))\,\omega_i(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\{\omega_i(x)\geqslant 0\}}\int_{\{\omega_i(y)\geqslant 0\}}\frac{(\omega_i(x)-\omega_i(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y
\end{aligned}\tag{91}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\omega_i^+(x)-\omega_i^+(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y&=\int_{\{\omega_i(x)\geqslant 0\}}\int_{\{\omega_i(y)<0\}}\frac{\omega_i^2(x)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\{\omega_i(x)<0\}}\int_{\{\omega_i(y)\geqslant 0\}}\frac{\omega_i^2(y)}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int_{\{\omega_i(x)\geqslant 0\}}\int_{\{\omega_i(y)\geqslant 0\}}\frac{(\omega_i(x)-\omega_i(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}\tag{92}$$

Following (91) and (92), it follows that

$$\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)(-\Delta)^{\frac{s}{2}}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\mathrm{d}x\geqslant\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\right|^2\mathrm{d}x.\tag{93}$$

It the follows from (88)–(90) and (93) that for $i = 1,2$ and $j = 1,2,\cdots,N$

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\right|^2\mathrm{d}x+\rho_0\int_{\mathbb{R}^N}\left|\left(\frac{\partial\bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+\right|^2\mathrm{d}x\leqslant 0\quad\text{as } k\to\infty,$$

which implies that $\left(\frac{\partial \bar{u}_{ik}}{\partial x_j}-\bar{v}\right)^+=0$ in $\mathbb{R}^3$. Thus, we have

$$\frac{\partial \bar{u}_{ik}}{\partial x_j}\leqslant \bar{v}\leqslant \frac{C}{1+|x|^{N+2s}}\quad \text{as } k\to\infty,\ i=1,2 \text{ and } j=1,2,\cdots,N.$$

By the same arguments as above, we can also obtain that $-\frac{\partial \bar{u}_{ik}}{\partial x_j}\leqslant \frac{C}{1+|x|^{N+2s}}$ as $k\to\infty$, $i=1,2$ and $j=1,2,\cdots,N$. This therefore completes the proof of lemma 5. □

To achieve the optimal decay frequency, denote

$$\hat{u}_{ik}(x):=\left(\frac{2\beta^*}{b}\right)^{\frac{N}{8s}}\bar{\varepsilon}_k^{\frac{N}{2}}u_{ik}\left(\bar{\varepsilon}_k^{\tau}x\right)\quad \text{for } i=1,2, \tag{94}$$

and

$$\hat{\eta}_k(x):=\frac{\hat{u}_{1k}(x)-\hat{u}_{2k}(x)}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}, \tag{95}$$

where $\tau<0$ satisfy

$$\begin{cases} \tau<\min\left\{-2s,\frac{-4s-6}{4s}\right\} & N=2,\\ -4s-4<\tau<\min\left\{-2s,\frac{-4s-5}{4s+2}\right\} & N=3.\end{cases} \tag{96}$$

In view of (76), (78), (94) and (95), it follows that

$$\begin{aligned}&\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right](-\Delta)^s\hat{u}_{ik}\\ &\quad+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}=\mu_{ik}\bar{\varepsilon}_k^{4s}\hat{u}_{ik}+\frac{b}{2}\hat{u}_{ik}^{\frac{8s}{N}+1}\quad \text{in } \mathbb{R}^N,\ i=1,2\end{aligned} \tag{97}$$

and

$$\begin{aligned}&2a_k\bar{\varepsilon}_k^{2s(2-\tau)}(-\Delta)^s\hat{\eta}_k+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\hat{u}_{2k}|^2\right)\mathrm{d}x\,(-\Delta)^s\hat{\eta}_k\\ &\quad+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}(\hat{u}_{1k}+\hat{u}_{2k})(-\Delta)^{\frac{s}{2}}\hat{\eta}_k\mathrm{d}x\,(-\Delta)^s(\hat{u}_{1k}+\hat{u}_{2k})+2\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k^{\tau}x|^2\hat{\eta}_k\\ &=\bar{\varepsilon}_k^{4s}(\mu_{1k}+\mu_{2k})\hat{\eta}_k+\frac{\bar{\varepsilon}_k^{4s}(\mu_{1k}-\mu_{2k})}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}(\hat{u}_{1k}+\hat{u}_{2k})+\frac{(8s+N)b}{N}\left[A\hat{u}_{1k}+(1-A)\hat{u}_{1k}\right]^{\frac{8s}{N}}\hat{\eta}_k.\end{aligned} \tag{98}$$

Following (76)–(78), we have

$$\begin{aligned}
&\frac{\bar{\varepsilon}_k^{4s}\left(\mu_{1k}-\mu_{2k}\right)}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}\left(\hat{u}_{1k}+\hat{u}_{2k}\right)\\
&\quad=\frac{b}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{2s}}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\bar{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\bar{u}_{2k}|^2\right)\mathrm{d}x\\
&\quad\times\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\left(\bar{u}_{1k}+\bar{u}_{2k}\right)(-\Delta)^{\frac{s}{2}}\bar{\eta}_k\mathrm{d}x\left(\hat{u}_{1k}+\hat{u}_{2k}\right)\\
&\quad-\frac{4s\beta^*}{N}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}}\int_{\mathbb{R}^N}\left(\bar{u}_{1k}^{\frac{4s+N}{N}}+\bar{u}_{2k}^{\frac{4s+N}{N}}\right)\left[A\bar{u}_{1k}+(1-A)\bar{u}_{2k}\right]^{\frac{4s}{N}}\bar{\eta}_k\mathrm{d}x\left(\hat{u}_{1k}+\hat{u}_{2k}\right).
\end{aligned}\tag{99}$$

We next claim that for any $x_0\in\mathbb{R}^N$, there exists a small constant $\delta>0$ such that

$$\int_{\partial B_\delta(x_0)}\left[\bar{\varepsilon}_k^{2s(1-\tau)}|(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2+\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k^{\tau}x|^2\hat{\eta}_k^2+\hat{\eta}_k^2\right]\mathrm{d}S\leqslant C\bar{\varepsilon}_k^{N(1-\tau)}\quad\text{as } k\to\infty.\tag{100}$$

In fact, multiplying (98) by $\hat{\eta}_k$ and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}
&\left[2a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\hat{u}_{2k}|^2\right)\mathrm{d}x\right]\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2\mathrm{d}x\\
&\quad+2\bar{\varepsilon}_k^{4s}\int_{\mathbb{R}^N}|\bar{\varepsilon}_k^{\tau}x|^2\hat{\eta}_k^2\mathrm{d}x-\bar{\varepsilon}_k^{4s}\left(\mu_{1k}+\mu_{2k}\right)\int_{\mathbb{R}^N}\hat{\eta}_k^2\mathrm{d}x:=A_1+A_2+A_3+A_4,
\end{aligned}\tag{101}$$

where

$$\begin{aligned}
A_1&:=-b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\left(\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\left(\hat{u}_{1k}+\hat{u}_{2k}\right)(-\Delta)^{\frac{s}{2}}\hat{\eta}_k\mathrm{d}x\right)^2,\\
A_2&:=\frac{b}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{2s}}\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\bar{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\bar{u}_{2k}|^2\right)\mathrm{d}x\\
&\quad\times\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\left(\bar{u}_{1k}+\bar{u}_{2k}\right)(-\Delta)^{\frac{s}{2}}\bar{\eta}_k\mathrm{d}x\int_{\mathbb{R}^N}\left(\bar{u}_{1k}+\bar{u}_{2k}\right)\bar{\eta}_k\mathrm{d}x,\\
A_3&:=-\frac{4s\beta^*}{N}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}}\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}\left(\bar{u}_{1k}^{\frac{4s+N}{N}}+\bar{u}_{2k}^{\frac{4s+N}{N}}\right)\\
&\quad\times\left[A\bar{u}_{1k}+(1-A)\bar{u}_{2k}\right]^{\frac{4s}{N}}\bar{\eta}_k\mathrm{d}x\int_{\mathbb{R}^N}\left(\bar{u}_{1k}+\bar{u}_{2k}\right)\bar{\eta}_k\mathrm{d}x,\\
A_4&:=\frac{(8s+N)b}{N}\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}\left[A\bar{u}_{1k}+(1-A)\bar{u}_{2k}\right]^{\frac{8s}{N}}\bar{\eta}_k^2\mathrm{d}x.
\end{aligned}$$

By the fact that $\|\bar{u}_{ik}\|_{H^s(\mathbb{R}^N)} \leqslant C$ as $k \to \infty$, we can derive from the Hölder inequality that

$$\begin{aligned} |A_1| &\leqslant C\bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\hat{u}_{1k}+\hat{u}_{2k})|^2 \mathrm{d}x \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x \\ &\leqslant C\bar{\varepsilon}_k^{(\tau-1)(N-4s)+(1-\tau)(N-2s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\bar{u}_{1k}+\bar{u}_{2k})|^2 \mathrm{d}x \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x \\ &\leqslant C\bar{\varepsilon}_k^{2s(1-\tau)} \|\bar{u}_{1k}+\bar{u}_{2k}\|^2_{H^s(\mathbb{R}^N)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x \\ &\leqslant C\bar{\varepsilon}_k^{2s(1-\tau)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x \quad \text{as } k\to\infty. \end{aligned} \tag{102}$$

Using the same argument of (102), we can obtain that

$$\begin{aligned} |A_2| &\leqslant C\bar{\varepsilon}_k^{(1-\tau)N} \|\bar{\eta}_k\|^2_{H^s(\mathbb{R}^N)} \\ &\leqslant C\bar{\varepsilon}_k^{(1-\tau)N} \left[\bar{\varepsilon}_k^{(N-2s)(\tau-1)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x + \bar{\varepsilon}_k^{N(\tau-1)} \int_{\mathbb{R}^N} \hat{\eta}_k^2 \mathrm{d}x\right] \\ &\leqslant C\bar{\varepsilon}_k^{2s(1-\tau)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x + C\int_{\mathbb{R}^N} \hat{\eta}_k^2 \mathrm{d}x \quad \text{as } k\to\infty. \end{aligned} \tag{103}$$

Noticing that $|\bar{\eta}_k| \leqslant 1$, we conclude from lemma 5 that

$$|A_3 + A_4| \leqslant C\bar{\varepsilon}_k^{(1-\tau)N} \quad \text{as } k\to\infty. \tag{104}$$

Thus, from (10), (37) and (101)–(104), we obtain that

$$\begin{aligned} &\bar{\varepsilon}_k^{2s(1-\tau)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x + \bar{\varepsilon}_k^{4s} \int_{\mathbb{R}^N} |\bar{\varepsilon}_k^{\tau} x|^2 \hat{\eta}_k^2 \mathrm{d}x - (\mu_{1k}+\mu_{2k})\bar{\varepsilon}_k^{4s} \int_{\mathbb{R}^N} \hat{\eta}_k^2 \mathrm{d}x \\ &\quad \leqslant C\left[\bar{\varepsilon}_k^{2s(1-\tau)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{\eta}_k|^2 \mathrm{d}x + \int_{\mathbb{R}^N} \hat{\eta}_k^2 \mathrm{d}x + \bar{\varepsilon}_k^{2(1-\tau)}\right] \quad \text{as } k\to\infty. \end{aligned}$$

Together with [6, lemma 4.5], we know that (100) holds. Thus we obtain following the estimate immediately

$$\|\bar{\eta}_k\|_{H^s(\mathbb{R}^N)} \leqslant C \quad \text{as } k\to\infty. \tag{105}$$

In order to deduce the limiting behaviors of $\bar{\eta}_k$, let us first study some basic properties of ground state φ, which will be used later.

Lemma 6. *Let $\varphi > 0$ be the unique radial positive solution of (5) and Γ denotes the corresponding linearized operator given by*

$$\Gamma := (-\Delta)^s + \frac{4s-N}{2N} - \frac{8s+N}{2N}\varphi^{\frac{8s}{N}}.$$

Then we have:

(i) φ is nondegenerate in $H^s(\mathbb{R}^N)$ in the sense that there holds

$$Ker\Gamma = span\left\{\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \ldots, \frac{\partial\varphi}{\partial x_N}\right\}.$$

(ii) $\Gamma\left(\frac{N}{4}\varphi + x\cdot\nabla\varphi\right) = -\frac{(4s-N)}{N}\varphi$ *and* $\Gamma(x\cdot\nabla\varphi) = 2s(-\Delta)^s\varphi$.

Proof. According to [15, theorem 3], we know that (i) holds. Let $w(x) > 0$ be the radial positive solution of

$$(-\Delta)^s w + w - w^{\frac{8s}{N}+1} = 0. \tag{106}$$

Define

$$\mathcal{L} := (-\Delta)^s + 1 - \frac{8s+N}{N} w^{\frac{8s}{N}}.$$

Following [15, (7.2) and (7.3)], we have that $\mathcal{L}\left(\frac{N}{4}w + x\cdot\nabla w\right) = -2sw$. Thus, (106) shows that

$$(-\Delta)^s (x\cdot\nabla w) + x\cdot\nabla w - \frac{8s+N}{N} w^{\frac{8s}{N}} (x\cdot\nabla w) = 2sw^{\frac{8s}{N}+1} - 2sw. \tag{107}$$

In view of (5) and (106), it follows that

$$w(x) = \left(\frac{N}{4s-N}\right)^{\frac{N}{8s}} \varphi\left[\left(\frac{2N}{4s-N}\right)^{\frac{1}{2s}} x\right], \tag{108}$$

then

$$(-\Delta)^s (x\cdot\nabla\varphi) + \frac{4s-N}{2N}(x\cdot\nabla\varphi) - \frac{8s+N}{2N}\varphi^{\frac{8s}{N}}(x\cdot\nabla\varphi) = s\varphi^{\frac{8s+N}{N}} - \frac{(4s-N)s}{N}\varphi. \tag{109}$$

Moreover, since $\varphi(x) > 0$ is the unique solution to (5), a straightforward calculation shows that

$$\begin{aligned}\Gamma\left(\frac{N}{4}\varphi + x\cdot\nabla\varphi\right) &= (-\Delta)^s (x\cdot\nabla\varphi) + \frac{4s-N}{2N}(x\cdot\nabla\varphi) - s\varphi^{\frac{8s+N}{N}} \\ &\quad - \frac{8s+N}{2N}\varphi^{\frac{8s}{N}}(x\cdot\nabla\varphi) = -\frac{(4s-N)s}{N}\varphi,\end{aligned}$$

in view of (109). This completes the proof of lemma 6. □

Lemma 7. *Under the assumptions of theorem 3, there exists $\bar{\eta}_0 \in C^1(\mathbb{R}^N)$ such that*

$$\bar{\eta}_k \to \bar{\eta}_0 \quad \text{in } C^1\left(\mathbb{R}^N\right) \text{ as } k \to \infty, \tag{110}$$

and

$$\bar{\eta}_0(x) = d_0\varphi + \bar{d}_0(x\cdot\nabla\varphi) + \sum_{i=1}^{N} d_i \frac{\partial\varphi}{\partial x_i}, \tag{111}$$

where $d_0, \bar{d}_0, d_1, \cdots, d_N$ are all constants.

Proof. By the fact that $\|\bar{u}_{ik}\|_{H^s(\mathbb{R}^N)} \leqslant C$ as $k\to\infty$ for $i = 1, 2$, we deduce from (105) that

$$\begin{aligned}&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}(\bar{u}_{1k}+\bar{u}_{2k})(-\Delta)^{\frac{s}{2}}\bar{\eta}_k \mathrm{d}x \\ &\leqslant \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\bar{u}_{1k}+\bar{u}_{2k})|^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\bar{\eta}_k|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant \|\bar{u}_{1k}+\bar{u}_{2k}\|_{H^s(\mathbb{R}^N)}\|\bar{\eta}_k\|_{H^s(\mathbb{R}^N)} \leqslant C \quad \text{as } k\to\infty.\end{aligned} \tag{112}$$

By the fact that $\bar{z}_{1k} \to 0$ and $\|\bar{u}_{1k}\|_\infty \leqslant C$ as $k \to \infty$, we derive from (61) and (78) that

$$|(-\Delta)^s \bar{u}_{1k}| = \left| \left(a_k \bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \bar{u}_{1k}|^2 \mathrm{d}x \right)^{-1} \times \left[-\bar{\varepsilon}_k^{4s} |\bar{\varepsilon}_k x + \bar{z}_{1k}|^2 \bar{u}_{1k} + \mu_{2k} \bar{\varepsilon}_k^{4s} \bar{u}_{1k} + \frac{b}{2} \bar{u}_{1k}^{\frac{8s}{N}+1} \right] \right| \leqslant C \quad \text{as } k \to \infty. \tag{113}$$

Combining (75) with (76) yields that

$$\mu_{ik} \bar{\varepsilon}_k^{4s} = e(a_k, \beta^*) \bar{\varepsilon}_k^{4s} + \frac{b}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{2s}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \bar{u}_{ik}|^2 \mathrm{d}x \right)^2 - \frac{4s\beta^*}{N+4s}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}} \int_{\mathbb{R}^N} \bar{u}_{ik}^{\frac{8s+2N}{N}} \mathrm{d}x \quad i = 1,2, \tag{114}$$

together with (77) implies that

$$\begin{aligned} &\bar{\varepsilon}_k^{4s} \bar{u}_{2k} \frac{\mu_{1k} - \mu_{2k}}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_\infty} \\ &= \frac{b}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{2s}} \int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} \bar{u}_{1k}|^2 + |(-\Delta)^{\frac{s}{2}} \bar{u}_{2k}|^2 \right) \mathrm{d}x \\ &\times \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} (\bar{u}_{1k} + \bar{u}_{2k}) (-\Delta)^{\frac{s}{2}} \bar{\eta}_k \mathrm{d}x \bar{u}_{2k} \\ &- \frac{4s\beta^*}{N}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}} \int_{\mathbb{R}^N} \left(\bar{u}_{1k}^{\frac{4s+N}{N}} + \bar{u}_{2k}^{\frac{4s+N}{N}} \right) [A\bar{u}_{1k} + (1-A)\bar{u}_{2k}]^{\frac{4s}{N}} \bar{\eta}_k \mathrm{d}x \bar{u}_{2k}. \end{aligned} \tag{115}$$

By the fact that $|\bar{\eta}_k| \leqslant 1$, $\|\bar{u}_{2k}\|_\infty \leqslant C$ and $\|\bar{u}_{ik}\|_{H^s(\mathbb{R}^N)} \leqslant C$ as $k \to \infty$ for $i = 1,2$, we conclude from (80) and (115) that

$$\left| \bar{\varepsilon}_k^{4s} \bar{u}_{2k} \frac{\mu_{1k} - \mu_{2k}}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_\infty} \right| \leqslant C \quad \text{as } k \to \infty. \tag{116}$$

We now rewrite problem (79) as

$$(-\Delta)^s \bar{\eta}_k = \bar{F}_k(x) \quad \text{in } \mathbb{R}^N,$$

where

$$\begin{aligned} \bar{F}_k(x) = &\left(a_k \bar{\varepsilon}_k^{2s} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \bar{u}_{2k}|^2 \mathrm{d}x \right)^{-1} \\ &\times \left[-b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} (\bar{u}_{1k} + \bar{u}_{2k}) (-\Delta)^{\frac{s}{2}} \bar{\eta}_k \mathrm{d}x (-\Delta)^s \bar{u}_{1k} \right. \\ &- \bar{\varepsilon}_k^{4s} |\bar{\varepsilon}_k x + \bar{z}_{1k}|^2 \bar{\eta}_k + \mu_{1k} \bar{\varepsilon}_k^{4s} \bar{\eta}_k + \frac{\bar{\varepsilon}_k^{4s} (\mu_{1k} - \mu_{2k}) \bar{u}_{2k}}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_\infty} \\ &\left. + \frac{(8s+N)b}{2N} [A\bar{u}_{1k} + (1-A)\bar{u}_{2k}]^{\frac{8s}{N}} \bar{\eta}_k \right]. \end{aligned}$$

By the fact that $|\bar{\eta}_k| \leqslant 1$, $\|\bar{u}_{ik}\|_\infty \leqslant C$ and $\|\bar{u}_{ik}\|_{H^s(\mathbb{R}^N)} \leqslant C$ as $k \to \infty$ for $i = 1,2$, we deduce from (112), (113) and (116) that $\bar{F}_k(x) \in L^\infty(\mathbb{R}^N)$ as $k \to \infty$. Thus, using [47, proposition 2.9], we know that $\|\bar{\eta}_k\|_{C^{1,\alpha}(\mathbb{R}^N)} \leqslant C$ for some $\alpha \in (0,1)$ as $k \to \infty$. Passing to a subsequence, there exists some function $\bar{\eta}_0 \in C^{1,\alpha}(\mathbb{R}^N)$ such that (110) holds.

Moreover, (55), (71) and (76) show that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \bar{u}_{ik}|^2 \mathrm{d}x \to \left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}} \quad \text{as } k \to \infty \text{ for } i = 1,2. \tag{117}$$

Taking $k \to \infty$ in (79), we then derive from (115) and (117) that

$$\begin{aligned}(-\Delta)^s \bar{\eta}_0 + \frac{4s-N}{2N}\bar{\eta}_0 - \frac{8s+N}{2N}\varphi^{\frac{8s}{N}}\bar{\eta}_0 = -2\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}\bar{\eta}_0 \mathrm{d}x (-\Delta)^s\varphi \\ +2\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}\bar{\eta}_0 \mathrm{d}x\varphi - \frac{8s\beta^*}{Nb}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}}\bar{\eta}_0 \mathrm{d}x\varphi,\end{aligned} \tag{118}$$

from which and lemma 6 we deduce that (111) holds. □

Until now, we have decomposed $\bar{\eta}_0$ and we need to verify that $d_0, \bar{d}_0, d_1, \cdots, d_N = 0$. Hereafter, we consider the relationships between the parameters.

Lemma 8. *Assume that $d_0, \bar{d}_0, d_1, \cdots, d_N$ are defined in (111), then*

$$\bar{d}_0 = \frac{2}{N} d_0.$$

Proof. We first claim that

$$d_1 \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \frac{\partial\varphi}{\partial x_1} (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x + \cdots + d_N \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \frac{\partial\varphi}{\partial x_N} (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x = 0. \tag{119}$$

Indeed, from (5) we have

$$\begin{aligned} d_i \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \frac{\partial\varphi}{\partial x_i} (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x + d_i \frac{4s-N}{2N} \int_{\mathbb{R}^N} \varphi \frac{\partial\varphi}{\partial x_i} \mathrm{d}x \\ - \frac{d_i}{2} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \frac{\partial\varphi}{\partial x_i} \mathrm{d}x = 0 \quad i = 1,2,\cdots,N. \end{aligned} \tag{120}$$

Applying lemma 6, we conclude that

$$\begin{aligned} d_1 \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \frac{\partial\varphi}{\partial x_1} (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x + \cdots + d_N \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \frac{\partial\varphi}{\partial x_N} (-\Delta)^{\frac{s}{2}} \varphi \mathrm{d}x \\ + d_1 \frac{4s-N}{2N} \int_{\mathbb{R}^N} \varphi \frac{\partial\varphi}{\partial x_1} \mathrm{d}x + \cdots + d_N \frac{4s-N}{2N} \int_{\mathbb{R}^N} \varphi \frac{\partial\varphi}{\partial x_N} \mathrm{d}x \\ - \frac{d_1(8s+N)}{2N} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \frac{\partial\varphi}{\partial x_1} \mathrm{d}x - \cdots - \frac{d_N(8s+N)}{2N} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \frac{\partial\varphi}{\partial x_N} \mathrm{d}x = 0. \end{aligned} \tag{121}$$

In view of (120) and (121), it follows that

$$d_1 \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \frac{\partial \varphi}{\partial x_1} \mathrm{d}x + \cdots + d_N \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \frac{\partial \varphi}{\partial x_N} \mathrm{d}x = 0. \tag{122}$$

Moreover, we know that $\int_{\mathbb{R}^N} \varphi \frac{\partial \varphi}{\partial x_i} \mathrm{d}x = 0$ for the season that φ is even in x_i while $\frac{\partial \varphi}{\partial x_i}$ is odd. Thus, from (121) and (122), we get that (119) holds. Following (111) and (119), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi (-\Delta)^{\frac{s}{2}} \bar{\eta}_0 \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi (-\Delta)^{\frac{s}{2}} \left(d_0 \varphi + \bar{d}_0 (x \cdot \nabla \varphi) + \sum_{i=1}^{N} d_i \frac{\partial \varphi}{\partial x_i} \right) \mathrm{d}x \\ &\quad = d_0 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi|^2 \mathrm{d}x + \bar{d}_0 \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi (-\Delta)^{\frac{s}{2}} (x \cdot \nabla \varphi) \mathrm{d}x. \end{aligned} \tag{123}$$

Taking advantage of lemma 6, we deduce from (6) that

$$\begin{aligned} &\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} \varphi (-\Delta)^{\frac{s}{2}} (x \cdot \nabla \varphi) \mathrm{d}x \\ &\quad = 2s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi|^2 \mathrm{d}x + \frac{N-4s}{2N} \int_{\mathbb{R}^N} \varphi (x \cdot \nabla \varphi) \mathrm{d}x + \frac{8s+N}{2N} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} (x \cdot \nabla \varphi) \mathrm{d}x \\ &\quad = 2s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi|^2 \mathrm{d}x + \frac{N-4s}{4N} \int_{\mathbb{R}^N} x \cdot \nabla \varphi^2 \mathrm{d}x + \frac{8s+N}{16s+4N} \int_{\mathbb{R}^N} x \cdot \nabla \varphi^{\frac{8s+2N}{N}} \mathrm{d}x \\ &\quad = 2s \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \varphi|^2 \mathrm{d}x - \frac{N-4s}{4} \int_{\mathbb{R}^N} \varphi^2 \mathrm{d}x + \frac{(8s+N)(-N)}{16s+4N} \int_{\mathbb{R}^N} \varphi^{\frac{8s+2N}{N}} \mathrm{d}x \\ &\quad = 2s \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}} + \frac{4s-N}{4} \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}} - \frac{(8s+N)(4s+N)}{16s+4N} \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}} \\ &\quad = \left(s - \frac{N}{2} \right) \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}}. \end{aligned} \tag{124}$$

Also, we get from (6) and (122) that

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \bar{\eta}_0 \mathrm{d}x &= \int_{\mathbb{R}^N} \varphi^{\frac{8s+N}{N}} \left(d_0 \varphi + \bar{d}_0 (x \cdot \nabla \varphi) + \sum_{i=1}^{N} d_i \frac{\partial \varphi}{\partial x_i} \right) \mathrm{d}x \\ &= \frac{N+4s}{N} d_0 \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}} - \frac{N^2 \bar{d}_0}{8s+2N} \int_{\mathbb{R}^N} \varphi^{\frac{8s+2N}{N}} \mathrm{d}x \\ &= \frac{N+4s}{N} d_0 \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}} - \frac{N \bar{d}_0}{2} \left(\frac{2\beta^*}{b} \right)^{\frac{N}{4s}}. \end{aligned} \tag{125}$$

Hence, we derive from lemma 6, (118), (123), (124) and (125) that

$$\begin{aligned}
\Gamma(\bar{\eta}_0) =& -2\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\left[d_0\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}+\left(s-\frac{N}{2}\right)\bar{d}_0\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\right](-\Delta)^s\varphi \\
&+2\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}}\left[d_0\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}+\left(s-\frac{N}{2}\right)\bar{d}_0\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\right]\varphi \\
&-\frac{8s\beta^*}{Nb}\left(\frac{2\beta^*}{b}\right)^{-1-\frac{N}{4s}}\left[\frac{N+4s}{N}\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}d_0-\frac{N}{2}\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\bar{d}_0\right]\varphi \\
=&\Gamma\left(d_0\varphi+\bar{d}_0(x\cdot\nabla\varphi)+\sum_{i=1}^N d_i\frac{\partial\varphi}{\partial x_i}\right) \\
=&d_0\Gamma(\varphi)+\bar{d}_0\Gamma(x\cdot\nabla\varphi) \\
=&\frac{4s(N-4s)}{N^2}d_0\varphi-\frac{8s}{N}d_0(-\Delta)^s\varphi+2s\bar{d}_0(-\Delta)^s\varphi,
\end{aligned}$$

which implies that

$$\bar{d}_0=\frac{2}{N}d_0,$$

this completes the proof of lemma 8. □

From [24], we give a technical result as follows.

Lemma 9. *Let $\rho>\theta>0$ be two constants. Suppose $(y-x)^2+t^2\geqslant\rho^2$, $t>0$ and $\alpha>N$. Then, when $\beta>N$, it holds that*

$$\int_{\mathbb{R}^N\setminus B_\theta(y-x)}\frac{1}{(t+|z|)^\alpha|y-z-x|^\beta}\leqslant C\left[\frac{1}{(1+|y-x|)^\beta}\frac{1}{t^{\alpha-N}}+\frac{1}{(1+|y-x|)^\beta}\frac{1}{\theta^{\beta-N}}\right],$$

where $C>0$ is a constant independent of θ. Moreover, for some $\epsilon\to 0$, we have

$$\int_{\mathbb{R}^N}\frac{1}{(t+|z|)^\alpha\left(1+\frac{|y-z-x|}{\epsilon}\right)^\beta}\leqslant C\epsilon^N\left[\frac{1}{(1+|y-x|)^\beta}\frac{1}{t^{\alpha-N}}+\frac{1}{(1+|y-x|)^\alpha}\right].$$

From above results, we now have the following result about the parameter d_i, $i=1,2,\cdots,N$.

Lemma 10. *Assume that d_i are defined in (111), where $i=1,2,\cdots,N$, it holds*

$$d_1=d_2=\cdots=d_N=0.$$

Proof. We first write the extension of $\hat{u}_{ik}$, it has

$$\begin{cases}
\operatorname{div}\left(t^{1-2s}\nabla\widetilde{\hat{u}_{ik}}\right)=0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
-\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\lim_{t\to 0}\partial_t\widetilde{\hat{u}_{ik}}(x,t) \\
=\quad -\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}+\mu_{ik}\bar{\varepsilon}_k^{4s}\hat{u}_{ik}+\frac{b}{2}\hat{u}_{ik}^{\frac{8s}{N}+1} \quad \text{in } \mathbb{R}^N,
\end{cases} \tag{126}$$

where $i = 1,2$. For writing convenience, we give the following definitions

$$
\begin{aligned}
&B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) := \left\{x \in \mathbb{R}^N : |x - \bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}| \leqslant \delta\right\} \subseteq \mathbb{R}^N,\\
&\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) := \left\{X = (x,t) : |X - \left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}, 0\right)| \leqslant \delta, t > 0\right\} \subseteq \mathbb{R}_+^{N+1},\\
&\partial'\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) := \left\{X = (x,t) : |X - \left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}, 0\right)| \leqslant \delta, t = 0\right\} \subseteq \mathbb{R}^N,\\
&\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) := \left\{X = (x,t) : |X - \left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}, 0\right)| = \delta, t > 0\right\} \subseteq \mathbb{R}_+^{N+1},\\
&\partial\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) := \partial'\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right) \cup \partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right),
\end{aligned}
$$

where $\delta > 0$ small is given by (100) and $\tau < 0$ is given by (96). Multiplying (126) by $\frac{\partial \widetilde{\hat{u}_{ik}}}{\partial x_j}$, where $i = 1,2$ and $j = 1,2,\cdots,N$, and integrating over $B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})$, we see that

$$
\begin{aligned}
&\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\quad\times\left[-\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial x_j} + \frac{1}{2}\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}|\nabla\widetilde{\hat{u}_{ik}}|^2\nu_j\right]\\
&\quad+\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}\frac{\partial\hat{u}_{ik}}{\partial x_j}\mathrm{d}x\\
&\quad=\frac{\mu_{ik}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^2\nu_j\mathrm{d}S + \frac{Nb}{2(8s+2N)}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^{\frac{8s+2N}{N}}\nu_j\mathrm{d}S.
\end{aligned}
\tag{127}
$$

Noticing that

$$
\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}\frac{\partial\hat{u}_{ik}}{\partial x_j}\mathrm{d}x = \frac{1}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}^2\nu_j\mathrm{d}S - \frac{1}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_j}\hat{u}_{ik}^2\mathrm{d}x,
\tag{128}
$$

from which and (127), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_j}\hat{u}_{ik}^2\mathrm{d}x\\
&\quad=\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\qquad\times\left[-\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial x_j} + \frac{1}{2}\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}|\nabla\widetilde{\hat{u}_{ik}}|^2\nu_j\right]\\
&\qquad+\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}^2\nu_j\mathrm{d}S - \frac{\mu_{ik}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^2\nu_j\mathrm{d}S\\
&\qquad-\frac{Nb}{2(8s+2N)}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^{\frac{8s+2N}{N}}\nu_j\mathrm{d}S.
\end{aligned}
\tag{129}
$$

Applying (76), (94) and lemma 5, we can obtain that

$$
\begin{aligned}
|\hat{u}_{ik}(x)|+|\nabla\hat{u}_{ik}(x)| &= |\bar{u}_{ik}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)|+\bar{\varepsilon}_k^{\tau-1}|\nabla\bar{u}_{ik}\left(\bar{\varepsilon}^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)| \\
&\leqslant \frac{C}{1+|\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}|^{N+2s}}+\frac{C\bar{\varepsilon}_k^{\tau-1}}{1+|\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}|^{N+2s}} \\
&\leqslant \frac{C\bar{\varepsilon}_k^{\tau-1}}{\left(1+|\frac{x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}}{\bar{\varepsilon}_k^{1-\tau}}|^2\right)^{\frac{N+2s}{2}}} \quad \text{as } k\to\infty \text{ and } i=1,2.
\end{aligned}
\tag{130}
$$

Following (12), (13), (130) and lemma 9, we derive that for $|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|+t^2=\delta^2$ and $t>0$, it has

$$
\begin{aligned}
|\widetilde{\hat{u}_{ik}}(x,t)| &\leqslant C\bar{\varepsilon}_k^{\tau-1}\int_{\mathbb{R}^N}\frac{t^{2s}}{(|x-\xi|+t)^{N+2s}}\frac{1}{\left(1+|\frac{\xi-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}}{\bar{\varepsilon}_k^{1-\tau}}|\right)^{N+2s}}\mathrm{d}\xi \\
&\leqslant \frac{C\bar{\varepsilon}_k^{(1-\tau)(N-1)}}{\left(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|\right)^{N+2s}} \quad \text{as } k\to\infty \text{ and } i=1,2.
\end{aligned}
\tag{131}
$$

Moreover, from (12), (13), (130), lemmas 5 and 9, we also obtain for $|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|+t^2=\delta^2$ and $t>0$

$$
\begin{aligned}
\left|\frac{\partial}{\partial x_j}\widetilde{\hat{u}_{ik}}(x,t)\right| &= \left|\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\frac{\partial}{\partial x_j}\hat{u}_{ik}(x-tz)\,\mathrm{d}z\right| \\
&\leqslant C\bar{\varepsilon}_k^{\tau-1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}\frac{1}{\left(1+|\frac{x-tz-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}}{\bar{\varepsilon}_k^{1-\tau}}|^2\right)^{\frac{N+2s}{2}}}\mathrm{d}z \\
&\leqslant C\bar{\varepsilon}_k^{\tau-1}\int_{\mathbb{R}^N}\frac{t^{2s}}{(t+|x-\xi|)^{N+2s}}\frac{1}{\left(1+|\frac{\xi-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}}{\bar{\varepsilon}_k^{1-\tau}}|\right)^{N+2s}}\mathrm{d}\xi \\
&\leqslant \frac{C\bar{\varepsilon}_k^{(1-\tau)(N-1)}}{\left(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|\right)^{N+2s}} \quad \text{as } k\to\infty,\ i=1,2 \text{ and } j=1,2,\cdots,N.
\end{aligned}
\tag{132}
$$

Under the same assumptions, we can also get

$$
\left|\frac{\partial}{\partial\nu}\widetilde{\hat{u}_{ik}}(x,t)\right|\leqslant\frac{C\bar{\varepsilon}_k^{(1-\tau)(N-1)}}{\left(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|\right)^{N+2s}} \quad \text{as } k\to\infty \text{ and } i=1,2. \tag{133}
$$

Clearly we have

$$\begin{aligned}|\hat{\xi}_k(x)|+|\nabla\hat{\xi}_k(x)| &= \left|\frac{\hat{u}_{1k}(x)-\hat{u}_{2k}(x)}{\left(\frac{2\beta^*}{b}\right)^{\frac{N}{8s}}\bar{\varepsilon}_k^{\frac{N}{2}}\|u_{1k}(\bar{\varepsilon}_k^{\tau}x)-u_{2k}(\bar{\varepsilon}_k^{\tau}x)\|_\infty}\right| \\ &\quad+\left|\frac{\nabla\hat{u}_{1k}(x)-\nabla\hat{u}_{2k}(x)}{\left(\frac{2\beta^*}{b}\right)^{\frac{N}{8s}}\bar{\varepsilon}_k^{\frac{N}{2}}\|u_{1k}(\bar{\varepsilon}_k^{\tau}x)-u_{2k}(\bar{\varepsilon}_k^{\tau}x)\|_\infty}\right| \\ &\leqslant \frac{C\bar{\varepsilon}_k^{\tau-1-\frac{N}{2}}}{\left(1+|\frac{x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}}{\bar{\varepsilon}_k^{1-\tau}}|^2\right)^{\frac{N+2s}{2}}}\quad \text{as } k\to\infty.\end{aligned}$$

We then deduce from (131)–(133) that for $|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|+t^2=\delta^2$ and $t>0$

$$|\widetilde{\eta}_k(x,t)|,\left|\frac{\partial}{\partial x_j}\widetilde{\eta}_k(x,t)\right|,\left|\frac{\partial}{\partial\nu}\widetilde{\eta}_k(x,t)\right|\leqslant\frac{C\bar{\varepsilon}_k^{(N-1)(1-\tau)-\frac{N}{2}}}{\left(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|\right)^{N+2s}}\quad\text{as } k\to\infty \text{ and } j=1,2,\cdots,N. \tag{134}$$

Following (129), we have

$$\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_j}(\hat{u}_{1k}+\hat{u}_{2k})\hat{\eta}_k\mathrm{d}x:=B_1+B_2+B_3+B_4+B_5+B_6, \tag{135}$$

where

$$\begin{aligned}B_1 &:= \frac{\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2(\hat{u}_{1k}+\hat{u}_{2k})\hat{\eta}_k\nu_j\mathrm{d}S,\\ B_2 &:= -\frac{\mu_{1k}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}(\hat{u}_{1k}+\hat{u}_{2k})\hat{\eta}_k\nu_j\mathrm{d}S-\frac{\bar{\varepsilon}_k^{4s}(\mu_{1k}-\mu_{2k})}{2\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{2k}^2\nu_j\mathrm{d}S,\\ B_3 &:= -\frac{b}{4}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(\hat{u}_{1k}^{\frac{4s+N}{N}}+\hat{u}_{2k}^{\frac{4s+N}{N}}\right)[A\hat{u}_{1k}+(1-A)\hat{u}_{2k}]^{\frac{4s}{N}}\hat{\eta}_k\nu_j\mathrm{d}S,\\ B_4 &:= -\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2\mathrm{d}x\right]\\ &\quad\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\left(\frac{\partial\widetilde{\eta}_k}{\partial x_j}\frac{\partial\widetilde{\hat{u}_{1k}}}{\partial\nu}+\frac{\partial\widetilde{\eta}_k}{\partial\nu}\frac{\partial\widetilde{\hat{u}_{2k}}}{\partial x_j}\right),\\ B_5 &:= \frac{1}{2}\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2\mathrm{d}x\right]\\ &\quad\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\nabla\left(\widetilde{\hat{u}_{1k}}+\widetilde{\hat{u}_{2k}}\right)\nabla\widetilde{\eta}_k\nu_j,\end{aligned}$$

$$B_6 := b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}(\hat{u}_{1k}+\hat{u}_{2k})(-\Delta)^{\frac{s}{2}}\hat{\eta}_k \mathrm{d}x$$
$$\times \left[\frac{1}{2}\int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s}|\nabla\widetilde{\hat{u}_{2k}}|^2\nu_j - \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s}\frac{\partial\widetilde{\hat{u}_{2k}}}{\partial\nu}\frac{\partial\widetilde{\hat{u}_{2k}}}{\partial x_j}\right].$$

Let us estimate each of term the above one by one. By the fact $\bar{z}_{1k}\to 0$ as $k\to\infty$, it follows from lemma 5 and (100) that

$$\begin{aligned}
|B_1| &\leqslant C\bar{\varepsilon}_k^{4s}\left(\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}|\bar{\varepsilon}_k^{\tau}x|^2\hat{\eta}_k \mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}|\bar{\varepsilon}_k^{\tau}x|^2|\hat{u}_{1k}+\hat{u}_{2k}|^2 \mathrm{d}S\right)^{\frac{1}{2}}\\
&\leqslant C\bar{\varepsilon}_k^{4s+\frac{(1-\tau)N-4s}{2}}\left(\int_{\partial B_\delta(0)}|\bar{\varepsilon}_k^{\tau}x+\bar{z}_{1k}|^2|\hat{u}_{1k}\left(x+\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)+\hat{u}_{2k}\left(x+\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)|^2 \mathrm{d}S\right)^{\frac{1}{2}}\\
&\leqslant C\bar{\varepsilon}_k^{4s+\frac{(1-\tau)N-4s}{2}}\left(\int_{\partial B_\delta(0)}|\bar{\varepsilon}_k^{\tau}x+\bar{z}_{1k}|^2|\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)+\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)|^2 \mathrm{d}S\right)^{\frac{1}{2}}\\
&\leqslant C\bar{\varepsilon}_k^{4s+\frac{(1-\tau)N-4s}{2}+\tau+(1-\tau)(N+2s)} \quad \text{as } k\to\infty.
\end{aligned} \tag{136}$$

Using the fact $\mu_{ik}\bar{\varepsilon}_k^{-4s}\to\frac{(N-4s)b}{2N}$ as $k\to\infty$, we deduce from lemma 5 and (100) that

$$\begin{aligned}
|B_2| &\leqslant C\left(\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}|\hat{u}_{1k}+\hat{u}_{2k}|^2 \mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}\hat{\eta}_k^2 \mathrm{d}S\right)^{\frac{1}{2}} + C\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}|\hat{u}_{2k}|^2 \mathrm{d}S\\
&\leqslant C\left(\int_{\partial B_\delta(0)}|\hat{u}_{1k}\left(x+\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)+\hat{u}_{2k}\left(x+\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)|^2 \mathrm{d}S\right)^{\frac{1}{2}}\left(\int_{\partial B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})}\hat{\eta}_k^2 \mathrm{d}S\right)^{\frac{1}{2}}\\
&\quad + C\int_{\partial B_\delta(0)}|\hat{u}_{2k}\left(x+\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)|^2 \mathrm{d}S\\
&\leqslant C\bar{\varepsilon}_k^{\frac{(1-\tau)N}{2}}\left(\int_{\partial B_\delta(0)}|\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)+\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)|^2 \mathrm{d}S\right)^{\frac{1}{2}} + C\int_{\partial B_\delta(0)}|\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)|^2 \mathrm{d}S\\
&\leqslant C\bar{\varepsilon}_k^{\frac{(1-\tau)N}{2}+(1-\tau)(N+2s)} \quad \text{as } k\to\infty.
\end{aligned} \tag{137}$$

Similarly, we also have

$$\begin{aligned}
|B_3| &\leqslant C\int_{\partial B_\delta(0)}\left|\bar{u}_{1k}^{\frac{4s+N}{N}}\left(\bar{\varepsilon}_k^{\tau-1}x\right)+\bar{u}_{2k}^{\frac{4s+N}{N}}\left(\bar{\varepsilon}_k^{\tau-1}x\right)\right||A\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)+(1-A)\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x\right)|^{\frac{4s}{N}} \mathrm{d}S\\
&\leqslant C\bar{\varepsilon}_k^{(1-\tau)(N+2s)\frac{8s+N}{N}} \quad \text{as } k\to\infty.
\end{aligned} \tag{138}$$

Noticing that $\|\bar{u}_{ik}\|_{H^s(\mathbb{R}^N)} \leqslant C$ as $k \to \infty$ for $i = 1,2$, it then follows from (132)–(134) that

$$\begin{aligned} |B_4| &\leqslant C\bar{\varepsilon}_k^{(1-\tau)2s} \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \left(\frac{\partial\widetilde{\eta}_k}{\partial x_j} \frac{\partial\widetilde{\hat{u}_{1k}}}{\partial\nu} + \frac{\partial\widetilde{\eta}_k}{\partial\nu} \frac{\partial\widetilde{\hat{u}_{2k}}}{\partial x_j} \right) \\ &\leqslant C\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}} \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \left(\frac{1}{(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|)^{N+2s}} \right)^2 \\ &\leqslant C\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}} \quad \text{as } k \to \infty. \end{aligned} \tag{139}$$

Similarly

$$|B_5| \leqslant C\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}} \quad \text{as } k \to \infty. \tag{140}$$

Applying the Hölder inequality, we derive from (132)–(134) that

$$\begin{aligned} |B_6| \leqslant\ & C\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} (\bar{u}_{1k}+\bar{u}_{2k})|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \bar{\eta}_k|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\ &\times \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \left(\frac{1}{(1+|x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}|)^{N+2s}} \right)^2 \\ \leqslant\ & C\|\bar{u}_{1k}+\bar{u}_{2k}\|_{H^s(\mathbb{R}^N)} \|\bar{\eta}_k\|_{H^s(\mathbb{R}^N)} \bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)} \leqslant C\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)} \quad \text{as } k \to \infty. \end{aligned} \tag{141}$$

For simplicity, we define

$$\begin{cases} q_1 := (1-\tau)\left(\frac{3N}{2}+2s\right)+\tau+2s, \\ q_2 := (1-\tau)\left(\frac{3N}{2}+2s\right), \\ q_3 := (1-\tau)(N+2s)\frac{8s+N}{N}, \\ q_4 := (1-\tau)(2s+2N-2)-\frac{N}{2}, \\ q_5 := (1-\tau)(2s+2N-2). \end{cases}$$

By (96), we can see that $q_1, q_2, q_3, q_4, q_5 > 0$, $0 < q_1 < q_2 < q_3$, $0 < q_4 < q_5$ and $0 < q_4 < q_1$. Then by the fact that $\frac{\bar{z}_{1k}}{\bar{\varepsilon}_k} \to 0$ and $\bar{\eta}_k \to \bar{\eta}_0$ in $C^1(\mathbb{R}^N)$ as $k \to \infty$, we can get from lemma 5, (7) and (135)–(141) that

$$\begin{aligned} &O\left(\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}}\right) \\ &= \frac{\bar{\varepsilon}_k^{4s}}{2} \int_{B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} \frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_j} (\hat{u}_{1k}+\hat{u}_{2k})\hat{\eta}_k \mathrm{d}x \\ &= \frac{\bar{\varepsilon}_k^{4s}}{2} \int_{B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} \frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_j} \left[\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right) + \bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right) \right] \bar{\eta}_k\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right) \mathrm{d}x \\ &= \frac{\bar{\varepsilon}_k^{4s+(1-\tau)N+\tau-1}}{2} \int_{B_{\bar{\varepsilon}_k^{\tau-1}\delta}(0)} \frac{\partial|\bar{\varepsilon}_k x+\bar{z}_{1k}|^2}{\partial x_j} (\bar{u}_{1k}+\bar{u}_{2k})\bar{\eta}_k \mathrm{d}x \\ &= \bar{\varepsilon}_k^{4s+(1-\tau)N+\tau+1} \int_{B_{\bar{\varepsilon}_k^{\tau-1}\delta}(0)} \left| x_j + \frac{\bar{z}_{1k}^j}{\bar{\varepsilon}_k} \right| (\bar{u}_{1k}+\bar{u}_{2k})\bar{\eta}_k \mathrm{d}x \\ &= O\left(\bar{\varepsilon}_k^{4s+(1-\tau)N+\tau+1}\right)(1+o(1)) \int_{\mathbb{R}^N} 2x_j\varphi\bar{\eta}_0 \mathrm{d}x, \end{aligned} \tag{142}$$

where $j = 1, 2, \cdots, N$ and $\bar{z}_{1k}^j = (\bar{z}_{1k}^1, \bar{z}_{1k}^2, \cdots, \bar{z}_{1k}^N) \in \mathbb{R}^N$. Applying (96) and the fact that $\frac{N}{4} < s < 1$, we can get that $(1-\tau)(2s+2N-2) - \frac{N}{2} > (1-\tau)N + 4s + 2 > 4s + (1-\tau)N + \tau + 1 > 0$. We thus obtain from (142), lemmas 7 and 8 that

$$0 = 2\int_{\mathbb{R}^N} x_j \varphi \left[d_0 \varphi + \bar{d}_0 (x \cdot \nabla \varphi) + \sum_{i=1}^{N} d_i \frac{\partial \varphi}{\partial x_i} \right] \mathrm{d}x = -d_j \int_{\mathbb{R}^N} \varphi^2 \mathrm{d}x \quad j = 1, 2, \cdots, N,$$

which shows that $d_j = 0$ for $j = 1, 2, \cdots, N$. □

Lemma 11. *Assume that d_0 and $\bar{d}_0$ are defined in (111), it holds $d_0 = \bar{d}_0 = 0$.*

Proof. Multiplying (126) by $[X - (\bar{\varepsilon}_k^{-\tau} \bar{z}_{1k}, 0)] \cdot \nabla \widetilde{\hat{u}_{ik}}$, where $i = 1, 2$, and integrating over $B_\delta(\bar{\varepsilon}_k^{-\tau} \bar{z}_{1k})$, other assumptions are same as before, then we have

$$\begin{aligned}
&-\left[a_k \bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_{ik}|^2 \mathrm{d}x \right] \\
&\times \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \frac{\partial \widetilde{\hat{u}_{ik}}}{\partial \nu} \left[X - \left(\bar{\varepsilon}_k^{-\tau} \bar{z}_{1k}, 0\right) \right] \cdot \nabla \widetilde{\hat{u}_{ik}} \\
&+ \frac{1}{2} \left[a_k \bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_{ik}|^2 \mathrm{d}x \right] \\
&\times \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} |\nabla \widetilde{\hat{u}_{ik}}|^2 \left[X - \left(\bar{\varepsilon}_k^{-\tau} \bar{z}_{1k}, 0\right) \right] \cdot \nu \\
&+ \frac{2s-N}{2} \left[a_k \bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_{ik}|^2 \mathrm{d}x \right] \\
&\times \int_{\partial\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \frac{\partial \widetilde{\hat{u}_{ik}}}{\partial \nu} \widetilde{\hat{u}_{ik}} \\
&\quad = \int_{B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} \left[-\bar{\varepsilon}_k^{4s} |\bar{\varepsilon}_k^{\tau} x|^2 \hat{u}_{ik} + \mu_{ik} \bar{\varepsilon}_k^{4s} \hat{u}_{ik} + \frac{b}{2} \hat{u}_{ik}^{\frac{8s}{N}+1} \right] \left(x - \bar{\varepsilon}_k^{-\tau} \bar{z}_{1k} \right) \cdot \nabla \hat{u}_{ik} \mathrm{d}x.
\end{aligned} \tag{143}$$

By direct calculations, we have

$$\begin{aligned}
&\frac{2s-N}{2} \left[a_k \bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_{ik}|^2 \mathrm{d}x \right] \\
&\quad \times \int_{\partial\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \frac{\partial \widetilde{\hat{u}_{ik}}}{\partial \nu} \widetilde{\hat{u}_{ik}} \\
&\qquad = \frac{2s-N}{2} \left[a_k \bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}_{ik}|^2 \mathrm{d}x \right] \\
&\qquad \times \int_{\partial''\mathbf{B}_\delta^+(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} t^{1-2s} \frac{\partial \widetilde{\hat{u}_{ik}}}{\partial \nu} \widetilde{\hat{u}_{ik}} \\
&\qquad + \frac{2s-N}{2} \int_{B_\delta(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})} \left[-\bar{\varepsilon}_k^{4s} |\bar{\varepsilon}_k^{\tau} x|^2 \hat{u}_{ik} + \mu_{ik} \bar{\varepsilon}_k^{4s} \hat{u}_{ik} + \frac{b}{2} \hat{u}_{ik}^{\frac{8s}{N}+1} \right] \hat{u}_{ik} \mathrm{d}x.
\end{aligned} \tag{144}$$

Integrating by parts, we see that

$$\begin{aligned}
&\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left[-\bar{\varepsilon}_k^{4s}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}+\mu_{ik}\bar{\varepsilon}_k^{4s}\hat{u}_{ik}+\frac{b}{2}\hat{u}_{ik}^{\frac{8s}{N}+1}\right]\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}\mathrm{d}x\\
&\quad=-\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^2\mathrm{d}x\\
&\qquad+\frac{\mu_{ik}\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^2\mathrm{d}x\\
&\qquad+\frac{bN}{2\left(8s+2N\right)}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x, \qquad (145)
\end{aligned}$$

$$\begin{aligned}
&\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^2\mathrm{d}x\\
&\quad=\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S\\
&\qquad-\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left[\nabla|\bar{\varepsilon}_k^{\tau}x|^2\cdot\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)+N|\bar{\varepsilon}_k^{\tau}x|^2\right]\hat{u}_{ik}^2\mathrm{d}x, \qquad (146)
\end{aligned}$$

$$\begin{aligned}
&\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^2\mathrm{d}x\\
&\quad=\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S-N\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^2\mathrm{d}x, \qquad (147)
\end{aligned}$$

$$\begin{aligned}
&\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nabla\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x\\
&\quad=\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}S-N\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x. \qquad (148)
\end{aligned}$$

Following (143)–(148), we derive that

$$\begin{aligned}
&-\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\quad\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\left[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)\right]\cdot\nabla\widetilde{\hat{u}_{ik}}\\
&\quad+\frac{1}{2}\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\quad\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}|\nabla\widetilde{\hat{u}_{ik}}|^2\left[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)\right]\cdot\nu\\
&\quad+\frac{2s-N}{2}\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\quad\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\widetilde{\hat{u}_{ik}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} |\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S\\
&\quad+\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\nabla|\bar{\varepsilon}_k^{\tau}x|^2\cdot\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\hat{u}_{ik}^2\mathrm{d}x-\frac{b(8s^2-2sN)}{16s+4N}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x\\
&\quad+s\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}^2\mathrm{d}x+\frac{\mu_{ik}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S\\
&\quad-s\mu_{ik}\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^2\mathrm{d}x+\frac{bN}{2(8s+2N)}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}S. \qquad (149)
\end{aligned}
$$

Moreover, we deduce from (114) that

$$
\begin{aligned}
\mu_{ik}\bar{\varepsilon}_k^{4s}\int_{\mathbb{R}^N}\hat{u}_{ik}^2\mathrm{d}x &= \left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(1-\tau)N+4s}e\left(a_k,\beta^*\right)\\
&\quad+\frac{b}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right)^2-\frac{2sb}{N+4s}\int_{\mathbb{R}^N}\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x, \qquad (150)
\end{aligned}
$$

from which and (149), we conclude that

$$
\begin{aligned}
&-\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)]\cdot\nabla\widetilde{\hat{u}_{ik}}\\
&+\frac{1}{2}\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\Bigg]\\
&\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}|\nabla\widetilde{\hat{u}_{ik}}|^2[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)]\cdot\nu\\
&+\frac{2s-N}{2}\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)}+b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right]\\
&\times\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\widetilde{\hat{u}_{ik}}+\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S\\
&-\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\nabla|\bar{\varepsilon}_k^{\tau}x|^2\cdot\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\hat{u}_{ik}^2\mathrm{d}x-s\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\hat{u}_{ik}^2\mathrm{d}x\\
&=\frac{\mu_{ik}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^2\mathrm{d}S+s\mu_{ik}\bar{\varepsilon}_k^{4s}\int_{\mathbb{R}^N\setminus B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^2\mathrm{d}x\\
&\quad+\frac{bN}{2(8s+2N)}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}S\\
&\quad+\frac{b(8s^2-2sN)}{16s+4N}\int_{\mathbb{R}^N\setminus B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x-s\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}e(a_k,\beta^*)\\
&\quad-\frac{sb}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\left(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{ik}|^2\mathrm{d}x\right)^2+\frac{Nsb}{8s+2N}\int_{\mathbb{R}^N}\hat{u}_{ik}^{\frac{8s+2N}{N}}\mathrm{d}x.
\end{aligned}
$$

This shows there exists $A \in (0,1)$ such that

$$
\begin{aligned}
&\frac{sb}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}} \bar{\varepsilon}_k^{(\tau-1)(N-4s)} \int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2 + |(-\Delta)^{\frac{s}{2}}\hat{u}_{2k}|^2\right)\mathrm{d}x \\
&\quad\times \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\hat{u}_{1k}+\hat{u}_{2k})(-\Delta)^{\frac{s}{2}}\hat{\eta}_k \mathrm{d}x \\
&\quad - \frac{sb}{2}\int_{\mathbb{R}^N}\left(\hat{u}_{1k}^{\frac{4s+N}{N}} + \hat{u}_{2k}^{\frac{4s+N}{N}}\right)[A\hat{u}_{1k} + (1-A)\hat{u}_{2k}]^{\frac{4s}{N}}\hat{\eta}_k \mathrm{d}x \\
&\quad := D_1 + D_2 + D_3 + D_4,
\end{aligned}
\tag{151}
$$

where

$$
\begin{aligned}
D_1 :=& \frac{\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{2k}|^2\mathrm{d}x\right]F_2}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty} \\
&- \frac{\left[a_k\bar{\varepsilon}_k^{2s(2-\tau)} + b\left(\frac{b}{2\beta^*}\right)^{\frac{N}{4s}}\bar{\varepsilon}_k^{(\tau-1)(N-4s)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2\mathrm{d}x\right]F_1}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty} \\
&- \frac{\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}S \\
&+ \frac{b\left(8s^2-2sN\right)}{4N}\int_{\mathbb{R}^N\setminus B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(\hat{u}_{1k}^{\frac{4s+N}{N}}+\hat{u}_{2k}^{\frac{4s+N}{N}}\right)[A\hat{u}_{1k}+(1-A)\hat{u}_{2k}]^{\frac{4s}{N}}\hat{\eta}_k\mathrm{d}x \\
&+ \frac{\mu_{1k}\bar{\varepsilon}_k^{4s}}{2}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}S \\
&+ \frac{\bar{\varepsilon}_k^{4s}\left(\mu_{1k}-\mu_{2k}\right)}{2\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\cdot\nu\hat{u}_{2k}^2\mathrm{d}S \\
&+ s\mu_{1k}\bar{\varepsilon}_k^{4s}\int_{\mathbb{R}^N\setminus B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}(\hat{u}_{2k}+\hat{u}_{1k})\hat{\eta}_k\mathrm{d}x + \frac{s\bar{\varepsilon}_k^{4s}(\mu_{1k}-\mu_{2k})}{\|\hat{u}_{1k}-\hat{u}_{2k}\|_\infty}\int_{\mathbb{R}^N\setminus B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\hat{u}_{2k}^2\mathrm{d}x \\
&+ \frac{b}{4}\int_{\partial B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}(x-\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k})\cdot\nu\left(\hat{u}_{1k}^{\frac{4s+N}{N}}+\hat{u}_{2k}^{\frac{4s+N}{N}}\right)[A\hat{u}_{1k}+(1-A)\hat{u}_{2k}]^{\frac{4s}{N}}\hat{\eta}_k\mathrm{d}S,
\end{aligned}
$$

$$
\begin{aligned}
F_i :=& -\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\left[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)\right]\cdot\nabla\widetilde{\hat{u}_{ik}} \\
&+ \frac{1}{2}\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}|\nabla\widetilde{\hat{u}_{ik}}|^2\left[X-\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k},0\right)\right]\cdot\nu + \frac{2s-N}{2}\int_{\partial''\mathbf{B}_\delta^+\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} t^{1-2s}\frac{\partial\widetilde{\hat{u}_{ik}}}{\partial\nu}\widetilde{\hat{u}_{ik}},
\end{aligned}
$$

$$
D_2 := \frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left[\nabla|\bar{\varepsilon}_k^{\tau}x|^2\cdot x\right]\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}x,
$$

$$
D_3 := -\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\left[\nabla|\bar{\varepsilon}_k^{\tau}x|^2\cdot\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\right]\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}x,
$$

$$
D_4 := s\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}x.
$$

The same arguments of $|B_1| - |B_6|$ give that

$$D_1 = O\left(\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}}\right) \quad \text{as } k \to \infty. \tag{152}$$

By the fact $\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k} \to 0$ as $k \to \infty$, we conclude from (142) that

$$\begin{aligned}
&\frac{\bar{\varepsilon}_k^{4s}}{2}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)} \left[\nabla|\bar{\varepsilon}_k^{\tau}x|^2 \cdot \left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)\right]\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k \mathrm{d}x \\
&\quad = \frac{\bar{\varepsilon}_k^{4s}}{2}\sum_{i=1}^{N}\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}^{(i)}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}\frac{\partial|\bar{\varepsilon}_k^{\tau}x|^2}{\partial x_i}\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}x \\
&\quad = O\left(\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}}\right) \quad \text{as } k\to\infty,
\end{aligned}$$

which shows that

$$D_3 = O\left(\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}}\right)) \quad \text{as } k \to \infty. \tag{153}$$

By the fact that $\nabla|\bar{\varepsilon}_k^{\tau}x|^2 \cdot x = 2|\bar{\varepsilon}_k^{\tau}x|^2$ and $\frac{\bar{z}_{1k}}{\bar{\varepsilon}_k} \to 0$ as $k \to \infty$, we deduce from lemmas 5 and 7 that

$$\begin{aligned}
D_2 =& \bar{\varepsilon}_k^{4s+2\tau}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|x|^2\left(\hat{u}_{2k}+\hat{u}_{1k}\right)\hat{\eta}_k\mathrm{d}x \\
=& \bar{\varepsilon}_k^{4s+2\tau}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|x|^2\left(\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)+\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)\right)\bar{\eta}_k\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)\mathrm{d}x \\
=& \bar{\varepsilon}_k^{4s+2+(1-\tau)N}\int_{B_{\bar{\varepsilon}_k^{\tau-1}\delta}(0)}\left|x+\frac{\bar{z}_{1k}}{\bar{\varepsilon}_k}\right|^2\left(\bar{u}_{1k}+\bar{u}_{2k}\right)\bar{\eta}_k\mathrm{d}x \\
=& 2\left(1+o\left(1\right)\right)\bar{\varepsilon}_k^{4s+2+(1-\tau)N}\int_{\mathbb{R}^N}|x|^2\varphi\,\bar{\eta}_0\mathrm{d}x
\end{aligned} \tag{154}$$

and

$$\begin{aligned}
D_4 =& s\bar{\varepsilon}_k^{4s}\int_{B_\delta\left(\bar{\varepsilon}_k^{-\tau}\bar{z}_{1k}\right)}|\bar{\varepsilon}_k^{\tau}x|^2\left(\bar{u}_{1k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)+\bar{u}_{2k}\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)\right)\bar{\eta}_k\left(\bar{\varepsilon}_k^{\tau-1}x-\bar{\varepsilon}_k^{-1}\bar{z}_{1k}\right)\mathrm{d}x \\
=& s\bar{\varepsilon}_k^{4s+2+(1-\tau)N}\int_{B_{\bar{\varepsilon}_k^{\tau-1}\delta}(0)}\left|x+\frac{\bar{z}_{1k}}{\bar{\varepsilon}_k}\right|^2\left(\bar{u}_{1k}+\bar{u}_{2k}\right)\bar{\eta}_k\mathrm{d}x \\
=& 2s\left(1+o\left(1\right)\right)\bar{\varepsilon}_k^{4s+2+(1-\tau)N}\int_{\mathbb{R}^N}|x|^2\varphi\,\bar{\eta}_0\mathrm{d}x.
\end{aligned} \tag{155}$$

Using (6) and lemma 7, we conclude that

$$\begin{aligned}
&\frac{bs}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}}\bar{\varepsilon}_k^{(N-4s)(\tau-1)}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\hat{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\hat{u}_{2k}|^2\right)\mathrm{d}x\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}(\hat{u}_{1k}+\hat{u}_{2k})(-\Delta)^{\frac{s}{2}}\hat{\eta}_k\mathrm{d}x\\
&=\frac{bs}{2}\left(\frac{2\beta^*}{b}\right)^{-\frac{N}{4s}}\bar{\varepsilon}_k^{(N-4s)(\tau-1)+2(1-\tau)(N-2s)}\int_{\mathbb{R}^N}\left(|(-\Delta)^{\frac{s}{2}}\bar{u}_{1k}|^2+|(-\Delta)^{\frac{s}{2}}\bar{u}_{2k}|^2\right)\mathrm{d}x\\
&\quad\times\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}(\bar{u}_{1k}+\bar{u}_{2k})(-\Delta)^{\frac{s}{2}}\bar{\eta}_k\mathrm{d}x\\
&=2bs(1+o(1))\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}\varphi(-\Delta)^{\frac{s}{2}}\bar{\eta}_0\mathrm{d}x
\end{aligned}\tag{156}$$

and

$$\begin{aligned}
&\frac{bs}{2}\int_{\mathbb{R}^N}\left(\hat{u}_{1k}^{\frac{4s+N}{N}}+\hat{u}_{2k}^{\frac{4s+N}{N}}\right)[A\hat{u}_{1k}+(1-A)\hat{u}_{2k}]^{\frac{4s}{N}}\hat{\eta}_k\mathrm{d}x\\
&\quad=\frac{bs}{2}\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}\left(\bar{u}_{1k}^{\frac{4s+N}{N}}+\bar{u}_{2k}^{\frac{4s+N}{N}}\right)[A\bar{u}_{1k}+(1-A)\bar{u}_{2k}]^{\frac{4s}{N}}\bar{\eta}_k\mathrm{d}x\\
&\quad=bs(1+o(1))\bar{\varepsilon}_k^{(1-\tau)N}\int_{\mathbb{R}^N}\varphi^{\frac{8s+N}{N}}\bar{\eta}_0\mathrm{d}x.
\end{aligned}\tag{157}$$

Following (123)–(125) and (151)–(157), we have

$$\begin{aligned}
&2bs(1+o(1))\bar{\varepsilon}_k^{(1-\tau)N}\left[\left(\frac{2\beta^*}{b}\right)^{\frac{4s}{N}}d_0+\left(s-\frac{N}{2}\right)\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\bar{d}_0\right]\\
&\quad-bs(1+o(1))\bar{\varepsilon}_k^{(1-\tau)N}\left[\frac{N+4s}{N}\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}d_0-\frac{N}{2}\left(\frac{2\beta^*}{b}\right)^{\frac{N}{4s}}\bar{d}_0\right]\\
&\quad\quad=O\left(\bar{\varepsilon}_k^{(1-\tau)(2s+2N-2)-\frac{N}{2}}\right)+2(1+s)(1+o(1))\bar{\varepsilon}_k^{(1-\tau)N+4s+2}\int_{\mathbb{R}^N}|x|^2\varphi\bar{\eta}_0\mathrm{d}x.
\end{aligned}\tag{158}$$

By (96), we know that $(1-\tau)(2s+2N-2)-\frac{N}{2}>(1-\tau)N+4s+2>0$. We thus derive from lemma 8 and (158) that

$$\int_{\mathbb{R}^N}|x|^2\varphi\bar{\eta}_0\mathrm{d}x=0,$$

from which and lemmas 7 and 10, we obtain that

$$0=\int_{\mathbb{R}^N}|x|^2\varphi\left[\frac{N}{2}\bar{d}_0\varphi+\bar{d}_0(x\cdot\varphi)\right]\mathrm{d}x=\frac{\bar{d}_0}{2}\int_{\mathbb{R}^N}\left[N|x|^2\varphi^2+|x|^2\left(x\cdot\nabla\varphi^2\right)\right]\mathrm{d}x=-\bar{d}_0\int_{\mathbb{R}^N}|x|^2\varphi^2\mathrm{d}x,$$

which shows that $\bar{d}_0=0$, thus, $d_0=0$. □

Proof of theorem 3. Let q_k be a point satisfying $|\bar{\eta}_k(q_k)|=\|\bar{\eta}_k\|_\infty=1$, we know that $|q_k|\leqslant C$ uniformly in k. Then lemma 7 implies that $\bar{\eta}_0\not\equiv 0$ on $\mathbb{R}^N$. From lemma 7, lemmas 10 and 11, we know that $\bar{\eta}_0\equiv 0$. Thus, our assumption that $u_{1k}\not\equiv u_{2k}$ is false. This completes the proof of theorem 3. □

Data availability statement

No new data were created or analysed in this study.

Ethics approval and consent to participate

Ethics and the consent to participate are approved by all authors.

Consent for publication

The authors express their consent for publication.

Conflict of interest

No interests of a financial or personal nature.

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References

[1] Applebaum D 2004 Lévy processes-from probability to finance and quantum groups *Not. Am. Math. Soc.* **51** 1336–47
[2] Alves C O, Corrêa F J S A and Ma T F 2005 Positive solutions for a quasilinear elliptic equation of Kirchhoff type *Comput. Math. Appl.* **49** 85–93
[3] Alves C O and Corrêa F J S A 2001 On existence of solutions for a class of problem involving a nonlinear operator *Commun. Appl. Nonlinear Anal.* **8** 43–56
[4] Blumenthal R M and Getoor R K 1960 Some theorems on stable processes *Trans. Am. Math. Soc.* **95** 263–73
[5] Caffarelli L and Silvestre L 2007 An extension problem related to the fractional Laplacian *Commun. PDE* **32** 1245–60
[6] Cao D M, Li S L and Luo P 2015 Uniqueness of positive bound states with multi-bump for nonlinear Schrödinger equations *Calc. Var. Partial Differ. Equ.* **54** 4037–63
[7] Chang S Y A and del Mar González M 2011 Fractional Laplacian in conformal geometry *Adv. Math.* **226** 1410–32
[8] Chen W J and Huang X M 2022 The existence of normalized solutions for a fractional Kirchhoff-type equation with doubly critical exponents *Z. Angew. Math. Phys.* **73** 226
[9] Chen S T, Rădulescu V D, Tang X H and Yuan S 2023 Normalized solutions for Schrödinger equations with critical exponential growth in $\mathbb{R}^2$ *SIAM J. Math. Anal.* **55** 7704–40
[10] Cheng M 2012 Bound state for the fractional Schrödinger equation with unbounded potential *J. Math. Phys.* **53** 043507

[11] Cingolani S, Gallo M and Tanaka K 2021 Normalized solutions for fractional nonlinear scalar field equations via Lagrangian formulation *Nonlinearity* **34** 4017–56
[12] Du M, Tian L X, Wang J and Zhang F B 2019 Existence of normalized solutions for nonlinear fractional Schrödinger equations with trapping potentials *Proc. R. Soc. Edinburgh* A **149** 617–53
[13] Di Nezza E, Palatucci G and Valdinoci E 2012 Hitchhiker's guide to the fractional Sobolev spaces *Bull. Sci. Math.* **136** 521–73
[14] Fareh S, Akrout K, Ghanmi A and Repovš D D 2023 Multiplicity results for fractional Schrödinger-Kirchhoff systems involving critical nonlinearities *Adv. Nonlinear Anal.* **12** 16
[15] Frank R, Lenzmann E and Silvestre L 2016 Uniqueness of radial solutions for the fractional Laplacian *Commun. Pure Appl. Math.* **69** 1671–726
[16] Felmer P, Quaas A and Tan J G 2012 Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian *Proc. R. Soc. Edinburgh* A **142** 1237–62
[17] Gao Y H, Luo X and Zhen M D 2024 Existence and classification of positive solutions for coupled purely critical Kirchhoff system *Bull. Math. Sci.* **14** 2450002
[18] Guo H L, Zhang Y M and Zhou H-S 2018 Blow-up solutions for a Kirchhoff type elliptic equation with trapping potential *Commun. Pure Appl. Anal.* **17** 1875–97
[19] Guo H L and Zhou H-S 2021 Properties of the minimizers for a constrained minimization problem arising in Kirchhoff equation *Discrete Contin. Dyn. Syst.* **41** 1023–50
[20] Guo H L, Liu H L and Zhao L L 2024 Concentration behavior and local uniqueness of normalized solutions for Kirchhoff type equation *Z. Angew. Math. Phys.* **75** 89
[21] Guo Y J and Seiringer R 2013 On the mass concentration for Bose-Einstein condensation with attractive interactions *Lett. Math. Phys.* **104** 141–56
[22] Guo Y J, Zeng X Y and Zhou H-S 2016 Energy estimates and symmetry breaking in attractive Bose-Einstein condensation with ring-shaped potential *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **33** 809–28
[23] Guo Y J, Lin C S and Wei J C 2017 Local uniqueness and refined spike profiles of ground states for two-dimensional attractive Bose-Einstein condensation *SIAM J. Math. Anal.* **49** 3671–715
[24] Guo Y X, Nie J J, Niu M M and Tang Z W 2017 Local uniqueness and periodicity for the prescribed scalar curvature problem of fractional operator in $\mathbb{R}^N$ *Calc. Var. PDE* **56** 1–41
[25] He X M, Rădulescu V D and Zuo W M 2022 Normalized ground states for the critical fractional Choquard equation with a local perturbation *J. Geom. Anal.* **32** 252
[26] He X M and Zuo W M 2016 Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities *Calc. Var. PDE* **55** 39
[27] He X M and Zou W M 2019 Ground state solutions for a class of fractional Kirchhoff equations with critical growth *Sci. China Math.* **62** 853–90
[28] He X M and Zuo W M 2019 Multiplicity of concentrating solutions for a class of fractional Kirchhoff equation *Manuscr. Math.* **158** 159–203
[29] Hu T X and Tang C L 2021 Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations *Calc. Var. Partial Differ. Equ.* **60** 210
[30] Huang X M and Zhang Y M 2020 Existence and uniqueness of minimizers for L^2-constrained problems related to fractional Kirchhoff equation *Math. Methods Appl. Sci.* **43** 8763–75
[31] Kirchhoff G 1883 *Mechanik* (Teubner)
[32] Kong L Z and Chen H B 2023 Normalized ground states for fractional Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities *J. Math. Phys.* **64** 061501
[33] Lan J L, He X M and Meng Y X 2023 Normalized solutions for a critical fractional Choquard equation with a nonlocal perturbation *Adv. Nonlinear Anal.* **12** 20230112
[34] Laskin N 2000 Fractional quantum mechanics and Lévy path integrals *Phys. Lett.* A **268** 298–305
[35] Laskin N 2002 Fractional Schrödinger equation *Phys. Rev.* **66** 56–108
[36] Li Y M and Bao J G 2019 Fractional Hardy-Hénon equations on exterior domains *J. Differ. Equ.* **266** 1153–75
[37] Li S and Zhu X C 2019 Mass concentration and local uniqueness of ground states for L^2-subcritical nonlinear Schrödinger equations *Z. Angew. Math. Phys.* **70** 34
[38] Li Q Q, Nie J J, Wang W B and Zhou J W 2024 Normalized solutions for Sobolev critical fractional Schrödinger equation *Adv. Nonlinear Anal.* **13** 20240027
[39] Liu L T, Chen H B and Yang J 2023 Normalized solutions to the fractional Kirchhoff equations with a perturbation *Appl. Anal.* **102** 1229–49

[40] Liu Z S, Luo Y J and Zhang J J 2022 A perturbation approach to studying sign-changing solutions of Kirchhoff equations with a general nonlinearity *Ann. Math. Pura Appl.* **201** 1229–55
[41] Liu Z S, Luo H J and Zhang J J 2022 Existence and multiplicity of bound state solutions to a Kirchhoff type equation with a general nonlinearity *J. Geom. Anal.* **32** 25
[42] Luo H J and Zhang Z T 2020 Normalized solutions to the fractional Schrödinger equations with combined nonlinearities *Calc. Var. Partial Differ. Equ.* **59** 1–35
[43] Luo X and Wang Q F 2017 Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in $\mathbb{R}^3$ *Nonlinear Anal. Real World Appl.* **33** 19–32
[44] Luo Y 2018 Uniqueness of ground states for nonlinear Hartree equations *J. Math. Phys.* **59** 081506
[45] Molica Bisci G, Rădulescu V D and Servadei R 2016 Variational methods for nonlocal fractional problems *Encyclopedia of Mathematics and its Applications* vol 162 (Cambridge University Press) (https://doi.org/10.1017/CBO9781316282397)
[46] Ni W M and Takagi I 1991 On the shape of least-energy solutions to a semilinear Neumann problem *Commun. Pure Appl. Math.* **44** 819–51
[47] Silvestre L 2007 Regularity of the obstacle problem for a fractional power of the Laplace operator *Commun. Pure Appl. Math.* **60** 67–112
[48] Sun X Q, Fu Y Q and Liang S H 2024 Multiplicity and concentration of solutions for Kirchhoff equations with exponential growth *Bull. Math. Sci.* **14** 2450004
[49] Tang X H and Cheng B T 2016 Ground state sign-changing solutions for Kirchhoff type problems in bounded domains *J. Differ. Equ.* **261** 2384–402
[50] Teng K M and Agarwal R P 2018 Existence and concentration of positive ground state solutions for nonlinear fractional Schrödinger-Poisson system with critical growth *Math. Methods Appl. Sci.* **42** 8258–93
[51] Teng K M and Cheng Y Q 2021 Multiplicity and concentration of nontrivial solutions for fractional Schrödinger-Poisson system involving critical growth *Nonlinear Anal.* **202** 112144
[52] Wang C and Sun J T 2023 Normalized solutions for the p-Laplacian equation with a trapping potential *Adv. Nonlinear Anal.* **12** 20220291
[53] Wei J C and Wu Y Z 2022 Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities *J. Funct. Anal.* **283** 109574
[54] Yao S, Chen H B and Sun J T 2023 Normalized solutions to the Chern-Simons-Schrödinger system under the nonlinear combined effect *Sci. China Math.* **66** 2057–80
[55] Ye H Y 2015 The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations *Math. Methods Appl. Sci.* **38** 2663–79
[56] Ye H Y 2015 The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations *Z. Angew. Math. Phys.* **66** 1483–97
[57] Ye H Y 2016 The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations *Z. Angew. Math. Phys.* **67** 29
[58] Zeng X Y, Zhang J J, Zhang Y M and Zhong X X 2023 Positive normalized solution to the Kirchhoff equation with general nonlinearities *J. Differ. Equ.* **365** 375–406
[59] Zhen M D and Zhang B L 2022 Normalized ground states for the critical fractional NLS equation with a perturbation *Rev. Mat. Complut.* **35** 89–132