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Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions

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Abstract

In this paper, we study a diffusion model of Kirchhoff-type driven by a nonlocal integro-differential operator. As a particular case, we consider the following diffusion problem

$$\begin{cases} \partial_t u + M\left([u]_s^2\right)(-\Delta)^s u = |u|^{p-2}u & \text{in } \Omega\times\mathbb{R}^+, \quad \partial_t u = \partial u/\partial t,\\ u(x,t)=0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times\mathbb{R}^+,\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

where $[u]_s$ is the Gagliardo seminorm of u, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $(-\Delta)^s$ is the fractional Laplacian with $0<s<1<p<\infty$, $u_0:\Omega\to\mathbb{R}^+$ is the initial function, and $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is continuous. Under some appropriate conditions, the local existence of nonnegative solutions is obtained by employing the Galerkin method. Then, by virtue of a differential inequality technique, we prove that the local nonnegative solutions blow-up in finite time with arbitrary negative initial energy and suitable initial values. Moreover, we give an estimate for the lower and upper bounds of the blow-up time. The main novelty is that our results cover the degenerate case, that is, the coefficient of $(-\Delta)^s$ could be zero at the origin.

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1. Introduction and main results

In this paper, we study the local existence and blow-up phenomena for the following fractional Kirchhoff-type parabolic problem

$$\begin{cases} \partial_t u + M\left([u]_s^2\right)\mathcal{L}_K u = |u|^{p-2}u & \text{in } \Omega\times\mathbb{R}^+, \quad \partial_t u = \partial u/\partial t,\\ u(x,t) = 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times\mathbb{R}^+,\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $[u]_s^2 = \iint_{\mathbb{R}^{2N}} |u(x,t)-u(y,t)|^2 K(x-y)\mathrm{d}x\mathrm{d}y$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $M:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a continuous function, $u_0\geqslant 0$ is the initial function on Ω, and $\mathcal{L}_K$ is a nonlocal integro-differential operator, which (up to normalization factors) may be defined by

$$\mathcal{L}_K\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^N}(2\varphi(x)-\varphi(x+y)-\varphi(x-y))K(y)\,\mathrm{d}y, \tag{1.2}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$, where $K:\mathbb{R}^N\setminus\{0\}\to\mathbb{R}^+$ is a function with the following properties:

(κ_1) $\gamma K\in L^1(\mathbb{R}^N)$, with $\gamma(x) = \min\{|x|^2,1\}$;
(κ_2) there exists $K_0>0$ such that $K(x)\geqslant K_0|x|^{-N-2s}$ for all $x\in\mathbb{R}^N\setminus\{0\}$.

A typical example for K is given by $K(x) = |x|^{-N-2s}$. In this case, $\mathcal{L}_K$ becomes the fractional Laplace operator $(-\Delta)^s$; see [25] and the references therein for recent results on the fractional Laplace operator, and [11] for further details on the fractional Laplacian and on the fractional Sobolev space. Throughout the paper, without further mentioning, we always assume that $s\in(0,1)$, $N>2s$ and K satisfies (κ_1) and (κ_2).

The interest in studying problems like (1.1) relies not only on mathematical purposes but also on their significance in real models, as explained by *Caffarelli* in [7] and *Laskin* in [21]. Actually, *Applebaum* [2] stated that the fractional Laplacian operator of the form $(-\Delta)^s$, $s\in(0,1)$, is the infinitesimal generator of a stable Lévy process. Recently, *Fiscella* and *Valdinoci* [16] proposed a stationary Kirchhoff variational equation which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Indeed, the stationary problem (1.1) is a fractional version of a model, the so-called stationary Kirchhoff equation, introduced by *Kirchhoff* in [20].

A usual model for anomalous diffusion is the following linear evolution equation involving the fractional Laplacian: $\partial_t u + (-\Delta)^s u = 0$, which derives asymptotically from basic random walk models, see [35, 41] and their references. Another nonlinear anomalous diffusion equation is the fractional porous medium equation

$$\partial_t u + (-\Delta)^s(u^m) = 0 \tag{1.3}$$

with $0<s<1$ and $m>0$, which was first proposed by *De Pablo et al* [13]. Many important results on these equations have been obtained, see the overview paper [41] and the references therein.

To explain the motivation of our problem (1.1), let us shortly introduce a prototype of nonlocal problem like (1.1). Nonlocal evolution equations of the form

$$\partial_t u = \int_{\mathbb{R}^N} (u(y,t) - u(x,t))K(x-y)\mathrm{d}y, \tag{1.4}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [14], if $u(x,t)$ is thought of as a density of population at the point x at time t and $K(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} u(y,t)K(x-y)\mathrm{d}y$ is the rate at which individuals are arriving at position x from all other places and $-\int_{\mathbb{R}^N} u(x,t)K(x-y)\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies (1.4). For recent references on nonlocal diffusion problems, see [3, 9, 12, 34].

If we consider the effects of total population, then equation (1.4) becomes

$$\partial_t u = M\left(\iint_{\mathbb{R}^N} |u(x,t)-u(y,t)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\int_{\mathbb{R}^N} (u(y,t)-u(x,t))K(x-y)\mathrm{d}y, \tag{1.5}$$

where the coefficient $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ denotes the possible changes of total population in $\mathbb{R}^N$. This describes the behaviour of individual subject to total population, such as the diffusion process of bacteria. Model (1.5) is also meaningful, since the way of measurements are usually taken in average sense. In particular, if $s \nearrow 1^-$ and $K(x) = |x|^{-N-2s}$, then equation (1.5) reduces to

$$\partial_t u = -M\left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u. \tag{1.6}$$

Models of the type (1.6) have been studied by many authors, see [18] and the references therein; see also [5, 30] for wave equations of Kirchhoff-type.

In [32], by using the sub-differential approach, *Pucci et al* obtained the well-posedness of solutions for problem (1.1) with $f(x,t)$ instead of $|u|^{p-2}u$. Moreover, the large-time behavior and extinction of solutions also are considered. With the help of potential well theory, *Fu* and *Pucci* [17] studied the existence of global weak solutions and established the vacuum isolating and blow-up of strong solutions, provided that $M \equiv 1$ and $2 < p \leqslant 2_s^* = 2N/(N-2s)$. However, the Kirchhoff function M is assumed to satisfy the non-degenerate condition in the above papers. In [26], *Pan et al* investigated for the first time the existence of global weak solutions for degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian, by combining the Galerkin method with potential well theory. In essence, the authors in [26] just considered that M is a special function, namely

$$M(t) = t^{\theta-1} \text{ with } \theta \in (1, 2_s^*/2), \quad \text{for all } t \in \mathbb{R}_0^+.$$

Motivated by the above works, we would like to consider more general conditions on M which cover the degenerate case $M(0) = 0$. More precisely, we assume that

(M_1) $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and there exist constants $m_0 > 0$ and $\theta > 1$ such that

$$M(\sigma) \geqslant m_0 \sigma^{\theta-1} \text{ for all } \sigma \in \mathbb{R}_0^+.$$

A typical prototype for the Kirchhoff function M is given by

$$M(\sigma) = a + b\sigma^{\theta-1}, \quad a \geqslant 0,\ b > 0,$$

for all $\sigma \in \mathbb{R}_0^+$. If $M(0) = 0$, then problem (1.1) is called degenerate Kirchhoff-type problem; if $M(0) > 0$, then problem (1.1) is non-degenerate. So the Kirchhoff-type problem (1.1) studied in this paper may be degenerate. Recently, a great attention has been paid to study the degenerate stationary Kirchhoff problems, see, e.g. [4, 8, 17, 23, 24, 31, 33, 42–44].

Definition 1.1. A function $u \in L^\infty(0,T;L^2(\Omega))\bigcap L^2(0,T;Z)\bigcap C(0,T;L^2(\Omega))$ is called a weak solution of problem (1.1), if $\partial_t u \in L^2(0,T;L^2(\Omega))$ and the following equality holds

$$\int_0^T\int_\Omega \frac{\partial u}{\partial t}\varphi \mathrm{d}x\mathrm{d}t + \int_0^T M(\|u\|_Z^2)\int_Q (u(x,t)-u(x,y))(\varphi(x,t)-\varphi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$= \int_0^T\int_\Omega |u|^{p-2}u\varphi \mathrm{d}x\mathrm{d}t,$$

for all $\varphi \in L^2(0,T;Z)$, where the space Z will be introduced in section 2.

Here we call u a global weak solution of problem (1.1), if the equality in definition 1.1 holds for any $0 < T < \infty$; u is a local weak solution, if there exists $T_0 > 0$ such that the equality in definition 1.1 holds for $0 < T \leqslant T_0$.

The first result of our paper is the following.

Theorem 1.1. *Let (M_1) holds. Suppose that $0 \leqslant u_0 \in Z$ and $2 < p < \frac{(2+2\theta)2_s^*-4\theta}{2_s^*} < 2_s^*$. Then there exists $T_0 > 0$ such that problem (1.1) admits at least a nontrivial, nonnegative weak solutions for all $t \in (0, T_0]$.*

In order to describe our second result, we need the following assumption:

(M_2) There exists a constant $\mu \geqslant 1$ such that

$$\mu\mathscr{M}(\sigma) \geqslant M(\sigma)\sigma \quad \text{for all } \sigma \in \mathbb{R}_0^+,$$

where $\mathscr{M}(\sigma) = \int_0^\sigma M(\tau)\mathrm{d}\tau$.

Note that condition (M_2) has been used to get the existence of solutions for the stationary $p(x)$-Kirchhoff problems, see for example [10].

Set

$$\mathcal{I}(u) = \frac{1}{2}\mathscr{M}(\|u\|_Z^2) - \frac{1}{p}\int_\Omega |u|^p \mathrm{d}x, \tag{1.7}$$

for all $u \in Z\bigcap L^p(\Omega)$.

The second result of our paper reads as follows.

Theorem 1.2. *Let (M_1) and (M_2) hold. Assume that $0 \leqslant u_0 \in Z$ and u is a nontrivial, nonnegative weak solution of problem (1.1). If $2\mu < p$ and $\mathcal{I}(u_0) < 0$, then the solution u blows up in finite time t^*, where t^* satisfies*

$$0 < t^* \leqslant \frac{\|u_0\|_{L^2(\Omega)}^2}{p(2-p)\mathcal{I}(u_0)}.$$

Furthermore, we give a precise estimate for the lower bounds of the blow-up time t^*.

Theorem 1.3. *Suppose that all conditions in theorem 1.2 hold. In addition, if $8s/3 < N < 4s$,*

$$1 < \mu < \min\left\{\frac{2_s^*}{2}, \frac{s2_s^*}{4(N-2s)} + \frac{1}{2}\right\}, \quad 2\mu < p < \frac{s2_s^*}{2(N-2s)} + 1,$$

then

$$\int_{\Phi(0)}^\infty \frac{\mathrm{d}\tau}{k_1 + k_2\tau^\Lambda} \leqslant t^*,$$

where

$$\Phi(0)=\int_\Omega u_0(x)^k\mathrm{d}x>0,\ \ \max\left\{2,\frac{2(N-2s)(p-1)}{s},\frac{N(\theta-1)}{s}\right\}<k\leqslant 2_s^*,$$
$$k_1=\left(1-\frac{2(N-2s)(p+k-1)}{k(2N-3s)}\right)|\Omega|,\ \ k_2=\frac{S^{-\frac{N}{3N-8s}}(3N-8s)}{4(N-2s)\varepsilon^{\frac{N}{3N-8s}}},$$
$$\varepsilon=\frac{8(N-2s)}{N}(k-1)m_0|\Omega|^{-\frac{2(\theta-1)(2_s^*-k)}{k2_s^*}}S^{\frac{1}{2}},\ \ \Lambda=\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)},$$

and S is the best constant of the embedding $Z\hookrightarrow L^{2_s^*}(\Omega)$.

Remark 1.1. To the best of our knowledge, there are no results to investigate the blow-up of solutions in the study of fractional diffusion problems of Kirchhoff-type.

The rest of the paper is organized as follows. In section 2, we recall some necessary definitions and properties of the fractional Sobolev spaces. In section 3, we obtain the local existence of weak solutions of problem (1.1) by the Galerkin method. In section 4, we show that the weak solutions of problem (1.1) blow-up in finite time under some appropriate conditions. Moreover, we give an estimate for the upper and lower bounds of the blow-up time.

2. Preliminaries

In this section, we first recall some necessary properties of fractional Sobolev spaces which will be used later, see [11, 37–39] for more details.

Let X be the linear space of Lebesgue measurable function $u:\mathbb{R}^N\to\mathbb{R}$ whose restrictions to Ω belong to $L^2(\Omega)$ and such that

$$\text{the map } (x,y)\mapsto|u(x)-u(y)|^2K(x-y) \text{ is in } L^1(\mathcal{Q},\mathrm{d}x\mathrm{d}y),$$

where $\mathcal{Q}=\mathbb{R}^{2N}\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega)$ and $\mathcal{C}\Omega=\mathbb{R}^N\setminus\Omega$. The space X is endowed with the norm

$$\|\varphi\|_X=\left(\|\varphi\|^2_{L^2(\Omega)}+\iint_{\mathcal{Q}}|u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/2}, \tag{2.1}$$

for all $\varphi\in X$. We observe that bounded and Lipschitz functions belong to X, thus X is not reduced to $\{0\}$; see [37] for further details on the space X.

The functional space Z denotes the closure of $C_0^\infty(\Omega)$ in X. The scalar product defined for any $\varphi,\psi\in Z$ as

$$\langle\varphi,\psi\rangle_Z=\iint_{\mathcal{Q}}(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))K(x-y)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

makes Z a Hilbert space. The norm

$$\|\varphi\|_Z=\left(\iint_{\mathcal{Q}}|\varphi(x)-\varphi(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\right)^{1/2} \tag{2.3}$$

is equivalent to the usual norm defined in (2.1), as proved in [36, lemma 6]. Note that in (2.1)–(2.3) the integrals can be extended to all $\mathbb{R}^N$ and $\mathbb{R}^{2N}$, since $u=0$ a.e. in $\mathcal{C}\Omega$. By lemma 6 of [36] and (κ_1), the Hilbert space $Z=(Z,\|\cdot\|_Z)$ is continuously embedded in $L^r(\Omega)$ for any $r\in[1,2_s^*]$. Hence there exists $C_r>0$ such that

$$\|u\|_{L^r(\Omega)} \leq C_r \|u\|_Z \text{ for all } u \in Z \text{ and } r \in [1, 2_s^*]. \tag{2.4}$$

Throughout the paper, the letters $c, c_i, C, C_i, i = 1, 2, \cdots$, denote positive constants which vary from line to line, but are independent of the terms that take part in any limit process.

We shall work on the Banach space $L^2(0,T;Z)$, endowed with the norm

$$\|u\|_{L^2(0,T;Z)} = \left(\int_0^T \|u\|_Z^2 \mathrm{d}t \right)^{1/2},$$

where $T \in (0, \infty)$ is a given constant. Obviously, $L^2(0, T, Z)$ is a Hilbert space, with the scalar product

$$\langle u, v \rangle_{L^2(0,T;Z)} = \int_0^T \langle u, v \rangle_Z \mathrm{d}t \text{ for all } u, v \in L^2(0, T; Z).$$

It follows from [40, theorem 1.5] that the dual space of $L^2(0, T;Z)$ can be identified with $L^2(0, T; Z')$.

Next, we consider the eigenvalue of the operator $\mathcal{L}_\mathcal{K}$ with homogeneous Dirichlet boundary data, namely the eigenvalue of the problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.5}$$

More precisely, the following weak formulation of (2.5) is discussed: there is a function $u \in Z$ such that

$$\iint_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x - y)\mathrm{d}x\mathrm{d}y = \lambda \int_\Omega u(x)\varphi(x)\mathrm{d}x, \tag{2.6}$$

for any $\varphi \in Z$. We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_K$ if there exists a non-trivial solution $u \in Z$ of problem (2.5) or its weak formulation (2.6), and any solution will be called an eigenfunction corresponding to the eigenvalue λ. Moreover, it easily follows from continuous embedding that the following Rayleigh quotient satisfies

$$\lambda_1 = \inf_{u \in Z \setminus \{0\}} \frac{\|u\|_Z^2}{\|u\|_{L^2(\Omega)}^2} \in (0, \infty). \tag{2.7}$$

Moreover, the sequence $\{e_k\}_k$ of eigenfunctions corresponding to $\{\lambda_k\}_k$ is an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis of Z, see [37, proposition 9] for more details.

Proposition 2.1 ([40, proposition 1.2]). *Let V be a Banach space which is dense and continuously embedded in the Hilbert space H. We identify $H = H'$ so that $V \hookrightarrow H = H' \hookrightarrow V'$. Then the Banach space $W_p = \{u \in L^p(0, T; V) : u' \in L^{p'}(0, T; V')\}$ is contained in $C(0, T; H)$. Moreover, if $u \in W_p$ then $\|u(\cdot)\|_{L^2(\Omega)}$ is absolutely continuous on $[0, T]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot)\|_{L^2(\Omega)}^2 = 2\langle u'(\cdot), u(\cdot)\rangle_{L^2(\Omega)} \text{ a.e. on } [0, T],$$

and there is a constant $C > 0$ such that

$$\|u\|_{C(0,T;H)} \leqslant C\|u\|_{W_p} \text{ for all } u \in W_p.$$

Proposition 2.2 ([40, proposition 1.3]). *Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Assume that $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, let B_0 and B_1 be reflexive, and define*

$$W = \{v : v \in L^p(0, T; B_0), \partial_t u \in L^q(0, T; B_1)\}.$$

Then the embedding $W \hookrightarrow L^p(0, T; B)$ is compact.

3. Proof of theorem 1.1

In this section, we prove the local existence of nonnegative weak solutions for problem (1.1).

Let $\{e_j\}_j$ denote the eigenfunctions of problem (2.5). Then $\|e_j\|_{L^2(\Omega)} = 1$, $\{e_j\}_j$ is an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis of Z. Set $V_n = \operatorname{span}\{e_1, \cdots, e_n\}$. Then $\{V_n\}_n$ is a dense subset of Z. Furthermore, we have the following property.

Lemma 3.1. *For $u_0 \in Z$, there exists a sequence $\{u_{0n}\}_n$ with $u_{0n} \in V_n$, such that $u_{0n} \to u_0$ in Z as $n \to \infty$.*

In order to prove the existence of weak solutions for problem (1.1) by applying the Galerkin method, we shall find the approximate solutions of the following equality:

$$u_n(x, t) = \sum_{j=1}^{n} (\eta_n(t))_j e_j(x) \quad \text{for all } n \in \mathbb{N},$$

with the unknown functions $(\eta_n(t))_j$ are determined by the system of ordinary differential equations

$$\begin{cases} \eta_n'(t) = I_n(t, \eta_n(t)), & t \in \mathbb{R}^+, \\ \eta_n(0) = U_{0n}, \end{cases} \tag{3.1}$$

where $U_{0n} = \left(\int_\Omega u_{0n}(x)e_1(x)\mathrm{d}x, \cdots, \int_\Omega u_{0n}(x)e_n(x)\mathrm{d}x\right)$, u_{0n} comes from lemma 3.1 and

$$\begin{aligned}(I_n(t, \eta_n))_j = &-M(\|u_n\|_Z^2) \iint_Q (u_n(x, t) - u_n(y, t))(e_j(x) - e_j(y))K(x - y)\mathrm{d}x\mathrm{d}y \\ &+ \int_\Omega |u_n^+(x, t)|^{p-2} u_n^+(x, t) e_j(x)\mathrm{d}x, \quad j = 1, 2, \cdots, n.\end{aligned}$$

It follows from the continuity of M and the definition of I_n that I_n is continuous on $\mathbb{R}_0^+ \times \mathbb{R}^n$. The Peano theorem (see [19]) implies that there exists a local solution of problem (3.1) on $[0,T_n]$ $(0 < T_n < \infty)$.

Lemma 3.2 (*A priori estimate*). *If $2 < p < \frac{(2+2\theta)2_s^* - 4\theta}{2_s^*} < 2_s^*$, then there exists $T^* > 0$ depending on $N, p, m_0, \theta, \|u_0\|_{L^2(\Omega)}$, such that*

$$\int_\Omega |u_n(x, t)|^2 \,\mathrm{d}x + \int_0^t \|u_n(x, t)\|_Z^{2\theta} \mathrm{d}t \leqslant C,$$

for all $t \in [0, T_0]$ and $T_0 = T^*/2$, where $C > 0$ independent of t and n.

Proof. Multiplying (3.1) by $(\eta_n(t))_j$ and summing with respect to j from 1 to n, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u_n(x, t)|^2 \mathrm{d}x + M(\|u_n\|_Z^2) \iint_Q |u_n(x, t) - u_n(y, t)|^2 K(x - y)\mathrm{d}x\mathrm{d}y = \int_\Omega |u_n^+(x, t)|^p \mathrm{d}x. \tag{3.2}$$

Since $2 < p < \frac{(2+2\theta)2_s^* - 4\theta}{2_s^*} < 2_s^*$ and using the classical interpolation inequality (see [1, theorem 2.11]), we have

$$\|u_n(x,t)\|_{L^p(\Omega)} \leqslant \|u_n(x,t)\|_{L^2(\Omega)}^{\kappa} \|u_n(x,t)\|_{L^{2_s^*}(\Omega)}^{1-\kappa},$$

for all $t \in [0, T_n]$, where $\kappa \in (0,1)$ satisfies

$$\frac{1}{p} = \frac{\kappa}{2} + \frac{1-\kappa}{2_s^*}.$$

We observe that

$$(1-\kappa)p = \frac{2_s^*(p-2)}{2_s^* - 2} < 2\theta$$

and

$$\alpha := \frac{2\theta\kappa p}{2\theta - (1-\kappa)p} = \frac{4\theta(2_s^* - p)}{2_s^*(2\theta - p + 2) - 4\theta} > 2.$$

Hence (2.4) implies that

$$\begin{aligned} \int_\Omega |u_n(x,t)|^p \mathrm{d}x &\leqslant \|u_n(x,t)\|_{L^2(\Omega)}^{\kappa p} \|u_n(x,t)\|_{L^{2_s^*}(\Omega)}^{(1-\kappa)p} \\ &\leqslant C_{2_s^*}^p \|u_n(x,t)\|_{L^2(\Omega)}^{\kappa p} \|u_n(x,t)\|_Z^{(1-\kappa)p}. \end{aligned} \tag{3.3}$$

For any $\varepsilon \in (0,1)$, the Young inequality yields

$$\|u_n(x,t)\|_{L^2(\Omega)}^{\kappa p} \|u_n(x,t)\|_Z^{(1-\kappa)p} \leqslant \varepsilon \|u_n(x,t)\|_Z^{2\theta} + C(\varepsilon)\|u_n(x,t)\|_{L^2(\Omega)}^{\alpha}.$$

Combining this inequality with (3.3), we get

$$\int_\Omega |u_n(x,t)|^p \mathrm{d}x \leqslant C_{2_s^*}^p \varepsilon \|u_n(x,t)\|_Z^{2\theta} + C(\varepsilon)\|u_n(x,t)\|_{L^2(\Omega)}^{\alpha}. \tag{3.4}$$

Inserting (3.4) in (3.2), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|u_n(x,t)\right|^2 \mathrm{d}x + M(\|u_n\|_Z^2) \iint_{\mathcal{Q}} |u_n(x,t) - u_n(y,t)|^2 K(x-y)\mathrm{d}x\mathrm{d}y \\ &\leqslant C_{2_s^*}^p \varepsilon \|u_n(x,t)\|_Z^{2\theta} + C(\varepsilon)\|u_n(x,t)\|_{L^2(\Omega)}^{\alpha}. \end{aligned} \tag{3.5}$$

Taking $\varepsilon = m_0/(2C_{2_s^*}^p)$ in (3.5), we conclude from (M_1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|u_n(x,t)\right|^2 \mathrm{d}x + \frac{m_0}{2}\|u_n(x,t)\|_Z^{2\theta} \leqslant C_1 \|u_n(x,t)\|_{L^2(\Omega)}^{\alpha}, \tag{3.6}$$

where $C_1 > 0$ is a constant. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(x,t)\|_{L^2(\Omega)}^2 \leqslant C_1 \|u_n(x,t)\|_{L^2(\Omega)}^{\alpha}.$$

Since $\alpha > 2$, the above inequality yields

$$\|u_n(x,t)\|^2_{L^2(\Omega)} \leqslant \left[C_2^{1-\frac{\alpha}{2}} - C_1\left(\frac{\alpha}{2}-1\right)t\right]^{\frac{2}{2-\alpha}},$$

only if $t < T^* := \frac{2C_2^{(2-\alpha)/2}}{C_1(\alpha-2)}$, where $C_2 = \sup_n \int_\Omega u_n^2(x,0)\mathrm{d}x \in (0,\infty)$. It follows that

$$\|u_n(x,t)\|^2_{L^2(\Omega)} \leqslant 2^{\frac{2}{\alpha-2}} C_2 \ \text{ for all } t \leqslant \min\{T_n, T^*/2\}.$$

Therefore

$$\int_\Omega u_n^2(x,T_n)\mathrm{d}x \leqslant 2^{2/(\alpha-2)}(C_2+1).$$

Thus, we can replace u_{0n} in (3.1) with $u_n(x,T_n)$ and extend the solution to the interval $[0, T^*/2]$ by repeating the above process. We obtain

$$\int_\Omega u_n^2(x,t)\mathrm{d}x \leqslant 2^{2/(\alpha-2)}C_2 \ \text{ for all } t \in [0, T^*/2].$$

Combining this inequality with (3.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|u_n(x,t)\right|^2\mathrm{d}x + \frac{a_0}{2}\|u_n(x,t)\|_Z^{2\theta} \leqslant C_3. \tag{3.7}$$

Fixing $\tau \in [0, T^*/2]$ and integrating (3.7) with respect to t over $[0,\tau]$, we get

$$\int_\Omega \left|u_n(x,\tau)\right|^2\mathrm{d}x + \frac{a_0}{2}\int_0^\tau \|u_n(x,t)\|_Z^{2\theta}\mathrm{d}t \leqslant C_3T^*.$$

The assertion follows by the arbitrary of τ. □

Next, by lemma 3.2, we obtain the following property.

Lemma 3.3. *There exists a constant $C > 0$ independent of n such that*

$$\int_0^{T_0}\int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2\mathrm{d}x\mathrm{d}t \leqslant C$$

and

$$\iint_{\mathcal{Q}} |u_n(x,t) - u_n(y,t)|^2 K(x-y)\mathrm{d}x\mathrm{d}y \leqslant C \ \text{ for all } t \in [0, T_0].$$

Proof. Multiplying (3.1) by $(\eta_n'(t))_j$ and summing with respect to j from 1 to n, we obtain

$$\begin{aligned}&\int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2\mathrm{d}x + M(\|u_n\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t))\\&\Big(\frac{\partial u_n(x,t)}{\partial t} - \frac{\partial u_n(y,t)}{\partial t}\Big)K(x-y)\mathrm{d}x\mathrm{d}y\\&= \int_\Omega |u_n^+(x,t)|^{p-2}u_n^+(x,t)\frac{\partial u_n(x,t)}{\partial t}\mathrm{d}x.\end{aligned} \tag{3.8}$$

Integrating (3.8) with respect to t over $[0,\tau]$ ($\tau \in [0, T_0]$), we have

$$\begin{aligned}&\int_0^\tau \int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathscr{M}(\|u_n(x,\tau)\|_Z^2) - \frac{\lambda}{p}\int_\Omega |u_n^+(x,\tau)|^p \mathrm{d}x \\ &= \frac{1}{p}\int_\Omega |u_n^+(x,0)|^p \mathrm{d}x + \frac{1}{2}\mathscr{M}(\|u_n(x,0)\|_Z^2),\end{aligned} \tag{3.9}$$

where $\mathscr{M}(\|u_n(x,\tau)\|_Z^2) = \int_0^{\|u_n(x,\tau)\|_Z^2} M(\sigma)\mathrm{d}\sigma$. Here we have used the following fact:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}(\|u_n(x,t)\|_Z^2) \\ &= 2M(\|u_n\|_Z^2)\iint_Q (u_n(x,t)-u_n(y,t))\Big(\frac{\partial u_n(x,t)}{\partial t} - \frac{\partial u_n(y,t)}{\partial t}\Big)K(x-y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Notice that by lemma 3.1

$$u_n(x,0) = \sum_{j=1}^n \left(\int_\Omega u_{0n}(x)e_j(x)\mathrm{d}x\right)e_j(x) = u_{0n}(x) \to u_0, \ \text{ in } Z,$$

as $n \to \infty$. Thus, there exists a constant $C_3 > 0$ such that $\|u_n(x,0)\|_Z^2 \leqslant C_3$. Furthermore, the continuity of M implies that there exists a constant $C_4 > 0$ such that

$$\mathscr{M}(\|u_n(x,0)\|_Z^2) \leqslant C_4.$$

Since $2 < p < \frac{(2+2\theta)2_s^* - 4\theta}{2_s^*} < 2_s^*$, there exists $C_p > 0$ such that

$$\|u_n(x,0)\|_{L^p(\Omega)} \leqslant C_p\|u_n(x,0)\|_Z,$$

where C_p is the embedding constant of $Z \hookrightarrow L^p(\Omega)$. Thus, there exists $C_5 > 0$ such that

$$\int_\Omega |u_n^+(x,0)|^p \mathrm{d}x \leqslant \int_\Omega |u_n(x,0)|^p \mathrm{d}x \leqslant C_5.$$

Combining the above inequalities with (3.9), we get

$$\int_0^\tau \int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathscr{M}(\|u_n(x,\tau)\|_Z^2) - \frac{1}{p}\int_\Omega |u_n^+(x,\tau)|^p \mathrm{d}x \leqslant C_6, \tag{3.10}$$

where $C_6 = C_4/2 + C_5/p$. Inserting (3.4) into (3.10), we have for any $\varepsilon \in (0,1)$

$$\begin{aligned}&\int_0^\tau \int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathscr{M}(\|u_n(x,\tau)\|_Z^2) \\ &\leqslant \frac{C_{2_s^*}^p}{p}\varepsilon\|u_n(x,\tau)\|_Z^{2\theta} + C(\varepsilon)\|u_n(x,\tau)\|_{L^2(\Omega)}^\alpha + C_6.\end{aligned} \tag{3.11}$$

Taking $\varepsilon = a_0/(4p\lambda C_{2_s^*}^p)$ in (3.11), we conclude from (M_1) and lemma 3.2 that

$$\int_0^\tau \int_\Omega \Big|\frac{\partial u_n(x,t)}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t + \frac{m_0}{4}\|u_n(x,\tau)\|_Z^{2\theta} \leqslant C_7\left(\|u_n(x,\tau)\|_{L^2(\Omega)}^\alpha + 1\right) \leqslant C_8.$$

Hence the lemma is proved. □

Remark 3.1. By (3.1), the following equality holds for all $\varphi \in Z$

$$\int_\Omega \frac{\partial u_n(x,t)}{\partial t}\varphi \mathrm{d}x + M(\|u_n\|_Z^2) \iint_Q (u_n(x,t)-u_n(y,t))(\varphi(x)-\varphi(y))K(x-y)\mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega |u_n^+|^{p-2}u_n^+\varphi \mathrm{d}x,$$

since $\{e_j\}_j$ is an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis of Z.

Proof of theorem 1.1. By lemma 3.2, lemma 3.3 and the reflexivity of $L^2(0,T_0;Z)$, there exist a subsequence of $\{u_n\}_n$ still denoted by $\{u_n\}_n$ and $u \in L^2(0,T_0;Z) \bigcap L^\infty(0,T_0;L^2(\Omega))$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(0,T_0;L^2(\Omega)), \\ u_n \rightharpoonup u & \text{weakly in } L^2(0,T_0;Z), \\ u_n \rightharpoonup u & \text{weakly} * \text{ in } L^\infty(0,T_0;Z), \\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{weakly in } L^2(0,T_0;L^2(\Omega)). \end{cases} \tag{3.12}$$

Next, we show that $u_n^+ \to u^+$ strongly in $L^p(0,T_0;L^p(\Omega))$. Since $\{u_n\}_n \subset L^2(0,T_0;Z)$ and $\{\frac{\partial u_n}{\partial t}\}_n \subset L^2(0,T_0;L^2(\Omega))$, proposition 2.2 implies that up to a subsequence, $u_n \to u$ strongly in $L^2(0,T_0;L^2(\Omega))$. Without loss of generality, we assume that $u_n \to u$ a.e. on $\Omega \times [0,T_0]$. Hence $u_n^+ \to u^+$ a.e. on $\Omega \times [0,T_0]$. Since $p < 2_s^*$, we have $\|u_n(x,t)-u(x,t)\|_{L^{2_s^*}(\Omega)} \leqslant C_{2_s^*}\|u_n(x,t)-u(x,t)\|_Z$ for all $t \in [0,T_0]$. Moreover, by $u_n \in L^\infty(0,T_0;Z)$, we deduce that

$$\int_0^{T_0}\int_\Omega |u_n(x,t)-u(x,t)|^{2_s^*}\mathrm{d}x\mathrm{d}t \leqslant C,$$

where $C > 0$ is a constant independent of n. For any measurable subset $U \subset \Omega \times [0,T_0]$, the Hölder inequality implies that

$$\begin{aligned} \int_U |u_n^+ - u^+|^p \mathrm{d}x\mathrm{d}t \leqslant \int_U |u_n - u|^p \mathrm{d}x\mathrm{d}t &\leqslant \big\||u_n-u|^p\big\|_{L^{\frac{2_s^*}{p}}(U)} \|1\|_{L^{\frac{2_s^*}{2_s^*-p}}(U)} \\ &= \|u_n-u\|_{L^{2_s^*}(U)}^p \|1\|_{L^{\frac{2_s^*}{2_s^*-p}}(U)} \\ &\leqslant C^p|U|^{(2_s^*-p)/2_s^*}, \end{aligned}$$

where $|U|$ is the Lebesgue measure of set U. This yields that the sequence $\{|u_n-u|^p\}_n$ is equi-integrable in $\Omega \times [0,T_0]$. By $u_n^+ \to u^+$ a.e. on $\Omega \times [0,T_0]$, we have $|u_n^+ - u^+|^p \to 0$ a.e. on $\Omega \times [0,T_0]$. Therefore, the Vitali convergence theorem implies

$$\lim_{n\to\infty}\int_0^{T_0}\int_\Omega |u_n^+(x,t)-u^+(x,t)|^p\mathrm{d}x\mathrm{d}t = 0. \tag{3.13}$$

Hence, the Brézis–Lieb lemma yields

$$\int_0^{T_0}\int_\Omega |u_n^+(x,t)|^p\mathrm{d}x\mathrm{d}t = \lim_{n\to\infty}\int_0^{T_0}\int_\Omega |u^+(x,t)|^p\mathrm{d}x\mathrm{d}t. \tag{3.14}$$

Since $\{|u_n^+|^{p-2}u_n^+\}_n$ is bounded in $L^{p/(p-1)}(0,T_0;L^{p/(p-1)}(\Omega))$, $|u_n^+|^{p-2}u_n^+ \to |u^+|^{p-2}u^+$ a.e. on $\Omega\times[0,T_0]$, we conclude from the Brézis–Lieb lemma that

$$\lim_{n\to\infty}\int_0^{T_0}\int_\Omega\left(\left||u_n^+|^{p-2}u_n^+\right|^{p/(p-1)}-\left||u_n^+|^{p-2}u_n^+-|u^+|^{p-2}u^+\right|^{p/(p-1)}\right)\mathrm{d}x\mathrm{d}t$$
$$=\int_0^{T_0}\int_\Omega\left||u^+|^{p-2}u^+\right|^{p/(p-1)}\mathrm{d}x\mathrm{d}t.$$

Combining this information with (3.14) we conclude that

$$\lim_{n\to\infty}\int_0^{T_0}\int_\Omega\left||u_n^+(x,t)|^{p-2}u_n^+(x,t)-|u^+(x,t)|^{p-2}u^+(x,t)\right|^{p/(p-1)}\mathrm{d}x\mathrm{d}t=0, \tag{3.15}$$

that is, $|u_n^+(x,t)|^{p-2}u_n^+(x,t)\to|u^+(x,t)|^{p-2}u^+(x,t)$ strongly in $L^{p/(p-1)}(0,T_0;L^{p/(p-1)}(\Omega))$.

Finally, we prove that $u_n\to u$ strongly in $L^2(0,T_0;Z)$.

To this aim, we define the operator $\mathscr{F}: L^2(0,T_0;Z)\times L^2(0,T_0;Z)\to\mathbb{R}$ by

$$\mathscr{F}(\varphi,\psi)=\int_0^{T_0}\iint_Q(\varphi(x,t)-\varphi(y,t))(\psi(x,t)-\psi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t,$$

for all $\varphi,\ \psi\in L^2(0,T_0;Z)$. Obviously, $\mathscr{F}$ is a bilinear operator in $L^2(0,T_0;Z)\times L^2(0,T;Z)$. The following fact implies that $\mathscr{F}$ is continuous in $L^2(0,T_0;Z)\times L^2(0,T_0;Z)$:

$$|\mathscr{F}(\varphi,\psi)|\leqslant\|\varphi\|_{L^2(0,T;Z)}\|\psi\|_{L^2(0,T_0;Z)}\ \text{ for all }\varphi\text{ and }\psi\in L^2(0,T;Z),$$

by the Hölder inequality. Hence the fact that u_n converges to u weakly in $L^2(0,T_0;Z)$ means that

$$\lim_{n\to\infty}\mathscr{F}(u_n,\psi)=\mathscr{F}(u,\psi)\ \text{ for all }\psi\in L^2(0,T_0;Z).$$

Let us recall that a sequence $\{f_j(t)\}_j$ is relatively compact in $L^1(0,T_0)$ if and only if:

(i) there exists a constant $C>0$ such that $\|f_j\|_{L^1(0,T_0)}\leqslant C$ for all j;
(ii) for every $\varepsilon>0$ there exists a constant $\delta=\delta(\varepsilon)>0$ such that for any measurable subset E with $|E|<\delta$, we have

$$\int_E|f_j(t)|\mathrm{d}t<\varepsilon,$$

uniformly for all j, see [29, proposition 1.3].

In the following, we show that $\{M(\|u_n\|_Z^2)\}_n$ is relatively compact in $L^1(0,T_0)$. By lemma 3.3 and the continuity of M, there exists $C>0$ such that $M(\|u_n\|_Z^2)\leqslant C$ for all n and t. Hence $\int_0^{T_0}M(\|u_n\|_Z^2)\mathrm{d}t\leqslant CT_0$ for all n, that is, the assertion (i) holds. For any $\varepsilon>0$ there exists $\delta=\varepsilon/C$ such that for any measurable subset E with $|E|<\delta$

$$\int_E M(\|u_n\|_Z^2)\mathrm{d}t\leqslant C|E|<\varepsilon.$$

Thus, the assertion (ii) is satisfied. It follows that $\{M(\|u_n\|_Z^2)\}_n$ is relatively compact in $L^1(0,T_0)$. Therefore, up to a subsequence, $M(\|u_n\|_Z^2)$ converges to some function $\xi(t)\in L^1(0,T_0)$ for a.e. $t\in[0,T_0]$. Furthermore, the Lebesgue dominated convergence theorem implies that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^{T_0}M(\|u_n\|_Z^2)\iint_Q(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&=\int_0^{T_0}\xi(t)\iint_Q(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t,\end{aligned}\tag{3.16}$$

for all $\varphi\in L^2(0,T_0;Z)$. Similarly,

$$\lim_{n\to\infty}\int_0^{T_0}\iint_Q[M(\|u_n\|_Z^2)-\xi(t)]^2(u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t=0,$$

which means that

$$M(\|u_n\|_Z^2)u\to\xi(t)u\ \text{ stongly in } L^2(0,T;Z).\tag{3.17}$$

Combining (3.17) with the fact that u_n converges weakly to u in $L^2(0,T;Z)$, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int_0^{T_0}M(\|u_n\|_Z^2)\iint_Q(u_n(x,t)-u_n(y,t))(u(x,t)-u(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&=\int_0^{T_0}\xi(t)\iint_Q(u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}\tag{3.18}$$

By remark 3.1, we have

$$\begin{aligned}&\int_0^{T_0}\int_\Omega\frac{\partial u_n}{\partial t}\varphi\mathrm{d}x\mathrm{d}t\\&+\int_0^{T_0}M(\|u_n\|_Z^2)\iint_Q(u_n(x,t)-u_n(y,t))(\varphi(x,t)-\varphi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&=\int_0^{T_0}\int_\Omega|u_n^+|^{p-2}u_n^+\varphi\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.19}$$

for all $\varphi\in L^2(0,T;Z)$. Letting $n\to\infty$ in (3.19) and using (3.16), we have

$$\begin{aligned}&\int_0^{T_0}\int_\Omega\frac{\partial u}{\partial t}\varphi\mathrm{d}x\mathrm{d}t\\&+\int_0^{T_0}\xi(t)\iint_Q(u(x,t)-u(y,t))(\varphi(x,t)-\varphi(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&=\int_0^{T_0}\int_\Omega|u^+|^{p-2}u^+\varphi\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.20}$$

for all $\varphi\in L^2(0,T;Z)$. Taking $\varphi=u$ in (3.20) and using proposition 2.1, we arrive at the equality

$$\frac{1}{2}\int_\Omega |u(x,T_0)|^2-|u_0(x)|^2\mathrm{d}x+\int_0^{T_0}\xi(t)\iint_Q(u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$=\int_0^{T_0}\int_\Omega |u^+(x,t)|^p\mathrm{d}x\mathrm{d}t. \tag{3.21}$$

Since $\int_\Omega u_n^2(x,T_0)\mathrm{d}x\leqslant C$, there exist a subsequence of $\{u_n\}_n$ (still denoted by $\{u_n\}_n$) and a function $\tilde{u}$ in $L^2(\Omega)$ such that $u_n(x,T_0)\rightharpoonup\tilde{u}$ weakly in $L^2(\Omega)$. Then for any $\varphi(x)\in C_0^\infty(\Omega)$ and $\eta(t)\in C^1[0,T_0]$, there holds

$$\int_0^{T_0}\int_\Omega\frac{\partial u_n}{\partial t}\varphi\eta\mathrm{d}x\mathrm{d}t=\int_\Omega u_n(x,T_0)\varphi\eta(T_0)\mathrm{d}x-\int_\Omega u_n(x,0)\varphi\eta(0)\mathrm{d}x$$
$$-\int_0^{T_0}\int_\Omega u_n\varphi\frac{\mathrm{d}\eta(t)}{\mathrm{d}t}\mathrm{d}x\mathrm{d}t.$$

Letting $n\to\infty$, we get by (3.12) and $u_n(x,0)\to u_0$ strongly in Z,

$$\int_0^{T_0}\int_\Omega\frac{\partial u}{\partial t}\varphi\eta\mathrm{d}x\mathrm{d}t=\int_\Omega\tilde{u}\varphi\eta(T_0)\mathrm{d}x-\int_\Omega u_0(x)\varphi\eta(0)\mathrm{d}x-\int_0^{T_0}\int_\Omega u\varphi\frac{\mathrm{d}\eta(t)}{\mathrm{d}t}\mathrm{d}x\mathrm{d}t.$$

Integrating by parts, the left-hand side of above equality can be written as

$$\int_\Omega u(x,T_0)\varphi\eta(T_0)\mathrm{d}x-\int_\Omega u(0,x)\varphi\eta(0)\mathrm{d}x-\int_0^{T_0}\int_\Omega u\varphi\frac{\mathrm{d}\eta(t)}{\mathrm{d}t}\mathrm{d}x\mathrm{d}t.$$

Hence, we deduce that

$$\int_\Omega(\tilde{u}-u(x,T_0))\eta(T_0)\varphi\mathrm{d}x-\int_\Omega(u_0(x)-u(x,0))\eta(0)\varphi\mathrm{d}x=0.$$

Choosing $\eta(T_0)=1,\eta(0)=0$ or $\eta(T_0)=0,\eta(0)=1$, by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$ we have $\tilde{u}=u(x,T_0)$ and $u(x{,}0)=u_0(x)$ a.e. in Ω. It follows that $u_n(x,T_0)\rightharpoonup u(x,T_0)$ weakly in $L^2(\Omega)$ as $n\to\infty$. Therefore

$$\int_\Omega u^2(x,T_0)\mathrm{d}x\leqslant\liminf_{n\to\infty}\int_\Omega u_n^2(x,T_0)\mathrm{d}x. \tag{3.22}$$

On the other hand, taking $\varphi=u_n$ in (3.1), we have

$$\frac{1}{2}\int_\Omega|u_n(x,T_0)|^2-|u_n(x,0)|^2\mathrm{d}x$$
$$+\int_0^{T_0}M(\|u_n\|_Z^2)\iint_Q(u_n(x,t)-u_n(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$
$$=\int_0^{T_0}\int_\Omega|u_n^+(x,t)|^p\mathrm{d}x\mathrm{d}t.$$

By (3.14), (3.16), (3.21), (3.22) and (3.18), we obtain

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_0^{T_0} M(\|u_n\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\leqslant -\frac{1}{2}\int_\Omega |u(x,T_0)|^2+|u_0(x)|^2\mathrm{d}x+\int_0^{T_0}\int_\Omega |u(x,t)|^p\mathrm{d}x\mathrm{d}t\\
&=\int_0^{T_0}\xi(t)\iint_{\mathcal{Q}}(u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&=\lim_{n\to\infty}\int_0^{T_0} M(\|u_n\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t))(u(x,t)-u(y,t))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t.
\end{aligned}
$$

Hence,

$$
\limsup_{n\to\infty}\int_0^{T_0} M(\|u_n\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t)-u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\leqslant 0.
$$

Since $M(\sigma)\geqslant 0$ for all $\sigma\geqslant 0$, the above inequality yields

$$
\lim_{n\to\infty}\int_0^{T_0} M(\|u_n\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t)-u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t=0.
$$

Thus, there exists a subsequence of $\{u_n\}_n$ still denoted by $\{u_n\}_n$ such that for a.e. $t\in[0,T_0]$

$$
\lim_{n\to\infty} M(\|u_n(x,t)\|_Z^2)\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t)-u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y=0.
$$

Furthermore, we obtain

$$
\lim_{n\to\infty}\iint_{\mathcal{Q}}(u_n(x,t)-u_n(y,t)-u(x,t)-u(y,t))^2K(x-y)\mathrm{d}x\mathrm{d}y=0\ \text{ for a.e. } t\in[0,T_0],
$$

since $M(\sigma)\geqslant a_0\sigma^{m_0}$ for all $\sigma\geqslant 0$. Thus, for a.e. $t\in[0,T_0]$, we have $u_n(x,t)\to u(x,t)$ strongly in Z. It is easy to see that $\|u_n(x,t)\|_Z^2\to\|u(x,t)\|_Z^2$ a.e. on $[0,T_0]$. Hence $M(\|u_n\|_Z^2)\to M(\|u\|_Z^2)$ a.e. on $[0,T_0]$, by the continuity of M. Therefore, we obtain that $\xi(t)=M(\|u\|_Z^2)$ a.e. on $[0,T_0]$.

It remains to show that $u\in C(0,T_0;L^2(\Omega))$ and $u\geqslant 0$ a.e. on $\Omega\times[0,T_0]$. Since

$$
u\in L^2(0,T_0;L^2(\Omega))\ \text{ and }\ \frac{\partial u}{\partial t}\in L^2(0,T_0;L^2(\Omega),
$$

up to a set of measure zero, we obtain that $u\in C(0,T_0;L^2(\Omega))$ by proposition 2.1.

Note that $u^-=\max\{0,-u\}\in L^2(0,T_0;Z)$. Taking $\varphi=-u^-\chi_{(0,\tau)}$ for $\tau\in(0,T_0]$ as test function in (3.20), where $\chi_{(0,\tau)}$ denotes the characteristic function of the set $(0,\tau)$, we obtain

$$
\int_0^\tau\int_\Omega\frac{\partial u}{\partial t}(-u^-)\mathrm{d}x\mathrm{d}t+\int_0^\tau M(\|u\|_Z^2)\iint_{\mathcal{Q}}|u^-(x,t)-u^-(y,t)|^2K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t\leqslant 0.\tag{3.23}
$$

It follows from (3.23) that

$$
\int_\Omega |u^-(x,\tau)|^2\mathrm{d}x\leqslant\int_\Omega |u_0^-(x)|^2\mathrm{d}x\ \text{ for all } \tau\in[0,T_0].
$$

Combining this inequality with $u_0 \geqslant 0$ in Ω, we get $u^- = 0$ a.e. on $\Omega \times [0, T_0]$. Hence $u \geqslant 0$ a.e. on $\Omega \times [0, T_0]$. In conclusion, it follows from (3.20) that the theorem is proved. □

Corollary 3.1. *The local solution $u \in L^2(0, T_0; Z) \bigcap C(0, T_0; L^2(\Omega))$ obtained by theorem 1.1 satisfies*

$$\begin{aligned}&\int_\Omega \frac{\partial u(x,\tau)}{\partial t}\phi \mathrm{d}x + M(\|u\|_Z^2) \iint_Q (u(x,\tau) - u(y,\tau))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega u(x,\tau)^{p-1}\phi \mathrm{d}x\end{aligned} \tag{3.24}$$

for almost every $0 < \tau \leqslant T_0$ and all $\phi \in Z$.

Proof. For $\tau \in (0, T_0)$, we choose $\Delta\tau$ such that $\tau + \Delta\tau \in (0, T_0]$. Without loss of generality, we assume that $\Delta\tau > 0$. To prove (3.24), let us take $\varphi = \phi\chi_{(\tau,\tau+\Delta\tau)}$ as a test function in (1.1), where $\chi_{(\tau,\tau+\Delta\tau)}$ is the characteristic function of the set $(\tau, \tau + \Delta\tau)$ and $\phi \in Z$. We obtain

$$\begin{aligned}&\int_\tau^{\tau+\Delta\tau} \int_\Omega \frac{\partial u}{\partial t}\phi \mathrm{d}x\mathrm{d}t + \int_\tau^{\tau+\Delta\tau} M(\|u(x,t)\|_Z^2) \iint_Q (u(x,t) - u(y,t))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \int_\tau^{\tau+\Delta\tau} \int_\Omega |u(x,t)|^p \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.25}$$

Taking into account that

$$\begin{aligned}&\left|\int_0^{T_0} \int_\Omega \frac{\partial u(x,t)}{\partial t} u(x,t)\mathrm{d}x\mathrm{d}t\right| \leqslant \left\|\frac{\partial u(x,t)}{\partial t}\right\|_{L^2(0,T_0;L^2(\Omega))} \|u\|_{L^2(0,T_0;L^2(\Omega))} < \infty, \\ &\int_\tau^{\tau+\Delta\tau} M(\|u(x,t)\|_Z^2) \iint_Q (u(x,t) - u(y,t))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\leqslant \int_\tau^{\tau+\Delta\tau} M(\|u\|_Z^2)\|u\|_Z\|\phi\|_Z \mathrm{d}t < \infty,\end{aligned}$$

and

$$\int_0^{T_0} \int_\Omega |u(x,t)|^p \mathrm{d}x\mathrm{d}t < \infty,$$

we get with the aid of Lebesgue's differential theorem

$$\begin{aligned}&\lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau} \int_\tau^{\tau+\Delta\tau} \int_\Omega \frac{\partial u(x,t)}{\partial t} u(x,t)\mathrm{d}x\mathrm{d}t = \int_\Omega \frac{\partial u(x,\tau)}{\partial t} u(x,\tau)\mathrm{d}x, \\ &\lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau} \int_\tau^{\tau+\Delta\tau} M(\|u(x,t)\|_Z^2) \iint_Q (u(x,t) - u(y,t))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= M(\|u(x,\tau)\|_Z^2) \iint_Q (u(x,\tau) - u(y,\tau))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y, \\ &\lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau} \int_\tau^{\tau+\Delta\tau} \int_\Omega |u(x,t)|^p \mathrm{d}x\mathrm{d}t = \int_\Omega |u(x,\tau)|^p \mathrm{d}x,\end{aligned}$$

for a.e. $\tau \in (0, T_0)$. Dividing (3.25) by $\Delta\tau$ and letting $\Delta\tau \to 0$, we arrive at

$$\int_\Omega \frac{\partial u}{\partial t}\phi \mathrm{d}x + M(\|u(x,\tau)\|_Z^2) \iint_Q (u(x,\tau) - u(y,\tau))(\phi(x) - \phi(y))K(x-y)\mathrm{d}x\mathrm{d}y$$
$$= \int_\Omega |u(x,\tau)|^p \mathrm{d}x,$$

for a.e. $\tau \in (0, T_0)$. The proof is now complete. □

4. Global nonexistence via blow-up analysis

In this section, by means of a differential inequality technique, we prove that the local weak solutions of problem (1.1) blow up in finite time; see [15, 22, 28] and the references therein for some results on blow-up of solutions. In the following, we shortly use u_t to denote $\frac{\partial u}{\partial t}$ for convenience.

Definition 4.1. We say that the solution $u(x,t)$ blows up in finite time if there exists $t^* \in (0, \infty)$ such that

$$\|u(x,t)\|_{L^2(\Omega)} \to \infty \quad \text{as } t \to t^*.$$

Proof of theorem 1.2. Let u be a nonnegative solution of problem (1.1). Set

$$f(t) = \|u(x,t)\|^2_{L^2(\Omega)}$$

and

$$\mathcal{I}(u) = \frac{1}{2}\mathscr{M}(\|u\|_Z^2) - \frac{1}{p}\int_\Omega u^p \mathrm{d}x.$$

By corollary 3.1 and (M_2), we obtain

$$f'(t) = 2\int_\Omega u u_t \mathrm{d}x = -2M(\|u\|_Z^2)\|u\|_Z^2 + 2\int_\Omega u^p \mathrm{d}x$$
$$\geqslant -2\mu\mathscr{M}(\|u\|_Z^2) + 2\int_\Omega u^p \mathrm{d}x.$$

It follows from $p > 2\mu$ that

$$f'(t) \geqslant -2\mu\mathscr{M}(\|u\|_Z^2) + 2\int_\Omega u^p \mathrm{d}x \geqslant E(t), \tag{4.1}$$

where

$$E(t) = -2p\mathcal{I}(u) = -p\mathscr{M}(\|u\|_Z^2) + 2\int_\Omega u^p \mathrm{d}x.$$

Through straightforward computation we deduce that

$$\begin{aligned} E'(t) &= 2p\left[-M(\|u\|_Z^2)\int_{\Omega}(u(x,t)-u(y,t))(u_t(x,t)-u_t(y,t))K(x-y)\mathrm{d}x\mathrm{d}y+\int_{\Omega}u^p u_t\mathrm{d}x\right] \\ &= 2p\left[-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{M}(\|u\|_Z^2)+\frac{1}{p}\frac{d}{dt}\int_{\Omega}u^p dx\right] \\ &= 2p\int_{\Omega}u_t^2\mathrm{d}x. \end{aligned} \tag{4.2}$$

Using the Hölder inequality and (4.2), we get

$$f(t)E'(t) = 2p\|u\|^2_{L^2(\Omega)}\|u_t\|^2_{L^2(\Omega)} \geqslant 2p\left(\int_{\Omega}uu_t\mathrm{d}x\right)^2 = \frac{p}{2}(f'(t))^2. \tag{4.3}$$

By (4.2), we know that $E(t)$ is a nondecreasing function with respect to t. Note that $E(t) = -2p\mathcal{I}(u(x,t))$. Thus, if $\mathcal{I}(u_0) < 0$, $E(0) > 0$, then $E(t) > 0$ for all $t > 0$. Hence, it follows from (4.1) that

$$f(t)E'(t) \geqslant \frac{p}{2}f'(t)E(t),$$

which can be rewritten as

$$\frac{E'(t)}{E(t)} \geqslant \frac{p}{2}\frac{f'(t)}{f(t)}. \tag{4.4}$$

Integrating (4.4) from 0 to t and using (4.1), we obtain the inequality

$$\frac{f'(t)}{[f(t)]^{\frac{p}{2}}} \geqslant \frac{E(0)}{[f(0)]^{\frac{p}{2}}}.$$

Integrating this inequality from 0 to t, we deduce

$$[f(t)]^{1-\frac{p}{2}} \leqslant [f(0)]^{1-\frac{p}{2}} - \left(\frac{p}{2}-1\right)\frac{E(0)}{[f(0)]^{\frac{p}{2}}}t. \tag{4.5}$$

Since $p > 2\mu \geqslant 2$, inequality (4.5) cannot hold for all $t > 0$ and we conclude that u blows up at finite time t^*, where

$$t^* \leqslant \frac{2f(0)}{(p-2)E(0)} = \frac{\|u_0\|^2_{L^2(\Omega)}}{p(2-p)\mathcal{I}(u_0)}.$$

This ends the proof. □

Finally, we estimate the lower bound of blow-up time t^*. Here we mainly employ some techniques from [6], see also [27].

Proof of theorem 1.3. Let $8s/3 < N < 4s$ and $\max\{2, 2(N-2s)(p-1)/s, \frac{N(\theta-1)}{s}\} < k \leqslant 2_s^*$ and $g(t) = t^{k-1}$ for all $t \geqslant 0$. Next we prove that

$$(a-b)(g(a)-g(b)) \geqslant \frac{2(k-1)}{k}(a^{\frac{k}{2}}-b^{\frac{k}{2}})^2 \quad \text{for all } a, b \geqslant 0. \tag{4.6}$$

For all $a \geqslant b \geqslant 0$, we have

$$\begin{aligned}(a-b)(g(a)-g(b)) &= (a-b)\int_b^a g'(\tau)\mathrm{d}\tau \\ &= (a-b)\int_b^a [(g'(\tau))^{\frac{1}{2}}]^2\mathrm{d}\tau \\ &\geqslant \int_b^a (g'(\tau))^{\frac{1}{2}}\mathrm{d}\tau = \frac{2(k-1)}{k}(a^{\frac{k}{2}}-b^{\frac{k}{2}})^2.\end{aligned}$$

Since g is a increasing function, the above inequality still holds for $0 \leqslant a < b$. Thus, (4.6) holds true. Taking $\phi = u^{k-1}$ as a test function in corollary 3.1, we get

$$\begin{aligned}&\int_\Omega u_t u^{k-1}\mathrm{d}x + M(\|u\|_Z^2)\iint_Q (u(x,t)-u(y,t))(u(x,t)^{k-1}-u(y,t)^{k-1})K(x-y)\mathrm{d}x\mathrm{d}y \\ &= \int_\Omega u^{p+k-1}\mathrm{d}x.\end{aligned} \tag{4.7}$$

Let

$$\Phi(t) = \int_\Omega u(x,t)^k \mathrm{d}x.$$

Then a direct computation yields that

$$\begin{aligned}\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} &= k\int_\Omega u^{k-1}u_t\mathrm{d}x \\ &= -kM(\|u\|_Z^2)\iint_Q (u(x,t)-u(y,t))(u(x,t)^{k-1} \\ &\quad - u(y,t)^{k-1})K(x-y)\mathrm{d}x\mathrm{d}y + k\int_\Omega u^{p+k-1}\mathrm{d}x.\end{aligned}$$

It follows from (4.6) and (M_1) that

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \leqslant -2(k-1)m_0\|u\|_Z^{2(\theta-1)}\iint_Q \left(u(x,t)^{\frac{k}{2}}-u(y,t)^{\frac{k}{2}}\right)^2 K(x-y)\mathrm{d}x\mathrm{d}y + k\int_\Omega u^{p+k-1}\mathrm{d}x. \tag{4.8}$$

Since $\max\{2, 2(N-2s)(p-1)/s, \frac{N(\theta-1)}{s}\} < k \leqslant 2_s^*$, we have

$$\int_\Omega u^k\mathrm{d}x \leqslant |\Omega|^{\frac{2_s^*-k}{2_s^*}}\left(\int_\Omega u^{2_s^*}\mathrm{d}x\right)^{\frac{k}{2_s^*}} \leqslant |\Omega|^{\frac{2_s^*-k}{2_s^*}}S^{-\frac{k}{2}}\|u\|_Z^k,$$

where S is the best constant of the embedding $Z \hookrightarrow L^{2_s^*}(\Omega)$. Thus,

$$\|u\|_Z^{2(\theta-1)} \geqslant |\Omega|^{-\frac{2(\theta-1)(2_s^*-k)}{k2_s^*}}S^{\theta-1}\left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{2(\theta-1)}{k}}.$$

Inserting this inequality into (4.8), we get

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \leqslant -2(k-1)m_0|\Omega|^{-\frac{2(\theta-1)(2_s^*-k)}{k2_s^*}} S^{\frac{1}{2}} \left(\int_\Omega u^k \mathrm{d}x\right)^{\frac{2(\theta-1)}{k}} \|u^{\frac{k}{2}}\|_Z^2 + k\int_\Omega u^{p+k-1}\mathrm{d}x. \tag{4.9}$$

Applying the Hölder and Young inequalities, we deduce that

$$\int_\Omega u^{p+k-1}\mathrm{d}x \leqslant |\Omega|^{m_1} \left(\int_\Omega u^{\frac{k(2N-3s)}{2(N-2s)}}\mathrm{d}x\right)^{m_2} \leqslant m_1|\Omega| + m_2\int_\Omega u^{\frac{k(2N-3s)}{2(N-2s)}}\mathrm{d}x, \tag{4.10}$$

where

$$m_1 = 1 - \frac{2(N-2s)(p+k-1)}{k(2N-3s)}, \quad m_2 = \frac{2(N-2s)(p+k-1)}{k(2N-3s)}.$$

Putting (4.10) into (4.9), we arrive at the inequality

$$\begin{aligned}\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} &\leqslant -2(k-1)m_0|\Omega|^{-\frac{2(\theta-1)(2_s^*-k)}{k2_s^*}} S^{\frac{1}{2}} \left(\int_\Omega u^k \mathrm{d}x\right)^{\frac{1}{k}} \|u^{\frac{k}{2}}\|_Z^2 \\ &\quad + m_1|\Omega| + m_2\int_\Omega u^{\frac{k(2N-3)}{2(N-2s)}}\mathrm{d}x.\end{aligned} \tag{4.11}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega u^{\frac{k(2N-3s)}{2(N-2s)}}\mathrm{d}x &\leqslant \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u^{\frac{k(N-s)}{N-2s}}\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{3}{4}} \left(\int_\Omega \left(u^{\frac{k}{2}}\right)^{\frac{2N}{(N-2s)}}\mathrm{d}x\right)^{\frac{1}{4}}.\end{aligned}$$

From the fractional Sobolev embedding, we have

$$\|u^{\frac{k}{2}}\|_{2_s^*}^{\frac{N}{2(N-2s)}} \leqslant S^{-\frac{N}{4(N-2s)}} \|u^{\frac{k}{2}}\|_Z^{\frac{N}{2(N-2s)}}.$$

Thus,

$$\begin{aligned}\int_\Omega u^{\frac{k(2N-3s)}{2(N-2s)}}\mathrm{d}x &\leqslant S^{-\frac{N}{4(N-2s)}} \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{3}{4}} \|u^{\frac{k}{2}}\|_Z^{\frac{N}{2(N-2s)}} \\ &= S^{-\frac{N}{4(N-2s)}} \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{3}{4}-\frac{(\theta-1)N}{2k(N-2s)}} \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{(\theta-1)N}{2k(N-2s)}} \|u^{\frac{k}{2}}\|_Z^{\frac{N}{2(N-2s)}}.\end{aligned}$$

Now the Young inequality means that

$$\begin{aligned}\int_\Omega u^{\frac{k(2N-3s)}{2(N-2s)}}\mathrm{d}x &\leqslant \frac{S^{-\frac{N}{3N-8s}}(3N-8s)}{4(N-2s)\varepsilon^{\frac{N}{3N-8s}}} \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)}} \\ &\quad + \frac{N\varepsilon}{4(N-2s)} \left(\int_\Omega u^k\mathrm{d}x\right)^{\frac{2(\theta-1)}{k}} \|u^{\frac{k}{2}}\|_Z^2,\end{aligned} \tag{4.12}$$

where ε is a positive constant to be determined later. Combining (4.11) with (4.12), we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leqslant k_1 + k_2\Phi(t)^{\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)}} + k_3\left(\int_\Omega u^k \mathrm{d}x\right)^{\frac{2(\theta-1)}{k}} \|u^{\frac{k}{2}}\|_Z^2,$$

where

$$k_1 = m_1|\Omega|,$$
$$k_2 = \frac{S^{-\frac{N}{3N-8s}}(3N-8s)}{4(N-2s)\varepsilon^{\frac{N}{3N-8s}}},$$
$$k_3 = \frac{N\varepsilon}{4(N-2s)} - 2(k-1)m_0|\Omega|^{-\frac{2(\theta-1)(2_s^*-k)}{k2_s^*}} S^{\frac{1}{2}}.$$

Now we choose ε such that $k_3 = 0$, then we arrive at the inequality

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leqslant k_1 + k_2\Phi^{\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)}}.$$

An integration of the above differential inequality from 0 to t yields

$$\int_{\Phi(0)}^{\Phi(t)} \frac{\mathrm{d}\tau}{k_1 + k_2\tau^{\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)}}} \leqslant t,$$

which together with $\lim_{t\to t^*}\Phi(t) = \infty$ implies that

$$\int_{\Phi(0)}^{\infty} \frac{\mathrm{d}\tau}{k_1 + k_2\tau^{\Lambda}} \leqslant t^*,$$

where $\Lambda = \frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)}$ and

$$\Phi(0) = \int_\Omega u_0^k \mathrm{d}x > 0.$$

Note that $\frac{3k(N-2s)-2N(\theta-1)}{k(3N-8s)} > 1$ if $k > \frac{N(\theta-1)}{s}$, hence the right-hand side of the above inequality is finite. Thus, the proof is complete. □

Remark 4.1. If $M(\sigma) \geqslant m_0$ for all $\sigma \geqslant 0$, that is, if the problem is non-degenerate, then the restriction $k \leqslant 2_s^*$ is not necessary. In this case, k satisfies $\max\{2, 2(N-2s)(p-1)/s\} < k < \infty$, hence we can relax the condition $2\mu < p < \frac{2_s^* s}{2(N-2s)} + 1$ to $2\mu < p < \infty$.

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