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Hardy-type inequalities for the drifting p-Laplace operator and applications

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We consider the drifting p-Laplace operator

$$\Delta_{p,v} u = e^{-v} \mathrm{div}\,(e^{v} |\nabla u|^{p-2} \nabla u)$$

and discuss generalized weighted Hardy-type inequalities associated with the measure $d\mu = e^{v(x)} dx$. As an application, we obtain several Liouville-type results for positive solutions of the non-linear elliptic problem with singular lower order term

$$-\Delta_{p,v} u \geq c(x) u^{p-1} + B \frac{|\nabla u|^p}{u} \quad \text{in } \Omega,$$

where Ω is a bounded or an unbounded exterior domain in $\mathbb{R}^N$, $N > p > 1$, $B + p - 1 > 0$, as well as of the non-autonomous quasilinear elliptic problem

$$-\Delta_{p,v} u + b(x) |\nabla u|^{p-1} \geq c(x) u^{p-1} \quad \text{in } \Omega,$$

with general weights $b \geq 0$ and $c > 0$. Liouville-type results are also discussed for a class of higher order differential equations.

Keywords: Liouville-type theorems; quasilinear elliptic equations;
Q5 weighted p-Laplace operator

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1. Introduction

For a smooth function $v : \Omega \to \mathbb{R}$ on a domain Ω in $\mathbb{R}^N$, we consider the drifting p-Laplace operator

$$\Delta_{p,v} u = e^{-v} \mathrm{div}\, (e^v |\nabla u|^{p-2} \nabla u), \quad 1 < p < \infty,$$

and the related measure

$$d\mu(x) = e^{v(x)}\, dx.$$

Note that we have

$$\Delta_{p,v} u = \Delta_p u + |\nabla u|^{p-2} \nabla u \cdot \nabla v,$$

where $\Delta_p u = \mathrm{div}\, (|\nabla u|^{p-2} \nabla u)$ is the standard p-Laplace operator. In particular, if v is a constant, then $\Delta_{p,v} u$ coincides with $\Delta_p u$. There has been interest in the study of the drifting p-Laplace equation, mostly the case $p = 2$ on a metric measure space $(M, \langle \cdot, \cdot \rangle, e^v\, dw)$, where $(M, \langle \cdot, \cdot \rangle)$ is an N-dimensional Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$, v is a smooth real-valued function defined on M and dw is the Riemannian volume element related to $\langle \cdot, \cdot \rangle$. Several interesting related results including Liouville theorems, estimates on the lowest eigenvalue, gradient estimates, and Harnack inequalities to the solutions of the problem $-\Delta_{2,v} u = \lambda u$ can be found, for instance, in [28, 40–42] and the references therein.

The aim of this work is to study Hardy-type inequalities related to the drifting p-Laplace operator. As an application, we discuss Liouville-type theorems for positive classical solutions of several quasilinear elliptic problems. One of the novelties is that we do not make any assumptions on the asymptotic behaviour of solutions at infinity, nor on whether they are bounded or radial. For instance, we consider positive solutions $u \in C^2(\Omega, \mathbb{R}_+)$ to

$$-\Delta_{p,v} u \geq c(x) u^{p-1} + B \frac{|\nabla u|^p}{u} \quad \text{in } \Omega, \tag{P}$$

where Ω is a bounded domain or an unbounded exterior domain in $\mathbb{R}^N$, $B + p - 1 > 0$ and $1 < p < N$.

The motivation for the problem (P) comes from the singular quasilinear elliptic equation

$$-\Delta_p u = \lambda u^{p-1} + B \frac{|\nabla u|^p}{u} + f \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$, $B \in \mathbb{R}$, $f \geq 0$ is an arbitrary locally integrable function or a measure. Equations of the type (1.1), in the case $p = 2$, have been studied in [1, 7, 8, 17, 21], mostly in bounded domains $\Omega \subset \mathbb{R}^N$, with $f \in L^r(\Omega)$ for some $r > 1$.

Arcoya et al. considered (1.1) in [7] on a bounded domain Ω with zero Dirichlet data, when $\lambda=0$, $p=2$, $0\le B<1$, $f\ge 0$, and proved the existence of positive solutions in suitable Sobolev spaces (depending on f and B). In the general case $p>1$, Cao-Verbitsky [21] characterized the existence of positive solutions u of (1.1) in $\Omega=\mathbb{R}^N$ with $\liminf_{x\to\infty} u(x)=0$, $\lambda=0$ and $B=\frac{q(p-1)}{p-1-q}<0$ for some $0<q<p-1$. In [9], Arcoya and Moreno-Mérida considered (1.1) with $\lambda\neq 0$ and $p=2$ and proved the existence (resp., non-existence) of positive solutions provided that $\lambda<\frac{\lambda_1}{1+B}$ (resp., $\lambda\ge\frac{\lambda_1}{1+B}$), where λ_1 denotes the lowest eigenvalue of the Laplace operator.

Notice that every solution of (1.1) is a solution of the problem (P) with $c(x)\equiv\lambda$. In this case, our results for (P) imply the non-existence of positive solutions for a wider range of $1-p<B<\infty$, if

$$\lambda>\left(\frac{p-1}{p-1+B}\right)^{p-1}\lambda_1^{\mu}(\Omega,p),$$

where

$$\lambda_1^{\mu}(\Omega,p)=\inf\left\{\frac{\int_\Omega|\nabla\phi|^p\,d\mu}{\int_\Omega|\phi|^p\,d\mu}:\phi\in W_0^{1,p}(\Omega;\mu),\ \phi\neq 0\right\}\tag{1.2}$$

is the lowest eigenvalue of the weighted p-Laplace operator. Here $W_0^{1,p}(\Omega;\mu)$ is the completion of $C_c^\infty(\Omega)$ under the weighted Sobolev norm

$$\|\phi\|_{1,p;\mu}=\left(\int_\Omega|\phi|^p\,d\mu+\int_\Omega|\nabla\phi|^p\,d\mu\right)^{\frac{1}{p}}.$$

We also consider the problem (P) in unbounded exterior domains Ω with a general potential $c\ge 0$ and v satisfying

$$v_1=\limsup_{|x|\to\infty}|x||\nabla v(x)|<\infty\quad\text{and}\quad v_2=\limsup_{|x|\to\infty}x\cdot\nabla v(x).\tag{1.3}$$

We prove the non-existence of positive solutions of (P), if

$$\liminf_{x\to\infty}|x|^p c(x)>\left(\frac{p-1}{p-1+B}\right)^{p-1}\left(\beta^2+\frac{v_1^2}{p^2}+\frac{2\beta}{p}v_2\right)^{\frac{p}{2}},\quad \beta=\frac{N-p}{p}.\tag{1.4}$$

We also show that the condition (1.4) is essentially sharp under appropriate conditions (see remark 3.3).

As an another application of the Hardy-type inequalities, we examine positive solutions $u\in C^2(\Omega)$ to the non-autonomous quasilinear elliptic problem

$$-\Delta_{p,v}u+b(x)|\nabla u|^{p-1}\ge c(x)u^{p-1}\quad\text{in }\Omega,\tag{Q}$$

with general continuous weights $b\ge 0$ and $c>0$ (not necessarily bounded), where Ω is a bounded domain or an exterior domain in $\mathbb{R}^N$, $1<p<N$. Problems similar to (Q), mostly in the case $p=2$ and $v=0$, have been studied in [2, 5, 6, 12, 13, 15, 16, 38].

Berestycki, Hamel, and Nadirashvili [12] proved that $u \equiv 0$ is the unique non-negative solution of

$$-\Delta u - b \cdot \nabla u - cu = 0 \quad \text{in}\, \mathbb{R}^N, \tag{1.5}$$

where $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ with $4c - |b|^2 > 0$. As a special case of the Liouville-type results for the problem (Q), we show that indeed the condition $p^p c - |b|^p > 0$ (which becomes $4c - |b|^2 > 0$ when $p = 2$) is also sufficient to rule out the existence of positive supersolutions to the more general equation

$$-\Delta_p u - |\nabla u|^{p-2}(b + \nabla v) \cdot \nabla u - cu^{p-1} = 0 \quad \text{in } \Omega,$$

where $\Omega = \mathbb{R}^N$ or any exterior domain in $\mathbb{R}^N$ $1 < p < N$, and v satisfying (1.3) (which also includes the case when v is constant).

Berestycki, Hamel, and Rossi [13] extended the results of [12] to elliptic equations with non-constant coefficients. In particular, they proved that if the vector field b and the function c are continuous, then the problem

$$-\Delta u + b(x) \cdot \nabla u \geq c(x)u \quad \text{in}\, \mathbb{R}^N \tag{1.6}$$

does not admit any positive solution, if the functions b and c are bounded and satisfy

$$\liminf_{|x|\to\infty} D(x) > 0, \quad D(x) = c(x) - \frac{|b(x)|^2}{4}. \tag{1.7}$$

In [38], Rossi generalized the non-existence results to the framework of fully non-linear elliptic equations in general unbounded domains, showing that the assumption (1.7) can be relaxed, in particular the case $\liminf_{|x|\to\infty} D(x) < 0$ is allowed (but all the above papers require $\limsup_{|x|\to\infty} D(x) > 0$). Note also that any non-existence result for positive solutions of the problem (Q) can be applied for positive solutions to

$$-\Delta_{p,v} u + |\nabla u|^{p-2} b(x) \cdot \nabla u \geq c(x)u^{p-1} \quad \text{in } \Omega, \tag{R}$$

where this time b is a smooth vector field, because by Cauchy–Schwarz inequality we have $|\nabla u|^{p-2} b(x) \cdot \nabla u \leq |b(x)||\nabla u|^{p-1}$. Recently in [3.14], as consequences of the study of problem (Q), the authors also extended some of the above non-existence results with some improvements to the general problem

$$-\Delta_p u + |\nabla u|^{p-2} b(x) \cdot \nabla u \geq c(x)u^{p-1} \quad \text{in } \Omega, \tag{1.8}$$

in exterior domains. In particular, the case

$$\limsup_{|x|\to\infty} D_p(x) = 0, \quad D_p(x) = c(x) - \frac{|b(x)|^p}{p^p},$$

is included.

We also discuss several Liouville-type theorems, extending the above results to the general problems (Q) and (R) both in bounded and unbounded exterior domains. In particular, applying our Liouville-type results to (1.8), we cover the case $\lim_{|x|\to\infty} D_p(x) = 0$. Also, applying our result for the general problem (Q), in the case when Ω is bounded and $b(x) = b \geq 0$ and $c(x) = c > 0$ are constant, we obtain the non-existence of positive solutions for (Q) if

$$c^{\frac{1}{p}} - \frac{b}{p} > \lambda_1^{\mu}(\Omega, p)^{\frac{1}{p}}.$$

This result also seems to be new even for the case $p = 2$ and v is a constant function. The crucial point in our proofs is that if the problem (Q) has a positive solution in an arbitrary domain $\Omega \subset \mathbb{R}^N$, then the functions b and c satisfy

$$\left(\int_\Omega c(x)|\phi|^p\,d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega |\nabla\phi|^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p\,d\mu\right)^{\frac{1}{p}},$$

for every $\phi \in C_c^\infty(\Omega)$, where $d\mu(x) = e^{v(x)}dx$. The interesting aspect of this estimate is that it does not depend on the solution itself.

At the end of this work, we briefly explain how our findings may be applied to obtain Liouville-type theorems for a class of higher order differential equations. In particular, we prove a Hardy-type inequality

$$\int_\Omega \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2\,dx \leq \int_\Omega |\nabla\phi|^2\,dx$$

for every $\phi \in C_c^\infty(\Omega)$, where $m \geq 1$ is an integer and u is a positive smooth polysuperharmonic function, that is, $(-\Delta)^i u \geq 0$ in Ω, $i = 1, \ldots, m$.

2. Hardy-type inequalities

This section discusses several Hardy-type inequalities. The proofs are based on the following lemma.

Lemma 2.1. *Let $V : \Omega \to \mathbb{R}^N$ be a smooth vector field and $\phi \in C_c^\infty(\Omega)$. Then*

$$\int_\Omega -\big(div\ V + V\cdot\nabla v + (p-1)|V|^{\frac{p}{p-1}}\big)|\phi|^p\,d\mu \leq \int_\Omega |\nabla\phi|^p\,d\mu. \tag{2.1}$$

Moreover, for any $t \in [0,1]$, we have

$$\left|\int_\Omega (-div\ V - V\cdot\nabla v)\phi^p\,d\mu\right|^p \leq p^p\left(\int_\Omega |V|^{\frac{tp}{p-1}}|\phi|^p\,d\mu\right)^{p-1}\left(\int_\Omega |V|^{p(1-t)}|\nabla\phi|^p\,d\mu\right). \tag{2.2}$$

Proof. Let $\phi \in C_c^\infty(\Omega)$ and $\phi_\varepsilon = \sqrt{|\phi|^2 + \varepsilon^2} - \varepsilon$. Then $\phi_\varepsilon \in C_c^\infty(\Omega)$ and by the divergence theorem, we have

$$\int_\Omega (-\mathrm{div}\ V)\phi_\varepsilon^p\,d\mu = \int_\Omega (-\mathrm{div}\ V)\phi_\varepsilon^p e^v\,dx = \int_\Omega V\cdot(p\phi_\varepsilon^{p-1}\nabla\phi_\varepsilon + \phi_\varepsilon^p\nabla v)\,d\mu.$$

Thus

$$\int_\Omega (-\mathrm{div}\ V - V\cdot\nabla v)\phi_\varepsilon^p\, d\mu = p\int_\Omega \phi_\varepsilon^{p-1} V\cdot\nabla\phi_\varepsilon\, d\mu \le p\int_\Omega |V|\phi_\varepsilon^{p-1}|\nabla\phi_\varepsilon|\, d\mu. \quad (2.3)$$

Since $0\le\phi_\varepsilon\le|\phi|$, we have

$$\nabla\phi_\varepsilon = \frac{|\phi|\nabla|\phi|}{\sqrt{|\phi|^2+\varepsilon^2}}.$$

By the fact that $|\nabla|\phi||\le|\nabla\phi|$ a.e., we have

$$\phi_\varepsilon^{p-1}|\nabla\phi_\varepsilon| \le |\phi|^{p-1}|\nabla|\phi|| \le |\phi|^{p-1}|\nabla\phi|.$$

Thus from (2.3) and Young's inequality, we obtain

$$\begin{aligned}\int_\Omega(-\mathrm{div}\ V - V\cdot\nabla v)\phi_\varepsilon^p\, d\mu &\le p\int_\Omega |V||\phi|^{p-1}|\nabla\phi|\, d\mu\\ &\le (p-1)\int_\Omega |V|^{\frac{p}{p-1}}\phi^p d\mu + \int_\Omega|\nabla\phi|^p\, d\mu.\end{aligned} \quad (2.4)$$

By the dominated convergence theorem as $\varepsilon\to 0$ we arrive at (2.1).

For any $t\in[0,1]$, Hölder's inequality implies that

$$\int_\Omega |V|\phi^{p-1}|\nabla\phi|\, d\mu \le \left(\int_\Omega |V|^{\frac{tp}{p-1}}\phi^p d\mu\right)^{\frac{p-1}{p}} \left(\int_\Omega |V|^{p(1-t)}|\nabla\phi|^p d\mu\right)^{\frac{1}{p}}.$$

Applying this in (2.4), we have

$$\left|\int_\Omega (-\mathrm{div}\ V - V\cdot\nabla v)\,\phi^p\, d\mu\right|^p \le p^p\left(\int_\Omega |V|^{\frac{tp}{p-1}}|\phi|^p\, d\mu\right)^{p-1}\left(\int_\Omega |V|^{p(1-t)}|\nabla\phi|^p\, d\mu\right).$$

□

Proposition 2.2. *Let Ω be a domain in $\mathbb{R}^N$ and assume that $E:\Omega\to\mathbb{R}$ is a positive smooth function. Then we have*

$$\int_\Omega \frac{-\Delta_{p,v}E}{E^{p-1}}\,|\phi|^p\, d\mu \le \int_\Omega |\nabla\phi|^p\, d\mu, \quad (2.5)$$

for every $\phi\in C_c^\infty(\Omega)$. Moreover, for any smooth function $F:\Omega\to\mathbb{R}$ and $t\in[0,1]$, we have

$$\left|\int_\Omega(-\Delta_{p,v}F)|\phi|^p\, d\mu\right|^p \le p^p\left(\int_\Omega |\nabla F|^{tp}|\phi|^p\, d\mu\right)^{p-1}\int_\Omega |\nabla F|^{(1-t)p(p-1)}|\nabla\phi|^p\, d\mu \quad (2.6)$$

for every $\phi\in C_c^\infty(\Omega)$.

Proof. We apply (2.1) with $V = |\nabla w|^{p-2}\nabla w$. With this choice of V, we have

$$\begin{aligned} \text{div } V + V\cdot\nabla v + (p-1)|V|^{\frac{p}{p-1}} &= \Delta_p w + |\nabla W|^{p-2}\nabla w\cdot\nabla v + (p-1)|\nabla w|^p \\ &= \Delta_{p,v} w + (p-1)|\nabla w|^p. \end{aligned}$$

From (2.1), we obtain

$$\int_\Omega -(\Delta_{p,v} w + (p-1)|\nabla w|^p)\phi^p \, d\mu \le \int_\Omega |\nabla\phi|^p \, d\mu.$$

By setting $w = \log E$, we have

$$\Delta_{p,v} w + (p-1)|\nabla w|^p = \frac{\Delta_{p,v} E}{E^{p-1}},$$

which leads to

$$\int_\Omega \frac{-\Delta_{p,v} E}{E^{p-1}} \, d\mu \le \int_\Omega |\nabla\phi|^p \, d\mu.$$

Moreover, by substituting $V = |\nabla F|^{p-2}\nabla F$ in (2.2) and applying div $V + V\cdot\nabla v = \Delta_{p,v} F$, we arrive at (2.6). □

Remark 2.3. Notice that if $W : \Omega \to \mathbb{R}$ is a function, which is measurable with respect to the measure $d\mu = e^{v(x)}dx$ and such that for smooth functions $E_\varepsilon > 0$, $\varepsilon > 0$, with $-\Delta_{p,v} E_\varepsilon > 0$, we have

$$\frac{-\Delta_{p,v} E_\varepsilon}{E_\varepsilon^{p-1}} \xrightarrow{\varepsilon\to 0} W \quad \text{pointwise a.e. in } \Omega, \tag{2.7}$$

then by (2.5) and Fatou's lemma we obtain

$$\int_\Omega W|\phi|^p d\mu \le \int_\Omega |\nabla\phi|^p d\mu \tag{2.8}$$

for every $\phi \in C_c^\infty(\Omega)$. Note also that (2.8) holds for any $W \in L^1_{\text{loc}}(\Omega;\mu)$ such that (2.7) holds (in this case we do not need $-\Delta_{p,v} E_\varepsilon > 0$). Moreover, if W satisfies (2.8) for measures $d\mu_\varepsilon = e^{v_\varepsilon(x)}dx$, where v_ε is smooth and $v_\varepsilon \to v$ pointwise a.e. on Ω as $\varepsilon \to 0$, then (2.8) also holds true with the measure $d\mu = e^{v(x)}dx$ provided $W \in L^1_{\text{loc}}(\Omega;\mu)$.

The following elementary inequalities will be useful later. For the proof of part (i) see lemma 2.1 in [30] or theorem 1 in [34] and for (ii) see [31].

Lemma 2.4. *Let* q > 2.

(i) *There exists a constant* $k_q > 0$ *such that*

$$(a+b)^q \ge a^q + b^q + qa^{q-1}b + k_q ab^{q-1} \tag{2.9}$$

for every $a, b \ge 0$, *where* $k_q \in (0,q)$ *when* $2 < q < 3$, $k_q = q$ *when* $q \ge 3$.

(ii) *There exists a constant* $c_q > 0$ *such that*

$$|a-b|^q \geq |a|^q - q|a|^{q-2}ab + c_q|b|^q, \tag{2.10}$$

for every $a, b \in \mathbb{R}$, *where* $c_q = \min_{0\leq t\leq \frac{1}{2}}((1-t)^q - t^q + qt^{q-1})$ *is the optimal constant.*

Recall the classical Hardy inequality

$$\int_\Omega |\nabla\phi|^p\,dx \geq \left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|\phi|^p}{|x|^p}\,dx,$$

for every $\phi \in C_c^\infty(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$, $1 < p < N$. Moreover, the constant $(\frac{N-p}{p})^p$ is best possible.

Many authors have studied Hardy-type inequalities and their generalizations to derivatives of higher order with weights, for example see [19, 20, 24, 25, 32, 33, 36, 37]. As a consequence of proposition 2.2, we have the following version of Hardy's inequality.

Corollary 2.5. *The following generalization of the Hardy inequality holds with the measure* $d\mu = e^{v(x)}dx$. *When* $0 \in \Omega$ *is a domain in* $\mathbb{R}^N$, $1 < p < N$, *then we have*

$$\left(\int_\Omega |\nabla\phi|^p\,d\mu\right)^{\frac{1}{p}} \geq \left|\frac{N-p}{p}\left(\int_\Omega \frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{1}{p}} + \frac{1}{p}\frac{\int_\Omega \frac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu}{\left(\int_\Omega \frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{p-1}{p}}}\right| \tag{2.11}$$

for every $\phi \in C_c^\infty(\Omega)$. *In particular, the following assertions hold true.*

(a) *If* $p \geq 2$ *and*

$$\int_\Omega \frac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu \geq 0$$

for some $\phi \in C_c^\infty(\Omega)$, *then*

$$\begin{aligned}\int_\Omega |\nabla\phi|^p\,d\mu \geq{}& \left(\tfrac{N-p}{p}\right)^p \int_\Omega \tfrac{|\phi|^p}{|x|^p}\,d\mu + \left(\tfrac{N-p}{p}\right)^{p-1}\int_\Omega \tfrac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu \\ &+ \tfrac{1}{p^p}\frac{\left(\int_\Omega \frac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu\right)^p}{\left(\int_\Omega \frac{|\phi|^p}{|x|^p}\,d\mu\right)^{p-1}} + \frac{k_p(N-p)}{p^p}\frac{\left(\int_\Omega \frac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu\right)^{p-1}}{\left(\int_\Omega \frac{|\phi|^p}{|x|^p}\,d\mu\right)^{p-2}},\end{aligned}$$

for some constant $\mathrm{k_p}$, *where* $k_p \in (0,p)$ *when* $2 < p < 3$, $k_p = p$ *when* $p \geq 3$ *and* $k_2 = 0$.

(b) *If* $p \geq 2$ *and*

$$\int_\Omega \frac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu \leq 0$$

for some $\phi \in C_c^\infty(\Omega)$, *then*

$$\begin{aligned} \int_\Omega |\nabla \phi|^p \, d\mu \quad & \geq \left(\tfrac{N-p}{p}\right)^p \int_\Omega \tfrac{|\phi|^p}{|x|^p} \, d\mu - \left(\tfrac{N-p}{p}\right)^{p-1} \int_\Omega \tfrac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu \\ & + \tfrac{c_p}{p^p} \frac{\left(\int_\Omega \frac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu \right)^p}{\left(\int_\Omega \frac{|\phi|^p}{|x|^p} \, d\mu \right)^{p-1}}, \end{aligned}$$

where $c_p = \min_{0 \leq t \leq \frac{1}{2}} ((1-t)^p - t^p + p t^{p-1})$.

(c) *If*

$$\int_\Omega \frac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu = 0$$

for some $\phi \in C_c^\infty(\Omega)$, *then*

$$\left(\frac{N-p}{p} \right)^p \int_\Omega \frac{|\phi|^p}{|x|^p} \, d\mu \leq \int_\Omega |\nabla \phi|^p \, d\mu.$$

Proof. First we assume that $0 \notin \Omega$ and apply proposition 2.2 with $E(x) = |x|^{-m}$. If $0 \in \Omega$ we can mimic the proof starting with $E(x) = (|x|^2 + \varepsilon)^{-m/2}$ and then pass to the limit as $\varepsilon \to 0$ using the fact that $|x|^{-p} \in L^1_{\mathrm{loc}}(\Omega; \mu)$ for $1 < p < N$. Since

$$\Delta_p(|x|^\alpha) = \alpha |\alpha|^{p-2} (\alpha(p-1) + N - p) |x|^{\alpha(p-1)-p},$$

we have

$$\begin{aligned} \Delta_{p,v} E \quad & = \Delta_p E + |\nabla E|^{p-2} \nabla v \cdot \nabla E \\ & = -m|m|^{p-2} (-m(p-1) + N - p) |x|^{-m(p-1)-p} \\ & \quad -m|m|^{p-2} |x|^{-m(p-1)-p} x \cdot \nabla v. \end{aligned}$$

This implies that

$$\frac{-\Delta_{p,v} E}{E^{p-1}} = \frac{m|m|^{p-2}(-m(p-1) + N - p)}{|x|^p} + m|m|^{p-2} \frac{x \cdot \nabla v}{|x|^p}.$$

By (2.5), we then have

$$m|m|^{p-2} \left((-m(p-1) + N - p) \int_\Omega \frac{|\phi|^p}{|x|^p} \, d\mu + \int_\Omega \frac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu \right) \leq \int_\Omega |\nabla \phi|^p \, d\mu. \tag{2.12}$$

for every $\phi \in C_c^\infty(\Omega)$. For a fixed $\phi \in C_c^\infty(\Omega)$, we set $X = (N-p)A + B$, where

$$A = \int_\Omega \frac{|\phi|^p}{|x|^p} \, d\mu \quad \text{and} \quad B = \int_\Omega \frac{x \cdot \nabla v}{|x|^p} |\phi|^p \, d\mu,$$

then we note the left-hand side of (2.12) becomes

$$m|m|^{p-2}((-m(p-1)+N-p)A+B).$$

Then we note that $m=\frac{X}{pA}$ is the number that maximizes the left-hand side of (2.12) with the maximum value $(\frac{|X|}{p})^p\frac{1}{A^{p-1}}$. Hence with this choice of m in (2.12), we get

$$\left|\frac{N-p}{p}\left(\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{1}{p}}+\frac{1}{p}\frac{\int_\Omega\frac{x\cdot\nabla v}{|x|^p}|\phi|^p d\mu}{\left(\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{p-1}{p}}}\right|^p\le\int_\Omega|\nabla\phi|^p\,d\mu. \tag{2.13}$$

We apply lemma 2.4(i) with

$$a=\frac{N-p}{p}\left(\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{1}{p}}\quad\text{and}\quad b=\frac{1}{p}\frac{\int_\Omega\frac{x\cdot\nabla v}{|x|^p}|\phi|^p\,d\mu}{\left(\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu\right)^{\frac{p-1}{p}}}.$$

to prove (a) and lemma 2.4(ii) to prove (b). For (c), the above inequality (2.13) becomes

$$\left(\frac{N-p}{p}\right)^p\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu\le\int_\Omega|\nabla\phi|^p\,d\mu.$$

□

We mention that the above generalized Hardy type inequalities apply well when the function v is a homogeneous function of order of some $k\in\mathbb{R}$. In this case, Euler's formula yields $x\cdot\nabla v(x)=kv(x)$. If $k=0$ then part (c) of corollary 2.5 holds for every $\phi\in C_c^\infty(\Omega)$, also if $v>0$ then (a) and (b) hold for every $\phi\in C_c^\infty(\Omega)$ if $k>0$ or $k<0$, respectively.

Many arguments in this section are based on proposition 2.2 by choosing appropriate functions E and F. We discuss more sophisticated versions of Hardy's inequalities in corollaries 2.6 and 2.7.

Corollary 2.6. *If $\gamma>p-N$ and $d\mu_\gamma=|x|^\gamma dx$ then*

$$\left(\frac{N-p+\gamma}{p}\right)^p\int_\Omega\frac{|\phi|^p}{|x|^p}\,d\mu_\gamma\le\int_\Omega|\nabla\phi|^p\,d\mu_\gamma \tag{2.14}$$

for every $\phi\in C_c^\infty(\Omega)$.

Proof. Let $v_\varepsilon(x)=\frac{\gamma}{2}\log(|x|^2+\varepsilon)$, $\varepsilon>0$, and note that

$$e^{v_\varepsilon(x)}=(|x|^2+\varepsilon)^{\frac{\gamma}{2}}\quad\text{and}\quad x\cdot\nabla v_\varepsilon(x)=\frac{\gamma|x|^2}{|x|^2+\varepsilon}.$$

By (2.13) with v_ε and using the facts that

$$e^{v_\varepsilon(x)}\xrightarrow{\varepsilon\to0}|x|^\gamma\quad\text{and}\quad x\cdot\nabla v_\varepsilon(x)\xrightarrow{\varepsilon\to0}\gamma$$

together with the local integrability of $|x|^{\gamma-p}$ for $\gamma > p - N$ we obtain (2.14). □

COROLLARY 2.7. *Let* $1 < p < \frac{2(N+\gamma)}{N+\gamma+1}$ *and* $\gamma \in (p-N, p)$. *Then we have*

$$\left(\frac{N-p+\gamma}{p}\right)^p \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p}\, d\mu \quad +(p-1)\left(\frac{N-p+\gamma}{p}\right)^{p-2} \int_{\mathbb{R}^N} \frac{|x|^{2-p}|\phi|^p}{(1+|x|)^2\log(1+|x|)}\, d\mu \\ \leq \int_{\mathbb{R}^N} |\nabla\phi|^p\, d\mu$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$, *where* $d\mu = |x|^\gamma dx$.

Proof. Let $p>1$, $p-N<\gamma<p$, $a>1$, $v(x) = \gamma \log|x|$ and

$$E(x) = \frac{\log(1+|x|)}{|x|^a}.$$

After some computations, we obtain

$$\frac{-\Delta_{p,v}E(x)}{E(x)^{p-1}} = \left(\frac{a}{|x|} - \frac{1}{(1+|x|)\log(1+|x|)}\right)^{p-2} \\ \cdot\left(\frac{a(N+\gamma-p-a(p-1))}{|x|^2} - \frac{N+\gamma-1-2(p-1)a}{|x|(1+|x|)\log(1+|x|)} + \frac{p-1}{(1+|x|)^2\log(1+|x|)}\right).$$

Consider the case when $a>1$, $N+\gamma-p-a(p-1) \geq 0$ and $N+\gamma-1-2(p-1)a \geq 0$, which is the case if

$$p \leq \frac{N+\gamma+a}{1+a} \quad \text{and} \quad 1 < a \leq \frac{N+\gamma-1}{2(p-1)}. \tag{2.15}$$

Since $\log(1+t) \geq \frac{t}{1+t}$ for $t \geq 0$, we obtain

$$\frac{-\Delta_{p,v}E(x)}{E(x)^{p-1}} \geq \left(\frac{a-1}{|x|}\right)^{p-2}\left(\frac{(a-1)(N+\gamma-1-a(p-1))}{|x|^2} + \frac{p-1}{(1+|x|)^2\log(1+|x|)}\right) \\ = \frac{(a-1)^{p-1}(N+\gamma-1-a(p-1))}{|x|^p} + \frac{(p-1)(a-1)^{p-2}}{(1+|x|)^2\log(1+|x|)|x|^{p-2}}.$$

If we set $a = \frac{N+\gamma}{p}$ we can see that (2.15) holds for any $1 < p < \frac{2(N+\gamma)}{N+\gamma+1}$, and from the calculations above, we get

$$\frac{-\Delta_{p,v}E(x)}{E(x)^{p-1}} \geq \frac{(\frac{N-p+\gamma}{p})^p}{|x|^p} + \frac{(p-1)(\frac{N-p+\gamma}{p})^{p-2}}{(1+|x|)^2\log(1+|x|)|x|^{p-2}}.$$

The claim follows from proposition 2.2. □

The uncertainty principle can be stated as

$$\left(\int_{\mathbb{R}^N} |x|^2\phi^2\, dx\right)\left(\int_{\mathbb{R}^N} |\nabla\phi|^2\, dx\right) \geq \frac{N^2}{4}\left(\int_{\mathbb{R}^N} \phi^2\, dx\right)^2 \tag{2.16}$$

for all $\phi \in L^2(\mathbb{R}^N)$, see [36]. The uncertainty principle in quantum mechanics asserts that the momentum and position of a particle cannot be determined simultaneously, see Cazacu–Flynn–Lam [24] or the book of Balinsky–Evans–Lewis [11]. The following corollary of proposition 2.2 is a general form of L^p-uncertainty principle with the measure $d\mu(x) = e^{v(x)}dx$.

COROLLARY 2.8. *Let $1 < p < \infty$. Then any $\phi \in L^p(\mathbb{R}^N)$ satisfies*

$$\left|\int_{\mathbb{R}^N} \frac{N + x\cdot\nabla v}{p}|\phi|^p\,d\mu\right|^p \le \left(\int_{\mathbb{R}^N} |x|^{\frac{p}{p-1}}|\phi|^p\,d\mu\right)^{p-1}\int_\Omega |\nabla\phi|^p\,d\mu. \tag{2.17}$$

In particular, if v *is constant, we have*

$$\left(\frac{N}{p}\right)^p\left(\int_{\mathbb{R}^N}|\phi|^p\,dx\right)^p \le \left(\int_{\mathbb{R}^N} |x|^{\frac{p}{p-1}}|\phi|^p\,dx\right)^{p-1}\int_\Omega |\nabla\phi|^p\,dx,$$

which coincides with (2.16) when p = 2.

Proof. It suffices to prove (2.17) for $\phi \in C_c^\infty(\mathbb{R}^N)$, then the conclusion follows by a density argument. Let $\varepsilon > 0$. We apply (2.6) with $t = 1$ and $F_\varepsilon(x) = (|x|^2 + \varepsilon)^{\frac{p}{2(p-1)}}$ and obtain

$$\left|\int_\Omega \Delta_{p,v}F_\varepsilon|\phi|^p\,d\mu\right|^p \le p^p\left(\int_\Omega |\nabla F_\varepsilon|^p|\phi|^p\,d\mu\right)^{p-1}\int_\Omega |\nabla\phi|^p\,d\mu. \tag{2.18}$$

Using the facts that

$$\Delta_{p,v}F_\varepsilon(x) \xrightarrow{\varepsilon\to 0} \left(\frac{p}{p-1}\right)^{p-1}(N + x\cdot\nabla v(x)) \quad\text{and}\quad |\nabla F_\varepsilon(x)| \xrightarrow{\varepsilon\to 0} \frac{p}{p-1}|x|^{\frac{1}{p-1}},$$

and passing to the limit in (2.18), we obtain (2.17). □

The hydrogen uncertainty principle

$$\left(\int_{\mathbb{R}^N}\phi^2\,dx\right)\left(\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx\right) \ge \frac{(N-1)^2}{4}\left(\int_{\mathbb{R}^N}\frac{\phi^2}{|x|}\,dx\right)^2 \tag{2.19}$$

for all $\phi \in L^2(\mathbb{R}^N)$ is connected both to the uncertainty principle and Hardy's inequality. Moreover, it is related to the ground state of a system with a single fixed nucleus and one electron, or a hydrogen atom, see [24]. The following corollary of proposition 2.2 is a general form of the hydrogen uncertainty principle.

COROLLARY 2.9. *Let $1 < p < N + 1$. Then any $\phi \in L^p(\mathbb{R}^N)$ satisfies*

$$\left|\int_{\mathbb{R}^N}\frac{N-p+1+x\cdot\nabla v}{p}\frac{|\phi|^p}{|x|^{p-1}}\,d\mu\right|^p \le \left(\int_{\mathbb{R}^N}|x|^{\frac{p(2-p)}{p-1}}|\phi|^p\,d\mu\right)^{p-1}\int_\Omega|\nabla\phi|^p\,d\mu, \tag{2.20}$$

where $d\mu(x) = e^{v(x)}dx$. In particular, if v *is constant, we have*

$$\left(\frac{N-p+1}{p}\right)^p\left(\int_{\mathbb{R}^N}\frac{|\phi|^p}{|x|^{p-1}}\,dx\right)^p \le \left(\int_{\mathbb{R}^N}|x|^{\frac{p(2-p)}{p-1}}|\phi|^p\,dx\right)^{p-1}\int_\Omega|\nabla\phi|^p\,dx,$$

which coincides with (2.19) when p = 2.

Proof. It suffices to prove (2.20) for $\phi \in C_c^\infty(\mathbb{R}^N)$ and the conclusion follows by a density argument. Let $\varepsilon > 0$. As in the proof of corollary 2.8, we obtain (2.18). Using the facts that

$$\Delta_{p,v} F_\varepsilon(x) \xrightarrow{\varepsilon\to 0} \left(\frac{1}{p-1}\right)^{p-1} \frac{N-p+1+x\cdot\nabla v}{|x|^{p-1}} \quad \text{and} \quad |\nabla F_\varepsilon(x)| \xrightarrow{\varepsilon\to 0} \frac{|x|^{\frac{2-p}{p-1}}}{p-1},$$

and passing to the limit in (2.18), we obtain (2.20). □

We discuss the L^2-Caffarelli–Kohn–Nirenberg inequality

$$C^2(N,a,b)\left(\int_{\mathbb{R}^N} \frac{|\phi|^2}{|x|^{a+b+1}}\,dx\right)^2 \le \int_{\mathbb{R}^N} \frac{|\nabla\phi|^2}{|x|^{2b}}\,dx \int_{\mathbb{R}^N} \frac{|\phi|^2}{|x|^{2a}}\,dx \tag{2.21}$$

for every $\phi \in C_0^\infty(\mathbb{R}^N\setminus\{0\})$, where $C(N,a,b)$, $a,b \in \mathbb{R}\cup\{\infty\}$, is a constant independent of ϕ. The best constant $C^2(N,a,b) > 0$ is known and the minimizers are fully described, see [23]. For example, it is shown in [23] that $C(N,a,b) = \frac{1}{2}|N-(a+b+1)|$ when $(a,b)\in A$, where

$$A = \left\{(a,b)|b+1-a>0, b\le \tfrac{N-2}{2}\right\} \cup \left\{(a,b)|b+1-a<0, b\ge \tfrac{N-2}{2}\right\}.$$

The following corollary can be considered as a L^p form of (2.21) with the measure $d\mu(x) = e^{v(x)}dx$.

Corollary 2.10. *Let $1 < p < \infty$ and $a,b\in\mathbb{R}$. Then*

$$\left|\int_\Omega \frac{N-((p-1)a+b+1)+x\cdot\nabla v}{|x|^{(p-1)a+b+1}}|\phi|^p\,d\mu\right|^p \le p^p\left(\int_\Omega \frac{|\phi|^p}{|x|^{pa}}\,d\mu\right)^{p-1}\int_\Omega \frac{|\nabla\phi|^p}{|x|^{pb}}\,d\mu \tag{2.22}$$

for every $\phi\in C_0^\infty(\mathbb{R}^N\setminus\{0\})$. In particular, if v *is constant, we have*

$$\left|\frac{N-(p-1)a-b-1}{p}\right|^p\left(\int_\Omega \frac{|\phi|^p}{|x|^{(p-1)a+b+1}}\,dx\right)^p \le \left(\int_\Omega \frac{|\phi|^p}{|x|^{pa}}\,dx\right)^{p-1}\int_\Omega \frac{|\nabla\phi|^p}{|x|^{pb}}\,dx,$$

which coincides with (2.21) for p = 2.

Proof. For the proof, we apply (2.2) in lemma 2.1 with $V(x) = |x|^{-\beta}x$, $\beta\in\mathbb{R}$. Since $\operatorname{div} V(x) = (N-\beta)|x|^{-\beta}$ and $|V(x)| = |x|^{1-\beta}$ we obtain, for any $t\in[0,1]$,

$$\left|\int_\Omega \frac{N-\beta+x\cdot\nabla v}{|x|^\beta}|\phi|^p\,d\mu\right|^p \le p^p\left(\int_\Omega \frac{|\phi|^p}{|x|^{\frac{tp(\beta-1)}{p-1}}}\,d\mu\right)^{p-1}\int_\Omega \frac{|\nabla\phi|^p}{|x|^{(1-t)p(\beta-1)}}\,d\mu. \tag{2.23}$$

Set $a = \frac{t(\beta-1)}{p-1}$ and $b = (1-t)(\beta-1)$, hence $\beta = (p-1)a+b+1$, then from (2.23) we arrive at (2.22). □

3. Non-existence results in unbounded domains

This section discusses Liouville theorems for positive solutions of (P) and (Q) in unbounded domains by applying Hardy's inequalities. The following auxiliary result which may be of independent interest.

PROPOSITION 3.1. *Let Ω be an exterior domain in $\mathbb{R}^N$, $1 < p < N$, $d\mu(x) = e^{v(x)}dx$, where* v *satisfies (1.3). If a non-negative function* g *satisfies*

$$\int_\Omega g|\phi|^p \, d\mu \leq \int_\Omega |\nabla\phi|^p \, d\mu \tag{3.1}$$

for every non-negative $\phi \in C_c^\infty(\Omega)$, then

$$\liminf_{|x|\to\infty} |x|^p g(x) \leq \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}, \quad \beta = \frac{N-p}{p}. \tag{3.2}$$

Proof of proposition 3.1: Assume that g satisfies (3.1) and for simplicity let $\Omega = R^N \setminus \overline{B_{R_0}}$, for some $R_0 > 0$. Let $d>1$, $R > 2R_0$ and let ψ a smooth function in Ω such that $0 \leq \psi \leq 1$, $x \in \Omega$, $\psi = 0$ when $R_0 < |x| < \frac{R}{2}$ and $|x| > 2dR$, $\psi = 1$ in $R < |x| < dR$, $|\nabla\psi| \leq \frac{4}{R}$ when $\frac{R}{2} < |x| < R$ and $|\nabla\psi| \leq \frac{4}{dR}$ when $\gamma R < |x| < 2dR$. We take

$$\phi(x) = |x|^{-\beta} e^{\frac{-v(x)}{p}} \psi(x), \quad \beta = \frac{N-p}{p},$$

as a test function in (3.1) and observe that

$$\begin{aligned} \int_\Omega |\nabla\phi|^p \, d\mu &= \int_{\frac{R}{2}<|x|<2dR} |\nabla\phi|^p \, d\mu \\ &= \int_{\frac{R}{2}<|x|<R} |\nabla\phi|^p \, d\mu + \int_{R<|x|<dR} |\nabla\phi|^p \, d\mu + \int_{dR<|x|<2dR} |\nabla\phi|^p \, d\mu \\ &= I_1(R) + I_2(R) + I_3(R). \end{aligned}$$

We estimate each $I_i(R)$, $i = 1, 2, 3$, separately. We note that

$$\begin{aligned} \nabla\phi(x) &= -e^{\frac{-v(x)}{p}} \left(\beta|x|^{-\beta-2}\psi(x)x - |x|^{-\beta}\nabla\psi(x) + \tfrac{1}{p}|x|^{-\beta}\psi(x)\nabla v(x)\right) \\ &= -e^{\frac{-v(x)}{p}} |x|^{-(\beta+1)} \left(\beta\psi(x)\tfrac{x}{|x|} - |x|\nabla\psi(x) + \tfrac{1}{p}\psi(x)|x|\nabla v(x)\right). \end{aligned}$$

Since $0 \leq \psi \leq 1$, $|\nabla\psi| \leq \frac{4}{R}$ and $\beta + 1 = \frac{N}{p}$, we obtain

$$|\nabla\phi(x)|^p \leq e^{-v(x)}|x|^{-N} \left(\beta + \frac{4|x|}{R} + \frac{1}{p}|x||\nabla v(x)|\right)^p.$$

By setting

$$M = \frac{1}{p}\sup_{x\in\Omega} |x||\nabla v(x)|,$$

we have

$$|\nabla\phi(x)|^p \leq (\beta + 4 + M)^p|x|^{-N}e^{-v(x)} = C_1|x|^{-N}e^{-v(x)}, \quad \frac{R}{2} < |x| < R, \tag{3.3}$$

where C_1 is independent of R. Similarly as above we get

$$|\nabla\phi(x)|^p \leq C_1|x|^{-N}e^{-v(x)}, \quad dR < |x| < 2dR. \tag{3.4}$$

Since $\psi = 1$ in $R < |x| < dR$, we have

$$|\nabla\phi(x)| = |x|^{-(\beta+1)}e^{\frac{-v(x)}{p}}\left|\beta\frac{x}{|x|} + \frac{1}{p}|x|\nabla v(x)\right|, \quad R < |x| < dR.$$

We also note that

$$\begin{aligned}\left|\beta\tfrac{x}{|x|} + \tfrac{1}{p}|x|\nabla v(x)\right|^2 &= \beta^2 + \tfrac{1}{p^2}|x|^2|\nabla v(x)|^2 + \tfrac{2\beta}{p}x\cdot\nabla v(x)\\ &\leq \beta^2 + \tfrac{M_1(d,R)^2}{p^2} + \tfrac{2\beta}{p}M_2(d,R) = A(R,d),\end{aligned}$$

where

$$M_1(d,R) = \sup_{R<|x|<dR}|x||\nabla v(x)| \quad \text{and} \quad M_2(d,R) = \sup_{R<|x|<dR} x\cdot\nabla v(x).$$

It follows that

$$|\nabla\phi(x)|^p \leq A(R,d)^{\frac{p}{2}}e^{-v(x)}|x|^{-N}, \quad R < |x| < dR. \tag{3.5}$$

By the estimates above and using the fact that

$$\int_{R<|x|<T}|x|^{-N}\,dx = C_N\log\frac{T}{R}$$

together with (3.3) we obtain

$$\begin{aligned}I_1(R) &= \textstyle\int_{\frac{R}{2}<|x|<R}|\nabla\phi|^p d\mu = \int_{\frac{R}{2}<|x|<R}|\nabla\phi|^p e^{v(x)}dx\\ &\leq C_1\textstyle\int_{\frac{R}{2}<|x|<R}|x|^{-N} = C_N C_1\log 2.\end{aligned}$$

Similarly, by (3.4), we have

$$I_3(R) \leq C_N C_1\log 2$$

and by (3.5), we obtain

$$I_2(R) = \int_{R<|x|<dR}|\nabla\phi|^p\,d\mu \leq C_N A(R,d)^{\frac{p}{2}}\log d.$$

Hence, we conclude that

$$\int_\Omega|\nabla\phi|^p\,d\mu = I_1(R) + I_2(R) + I_3(R) \leq 2C_1C_N\log 2 + C_N A(R,d)^{\frac{p}{2}}\log d. \tag{3.6}$$

By the properties of ϕ, we have

$$\begin{aligned}\int_\Omega c(x)|\phi|^p\,d\mu &\geq \int_{R<|x|<dR} c(x)|\phi|^p\,d\mu = \int_{R<|x|<dR} c(x)|x|^{-\beta p}\,dx\\ &= \int_{R<|x|<dR} |x|^p c(x)|x|^{-\beta p-p}\,dx = \int_{R<|x|<dR} |x|^p c(x)|x|^{-N}\,dx\\ &\geq C_N \inf_{R<|x|<dR} |x|^p c(x)\log d.\end{aligned}$$

By the above estimate, (3.6) and (3.1), we obtain

$$\inf_{R<|x|<dR} |x|^p c(x) \leq \frac{2C_1\log 2}{\log d} + A(R,d)^{\frac{p}{2}}. \tag{3.7}$$

By first letting $d\to\infty$ and then $R\to\infty$ in (3.7), we obtain

$$\liminf_{x\to\infty} |x|^p c(x) \leq \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}},$$

which proves (3.2).

3.1. Liouville-type results for the problem (P)

By applying propositions 2.2 and 3.1, we have the following non-existence result for positive solutions to the problem (P).

THEOREM 3.2. *Consider the problem (P) in an exterior domain Ω in $\mathbb{R}^N$, $1<p<N$, where* v *satisfies (1.3).*

(i) *If $p-1+B>0$, then the problem (P) does not admit any positive solution provided*

$$\liminf_{|x|\to\infty} |x|^p c(x) > \left(\frac{p-1}{p-1+B}\right)^{p-1}\left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}, \quad \beta = \frac{N-p}{p}. \tag{3.8}$$

In particular, the problem

$$-\Delta_{p,v}u \geq c(x)u^{p-1} \quad \text{in } \Omega,$$

does not admit any positive solution, if

$$\liminf_{x\to\infty} |x|^p c(x) > \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}. \tag{3.9}$$

(ii) *Let* $\mathrm{E}>0$ *be a smooth function in an exterior domain Ω of $\mathbb{R}^N$, $1<p<N$, with $-\Delta_{p,v}E\geq 0$ in Ω, where* v *satisfies (1.3). Then*

$$\liminf_{|x|\to\infty} |x|^p \frac{-\Delta_{p,v}E(x)}{E(x)^{p-1}} \leq \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}. \tag{3.10}$$

Proof. Let u be a positive solution of (P) in Ω. Let $t>0$ and $u(x) = \frac{w^t(x)}{t}$. Since

$$\nabla u(x) = w(x)^{t-1}\nabla w(x)$$

and

$$\Delta_p u(x) = (t-1)(p-1)w(x)^{(t-1)(p-1)-1}|\nabla w(x)|^p + w^{(t-1)(p-1)}\Delta_p w(x),$$

we obtain

$$-\Delta_{p,v} w(x) \geq \frac{c(x)}{t^{p-1}} w(x)^{p-1} + ((t-1)(p-1)+Bt)\frac{|\nabla w(x)|^p}{w(x)}.$$

Set $t = \frac{p-1}{p-1+B}$ to arrive at

$$\frac{-\Delta_{p,v} w(x)}{w(x)^{p-1}} \geq \left(\frac{p-1+B}{p-1}\right)^{p-1} c(x).$$

By multiplying both sides by $|\phi|^p$, $\phi \in C_c^\infty(\Omega)$, and integration over Ω we get

$$\int_\Omega \frac{-\Delta_{p,v} w}{w^{p-1}}|\phi|^p \, d\mu \geq \left(\frac{p-1+B}{p-1}\right)^{p-1} \int_\Omega c(x)|\phi|^p \, d\mu.$$

Then proposition 2.2 implies that

$$\int_\Omega |\nabla\phi|^p \, d\mu \geq \left(\frac{p-1+B}{p-1}\right)^{p-1} \int_\Omega c(x)|\phi|^p \, d\mu.$$

By proposition 3.1, we see that $g(x) = (\frac{p-1-A}{p-1})^{p-1}c(x)$ must satisfy (3.2), hence there is no positive solution if

$$\liminf_{|x|\to\infty} |x|^p c(x) > \left(\frac{p-1}{p-1-B}\right)^{p-1}\left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}.$$

□

Remark 3.3. Notice that $v_1 \geq |v_2|$ implies

$$\left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}} \geq \left|\frac{N-p+v_2}{p}\right|^p,$$

and the equality holds if and only if $v_1 = |v_2|$. We claim that if

$$\lim_{|x|\to\infty} x\cdot\nabla v(x) = v_2,$$

then the condition

$$\alpha = \limsup_{x\to\infty} |x|^p c(x) < \left(\frac{p-1}{p-1+B}\right)^{p-1}\left|\frac{N-p+v_2}{p}\right|^p, \tag{3.11}$$

suffices for the problem (P) to have a solution in an exterior domain $\Omega_R = \mathbb{R}^N \setminus \overline{B_R}$ with R sufficiently large. Hence, in this case, (3.9) is essentially sharp. To prove this, we search for $t \neq 0$ such that $u(x) = |x|^t$ is a solution to (P) in $\mathbb{R}^N \setminus \overline{B_R}$ for R sufficiently large. A direct computation gives

$$\begin{aligned} &-\Delta_{p,v} u(x) - c(x)u(x)^{p-1} - B\tfrac{|\nabla u(x)|^p}{u(x)} \\ &\quad = |x|^{-p-t(p-1)} \big(-(p-1+B)|t|^p - (N-p+x\cdot\nabla v(x))t|t|^{p-2} - |x|^p c(x)\big) \\ &\quad \geq 0 \end{aligned}$$

for $|x|$ sufficiently large. Assume that $N-p+v_2 > 0$ (the other case is similar). By (3.11), we may choose $\alpha_1 > \alpha$ and $\delta < v_2$ so that $N-p+\delta > 0$ and

$$\alpha_1 < \left(\frac{p-1}{p-1+B}\right)^{p-1} \left(\frac{N-p+\delta}{p}\right)^p. \tag{3.12}$$

By the definitions of α, v_2, for $|x|$ sufficiently large, we have $|x|^p c(x) < \alpha_1$ and $x\cdot\nabla v(x) > \delta$, hence by the computation above we see that $u(x) = |x|^t$ for a $t<0$ is a solution of (P) in Ω_R for R large, if

$$h(t) = -(p-1+B)|t|^p + (N-p+\delta)|t|^{p-1} - \alpha_1 > 0$$

for some $t<0$. By (3.12), we have

$$h(t_0) = \left(\frac{p-1}{p-1+B}\right)^{p-1} \left(\frac{N-p+\delta}{p}\right)^p - \alpha_1 > 0,$$

with

$$t_0 = -\frac{(p-1)(N-p+\delta)}{p(p-1+B)} < 0.$$

Thus, for R large, $u(x) = |x|^{-t_0}$ is a solution to (P) in $\mathbb{R}^N \setminus \overline{B_R}$.

Example 3.4. Consider the problem

$$-\Delta_{p,v} u \geq |x|^a u^q \quad \text{in } \Omega, \tag{3.13}$$

where $a \in \mathbb{R}$, $1 < p < N$, $q > p-1$ and Ω is an exterior domain in $\mathbb{R}^N$. If u is a positive solution of (3.13) then by (3.9) in theorem 3.2 we get

$$\left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}} \geq \liminf_{|x|\to\infty} |x|^p \frac{-\Delta_{p,v}u(x)}{u(x)^{p-1}} \geq \liminf_{|x|\to\infty} |x|^{a+p} u(x)^{q-p+1}.$$

Let us additionally assume that u is p-superharmonic at infinity, i.e., $-\Delta_p u(x) \geq 0$ for $|x| > R$, R large. It is well known that a p-superharmonic function u in an exterior domain Ω satisfies

$$u(x) \geq C|x|^{\frac{p-N}{p-1}}, \quad x \in \Omega,$$

(see for instance [39]), hence we must have $a+p+(q-p+1)\frac{p-N}{p-1} \le 0$ or equivalently $q \ge \frac{(N+a)(p-1)}{N-p}$. Thus, Eq. (3.13) does not admit any positive p-superharmonic solution if $q < \frac{(N+a)(p-1)}{N-p}$. By a similar argument, we see that the equation

$$-\Delta_{p,v} u = \frac{\mu u}{|x|^p} \quad \text{in } \Omega, \quad \mu > 0,$$

where Ω an exterior domain, does not admit any positive supersolution, if

$$\mu > \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p} v_2\right)^{\frac{p}{2}}.$$

Remark 3.5. In [3.14], it is shown that if $v_1 = \limsup_{|x|\to\infty} |x||b(x)| < \infty$, then problem

$$-\Delta_p u + |\nabla u|^{p-2} b(x) \cdot \nabla u \ge c(x) u^{p-1} \quad \text{in } \Omega \tag{3.14}$$

does not have any positive solution in exterior domains provided

$$\liminf_{|x|\to\infty} |x|^p c(x) > \left(\frac{N-p+v_1}{p}\right)^p.$$

If $b = \nabla v$ for some smooth function v, then (3.14) can be written as

$$-\Delta_{p,v} u \ge c(x) u^{p-1} \quad \text{in } \Omega.$$

Then noticing that $|v_2| \le v_1$ (v_1, v_2 defined in (1.3)), we have

$$\left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p} v_2\right)^{\frac{p}{2}} \le \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p} v_1\right)^{\frac{p}{2}} = \left(\beta + \frac{v_1}{p}\right)^p = \left(\frac{N-p+v_1}{p}\right)^p.$$

Hence, theorem 3.2 improves the previous results in [2, 3.14] when $b = \nabla v$ for some smooth vector field v satisfying $|v_2| \neq v_1$. We see this in the next example.

Example 3.6. Consider (3.14) in an exterior domain $\Omega \subset \mathbb{R}^N$, $1 < p < N$, with $b = \nabla v$, where

$$v(x) = \frac{x_1^2}{|x|^2} e_1, \quad e_1 = (1, 0, 0, \dots, 0),$$

for $x = (x_1, \dots, x_N)$. Then we see that

$$x \cdot \nabla v(x) = 0 \quad \text{and} \quad |x||\nabla v(x)| = \frac{2|x_1|\sqrt{x_2^2 + \cdots + x_N^2}}{|x|^2} \le 1, \quad x \in \Omega.$$

Hence, $v_2 = 0$ and $v_1 = 1$. By theorem 3.2, we see that (3.14) does not admit any positive smooth solution, if

$$\liminf_{x\to\infty} |x|^p c(x) > \left(\beta^2 + \frac{1}{p^2}\right)^{\frac{p}{2}} = \left(\left(\frac{N-p}{p}\right)^2 + \frac{1}{p^2}\right)^{\frac{p}{2}}.$$

3.2. Results for the problem (Q)

In this subsection, we discuss Liouville-type results for positive solutions to the problem (Q).

Proposition 3.7. *Consider the problem (Q) is in an arbitrary domain $\Omega \subset \mathbb{R}^N$, $1 < p < N$, with $c \geq 0$ and* b *being continuous functions, and* v *satisfying (1.3). If the problem has a positive solution in Ω, then*

$$\left(\int_\Omega c(x)|\phi|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega |\nabla\phi|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p \, d\mu\right)^{\frac{1}{p}}, \tag{3.15}$$

and

$$\int_\Omega \left(\frac{c(x)}{(t+1)^{p-1}} - \frac{|b(x)|^p}{p^p t^{p-1}}\right)|\phi|^p \, d\mu \leq \int_\Omega |\nabla\phi|^p \, d\mu, \quad t > 0, \tag{3.16}$$

for every $\phi \in C_c^\infty(\Omega)$, which also implies that

$$\inf_{\operatorname{supp}\phi}\left(1 - \frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \int_\Omega c(x)|\phi|^p \, d\mu \leq \int_\Omega |\nabla\phi|^p \, d\mu. \tag{3.17}$$

Proof. Let $u > 0$ be a positive solution of (Q). As in the proof of theorem 3.2, let $t > 0$ and $u(x) = \frac{w(x)^{t+1}}{t+1}$. Since

$$\Delta_p u(x) = t(p-1)w(x)^{k(p-1)-1}|\nabla w(x)|^p + w(x)^{t(p-1)}\Delta_p w(x),$$

we have

$$\begin{aligned}&-t(p-1)w(x)^{t(p-1)-1}|\nabla w(x)|^p - w(x)^{t(p-1)}\Delta_{p,v}w(x) + b(x)w(x)^{tp}|\nabla w(x)|^p \\ &\geq c(x)\frac{w(x)^{(t+1)(p-1)}}{(t+1)^{p-1}}.\end{aligned}$$

Dividing both sides of the above inequality by $w(x)^{(t+1)(p-1)}$, we get

$$\begin{aligned}\frac{-\Delta_{p,v}w(x)}{w(x)^{p-1}} &\geq \frac{c(x)}{(t+1)^{p-1}} + t(p-1)\frac{|\nabla w(x)|^p}{w^p} - b(x)\frac{|\nabla w(x)|^{p-1}}{w(x)^{p-1}} \\ &= \frac{c(x)}{(t+1)^{p-1}} + t(p-1)T(x)^p - b(x)T(x)^{p-1},\end{aligned}$$

where

$$T(x) = \frac{|\nabla w(x)|}{w(x)}.$$

Then noticing that for $A, B > 0$, we have

$$\min_{T>0}(AT^p - BT^{p-1}) = -\frac{(p-1)^{p-1}}{p^p}\frac{B^p}{A^{p-1}}$$

we obtain

$$\frac{-\Delta_{p,v}w(x)}{w(x)^{p-1}} \geq \frac{c(x)}{(t+1)^{p-1}} - \frac{|b(x)|^p}{p^p t^{p-1}}. \tag{3.18}$$

Hence, from proposition 2.2, we have

$$\int_\Omega \left(\frac{c(x)}{(t+1)^{p-1}} - \frac{|b(x)|^p}{p^p t^{p-1}} \right) |\phi|^p \, d\mu \le \int_\Omega |\nabla \phi|^p \, d\mu, \quad t > 0, \tag{3.19}$$

which proves (3.16).

Note also that, for $x \in \operatorname{supp} \phi$, we have

$$\begin{aligned} \tfrac{c(x)}{(t+1)^{p-1}} - \tfrac{|b(x)|^p}{p^p t^{p-1}} &= c(x) \left(\tfrac{1}{(t+1)^{p-1}} - \tfrac{|b(x)|^p}{p^p c(x)} \tfrac{1}{t^{p-1}} \right) \\ &\ge c(x) \left(\tfrac{1}{(t+1)^{p-1}} - \tfrac{B}{t^{p-1}} \right), \end{aligned}$$

where

$$B = \sup_{\operatorname{supp} \phi} \frac{|b(x)|^p}{p^p c(x)}.$$

Using the fact that, for $0 < B < 1$, the function

$$g(t) = \frac{1}{(t+1)^{p-1}} - \frac{B}{t^{p-1}}, \quad t > 0,$$

achieves its maximum

$$g(t_0) = \left(1 - B^{\frac{1}{p}}\right)^p \quad \text{at} \quad t_0 = \frac{B^{\frac{1}{p}}}{1 - B^{\frac{1}{p}}},$$

then by the above computation we see that if $u = \frac{w^{t_0+1}}{t_0+1}$, then w satisfies

$$\frac{-\Delta_{p,v} w(x)}{w(x)^{p-1}} \ge (1 - B^{\frac{1}{p}})^p c(x).$$

From (3.19), we then have

$$\left(1 - B^{\frac{1}{p}}\right)^p \int_\Omega c(x) |\phi|^p \, d\mu \le \int_\Omega |\nabla \phi|^p \, d\mu$$

for every $\phi \in C_c^\infty(\Omega)$. This proves (3.17).

Moreover, if for a $\phi \in C_c^\infty(\Omega)$ we multiply both sides of (3.18) by $|\phi|^p$, integrating over Ω and applying the Hardy-type inequality (2.5) in proposition 2.2, we obtain

$$\int_\Omega c(x) |\phi|^p \, d\mu \le (t+1)^{p-1} \int_\Omega |\nabla \phi|^p \, d\mu + \frac{(t+1)^{p-1}}{t^{p-1}} \int_\Omega \frac{b(x)^p}{p^p} |\phi|^p \, d\mu, \tag{3.20}$$

which is true for all $t > 0$. Let $\phi \in C_c^\infty(\Omega)$ and set

$$A = \int_\Omega |\nabla \phi|^p \, d\mu \quad \text{and} \quad B = \int_\Omega \frac{b(x)^p}{p^p} |\phi|^p \, d\mu.$$

Then (3.20) can be rewritten as

$$\int_\Omega c(x)|\phi(x)|^p \, d\mu \le A(t+1)^{p-1} + B\frac{(t+1)^{p-1}}{t^{p-1}}, \quad t>0.$$

The best possible choice for t would be the one which minimizes the right-hand side of the above inequality. Note that for $A>0$ and $B \ge 0$, the function

$$f(t) = A(t+1)^{p-1} + B\frac{(t+1)^{p-1}}{t^{p-1}}, \quad t>0,$$

achieves its minimum

$$f(t_0) = (A^{\frac{1}{p}} + B^{\frac{1}{p}})^p \quad \text{at} \quad t_0 = \left(\frac{B}{A}\right)^{\frac{1}{p}}.$$

Therefore, by setting

$$t = \left(\frac{\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p \, d\mu}{\int_\Omega |\nabla \phi|^p \, d\mu}\right)^{\frac{1}{p}},$$

we arrive at

$$\int_\Omega c(x)|\phi|^p \, d\mu \le \left(\left(\int_\Omega |\nabla \phi|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega \frac{b(x)^p}{p^p}|\phi|^p \, d\mu\right)^{\frac{1}{p}}\right)^p,$$

which proves (3.15). □

Using the above results, we can formulate our non-existence results for the problem (Q).

THEOREM 3.8. *Let $1<p<N$, $b, c \in C(\Omega)$, where $\Omega = \mathbb{R}^N \setminus \overline{B_{R_0}}$, $R_0>0$, is an exterior domain in $\mathbb{R}^N$, with $c(x) - \frac{b(x)^p}{p^p} > 0$ for large $|x|$ and* v *is smooth and satisfies (1.3). Then the problem (Q) does not have any positive solution in Ω, if either*

$$\liminf_{|x|\to\infty} |x|^p \left(\frac{c(x)}{(t+1)^{p-1}} - \frac{b(x)^p}{p^p t^{p-1}}\right) > \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{p}{2}}, \quad \beta = \frac{N-p}{p}, \tag{3.21}$$

for some t > 0 *or*

$$\sup_{R>2R_0}\left(\inf_{\frac{R}{2}<|x|<2dR}\left(1 - \frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \inf_{R<|x|<dR} |x|^p c(x)\right) = \infty, \tag{3.22}$$

for some d > 1*. In particular, if $\tau = \limsup_{|x|\to\infty} |x| b(x) < \infty$, then the problem (Q) does not have any positive solution, if*

$$\liminf_{|x|\to\infty} |x|^p c(x) > \left(\frac{\tau}{p} + \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p}v_2\right)^{\frac{1}{2}}\right)^p. \tag{3.23}$$

Proof. The proof of non-existence of positive solution under condition (3.21) is a consequence of (3.16) and proposition 3.1. To prove the result under the condition (3.22) note that by (3.17), if (Q) has a solution $u > 0$, then we have

$$\inf_{\operatorname{supp}\phi}\left(1-\frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p\int_\Omega c(x)|\phi|^p\,d\mu \le \int_\Omega |\nabla\phi|^p\,d\mu \tag{3.24}$$

for every $\phi \in C_c^\infty(\Omega)$. Consider the same test function ϕ as in the proof of proposition 3.1. By applying (3.6) for ϕ, from (3.24), we get

$$\inf_{\operatorname{supp}\phi}\left(1-\frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p\int_\Omega c(x)|\phi|^p\,d\mu \le 2C_1C_N\log 2 + C_N A(R,d)^{\frac{p}{2}}\log d \le K_{d,N},$$

where $K_{d,N}$ is a constant independent of R. Since we have $\operatorname{supp}\phi = \{\frac{R}{2} \le |x| \le 2dR\}$ and $\phi \equiv 1$ in $R < |x| < dR$, we obtain

$$\inf_{\frac{R}{2}<|x|<2\gamma R}\left(1-\frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p\inf_{R<|x|<\gamma R}|x|^p c(x) \le K_{d,N}. \tag{3.25}$$

Hence, the problem does not admit any positive solution if (3.22) holds true.

To prove the last part, let

$$\alpha = \liminf_{|x|\to\infty}|x|^p c(x).$$

If $\alpha = \infty$, then (3.21) obviously holds and there is no positive solution. Thus, we assume that $\alpha < \infty$. By

$$\liminf_{|x|\to\infty}|x|^p\left(\frac{c(x)}{(t+1)^{p-1}}-\frac{b(x)^p}{p^pt^{p-1}}\right) \ge \frac{\alpha}{(t+1)^{p-1}}-\frac{\tau^p}{p^pt^{p-1}},$$

we see that (3.21) holds, if

$$\frac{\alpha}{(t+1)^{p-1}}-\frac{\tau^p}{p^pt^{p-1}} > \left(\beta^2+\frac{v_1^2}{p^2}+\frac{2\beta}{p}v_2\right)^{\frac{p}{2}}$$

for some $t > 0$. Taking $t = \tau(p\alpha^{\frac{1}{p}}-\tau)^{-1}$ we find that the inequality above becomes

$$\alpha > \left(\frac{\tau}{p}+\left(\beta^2+\frac{v_1^2}{p^2}+\frac{2\beta}{p}v_2\right)^{\frac{1}{2}}\right)^p.$$

This concludes the proof. □

Remark 3.9. By [12], the trivial solution $u \equiv 0$ is the unique non-negative solution of the equation

$$-\Delta u - b \cdot \nabla u - cu = 0 \quad \text{in } \mathbb{R}^N,$$

where $b \in \mathbb{R}^N$, $c \in \mathbb{R}$ with $4c - |b|^2 > 0$. By [3.14], this is true for

$$-\Delta_p u - |\nabla u|^{p-2} b \cdot \nabla u - cu = 0 \quad \text{in } \Omega,$$

where Ω is an exterior domain in $\mathbb{R}^N$, $1 < p < N$ and $c - \frac{|b|^p}{p^p} > 0$. A simple application of theorem 3.8 shows that the same result is true for the more general equation

$$-\Delta_p u - |\nabla u|^{p-2}(b + \nabla v) \cdot \nabla u - cu^{p-1} = 0 \quad \text{in } \Omega, \tag{3.26}$$

where Ω is an exterior domain in $\mathbb{R}^N$, $1 < p < N$, $c - \frac{|b|^p}{p^p} > 0$ and v satisfying(1.3). Indeed, since

$$\big||\nabla u|^{p-2} b \cdot \nabla u\big| \leq |b||\nabla u|^{p-1},$$

we note that any positive solution of (3.26) is also a solution of

$$-\Delta_{p,v} u + |b||\nabla u|^{p-1} \geq cu^{p-1} \quad \text{in } \Omega.$$

Note that if $c - \frac{|b|^p}{p^p} > 0$, then $\delta = 1 - \frac{|b|}{p\sqrt[p]{c}} > 0$. Thus we may apply (3.22) to conclude the result, since

$$\begin{aligned} \sup_{R>2R_0} \left(\inf_{\frac{R}{2}<|x|<2dR} \left(1 - \frac{b(x)}{p\sqrt[p]{c(x)}}\right)^p \inf_{R<|x|<dR} |x|^p c(x) \right) &= \delta c \sup_{R>2R_0} \inf_{R<|x|<dR} |x|^p \\ &= \infty. \end{aligned}$$

Example 3.10. Consider the equation

$$-\Delta_p u - |\nabla u|^{p-2} \left(b + \frac{\gamma x}{|x|^2}\right) \cdot \nabla u - cu^{p-1} = 0 \quad \text{in } \Omega, \tag{3.27}$$

where $b \in \mathbb{R}^N$, $\gamma \in \mathbb{R}$ and Ω is an exterior domain in $\mathbb{R}^N$, $1 < p < N$. Since $\frac{\gamma x}{|x|^2} = \nabla(\gamma \log|x|)$ and $v(x) = \gamma \log|x|$ satisfies (1.3), by the above remark, for any $\gamma \in \mathbb{R}$, the problem (3.27) does not have any positive supersolution provided $c - \frac{|b|^p}{p^p} > 0$.

Remark 3.11. As in remark 3.3, we note that, by the inequality $v_1 \geq |v_2|$, we have

$$\left(\frac{\tau}{p} + \left(\beta^2 + \frac{v_1^2}{p^2} + \frac{2\beta}{p} v_2\right)^{\frac{1}{2}}\right)^p \geq \left(\frac{\tau + |N - p + v_2|}{p}\right)^p$$

and the equality holds iff $v_1 = |v_2|$. We show that if $v_2 = \lim_{|x|\to\infty} x \cdot \nabla v$ and

$$\alpha = \limsup_{|x|\to\infty} |x|^p c(x) < \left(\frac{\tau + |N - p + v_2|}{p}\right)^p, \tag{3.28}$$

then the problem (Q) has a positive solution in all exterior domains $\mathbb{R}^N \setminus \overline{B_R}$ with R sufficiently large. To see this, assume (3.28) holds and consider the case $N-p+v_2>0$ (the other case is similar). Let $\alpha_1>\alpha$, $\tau_1<\tau$ and $\delta<v_2$ be so that so that $N-p+\delta>0$ and

$$\alpha_1 < \left(\frac{\tau_1+N-p+\delta}{p}\right)^p. \tag{3.29}$$

We search for $t>0$ such that $u(x)=|x|^{-t}$ is a solution to (Q) in $\mathbb{R}^N\setminus\overline{B_R}$ for R sufficiently large. A direct computation gives

$$\begin{aligned}&-\Delta_{p,v}u(x)+b(x)|\nabla u(x)|^{p-1}-c(x)u(x)^{p-1}\\ &\quad=|x|^{-p-t(p-1)}\left(-(p-1)t^p+t^{p-1}\left(N-p+x\cdot\nabla v+|x|b(x)\right)-|x|^pc(x)\right)\geq 0\end{aligned}$$

for $|x|$ sufficiently large. By the definitions of α,τ,v_2, we have $|x|^pc(x)<\alpha_1$, $|x|b(x)>\tau_1$, and $x\cdot\nabla v>\delta$ for $|x|$ sufficiently large, hence the inequality above holds once we have

$$g(t)=-(p-1)t^p+(N-p+\delta+\tau_1)t^{p-1}-\alpha_1\geq 0$$

for some $t>0$. Now if we set

$$t_1=\frac{N-p+\delta+\tau_1}{p}>0$$

then

$$g(t_1)=\left(\frac{N-p+\tau_1+\delta}{p}\right)^p-\alpha_1,$$

which is positive by (3.29). Thus, for R large, $u(x)=|x|^{-t_1}$ is a solution to (Q) in $\mathbb{R}^N\setminus\overline{B_R}$.

4. Non-existence results in bounded domains

In this section, we apply our main results to the problems (P) and (Q) in bounded domains $\Omega\subset\mathbb{R}^N$, $1<p<N$. For the sake of simplicity, we only discuss the case in which c and b are constant functions. We consider the lowest eigenvalue $\lambda_1^\mu(\Omega,p)$ for the weighted p-Laplace equation given by (1.2).

PROPOSITION 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $1<p<N$, and assume that* v *satisfies (1.3).*

(i) *The problem*

$$-\Delta_{p,v}u\geq\lambda u^{p-1}+B\frac{|\nabla u|^p}{u}\quad in\,\Omega, \tag{4.1}$$

where $p-1+B>0$ and $\lambda>0$, does not have any positive solution, if

$$\lambda\left(\frac{p-1+B}{p-1}\right)^{p-1}>\lambda_1^\mu(\Omega,p).$$

(ii) *The problem*

$$-\Delta_p u + b\nabla u|^{p-1} \geq cu^{p-1} \text{ in } \Omega, \tag{4.2}$$

where $b \geq 0$ *and* c $>$ *0 does not have any positive solution in* Ω*, if*

$$c^{\frac{1}{p}} - \frac{b}{p} > \lambda_1^\mu(\Omega, p)^{\frac{1}{p}}.$$

Proof. Let u be a positive solution of problem 4.1 in Ω. As in the proof of theorem 3.2, we see that B and λ have to satisfy

$$\lambda \left(\frac{p-1-A}{p-1}\right)^{p-1} \leq \frac{\int_\Omega |\nabla \phi|^p d\mu}{\int_\Omega |\phi|^p \, d\mu},$$

for every $\phi \in C_0^\infty(\Omega)$, $\phi \neq 0$, which implies that

$$\lambda \left(\frac{p-1-A}{p-1}\right)^{p-1} \leq \lambda_1^\mu(\Omega, p).$$

Hence, the problem (4.1) does not have any positive solution if the inequality above does not hold. This proves (i).

Then we consider (ii). Let u be a positive solution of (4.2) for some $c, b > 0$. From proposition 3.7, we must have

$$\left(\int_\Omega c|\phi|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega |\nabla \phi|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega \frac{b^p}{p^p} |\phi|^p \, d\mu\right)^{\frac{1}{p}},$$

or equivalently

$$\left(c^{\frac{1}{p}} - \frac{b}{p}\right)^p \leq \frac{\int_\Omega |\nabla \phi|^p \, d\mu}{\int_\Omega |\phi|^p \, d\mu},$$

for every $\phi \in C_c^\infty(\Omega)$, $\phi \neq 0$. This implies that

$$\left(c^{\frac{1}{p}} - \frac{b}{p}\right)^p \leq \lambda_1^\mu(\Omega, p).$$

Therefore, (4.2) does not have any positive solution if $c^{\frac{1}{p}} - \frac{b}{p} > \lambda_1^\mu(\Omega, p)^{\frac{1}{p}}$. □

Remark 4.2. Regarding the condition $p-1+B > 0$ in proposition 4.1, it is easy to see that if $p-1+B < 0$, then the problem (P) has a positive solution in an arbitrary proper domain $\Omega \subset \mathbb{R}^N$ assuming the functions $x \cdot \nabla v(x)$ and c are bounded in Ω. Also, when $p-1+B=0$, the same is true if for some $\delta>0$

$$N-p+(x-x_0)\cdot\nabla v(x) \geq \delta, \quad x\in\Omega, \text{ for some } x_0 \notin \Omega, \tag{4.3}$$

or

$$N-p+(x-x_0)\cdot\nabla v(x) \leq -\delta, \quad x\in\Omega \text{ for some } x_0 \notin \Omega. \tag{4.4}$$

To see this consider $u(x)=|x-x_0|^t$, $t\neq 0$, with $x_0 \notin \Omega$. As in remark 3.3, we see that u is a solution to (P), if

$$-(p-1+B)|t|^p + (N-p+(x-x_0)\cdot\nabla v(x))t|t|^{p-2} - |x-x_0|^p c(x) \geq 0, \quad x\in\Omega,$$

which is obviously true for all large $t>0$, if $p-1+B<0$. When $p-1+B=0$ then the above inequality holds true for large $t>0$ if v satisfies (4.3), and for $t<0$, $|t|$ large, if v satisfies (4.4).

5. Higher order differential equations

The final section illustrates how our findings can be applied to higher order differential equations of the type

$$(-\Delta)^m u \geq c(x)u^q \quad \text{in } \Omega.$$

In recent years, there has been a lot of interest in the existence or non-existence of solutions to several kinds of higher order differential equations and systems on $\mathbb{R}^N$. For instance, a differential equation or inequality of the form

$$(-\Delta)^m u \geq f(u) \quad \text{in } \Omega, \tag{5.1}$$

where $\Omega=\mathbb{R}^N$ or an exterior domain in $\mathbb{R}^N$. A relevant special case of (5.1) is $f(u)=u^p$ with $p>0$. It is well known that if $1<p<\frac{N}{N-2m}$ then the latter inequality in the whole space does not admit any non-negative polysuperharmonic solution u, that is,

$$(-\Delta)^i u \geq 0 \quad \text{in } \Omega, \quad i=1,\ldots,m,$$

see, for example, corollary 3.6 in Caristi, D'Ambrosio, and Mitidieri [22], where the authors have proved Liouville theorems for supersolutions of the polyharmonic Hénon–Lane–Emden system and also explored its connection with the Hardy–Littlewood–Sobolev systems. For more results on the structure of positive solutions or some related problems, we refer to [18, 26, 27] and the references therein.

We start with the following Hardy-type inequality involving the operator $(-\Delta)^m$ that can be proved using proposition 2.2.

Corollary 5.1. *Let $m \geq 1$ be an integer, Ω be a domain in $\mathbb{R}^N$ and $u \in C^{2m}(\Omega)$ a positive polysuperharmonic function. Then we have*

$$\int_\Omega \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2 \, dx \leq \int_\Omega |\nabla \phi|^2 \, dx \tag{5.2}$$

for every $\phi \in C_c^\infty(\Omega)$

Proof. Let u be a smooth positive polysuperharmonic function. Proposition 2.2 with $p = 2$ and v as a constant and $E = (-\Delta)^{i-1} u$, $i = 1, \ldots, m$, implies that

$$\int_\Omega \frac{(-\Delta)^i u}{(-\Delta)^{i-1} u} |\phi|^2 \, dx \leq \int_\Omega |\nabla \phi|^2 \, dx, \tag{5.3}$$

for every $\phi \in C_c^\infty(\Omega)$. We observe that

$$\left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2 = \prod_{i=1}^m \left(\frac{(-\Delta)^i u}{(-\Delta)^{i-1} u} \phi^2\right)^{\frac{1}{m}}$$

and by Holder's inequality and (5.3) we obtain

$$\int_\Omega \left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \phi^2 \, dx \leq \prod_{i=1}^m \left(\int_\Omega \frac{(-\Delta)^i u}{(-\Delta)^{i-1} u} |\phi|^2 \, dx\right)^{\frac{1}{m}} \leq \int_\Omega |\nabla \phi|^2 \, dx.$$

□

Corollary 5.2. *Assume that $u \in C^{2m}(\Omega)$ satisfies*

$$\begin{cases} (-\Delta)^m u \geq c(x) u & \text{in } \Omega \\ (-\Delta)^i u > 0 & \text{in } \Omega, \quad i = 0, 1, \ldots, m-1, \end{cases} \tag{5.4}$$

where $m \geq 1$ is an integer, Ω is an exterior domain in $\mathbb{R}^N$, $N > 2m$, and $c \geq 0$. Then

$$\liminf_{|x| \to \infty} |x|^{2m} c(x) \leq \left(\frac{N-2}{2}\right)^{2m}. \tag{5.5}$$

In particular, (5.4) does not admit any positive solution if

$$\liminf_{|x| \to \infty} |x|^2 c(x)^{\frac{1}{m}} > \frac{(N-2)^2}{4}.$$

Proof. If $u \in C^{2m}(\Omega)$ satisfies (5.4), then we have

$$\left(\frac{(-\Delta)^m u}{u}\right)^{\frac{1}{m}} \geq c(x)^{\frac{1}{m}} \quad \text{in } \Omega.$$

and by corollary 5.1 we obtain

$$\int_\Omega c(x)^{\frac{1}{m}}\ \phi^2\,dx \leq \int_\Omega |\nabla\phi|^2\,dx,$$

for every $\phi\in C_c^\infty(\Omega)$. By proposition 3.1 with $p=2$ and $v(x)=0$, we have

$$\liminf_{|x|\to\infty} |x|^2 c(x)^{\frac{1}{m}} \leq \frac{(N-2)^2}{4},$$

which is equivalent to (5.5). □

The above result together with the following lemma can be applied to obtain Liouville-type results for related higher order equations or inequalities in exterior domains.

Lemma 5.3. *Suppose that* u >0 *is a smooth function such that*

$$(-\Delta)^i u>0 \quad in\ \Omega, \quad i=1,\dots,m,$$

where $m\geq 1$ is an integer and Ω is an exterior domain $\Omega\subset\mathbb{R}^N$, $N>2m$. Then there exists a positive constant C*, depending only on* u*, Ω, and* N*, so that*

$$u(x)\geq C|x|^{2m-N} \quad for\ every x\in\Omega. \tag{5.6}$$

Proof. We prove the claim by induction. The case $m=1$ is well known, see for example [10], also see [3] for the case $m=2$. Assume the statement is true for $m-1$. Then since $w=-\Delta u$ satisfies the induction hypothesis we get

$$-\Delta u = w(x)\geq C|x|^{2(m-1)-N} \quad \text{in } \Omega. \tag{5.7}$$

Fix $r_0>0$ such that $\mathbb{R}^N\setminus B_{r_0}\subset\Omega$. Select a $\gamma>0$ so small that $\gamma<\frac{C}{2(m-1)(N-2m)}$ and $u(x)\geq\gamma|x|^{2m-N}$ in a neighbourhood of ∂B_{r_0}. Then for each $\varepsilon>0$, there exists $R_\varepsilon>r_0$ such that

$$u(x)+\varepsilon\geq\varepsilon\geq\gamma|x|^{2m-N} \quad \text{for every } x\in\mathbb{R}^N\setminus B_{R_\varepsilon}.$$

Notice that

$$\begin{aligned}-\Delta(u+\varepsilon) &= -\Delta u\geq C|x|^{2(m-1)-N}\\ &\geq 2\gamma(m-1)(N-2m)|x|^{2(m-1)-N}\\ &= -\Delta(\gamma|x|^{2m-N}).\end{aligned}$$

Applying the maximum principle in $B_R\setminus B_{r_0}$, for each $R>R_\varepsilon$, we get

$$u(x)+\varepsilon \geq \gamma |x|^{2m-N} \quad \text{for every } x \in \mathbb{R}^N \setminus B_{r_0}.$$

Letting $\varepsilon \to 0$, we obtain $u(x) \geq c|x|^{2m-N}$ in $\mathbb{R}^N \setminus B_{r_0}$, which proves (5.6). □

Corollary. *Consider the problem*

$$\begin{cases} (-\Delta)^m u \geq c(x)u^q & \text{in } \Omega, \\ (-\Delta)^i u > 0 & \text{in } \Omega, \quad i=0,1,\dots,m-1, \end{cases} \tag{5.8}$$

where $m \geq 1$ is an integer, Ω is an exterior domain in $\mathbb{R}^N$, $N > 2m$ and $q > 1$. This problem does not admit any positive smooth solution if

$$\lim_{|x|\to\infty} |x|^{2qm-N(q-1)} c(x) = \infty. \tag{5.9}$$

In particular, if $c(x)=|x|^\alpha$ then there is no positive solution when

$$q < \frac{N+\alpha}{N-2m}. \tag{5.10}$$

Moreover, the problem

$$\begin{cases} (-\Delta)^m u \geq \frac{\mu u}{|x|^{2m}} & \text{in } \Omega, \\ (-\Delta)^i u > 0 & \text{in } \Omega, \quad i=0,1,\dots,m-1, \end{cases} \tag{5.11}$$

does not have any positive smooth solution, if

$$\mu > \Big(\frac{N-2}{2}\Big)^{2m}.$$

Proof. Let u be a positive solution of (5.11). Thus

$$(-\Delta)^m u \geq c(x)u^q = (c(x)u^{q-1})u \quad \text{in } \Omega.$$

Then by corollary 5.2, we have

$$\liminf_{|x|\to\infty} |x|^2 (c(x)u(x)^{q-1})^{\frac{1}{m}} \leq \frac{(N-2)^2}{4}.$$

However, by lemma 5.3, we have

$$u(x) \geq C|x|^{2m-N} \quad \text{for every } x \in \Omega,$$

where C is independent of $x \in \Omega$, which together $q-1>0$ imply that

$$\liminf_{|x|\to\infty} |x|^{2qm-N(q-1)} c(x) < \infty.$$

This implies that (5.11) does not have any positive solution if (5.9) holds.

If $c(x) = |x|^\alpha$, then (5.9) reads as

$$\liminf_{|x|\to\infty} |x|^{\alpha+2qm-N(q-1)} = \infty,$$

which holds true if $q < \frac{N+\alpha}{N-2m}$.

The last assertion is an immediate consequence of corollary 5.2. □

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