

# CONCENTRATION AND MULTIPLICITY OF SOLUTIONS FOR FRACTIONAL DOUBLE PHASE PROBLEMS

YOUPEI ZHANG, VICENȚIU D. RĂDULESCU\*, JING CHEN, AND DONGDONG QIN

ABSTRACT. In the present paper, we consider the following fractional double phase problem with nonlocal reaction:

$$\left\{ \begin{array}{l} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 \end{array} \right\} \quad \text{in } \mathbb{R}^N,$$

where  $\varepsilon$  is a positive parameter,  $0 < s < 1$ ,  $2 \leq p < q < \min\{2p, N/s\}$ ,  $0 < \mu < sp$ ,  $(-\Delta)_t^s$  ( $t \in \{p, q\}$ ) is the fractional  $t$ -Laplace operator, the reaction term  $f : \mathbb{R} \mapsto \mathbb{R}$  is continuous and the potential  $V \in C(\mathbb{R}^N, \mathbb{R})$  satisfying a local condition. Using a variational approach and topological tools (the non-standard  $C^1$ -Nehari manifold analysis and the abstract category theory), multiplicity of positive solutions and concentration properties for the above problem are established. Our results extend and complement some previous contributions related to double phase variational integrals.

KEY WORDS: Fractional double phase problem; Nonlocal reaction; Nehari manifold; Category theory.

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## 1. INTRODUCTION

**1.1. Features of the paper and historical comments.** In this paper, we are concerned with the study of concentration and multiplicity properties of solutions for a class of fractional double phase problems with Choquard nonlinearity. The features of this paper are the following:

- (i) the presence of several nonlocal operators with different growth, which generates a double phase associated energy;
- (ii) the analysis developed in this paper is concerned with the *combined effects* of a non-standard fractional operator with unbalanced growth and a nonlocal Choquard term;
- (iii) the potential describing the absorption term satisfies a *local condition* and no information on the behavior of the potential at infinity is imposed;
- (iv) the main concentration property creates a bridge between the *global maximum point* of the solution and the *global minimum* of the potential;
- (v) since the nonlinearity is only assumed to be *continuous*, one cannot apply the standard  $C^1$ -Nehari manifold arguments due to the lack of differentiability of the associated Nehari manifold;
- (vi) our analysis combines the *nonlocal* nature of the fractional  $(p, q)$ -operator and of the Choquard nonlinearity with the *local* perturbation in the absorption term.

The problem studied in this paper combines both above mentioned features. More exactly, we are interested in concentration phenomena associated to a nonlinear Choquard problem driven by fractional Laplace operators with different power. This integro-differential operator appears in several relevant models of applied sciences. We only recall that the fractional power of the Laplace operator is the infinitesimal generator of Lévy stable diffusion process and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. Moreover, fractional Sobolev Spaces have been well known since the beginning of the last century, especially within the framework of harmonic analysis. The starting point in the study of fractional

problems is attributed to the pioneering papers of Caffarelli et al. [20, 21, 22]. For a comprehensive introduction to the study of fractional equations and the use of variational methods in the treatment of these problems, we refer to the monographs by Di Nezza, Palatucci & Valdinoci [27] and Molica Bisci, Rădulescu & Servadei [45]. Therefore, the nonlocal operators are becoming increasingly popular in applied sciences, theoretical research and real-world applications.

Since the content of this paper is at interplay between “double phase problems” and “Choquard problems”, we now recall some pioneering achievements in these fields.

We start with a short description on the development of double phase problems. To the best of our knowledge, the first contributions to this field are due to J. Ball [13, 14], in relationship with problems in nonlinear elasticity and composite materials. Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary. If  $u : \Omega \rightarrow \mathbb{R}^N$  is the displacement and if  $Du$  is the  $N \times N$  matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$(1) \quad I(u) = \int_{\Omega} f(x, Du(x)) dx,$$

where the energy function  $f = f(x, \xi) : \Omega \times \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$  is quasiconvex with respect to  $\xi$ , see Morrey [47]. A simple example considered by Ball is given by functions  $f$  of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where  $\det \xi$  is the determinant of the  $N \times N$  matrix  $\xi$ , and  $g, h$  are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geq c_1 |\xi|^p; \quad \lim_{t \rightarrow +\infty} h(t) = +\infty,$$

where  $c_1$  is a positive constant and  $1 < p < N$ . The condition  $p < N$  is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the variational integral (1) that are discontinuous at one point where a cavity forms; in fact, every  $u$  with finite energy belongs to the Sobolev space  $W^{1,p}(\Omega, \mathbb{R}^N)$ , and thus it is a continuous function if  $p > N$ . In accordance with these problems arising in nonlinear elasticity, Marcellini [41, 42] considered continuous functions  $f = f(x, u)$  with unbalanced growth that satisfy

$$c_1 |u|^p \leq |f(x, u)| \leq c_2 (1 + |u|^q) \quad \text{for all } (x, u) \in \Omega \times \mathbb{R},$$

where  $c_1, c_2$  are positive constants and  $1 \leq p \leq q$ . Recently, a great deal of works have enriched the mathematical analysis of nonlinear models with unbalanced growth, we refer to the works of Bahrouni, Rădulescu & Repovš [12], Crespo-Blanco, Gasínski, Harjulehto & Winkert [23], Liu & Papageorgiou [40], Papageorgiou, Pudelko & Rădulescu [49]. In addition, an overview of recent developments on the regularity of nonlocal double problems can be found in Byun, Ok & Kyeong [19] and De Filippis & Palatucci [25].

In the pioneering works of Fröhlich [30] and Pekar [50], the following Choquard equation was introduced,

$$(2) \quad -\Delta u + u = \left( \frac{1}{|x|} * |u|^2 \right) u \text{ in } \mathbb{R}^3,$$

which models quantum polaron and corresponds to the study of the free electrons in ionic lattices interacting with phonons associated to deformations of the lattices or with the polarisation created on the medium. Choquard also used it to study steady states of the one component plasma approximation in the Hartree-Fock theory describing an electron trapped in its own hole [37]. Equation (2) is also called Schrödinger-Newton equation which combines the Schrödinger equation of quantum physics with nonrelativistic Newtonian gravity. Such a model can also be deduced from the Einstein-Klein-Gordon and Einstein-Dirac systems relating to boson stars and the collapse of galaxy fluctuations of scalar field dark matter, we

bring the reader's attention to Elgart & Schlein [29], Jones [36], Lions [39], Penrose [51, 52] and Schunck & Mielke [55].

In conclusion, based on the historical research background of the fractional double problem and the Choquard problem and the related theoretical foundation, focusing on the characteristics of such problems and the combined effects produced by their combination, we will apply analytical, topological and variational methods, and develop some new techniques to devote ourselves to the study of new phenomena of nonlocal double problems with nonlocal reaction terms, when the potential  $V$  satisfies a general local condition and the nonlinearity  $f$  possesses only the continuity property.

**1.2. Statement of the problem and main result.** In this paper we focus on the existence, multiplicity and concentration behavior of positive solutions for the nonlinear fractional  $(p, q)$ -Choquard problem:

$$(3) \quad \left\{ \begin{array}{l} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N, \end{array} \right\}$$

where  $\varepsilon$  is a positive parameter,  $0 < s < 1$ ,  $2 \leq p < q < \min\{2p, N/s\}$ ,  $0 < \mu < sp$ ,  $(-\Delta)_t^s$  ( $t \in \{p, q\}$ ) is the fractional  $t$ -Laplace operator,  $V : \mathbb{R}^N \mapsto \mathbb{R}$  and  $f : \mathbb{R} \mapsto \mathbb{R}$  are continuous functions,  $F$  is the primitive function of  $f$ , and  $*$  denotes the convolution product.

In the local setting corresponding to  $s \rightarrow 1^-$  (up to normalization), the double phase problem (3) is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [16] that appears in electromagnetism:

$$-\operatorname{div} \left( \frac{Du}{(1 - 2|Du|^2)^{1/2}} \right) = h(u) \text{ in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1 - x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2} x^2 + \frac{5!!}{3! \cdot 2^3} x^3 + \cdots + \frac{(2n-3)!!}{(n-1)! 2^{n-1}} x^{n-1} + \cdots \text{ for } |x| < 1.$$

Taking  $x = 2|Du|^2$  and adopting the first order approximation, we get a particular case of the problem (3) for  $p = 2$  and  $q = 4$ . Furthermore, the  $n$ -th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2} \Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!} \Delta_{2n} u.$$

Finally, we also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div} \left( \frac{|\nabla u|^2}{(1 - |\nabla u|^4)^{3/4}} \nabla u \right)$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Clearly, we can use the Taylor formula to conclude that the fourth-order relativistic operator can be approximated by the following double phase operator

$$u \mapsto \Delta_4 u + \frac{3}{4} \Delta_8 u.$$

The purpose of this paper is to study a class of fractional unbalanced double phase problems in which the nonlocal term appears also in the nonlinear part. Problem (3) is closely related to the analysis of nonlinear problems and stationary waves for models arising in mathematical physics, such as phase transitions, anomalous diffusion, composite materials, image processing, fractional quantum mechanics in the study of particles on stochastic fields, fractional superdiffusion, fractional white-noise limit, etc., see Pucci & Saldi [53]. For more details, we refer interested readers to the preliminary introduction of this topic in [27, 45]. From the point of view of physics, in the semi-classical sense (namely, as  $\varepsilon \rightarrow 0$ ), it is of great significance to study the existence and shape of the standing wave solutions of problem (3), which

can be used to describe the transition relationship between quantum mechanics and classical mechanics. From a mathematical point of view, in the framework of this semi-classical state we can obtain more dynamic behavior information, such as concentration, convergence, decay, bifurcation and other properties.

Note that, if  $p = q = 2$ , after rescaling, problem (3) reduces to the following nonlinear fractional Choquard equation

$$(4) \quad \varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \text{ in } \mathbb{R}^N.$$

When  $\varepsilon = 1$ ,  $F(u) = |u|^p$  and  $f(u) = |u|^{p-2}u$ , d'Avenia, Siciliano & Squassina [24] dealt with the regularity, existence and non existence, symmetry and decay properties of solutions to problem (4). Under the potential  $V$  has a local minimum, Ambrosio [9] established the multiplicity and concentration of positive solutions to problem (4) via the penalization technique and Ljusternik-Schnirelmann theory.

When  $p = q \neq 2$ , after rescaling, problem (3) boils down to the following nonlinear fractional Choquard problem

$$(5) \quad \left\{ \begin{array}{l} \varepsilon^{sp}(-\Delta)_p^s u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N), u > 0 \end{array} \right\}$$

In [7] Ambrosio investigated the existence, multiplicity, and concentration of positive solutions for the fractional Choquard problem (5) assuming that the potential  $V : \mathbb{R}^N \mapsto \mathbb{R}$  fulfills the following condition due to Rabinowitz [54]:

$$(V) \quad V_\infty = \liminf_{|x| \rightarrow +\infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x).$$

These solutions concentrate at global minimum points of  $V$  under the global hypothesis (V). In the local sense  $s = 1$ , problem (5) becomes the following quasi-linear Choquard problem with the  $p$ -Laplace operator

$$(6) \quad \left\{ \begin{array}{l} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), u > 0 \end{array} \right\}$$

for which several existence, multiplicity and concentration results of positive solutions have been studied by different authors, under suitable assumptions on the potential function  $V$  and the reaction  $f$ ; see for instance, Alves & Yang [3, 4, 5]. On the other hand, if  $s = 1$  and  $1 < p < q < N$ , after rescaling, problem (3) reduces to the following  $(p, q)$ -Laplace problem:

$$(7) \quad \left\{ \begin{array}{l} -\varepsilon^p \Delta_p u - \varepsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = \varepsilon^{\mu-N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0 \end{array} \right\}$$

Very recently, by assuming that the potential  $V$  and the nonlinear reaction  $f$  satisfy the following conditions:

$$(\bar{V}) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \text{ and } +\infty > V_\infty = \liminf_{|x| \rightarrow +\infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) > 0;$$

$$(f) \quad f \in C^1(\mathbb{R}, \mathbb{R}),$$

Zhang, Zhang & Rădulescu studied the multiplicity and concentration behavior of positive solutions to problem (7) in [61].

For related concentration and multiplicity properties of solutions, we refer to the recent paper by Alves & de Moraes Filho [2], Alves, Ambrosio & Isernia [1], Ambrosio [8], Ambrosio, Isernia & Rădulescu [10], Ambrosio & Rădulescu [11], Del Pino & Felmer [26], Gao, Tang & Chen [31], Gu & Tang [32], Ji & Rădulescu [34, 35], Moroz & Van Schaftingen [46], Zhang, Zhang [59], Zhang, Zhang & Rădulescu [60] and Zhang, Tang & Rădulescu [62]. We also cite

Mingione and Rădulescu [44] for an overview of recent results concerning elliptic variational problems with nonstandard growth and nonuniform ellipticity.

Inspired by the above works, in this paper we are interested in studying the multiplicity and concentration behavior as  $\varepsilon \rightarrow 0$  of positive solutions for problem (3), when we suppose a local condition on the potential  $V$ , and the nonlinear term  $f$  is only assumed to be continuous (without  $C^1$  differentiability). Precisely, we first impose that  $V$  is a continuous map satisfying the following assumptions:

(V<sub>1</sub>) there exists  $V_0 > 0$  such that  $V_0 := \inf_{x \in \mathbb{R}^N} V(x)$ ;

(V<sub>2</sub>) there exists an open bounded domain  $\Lambda \subset \mathbb{R}^N$  such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Next, we assume that  $f$  is a merely continuous function that verifies the following conditions:

(f<sub>1</sub>)  $\lim_{t \rightarrow 0} \frac{|f(t)|}{|t|^{p-1}} = 0$ ;

(f<sub>2</sub>) there exists  $\nu \in (q, q_s^*(2 - \mu/N)/2)$  such that

$$\lim_{|t| \rightarrow +\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0,$$

where  $q_s^* = Nq/(N - sq)$ ;

(f<sub>3</sub>)  $0 < pF(t) := p \int_0^t f(\tau) d\tau \leq f(t)t$  for all  $t > 0$ ;

(f<sub>4</sub>) the map  $t \mapsto \frac{f(t)}{t^{q-1}}$  is increasing for all  $t > 0$ .

**Remark 1.** *We would like to point out that, since the hypotheses on  $V$  and  $f$  are different from [7, 54, 59, 60, 61], our arguments are totally distinct with them, and improve the previous results for the singularly perturbed fractional problem with nonlocal Choquard reaction, in the sense that we establish multiplicity results and concentration behavior for the fractional  $(p, q)$ -problems involving continuous nonlinearity and by imposing a local condition on the potential  $V$ . Compared with the local case  $s = 1$ , we point out that our result improves Theorem 1.2 in [61]. In addition, we believe that the ideas contained here can be used in other cases to study problems driven by more general operators, under local conditions on the potential  $V$  and without the differentiability of the nonlinearity  $f$ .*

In order to look for positive solutions of problem (3), we may assume that  $f(t) = 0$  for all  $t \leq 0$ .

Recall that if  $A$  is a closed subset of a topological space  $Y$ , then we use  $\text{cat}_Y(A)$  to denote the Ljusternik-Schnirelmann category of  $A$  in  $Y$ , that is, the smallest number of closed and contractible sets in  $Y$  which cover  $A$ . For more details, we refer to Willem [57].

Our main result in this paper establishes the following concentration and multiplicity properties.

**Theorem 1.** *Let  $0 < \mu < sp$ . Assume that the nonlinearity  $f$  fulfils (f<sub>1</sub>)–(f<sub>4</sub>) with  $\nu < (N - \mu)q/(N - sq)$  and the potential  $V$  verifies hypotheses (V<sub>1</sub>)–(V<sub>2</sub>). Then for all  $\delta > 0$  such that  $M_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \subset \Lambda$ , there exists  $\varepsilon_\delta > 0$  such that, for any  $\varepsilon \in (0, \varepsilon_\delta)$  problem (3) has at least  $\text{cat}_{M_\delta}(M)$  positive solutions. Moreover, if  $u_\varepsilon$  denotes one of these solutions and  $x_\varepsilon \in \mathbb{R}^N$  is global maximum point of  $u_\varepsilon$ , then  $\lim_{\varepsilon \rightarrow 0} V(\varepsilon x_\varepsilon) = V_0$ .*

The proof of Theorem 1 is based on topological and variational methods and refined analytic techniques. Let us now take the next steps to outline our strategies and methods for proving Theorem 1:

(1) concerning our variational approach: because there is no information on the behavior of  $V$  at infinity, as in [26], we first modify in a convenient way the nonlinear reaction outside

the set  $\Lambda$ , and then we will study an auxiliary problem. The main characteristic of the corresponding modified energy functional is that it verifies all the conditions of the mountain pass theorem [6]. Finally, we prove that, for  $\varepsilon > 0$  sufficiently small the solutions of the auxiliary problem are indeed solutions of the original one;

(2) topological techniques: in order to get multiple solutions of the auxiliary problem, we will use the generalized Nehari manifold method and some well-known topological techniques proposed by Benci and Cerami in [15] based on accurate comparisons between the category of some sub-level sets of the modified functional and the category of the set  $M$ . Note that the nonlinearity is only continuous, we stress that standard  $C^1$ -Nehari manifold arguments as in [61] can not employ in our setting due to the lack of differentiability of the associated Nehari manifold. To this end, we take inspiration by some ideas developed in [56] and make use of some abstract critical point results obtained in [56] to overcome this obstacle;

(3) the combination effects of nonlocality and nonhomogeneous: the lack of homogeneity caused by fractional  $(p, q)$ -Laplacian operator, together with the appearance of nonlocal Choquard reaction term, makes our analysis more refined and interesting than the above works and also brings some new difficulties to our analysis. In particular we need to develop a suitable Moser iteration scheme to obtain  $L^\infty$ -estimates and absorb some ideas from [11, 62] to build on the Hölder regularity results under this work.

Throughout this paper,  $C, C_0, C_1, C_2, \dots$  denote positive constants whose exact values are inessential and can change from line to line, and the same  $C, C_0, C_1, C_2, \dots$  may represent different constants;  $B_\rho(y)$  denotes the open ball centered at  $y \in \mathbb{R}^N$  with radius  $\rho > 0$  and  $B_\rho^c(y)$  denotes the complement of  $B_\rho(y)$  in  $\mathbb{R}^N$ . In particular,  $B_\rho$  and  $B_\rho^c$  denote  $B_\rho(0)$  and  $B_\rho^c(0)$ , respectively.

## 2. AUXILIARY RESULTS

**2.1. Notations.** Let  $u : \mathbb{R}^N \mapsto \mathbb{R}$ . For  $0 < s < 1$  and  $p > 1$ , we define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{ u : |u|_p^p := \int_{\mathbb{R}^N} |u|^p dx < +\infty, [u]_{s,p}^p := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < +\infty \right\}$$

equipped with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := ([u]_{s,p}^p + |u|_p^p)^{\frac{1}{p}}.$$

For all  $u, v \in W^{s,p}(\mathbb{R}^N)$ , we define

$$\langle u, v \rangle_{s,p} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx dy.$$

We now recall some fundamental embeddings. Let  $s \in (0, 1)$  and  $p \in (1, +\infty)$  such that  $N > sp$ . Then for any  $r \in [p, p_s^*]$  there exists a constant such that  $C_r > 0$  such that

$$(8) \quad |u|_r \leq C_r \|u\|_{W^{s,p}(\mathbb{R}^N)}$$

for all  $u \in W^{s,p}(\mathbb{R}^N)$ . Moreover,  $W^{s,p}(\mathbb{R}^N)$  is compactly embedded in  $L_{\text{loc}}^r(\mathbb{R}^N)$  for any  $r \in [1, p_s^*]$ .

In order to treat the nonlinear fractional  $(p, q)$ -Choquard problem, we use the following space

$$X = W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$$

endowed with the norm

$$\|u\|_X := \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

Additionally, since  $W^{s,r}(\mathbb{R}^N)$  ( $1 < r < +\infty$ ) is a separable reflexive Banach space, then  $X$  is also a separable reflexive Banach space.

**2.2. The penalized problem.** In order to overcome the lack of compactness of problem (3), we shall adapt the penalization method introduced by del Pino & Felmer [26] to deal with the nonlinear fractional  $(p, q)$ -Choquard problem. Furthermore, without loss of generality, we can assume that

$$0 \in \Lambda \quad \text{and} \quad V(0) = V_0.$$

So, we need to find a constant  $K > 0$  (which is determined later, see Lemma 6) and take a unique number  $a > 0$  such that  $f(a) = \frac{V_0}{K} (a^{p-1} + a^{q-1})$ .

Now, we define

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if } t \leq a, \\ \frac{V_0}{K} (t^{p-1} + t^{q-1}), & \text{if } t > a \end{cases}$$

and

$$g(x, t) = \begin{cases} \chi_\Lambda(x)f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t), & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

where  $\chi_\Omega$  is the characteristic function on  $\Omega \subset \mathbb{R}^N$ .

We can easily observe that  $g$  is a Carathéodory function and fulfills the following properties:

- (g<sub>1</sub>)  $\lim_{t \rightarrow 0^+} \frac{g(x, t)}{t^{p-1}} = 0$  uniformly for all  $x \in \mathbb{R}^N$ ;
- (g<sub>2</sub>)  $g(x, t) \leq f(t)$  for all  $x \in \mathbb{R}^N$  and  $t > 0$ ;
- (g<sub>3</sub>)<sub>i</sub>  $0 < pG(x, t) := p \int_0^t g(x, \tau) d\tau \leq g(x, t)t$  for all  $x \in \Lambda$  and  $t > 0$ ;
- (g<sub>3</sub>)<sub>ii</sub>  $0 < pG(x, t) \leq g(x, t)t \leq \frac{V_0}{K} (t^p + t^q)$  for all  $x \in \Lambda^c$  and  $t > 0$ ;
- (g<sub>4</sub>) the maps  $t \mapsto \frac{g(x, t)}{t^{\frac{q}{2}-1}}$  and  $t \mapsto \frac{G(x, t)}{t^{\frac{q}{2}}}$  are both increasing for all  $x \in \mathbb{R}^N$  and  $t > 0$ .

**Remark 2.** We shall consider the following penalized problem

$$(9) \quad \left\{ \begin{array}{l} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) g(\varepsilon x, u), \\ u \in X, \quad u > 0 \end{array} \right\}$$

in  $\mathbb{R}^N$ . If  $u_\varepsilon$  is a solution of problem (9) such that  $u_\varepsilon(x) \leq a$  for all  $x \in \Lambda_\varepsilon^c$ , where  $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$ , then  $g(\varepsilon x, u_\varepsilon) = f(u_\varepsilon)$ ,  $G(\varepsilon x, u_\varepsilon) = F(\varepsilon x, u_\varepsilon)$ . So,  $u_\varepsilon$  is also a solution of problem (3).

For any  $\varepsilon \geq 0$ , we define the space

$$X_\varepsilon := \left\{ u \in X : \int_{\mathbb{R}^N} V(\varepsilon x) (|u|^p + |u|^q) dx < +\infty \right\}$$

equipped with the norm

$$\|u\|_{X_\varepsilon} := \|u\|_{V_\varepsilon, p} + \|u\|_{V_\varepsilon, q},$$

where  $\|u\|_{V_\varepsilon, t}^t := [u]_{s, t}^t + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t dx$  for  $t \in \{p, q\}$ .

From now on, we focus to the critical points of the Euler-Lagrange functional  $J_\varepsilon : X_\varepsilon \mapsto \mathbb{R}$  defined by

$$J_\varepsilon(u) := \frac{1}{p} \|u\|_{V_\varepsilon, p}^p + \frac{1}{q} \|u\|_{V_\varepsilon, q}^q - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) G(\varepsilon x, u) dx$$

for all  $u \in X_\varepsilon$ . By a standard argument, we can infer that  $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$  and its derivative is given by

$$\begin{aligned} \langle J'_\varepsilon(u), v \rangle &= \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon x) (|u|^{p-2}u + |u|^{q-2}u) v dx \\ &\quad - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) g(\varepsilon x, u) v dx \text{ for all } u, v \in X_\varepsilon. \end{aligned}$$

Let us define the Nehari manifold associated with problem (9), that is,

$$\mathcal{N}_\varepsilon := \{u \in X_\varepsilon \setminus \{0\} : \langle J'_\varepsilon(u), u \rangle = 0\}.$$

We use  $X_\varepsilon^+$  to denote the open set defined by

$$X_\varepsilon^+ := \{u \in X_\varepsilon : |\text{supp}(u^+)| > 0\}$$

and we introduce the set  $S_\varepsilon^+ := S_\varepsilon \cap X_\varepsilon^+$ , where  $S_\varepsilon := \{u \in X_\varepsilon : \|u\|_{X_\varepsilon} = 1\}$ . We first observe that  $S_\varepsilon^+$  is an incomplete  $C^{1,1}$ -manifold of codimension one. So, for all  $u \in S_\varepsilon^+$  we have  $X_\varepsilon = T_u(S_\varepsilon^+) \oplus \mathbb{R}u$ , where

$$T_u(S_\varepsilon^+) := \left\{ v \in X_\varepsilon : \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + \int_{\mathbb{R}^N} V(\varepsilon x) (|u|^{p-2}u + |u|^{q-2}u) v dx = 0 \right\}.$$

Due to the fact that  $f$  is only continuous, the following result is crucial to bypass the non-differentiability of  $\mathcal{N}_\varepsilon$ .

**Lemma 2.** *Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$ – $(V_2)$  are fulfilled. Then the following properties hold true:*

- (a) *for all  $u \in X_\varepsilon^+$ , there exists a unique  $t_u > 0$  such that  $t_u u \in \mathcal{N}_\varepsilon$ . Furthermore,  $\hat{m}_\varepsilon(u) = t_u u$  is the unique maximum of  $\ell_u(t) := J_\varepsilon(tu)$ ;*
- (b) *there exists  $\tau > 0$  independent of  $u$  such that  $t_u \geq \tau$  for all  $u \in S_\varepsilon^+$ . Moreover, for each compact set  $\mathcal{W} \subset S_\varepsilon^+$ , there exists a constant  $C_{\mathcal{W}} > 0$  such that  $t_u \leq C_{\mathcal{W}}$  for all  $u \in \mathcal{W}$ ;*
- (c) *the mapping  $\hat{m}_\varepsilon : X_\varepsilon^+ \mapsto \mathcal{N}_\varepsilon$  is continuous and  $m_\varepsilon := \hat{m}_\varepsilon|_{S_\varepsilon^+}$  is a homeomorphism between  $S_\varepsilon^+$  and  $\mathcal{N}_\varepsilon$ , and the inverse of  $m_\varepsilon$  is given by  $m_\varepsilon^{-1}(u) := u/\|u\|_{X_\varepsilon}$ ;*
- (d)  *$c_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) \geq \varrho_0 > 0$  and  $J_\varepsilon$  is bounded below on  $\mathcal{N}_\varepsilon$ , where  $\varrho_0$  is independent of  $\varepsilon$ ,  $K$  and  $a$ ;*
- (e) *let  $0 < \mu < sp$  and  $\nu < (N-\mu)q/(N-sq)$ . If there exists a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset S_\varepsilon^+$  such that  $\text{dist}(u_n, \partial S_\varepsilon^+) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\|m_\varepsilon(u_n)\|_{X_\varepsilon} \rightarrow +\infty$  and  $J_\varepsilon(m_\varepsilon(u_n)) \rightarrow +\infty$  as  $n \rightarrow \infty$ .*

*Proof.* (a) For each  $u \in X_\varepsilon^+$  and  $t > 0$ ,  $\ell_u(0) = 0$ . From Theorem 4.3 of Lieb & Loss [38],  $(g_2)$ ,  $(f_1)$ – $(f_2)$ , (8) and hypothesis  $(V_1)$  we can deduce that there exists some constant  $C > 0$  such that

$$\ell_u(t) \geq \frac{1}{2q-1q} \|u\|_{X_\varepsilon}^q t^q - C \|u\|_{X_\varepsilon}^{2p} t^{2p} \text{ for } 0 < t < \frac{1}{\|u\|_{X_\varepsilon}}.$$

Due to  $2p > q$ , we see that  $\ell_u(t) > 0$  for  $t > 0$  sufficiently small. Using  $(g_3)_i$  and  $(g_3)_{ii}$ , we can find a constant

$$C_u = \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) G(\varepsilon x, u) dx > 0$$

such that

$$\ell_u(t) \leq \frac{1}{p} (\|u\|_{X_\varepsilon}^p + \|u\|_{X_\varepsilon}^q) t^q - C_u t^{2p}$$



for all  $t > 1$ . Applying  $2p > q$  again, we know that  $\ell_u(t) < 0$  for  $t > 1$  large enough. Hence  $\max_{t \geq 0} \ell_u(t)$  is attained at some  $t_u > 0$  verifying  $\ell'_u(t_u) = 0$  and  $t_u u \in \mathcal{N}_\varepsilon$ . We point out that

$$(10) \quad tu \in \mathcal{N}_\varepsilon \iff \|u\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(\varepsilon y, tu(y))}{t^{\frac{q}{2}}|x-y|^\mu} \frac{g(\varepsilon x, tu(x))}{t^{\frac{q}{2}-1}} u(x) dx dy - t^{p-q} \|u\|_{V_{\varepsilon,p}}^p.$$

According to  $q > p$  and  $(g_4)$  we conclude that the right-hand side in (10) is an increasing function with respect to  $t > 0$ . Therefore, the uniqueness of  $t_u$  is now obvious.

(b) For any  $u \in S_\varepsilon^+$ , in view of Lemma 2-(a), there exists  $t_u > 0$  such that

$$t_u^p \|u\|_{V_{\varepsilon,p}}^p + t_u^q \|u\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, t_u u) \right) g(\varepsilon x, t_u u) t_u u dx.$$

By Theorem 4.3 of Lieb & Loss [38], together with  $(g_2)$ ,  $(f_1)$ ,  $(f_2)$ , (8) and hypothesis  $(V_1)$ , we can infer that for any  $\sigma > 0$  there exists some constant  $C_\sigma > 0$  such that for all  $u \in S_\varepsilon^+$

$$t_u^p \|u\|_{V_{\varepsilon,p}}^p + t_u^q \|u\|_{V_{\varepsilon,q}}^q \leq \sigma t_u^{2p} \|u\|_{V_{\varepsilon,p}}^{2p} + C_\sigma t_u^{2\nu} \|u\|_{V_{\varepsilon,q}}^{2\nu}.$$

Assume that  $t_u \leq 1$ . Choosing  $\sigma = \frac{1}{2}$ , we have

$$\begin{aligned} C t_u^q &\leq \frac{1}{2} t_u^q \left( \|u\|_{V_{\varepsilon,p}}^q + \|u\|_{V_{\varepsilon,q}}^q \right) \text{ for some constant } C > 0 \\ &\leq \frac{1}{2} t_u^p \|u\|_{V_{\varepsilon,p}}^p + t_u^q \|u\|_{V_{\varepsilon,q}}^q \text{ (since } q > p, t_u \leq 1 \text{ and } 1 = \|u\|_{X_\varepsilon} \geq \|u\|_{V_{\varepsilon,p}}) \\ &\leq C_{1/2} t_u^{2\nu} \text{ (since } 1 = \|u\|_{X_\varepsilon} \geq \|u\|_{V_{\varepsilon,q}}), \\ \Rightarrow t_u &\geq \tau \text{ for some constant } \tau > 0 \text{ (since } \nu > q), \text{ where } \tau \text{ is independent of } u. \end{aligned}$$

Assume that  $t_u > 1$ . Taking  $\sigma = 1$  and applying  $1 = \|u\|_{X_\varepsilon} \geq \|u\|_{V_{\varepsilon,p}}$ , we get

$$\begin{aligned} C t_u^p &\leq t_u^p \left( \|u\|_{V_{\varepsilon,p}}^q + \|u\|_{V_{\varepsilon,q}}^q \right) \text{ for some constant } C > 0 \\ &\leq t_u^p \|u\|_{V_{\varepsilon,p}}^p + t_u^q \|u\|_{V_{\varepsilon,q}}^q \text{ (since } q > p \text{ and } 1 = \|u\|_{X_\varepsilon} \geq \|u\|_{V_{\varepsilon,p}}) \\ &\leq (1 + C_1) t_u^{2\nu} \text{ (since } q > p, t_u > 1 \text{ and } 1 = \|u\|_{X_\varepsilon} \geq \|u\|_{V_{\varepsilon,p}}, \|u\|_{V_{\varepsilon,q}}), \\ \Rightarrow t_u &\geq \tau \text{ for some constant } \tau > 0 \text{ (since } \nu > q), \text{ where } \tau \text{ is independent of } u. \end{aligned}$$

So, there exists  $\tau > 0$  independent of  $u$  such that  $t_u \geq \tau$  for all  $u \in S_\varepsilon^+$ .

Let  $\mathcal{W} \subset S_\varepsilon^+$  be a compact set. Arguing by contradiction, we may assume that there exists a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{W}$  such that  $1 \leq t_n := t_{u_n} \rightarrow +\infty$  as  $n \rightarrow \infty$ . Since  $\mathcal{W}$  is a compact set, there is  $u \in \mathcal{W}$  such that  $u_n \rightarrow u$  in  $X_\varepsilon$  as  $n \rightarrow \infty$ . As in the proof of Lemma 2-(a), together with Fatou's Lemma, we can see that

$$(11) \quad J_\varepsilon(t_n u_n) \rightarrow -\infty \text{ as } n \rightarrow \infty.$$

In addition, for any fixed  $\varphi \in \mathcal{N}_\varepsilon$ , we have  $\langle J'_\varepsilon(\varphi), \varphi \rangle = 0$ . Combining  $(g_3)_i$  with  $(g_3)_{ii}$ , we have

$$\begin{aligned} J_\varepsilon(\varphi) &= J_\varepsilon(\varphi) - \frac{1}{2p} \langle J'_\varepsilon(\varphi), \varphi \rangle \\ &\geq \left( \frac{1}{q} - \frac{1}{2p} \right) \left( \|\varphi\|_{V_{\varepsilon,p}}^p + \|\varphi\|_{V_{\varepsilon,q}}^q \right). \end{aligned}$$

Taking  $\varphi_n = t_n u_n \in \mathcal{N}_\varepsilon$  in the above inequality we get

$$\begin{aligned} J_\varepsilon(t_n u_n) &\geq \left(\frac{1}{q} - \frac{1}{2p}\right) \left(t_n^p \|u_n\|_{V_{\varepsilon,p}}^p + t_n^q \|u_n\|_{V_{\varepsilon,q}}^q\right) \\ &\geq \left(\frac{1}{q} - \frac{1}{2p}\right) \left(t_n^p \|u_n\|_{V_{\varepsilon,p}}^q + t_n^q \|u_n\|_{V_{\varepsilon,q}}^q\right) \\ &\quad (\text{since } q > p, t_n \geq 1 \text{ and } 1 = \|u_n\|_{X_\varepsilon} \geq \|u_n\|_{V_{\varepsilon,p}}) \\ &\geq C t_n^p \text{ for some constant } C > 0, \\ &\Rightarrow -\infty \geq +\infty \text{ (see (11) and use the assumption } t_n \rightarrow +\infty \text{ as } n \rightarrow \infty). \end{aligned}$$

This is a contradiction.

(c) As in Lemma 2-(a), we can define the maps  $\hat{m}_\varepsilon : X_\varepsilon^+ \mapsto \mathcal{N}_\varepsilon$  and  $m_\varepsilon : S_\varepsilon^+ \mapsto \mathcal{N}_\varepsilon$  by

$$(12) \quad \hat{m}_\varepsilon(u) = t_u u \quad \text{and} \quad m_\varepsilon = \hat{m}_\varepsilon|_{S_\varepsilon^+}.$$

Firstly we note that  $\hat{m}_\varepsilon$ ,  $m_\varepsilon$  and  $m_\varepsilon^{-1}$  are well-defined. In indeed, using Lemma 2-(a), for any fixed  $u \in X_\varepsilon^+$  it follows that there exists a unique  $\hat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$ . In addition, if  $u \in \mathcal{N}_\varepsilon$ , and so  $u \in X_\varepsilon^+$ . Otherwise, we obtain  $|\text{supp}(u^+)| = 0$ . The above equality, hypothesis  $(V_1)$ , the definition of  $g$  yield that  $u = 0$ . This is impossible since  $u \neq 0$ . Thus, the inverse of  $m_\varepsilon$  is given by  $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{X_\varepsilon}} \in S_\varepsilon^+$  for all  $u \in \mathcal{N}_\varepsilon$ . Consequently,  $m_\varepsilon^{-1}$  is well-defined and continuous. On the other hand, for all  $u \in S_\varepsilon^+$  we can deduce that

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{X_\varepsilon}} = \frac{u}{\|u\|_{X_\varepsilon}} = u.$$

This yields that  $m_\varepsilon$  is bijection.

Next, we show that  $\hat{m}_\varepsilon$  is continuous map. To this end, let  $\{u_n, u\}_{n \in \mathbb{N}} \subset X_\varepsilon^+$  such that  $u_n \rightarrow u$  in  $X_\varepsilon$  as  $n \rightarrow \infty$ . On account of the fact that  $\hat{m}_\varepsilon(tu) = \hat{m}_\varepsilon(u)$  for all  $t > 0$ , we can assume that  $\|u_n\|_{X_\varepsilon} = \|u\|_{X_\varepsilon} = 1$  for all  $n \in \mathbb{N}$ . According to Lemma 2-(b), we know that there exists  $t_n := t_{u_n} \rightarrow t_0 > 0$  as  $n \rightarrow \infty$  such that  $t_n u_n \in \mathcal{N}_\varepsilon$ , then we have

$$t_n^p \|u_n\|_{V_{\varepsilon,p}}^p + t_n^q \|u_n\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, t_n u_n) \right) g(\varepsilon x, t_n u_n) t_n u_n dx.$$

In the above relation we pass to the limit as  $n \rightarrow \infty$ . Then,

$$t_0^p \|u\|_{V_{\varepsilon,p}}^p + t_0^q \|u\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, t_0 u) \right) g(\varepsilon x, t_0 u) t_0 u dx.$$

This implies that  $t_0 u \in \mathcal{N}_\varepsilon$ . From Lemma 2-(a) we know that  $t_u = t_0$ . Hence it follows that  $\hat{m}_\varepsilon(u_n) \rightarrow \hat{m}_\varepsilon(u)$  in  $X_\varepsilon^+$  as  $n \rightarrow \infty$ . Thus,  $\hat{m}_\varepsilon$  and  $m_\varepsilon$  are continuous mappings.

(d) For  $\varepsilon > 0$ ,  $0 < t < 1$  and  $u \in S_\varepsilon^+$ , using Theorem 4.3 of Lieb & Loss [38],  $(g_2)$ ,  $(f_1)$ – $(f_2)$ , (8) and hypothesis  $(V_1)$  again, we can conclude that there exists  $C > 0$  such that

$$J_\varepsilon(tu) \geq \frac{1}{2^{q-1}q} t^q - C t^{2p}.$$

Thus, we can find  $\varrho_0 > 0$  such that  $J_\varepsilon(tu) \geq \varrho_0$  for  $0 < t < 1$  sufficiently small (since  $2p > q > 0$ ), where  $\varrho_0$  is independent of  $\varepsilon$ ,  $K$  and  $a$ . In addition, by Lemma 2-(a), (b), (c), we observe (see Szulkin & Weth [56]) that

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u \in X_\varepsilon^+} \max_{t > 0} J_\varepsilon(tu) = \inf_{u \in S_\varepsilon^+} \max_{t > 0} J_\varepsilon(tu).$$

So,  $J_\varepsilon(u)|_{\mathcal{N}_\varepsilon} \geq \varrho_0$ .

(e) Let  $\{u_n\}_{n \in \mathbb{N}} \subset S_\varepsilon^+$  be a sequence such that  $\text{dist}(u_n, \partial S_\varepsilon^+) \rightarrow 0$  as  $n \rightarrow \infty$ . For any  $\varphi \in \partial S_\varepsilon^+$  and  $n \in \mathbb{N}$ , then we obtain  $|u_n^+| \leq |u_n - \varphi|$  a.e. in  $\Lambda_\varepsilon$ . Hence, from  $(V_1)$  and Sobolev

embedding, for any  $r \in [p, q_s^*]$  and  $n \in \mathbb{N}$  it follows that

$$|u_n^+|_{L^r(\Lambda_\varepsilon)} \leq \inf_{\varphi \in \partial S_\varepsilon^+} |u_n - \varphi|_{L^r(\Lambda_\varepsilon)} \leq C_r \inf_{\varphi \in \partial S_\varepsilon^+} \|u_n - \varphi\|_{X_\varepsilon}.$$

Taking into account  $\|u_n\|_{X_\varepsilon} = 1$  and using hypothesis  $(V_1)$ , we have

$$\begin{aligned} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u_n\|_{W^{s,q}(\mathbb{R}^N)}^q &\leq \frac{1}{\min\{V_0, 1\}} \left( \|u_n\|_{V_{\varepsilon,p}}^p + \|u_n\|_{V_{\varepsilon,q}}^q \right) \\ &\leq \frac{1}{\min\{V_0, 1\}} \left( \|u_n\|_{X_\varepsilon}^p + \|u_n\|_{X_\varepsilon}^q \right) \\ &\leq \frac{2}{V_0} + 2. \end{aligned}$$

Note that  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Then, for all  $t > 0$  we can deduce from Lemma 6,  $(V_1)$ ,  $(g_3)_{ii}$ ,  $(g_2)$  and  $(f_1)$ – $(f_2)$  that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, tu_n) \right) G(\varepsilon x, tu_n) dx \\ &= \frac{K}{2} \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n) dx + \frac{K}{2} \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n) dx \\ &\leq \frac{V_0}{2p} \int_{\Lambda_\varepsilon^c} (t^p |u_n|^p + t^q |u_n|^q) dx + \frac{K}{2} \int_{\Lambda_\varepsilon} F(tu_n^+) dx \\ &\leq \frac{t^p}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx + \frac{t^q}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^q dx + C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p dx + C_2 t^\nu \int_{\Lambda_\varepsilon} (u_n^+)^p dx \\ &\leq \frac{t^p}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx + \frac{t^q}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^q dx + \hat{C}_p t^p \text{dist}(u_n, \partial S_\varepsilon^+)^p + \hat{C}_\nu t^\nu \text{dist}(u_n, \partial S_\varepsilon^+)^p, \end{aligned}$$

where  $C_1, C_2, \hat{C}_p$  and  $\hat{C}_\nu$  are some positive constants. So, we have

$$(13) \quad \begin{aligned} &\int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, tu_n) \right) G(\varepsilon x, tu_n) dx \\ &\leq \frac{t^p}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx + \frac{t^q}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^q dx + o_n(1), \end{aligned}$$

as  $n \rightarrow \infty$ . Moreover, for any  $t > 1$  we infer that

$$\begin{aligned} &\frac{t^p}{p} \|u_n\|_{V_{\varepsilon,p}}^p + \frac{t^q}{q} \|u_n\|_{V_{\varepsilon,q}}^q - \frac{t^p}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx - \frac{t^q}{2p} \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^q dx \\ &= \frac{t^p}{p} [u_n]_{s,p}^p + \frac{1}{2p} t^p \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^p dx + \frac{t^q}{q} [u_n]_{s,q}^q + \left( \frac{1}{q} - \frac{1}{2p} \right) t^q \int_{\mathbb{R}^N} V(\varepsilon x) |u_n|^q dx \\ &\geq \frac{1}{2p} t^p \|u_n\|_{V_{\varepsilon,p}}^p + \left( \frac{1}{q} - \frac{1}{2p} \right) t^q \|u_n\|_{V_{\varepsilon,q}}^q \quad (\text{since } 2p > q) \\ &\geq \frac{1}{2p} t^p \|u_n\|_{V_{\varepsilon,p}}^q + \left( \frac{1}{q} - \frac{1}{2p} \right) t^q \|u_n\|_{V_{\varepsilon,q}}^q \quad (\text{due to } q > p \text{ and } 1 = \|u_n\|_{X_\varepsilon} \geq \|u_n\|_{V_{\varepsilon,p}}) \\ (14) \quad &\geq \frac{1}{2q-1} \left( \frac{1}{q} - \frac{1}{2p} \right) t^p \quad (\text{since } 2p > q > p > 1, t > 1). \end{aligned}$$

Recalling that the definition of  $m_\varepsilon$  and invoking relations (13) and (14), for all  $t > 1$  we can deduce that

$$\liminf_{n \rightarrow \infty} J_\varepsilon(m_\varepsilon(u_n)) \geq \liminf_{n \rightarrow \infty} J_\varepsilon(tu_n) \geq \frac{1}{2q-1} \left( \frac{1}{q} - \frac{1}{2p} \right) t^p.$$

The above inequality combined with the definition of  $J_\varepsilon$  and the arbitrariness of  $t > 1$  means that

$$\liminf_{n \rightarrow \infty} \left( \frac{1}{p} \|m_\varepsilon(u_n)\|_{V_{\varepsilon,p}}^p + \frac{1}{q} \|m_\varepsilon(u_n)\|_{V_{\varepsilon,q}}^q \right) \geq \liminf_{n \rightarrow \infty} J_\varepsilon(m_\varepsilon(u_n)) = +\infty,$$

and so  $\|m_\varepsilon(u_n)\|_{X_\varepsilon} \rightarrow +\infty$  as  $n \rightarrow \infty$ . This proof is now complete.  $\square$

Now, we introduce the functionals

$$\hat{\psi}_\varepsilon : X_\varepsilon^+ \mapsto \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : S_\varepsilon^+ \mapsto \mathbb{R}$$

defined by  $\hat{\psi}_\varepsilon(u) := J_\varepsilon(\hat{m}_\varepsilon(u))$  for any  $u \in X_\varepsilon^+$  and  $\psi_\varepsilon := \hat{\psi}_\varepsilon|_{S_\varepsilon^+}$ .

Using Lemma 2 and Corollary 2.3 in Szulkin & Weth [56] we deduce that the following result holds true.

**Lemma 3.** *Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$ – $(V_2)$  are satisfied. Then,*

- (a)  $\hat{\psi}_\varepsilon \in C^1(X_\varepsilon^+, \mathbb{R})$  and  $\langle \hat{\psi}'_\varepsilon(u), v \rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{X_\varepsilon}}{\|u\|_{X_\varepsilon}} \langle J'_\varepsilon(\hat{m}_\varepsilon(u)), v \rangle$  for all  $u \in X_\varepsilon^+$ , all  $v \in X_\varepsilon$ ;
- (b)  $\psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$  and  $\langle \psi'_\varepsilon(u), v \rangle = \|m_\varepsilon(u)\|_{X_\varepsilon} \langle J'_\varepsilon(m_\varepsilon(u)), v \rangle$  for all  $u \in S_\varepsilon^+$ , all  $v \in T_u(S_\varepsilon^+)$ ;
- (c) if  $\{u_n\}_{n \in \mathbb{N}}$  is a Palais-Smale sequence for the functional  $\psi_\varepsilon$ , then  $\{m_\varepsilon(u_n)\}_{n \in \mathbb{N}}$  is a Palais-Smale sequence for the functional  $J_\varepsilon$ . If  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\varepsilon$  is bounded Palais-Smale sequence for the functional  $J_\varepsilon$ , then,  $\{m_\varepsilon^{-1}(u_n)\}_{n \in \mathbb{N}} \subset S_\varepsilon^+$  is a Palais-Smale sequence for  $\psi_\varepsilon$ ;
- (d)  $u \in S_\varepsilon^+$  is a critical point of the functional  $\psi_\varepsilon$  if and only if  $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$  is a critical point of the functional  $J_\varepsilon$ . Moreover, the corresponding critical values coincide and

$$\inf_{u \in S_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = c_\varepsilon.$$

**Lemma 4.** *The modified functional  $J_\varepsilon$  admits a Palais-Smale sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  at the level  $c_\varepsilon$ , that is,  $J_\varepsilon(u_n) \rightarrow c_\varepsilon$  in  $\mathbb{R}$  and  $J'_\varepsilon(u_n) \rightarrow 0$  in  $X_\varepsilon^*$  as  $n \rightarrow \infty$ , where  $c_\varepsilon$  is given in Lemma 2. Furthermore, there exists some constant  $\vartheta > 0$  (independent of  $\varepsilon$ ,  $K$  and  $a$ ) such that  $c_\varepsilon < \vartheta$  for all  $\varepsilon$  sufficiently small.*

*Proof.* In view of Lemma 2, we only need to verify that  $J_\varepsilon$  possesses a mountain pass geometry, that is, the functional  $J_\varepsilon$  satisfies the following properties:

- (i) there exist some constants  $\rho_1 > 0$  and  $\delta_1 > 0$  such that  $J_\varepsilon(u) \geq \delta_1$  for  $\|u\|_{X_\varepsilon} = \rho_1$ ;
- (ii) there exists an element  $e \in X_\varepsilon$  with  $\|e\|_{X_\varepsilon} > \rho_1$  such that  $J_\varepsilon(e) < 0$ .

(i) Arguing as in the proof of Lemma 2-(a), we find some constants  $C > 0$  and  $C' > 0$  such that

$$J_\varepsilon(u) \geq C \|u\|_{X_\varepsilon}^q - C' \|u\|_{X_\varepsilon}^{2p}$$

for  $\|u\|_{X_\varepsilon} = \rho_1 \in (0, 1)$ . So, using  $2p > q$  we obtain (i) if we take  $\rho_1$  small enough.

(ii) We choose a suitable function  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $\varphi \geq 0$ ,  $\varphi \not\equiv 0$  and  $\text{supp}(\varphi) \subset \Lambda$ . Then for all  $\varepsilon$  sufficiently small it is obvious that  $G(\varepsilon x, \varphi) = F(\varphi)$  for all  $x \in \mathbb{R}^N$ . Hence, with arguments as in the proof of Lemma 2-(a), there exist two constants

$$C_\varphi = \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\varphi) \right) F(\varphi) dx > 0$$

and  $C > 0$  such that

$$J_\varepsilon(t\varphi) \leq Ct^q - C_\varphi t^{2p}$$

for all  $t > 1$ . Hence (ii) holds true for  $e = t\varphi$  and for some  $t > 1$  sufficiently large.

According to the mountain pass theorem without the Palais-Smale condition (see Brezis & Nirenberg [18]), we establish the existence of a Palais-Smale sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  at the level  $c_\varepsilon$ . We recall that  $\text{supp}(\varphi) \subset \Lambda$  and then we can infer that there exists a constant  $\vartheta > 0$

such that  $c_\varepsilon < \vartheta$  for all  $\varepsilon$  small enough, where  $\vartheta$  is independent of  $\varepsilon$ ,  $K$  and  $a$ . This completes the proof of the lemma.  $\square$

**Lemma 5.** *Assume that  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  is Palais-Smale sequence of  $J_\varepsilon$  at the level  $c \leq \vartheta$ , where  $\varepsilon > 0$  small enough. Then, for  $\varepsilon > 0$  small enough, the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  is bounded and*

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u_n\|_{W^{s,q}(\mathbb{R}^N)}^q \leq 2 \left( \frac{1}{V_0} + 1 \right) \frac{pq(\theta + 1)}{2p - q} \text{ for all } n \in \mathbb{N} \text{ large enough.}$$

*Proof.* According to  $(g_3)_{i,ii}$  and using hypothesis  $(V_1)$ , for any  $\varepsilon > 0$  small enough we deduce that

$$\begin{aligned} \vartheta + o_n(1)\|u_n\|_{X_\varepsilon} &\geq c + o_n(1)\|u_n\|_{X_\varepsilon} \\ &\geq J_\varepsilon(u_n) - \frac{1}{2p} \langle J'_\varepsilon(u_n), u_n \rangle \\ &\geq \left( \frac{1}{q} - \frac{1}{2p} \right) \left( \|u_n\|_{V_{\varepsilon,p}}^p + \|u_n\|_{V_{\varepsilon,q}}^q \right) \\ &\quad + \frac{1}{2p} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) (g(\varepsilon x, u_n)u_n - pG(\varepsilon x, u_n)) dx \\ (15) \quad &\geq \min\{V_0, 1\} \left( \frac{1}{q} - \frac{1}{2p} \right) \left( \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u_n\|_{W^{s,q}(\mathbb{R}^N)}^q \right). \end{aligned}$$

So, for  $\varepsilon > 0$  small enough, we deduce that the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  is bounded and

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u_n\|_{W^{s,q}(\mathbb{R}^N)}^q \leq 2 \left( \frac{1}{V_0} + 1 \right) \frac{pq(\theta + 1)}{2p - q}$$

for all  $n \in \mathbb{N}$  large enough.  $\square$

According to (15), we now define the following set:

$$\mathcal{B} := \left\{ u \in X_\varepsilon : \|u\|_{W^{s,p}(\mathbb{R}^N)}^p + \|u\|_{W^{s,q}(\mathbb{R}^N)}^q \leq 2 \left( \frac{1}{V_0} + 1 \right) \left( \frac{(\vartheta + 1)pq}{2p - q} + 1 \right) \right\},$$

where  $\vartheta$  is given in Lemma 4. Using the above notation, we can show the following estimate.

**Lemma 6.** *Assume that  $(f_1)$ – $(f_4)$  hold true,  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ , then there is a constant  $K > 0$  such that*

$$\sup_{u \in \mathcal{B}} \left| \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right|_{L^\infty(\mathbb{R}^N)} < \frac{K}{2} \text{ for all } \varepsilon > 0.$$

*Proof.* Firstly,  $(g_2)$  and  $(f_1)$ – $(f_2)$  imply that there exists a constant  $C > 0$  (which is independent of  $\varepsilon$ ) such that

$$(16) \quad |G(\varepsilon x, t)| \leq |F(t)| \leq C(|t|^p + |t|^\nu) \text{ for all } t \in \mathbb{R} \text{ and for all } \varepsilon > 0.$$

Therefore

$$\begin{aligned} \left| \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right| &\leq \left| \frac{1}{|x|^\mu} * F(u) \right| \\ &\leq \int_{|x-y| \leq 1} \frac{|F(u(y))|}{|x-y|^\mu} dy + \int_{|x-y| > 1} \frac{|F(u(y))|}{|x-y|^\mu} dy \\ &\leq C \int_{|x-y| \leq 1} \frac{|u(y)|^p + |u(y)|^\nu}{|x-y|^\mu} dy + C \int_{\mathbb{R}^N} (|u(y)|^p + |u(y)|^\nu) dy \text{ (see (16))} \\ &\leq C \int_{|x-y| \leq 1} \frac{|u(y)|^p + |u(y)|^\nu}{|x-y|^\mu} dy + C \text{ (see (8) and use the definition of } \mathcal{B}) \end{aligned}$$

for some constant  $C > 0$  (which is independent of  $x \in \mathbb{R}^N$  and  $\varepsilon > 0$ ).

Let us choose

$$t_1 \in \left( \frac{N}{N-\mu}, \frac{N}{N-sp} \right] \quad \text{and} \quad t_2 \in \left( \frac{N}{N-\mu}, \frac{Nq}{(N-sq)\nu} \right],$$

since  $0 < \mu < sp$  and  $\nu < (N-\mu)q/(N-sq)$ . Then, combining the Hölder inequality, (8) and the definition of  $\mathcal{B}$  we can easily see that

$$\int_{|x-y| \leq 1} \frac{|u(y)|^p + |u(y)|^\nu}{|x-y|^\mu} dy \leq C \text{ for all } x \in \mathbb{R}^N.$$

for some constant  $C > 0$  (which is independent of  $x \in \mathbb{R}^N$  and  $\varepsilon > 0$ ). So, from the above information we complete the proof of this lemma.  $\square$

Now we show that the modified functional  $J_\varepsilon$  satisfies the Palais-Smale condition.

**Lemma 7.** *Let  $0 < \mu < sp$  and  $\nu < (N-\mu)q/(N-sq)$ . Then for all  $\varepsilon > 0$  sufficiently small the modified functional  $J_\varepsilon$  satisfies the Palais-Smale condition with  $c \leq \vartheta$ .*

*Proof.* For all  $\varepsilon > 0$  sufficiently small let  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  be a Palais-Smale sequence of the functional  $J_\varepsilon$  at the level  $c$ . From Lemma 5 we know that  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  is bounded for all  $\varepsilon > 0$  small enough. So, passing to a subsequence we may assume that there exists some  $u \in X_\varepsilon$  such that  $u_n \xrightarrow{w} u$  in  $X_\varepsilon$ ,  $u_n(x) \rightarrow u(x)$  a.e. in  $\mathbb{R}^N$  and  $u_n \rightarrow u$  in  $L^r_{\text{loc}}(\mathbb{R}^N)$  for all  $r \in [1, q_s^*)$  as  $n \rightarrow \infty$ . We first assert that the following property is fulfilled.

**Claim 1:** The following properties hold up to a subsequence:

- (a)  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * (G(\varepsilon x, u_n) - G(\varepsilon x, u)) \right) g(\varepsilon x, u) \varphi dx = 0$  for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ ;
- (b)  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) (g(\varepsilon x, u_n) - g(\varepsilon x, u)) \varphi dx = 0$  for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ ;
- (c) we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) g(\varepsilon x, u_n) \varphi dx \\ &= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) g(\varepsilon x, u) \varphi dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

(a) Note that

$$\begin{aligned} \int_{\mathbb{R}^N} |G(\varepsilon x, u_n)|^{\frac{2N}{2N-\mu}} dx &\leq \int_{\mathbb{R}^N} |C(|u_n|^p + |u_n|^\nu)|^{\frac{2N}{2N-\mu}} dx \text{ (see (16))} \\ &\leq C \int_{\mathbb{R}^N} \left( |u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\nu}{2N-\mu}} \right) dx \\ &\leq C \left( \|u_n\|_{X_\varepsilon}^{2Np/(2N-\mu)} + \|u_n\|_{X_\varepsilon}^{2N\nu/(2N-\mu)} \right) \text{ (see (8))} \\ &\leq C \text{ (due to the boundedness of the sequence } \{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon). \end{aligned}$$

On the other hand, since  $u_n(x) \rightarrow u(x)$  a.e. in  $\mathbb{R}^N$  as  $n \rightarrow \infty$  we obtain

$$G(\varepsilon x, u_n) \rightarrow G(\varepsilon x, u) \text{ a.e. in } \mathbb{R}^N \text{ as } n \rightarrow \infty.$$

From Proposition 5.4.7 of Willem [58] it follows that

$$(17) \quad G(\varepsilon \cdot, u_n) \xrightarrow{w} G(\varepsilon \cdot, u) \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

Let us define

$$H(w) := \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * w(x) \right) g(\varepsilon x, u) \varphi dx \text{ for all } w \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

Then Theorem 4.3 of Lieb & Loss [38],  $(g_2)$ ,  $(f_1)$ – $f_2$ ), Hölder's inequality, (8), and since  $u \in X_\varepsilon$  imply that  $H$  is linear bounded functional. Combining this with (17) we derive that (a) holds true.

(b) Using the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$ , together with Theorem 4.3 of Lieb & Loss [38],  $(g_2)$ ,  $(f_1)$ – $(f_2)$  and (8), we see that  $G(\varepsilon, u_n)$  is also bounded in  $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ . Then, applying the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  again and combining with the compact embeddings and Dominated Convergence Theorem, we deduce that (b) is also true.

(c) This result is a consequence of (a) and (b).

Next, we consider the following sequence

$$b_n(x, y) := \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x - y|^{\frac{N+sp}{p'}}$$

and we also introduce the following function

$$b(x, y) := \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\frac{N+sp}{p'}}$$

where  $p' = \frac{p}{p-1}$ . Then  $\{b_n\}_{n \in \mathbb{N}} \subset L^{p'}(\mathbb{R}^{2N})$  is bounded and  $b_n(x, y) \rightarrow b(x, y)$  a.e. in  $\mathbb{R}^{2N}$  as  $n \rightarrow \infty$ . Then we deduce from Proposition 5.4.7 of Willem [58] that

$$(18) \quad b_n \xrightarrow{w} b \text{ in } L^{p'}(\mathbb{R}^{2N}) \text{ as } n \rightarrow \infty.$$

On the other hand, for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$  we see that

$$\frac{\varphi(x) - \varphi(y)}{|x - y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^{2N}).$$

From this and (18) we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy. \end{aligned}$$

In a similar fashion we also have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} dx dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sq}} dx dy. \end{aligned}$$

Finally, it is easy to check that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon x, u_n) |u_n|^{p-2} u_n \varphi dx &= \int_{\mathbb{R}^N} V(\varepsilon x, u) |u|^{p-2} u \varphi dx, \\ \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(\varepsilon x, u_n) |u_n|^{q-2} u_n \varphi dx &= \int_{\mathbb{R}^N} V(\varepsilon x, u) |u|^{q-2} u \varphi dx. \end{aligned}$$

From the above information, together with the fact that  $\langle J'_\varepsilon(u_n), \varphi \rangle \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $\langle J'_\varepsilon(u), \varphi \rangle = 0$  for all  $\varphi \in C_0^\infty(\mathbb{R}^N)$ . Using the density of  $C_0^\infty(\mathbb{R}^N)$  in  $X_\varepsilon$  we know that  $u$  is a critical point of the functional  $J_\varepsilon$ . Consequently,  $\langle J'_\varepsilon(u), u \rangle = 0$ .

In order to show that the Palais-Smale sequence satisfies the Palais-Smale condition, we need to establish the following property.

**Claim 2:** We have

$$\limsup_{n \rightarrow \infty} \int_{B_R^c} \left( \int_{\mathbb{R}^N} \left( \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} + \frac{|u_n(x) - u_n(y)|^q}{|x - y|^{N+sq}} \right) dy + V(\varepsilon x) (|u_n|^p + |u_n|^q) \right) dx \rightarrow 0 \text{ as } R \rightarrow +\infty.$$

We first observe that, for all  $\varepsilon > 0$  small enough, there exists  $n_0 \in \mathbb{N}$  such that

$$\sup_{n \geq n_0} \left| \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right|_{L^\infty(\mathbb{R}^N)} < \frac{K}{2}.$$

For any  $R > 0$ , let  $\eta_R \in C^\infty(\mathbb{R}^N)$  be such that  $0 \leq \eta_R \leq 1$ ,  $\eta_R = 0$  in  $B_{\frac{R}{2}}$ ,  $\eta_R = 1$  in  $B_R^c$  and  $|\nabla \eta_R| \leq C/R$  for some constant  $C > 0$  (which is independent of  $R$ ). Taking into account the boundedness of the sequence  $\{\eta_R u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$ , we see that  $\langle J'_\varepsilon(u_n), \eta_R u_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$ . It follows that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p \eta_R(x)}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q \eta_R(x)}{|x - y|^{N+sq}} dx dy \\ & + \int_{\mathbb{R}^N} V(\varepsilon x) (|u_n|^p + |u_n|^q) \eta_R dx \\ & = o_n(1) - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sp}} dx dy \\ & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sq}} dx dy \\ & + \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) g(\varepsilon x, u_n) u_n \eta_R dx. \end{aligned}$$

Fix  $\varepsilon > 0$  small enough. Let  $R > 0$  sufficiently large such that  $\Lambda_\varepsilon \subset B_{\frac{R}{2}}$ . Using the definitions of  $\eta_R$  and  $K$ , together with  $(g_3)_{ii}$ , for  $n \geq n_0$  large enough we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p \eta_R(x)}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^q \eta_R(x)}{|x - y|^{N+sq}} dx dy \\ & + \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon x) (|u_n|^p + |u_n|^q) \eta_R dx \\ & \leq o_n(1) - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sp}} dx dy \\ (19) \quad & - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sq}} dx dy. \end{aligned}$$

From the Hölder inequality and the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  it follows that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sp}} dx dy \right| \\ (20) \quad & \leq C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_R(x) - \eta_R(y)|^p |u_n(x)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \text{ (for some constant } C > 0). \end{aligned}$$



In addition, by the definition of  $\eta_R$ , polar coordinates and the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$ , we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta_R(x) - \eta_R(y)|^p |u_n(x)|^p}{|x - y|^{N+sp}} dx dy \\
& \leq \int_{\mathbb{R}^N} \int_{|x-y|>R} \frac{|\eta_R(x) - \eta_R(y)|^p |u_n(x)|^p}{|x - y|^{N+sp}} dx dy \\
& \quad + \int_{\mathbb{R}^N} \int_{|x-y|\leq R} \frac{|\eta_R(x) - \eta_R(y)|^p |u_n(x)|^p}{|x - y|^{N+sp}} dx dy \\
& \leq C \int_{\mathbb{R}^N} \int_{|z|>R} \frac{|u_n(x)|^p}{|z|^{N+sp}} dx dz + \frac{C}{R^p} \int_{\mathbb{R}^N} \int_{|z|\leq R} \frac{|u_n(x)|^p}{|z|^{N+sp-p}} dx dz \\
& \leq \frac{C}{R^{sp}} \int_{\mathbb{R}^N} |u_n|^p dx + \frac{C}{R^p} R^{-sp+p} \int_{\mathbb{R}^N} |u_n|^p dx \\
(21) \quad & \leq \frac{C}{R^{sp}}.
\end{aligned}$$

Using (20) and (21), we see that

$$(22) \quad \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sp}} dx dy \right| \leq \frac{C}{R^s}.$$

Also, we have

$$(23) \quad \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2} (u_n(x) - u_n(y)) (\eta_R(x) - \eta_R(y)) u_n(y)}{|x - y|^{N+sq}} dx dy \right| \leq \frac{C}{R^s}.$$

So, we deduce from (19), (22) and (23) that Claim 2 holds true.

Using Claim 2 and applying the locally compact embedding  $X_\varepsilon \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N)$  we can derive that  $u_n \rightarrow u$  in  $L^p(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . In addition, we deduce from the interpolation inequality that  $u_n \rightarrow u$  in  $L^r(\mathbb{R}^N)$  as  $n \rightarrow \infty$  for all  $r \in [p, q_s^*)$ . Then, from Theorem 4.3 of Lieb-Loss [38], (g<sub>2</sub>), (f<sub>1</sub>)–(f<sub>2</sub>), the Dominated Convergence Theorem and the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  it follows that

$$(24) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) g(\varepsilon x, u_n) u_n dx = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) g(\varepsilon x, u) u dx.$$

Also, we have

$$\begin{aligned}
& \langle J'_\varepsilon(u_n), u_n \rangle = o_n(1) \text{ as } n \rightarrow \infty, \\
\Rightarrow & \|u_n\|_{V_{\varepsilon,p}}^p + \|u_n\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u_n) \right) g(\varepsilon x, u_n) u_n dx + o_n(1) \text{ as } n \rightarrow \infty
\end{aligned}$$

and

$$\begin{aligned}
& \langle J'_\varepsilon(u), u \rangle = 0, \\
\Rightarrow & \|u\|_{V_{\varepsilon,p}}^p + \|u\|_{V_{\varepsilon,q}}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon x, u) \right) g(\varepsilon x, u) u dx.
\end{aligned}$$

Hence the above fact and (24) imply that

$$\|u_n\|_{V_{\varepsilon,p}}^p + \|u_n\|_{V_{\varepsilon,q}}^q = \|u\|_{V_{\varepsilon,p}}^p + \|u\|_{V_{\varepsilon,q}}^q + o_n(1) \text{ as } n \rightarrow \infty.$$

Furthermore, according to Lemma 10-(a), (b) we see that

$$\|u_n - u\|_{V_{\varepsilon,p}}^p = \|u_n\|_{V_{\varepsilon,p}}^p - \|u\|_{V_{\varepsilon,p}}^p + o_n(1)$$

and

$$\|u_n - u\|_{V_{\varepsilon,q}}^q = \|u_n\|_{V_{\varepsilon,q}}^q - \|u\|_{V_{\varepsilon,q}}^q + o_n(1),$$

as  $n \rightarrow \infty$ . Thus we conclude that

$$\|u_n - u\|_{V_\varepsilon, p}^p + \|u_n - u\|_{V_\varepsilon, q}^q = o_n(1).$$

This fact means that  $\|u_n - u\|_{X_\varepsilon} \rightarrow 0$  as  $n \rightarrow \infty$ . Now, we complete the proof of the lemma.  $\square$

**Lemma 8.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$ – $(V_2)$  are fulfilled, then for all  $\varepsilon > 0$  small enough problem (9) possesses a nonnegative solution  $u_\varepsilon \in X_\varepsilon$ .*

*Proof.* Using Lemmas 4 and 7, we can employ the Mountain Pass Theorem to infer that for all  $\varepsilon > 0$  sufficiently small there exists  $u_\varepsilon \in X_\varepsilon$  such that  $J'_\varepsilon(u_\varepsilon) = 0$  and  $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ . Furthermore, choosing  $u_\varepsilon^- := \min\{u_\varepsilon, 0\}$  and recalling that  $g(\varepsilon, t) = 0$  for all  $t \leq 0$  and  $\langle J'_\varepsilon(u_\varepsilon), u_\varepsilon^- \rangle = 0$ , we can infer that

$$\begin{aligned} & \|u_\varepsilon^-\|_{V_\varepsilon, p}^p + \|u_\varepsilon^-\|_{V_\varepsilon, q}^q \leq 0, \\ \Rightarrow & u_\varepsilon^- = 0, \\ \Rightarrow & u_\varepsilon \geq 0 \text{ and } u_\varepsilon \not\equiv 0. \end{aligned}$$

This ends the proof of the lemma.  $\square$

**Corollary 9.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Then for all  $\varepsilon > 0$  sufficiently small, the functional  $\psi_\varepsilon$  satisfies the Palais-Smale condition at the level  $c \leq \vartheta$  on  $S_\varepsilon^+$ .*

*Proof.* Let  $\{u_n\}_{n \in \mathbb{N}} \subset S_\varepsilon^+$  be a Palais-Smale sequence for the functional  $\psi_\varepsilon$  at the level  $c$ , that is,

$$\psi_\varepsilon(u_n) \rightarrow c \text{ in } \mathbb{R} \quad \text{and} \quad \psi'_\varepsilon(u_n) \rightarrow 0 \text{ in } T_{u_n}(S_\varepsilon^+)^* \text{ as } n \rightarrow \infty.$$

Using Lemma 3-(c), we see that  $\{m_\varepsilon(u_n)\}_{n \in \mathbb{N}} \subset X_\varepsilon$  is also a Palais-Smale sequence for the functional  $J_\varepsilon$  at the level  $c$ . Therefore, we can deduce from Lemma 7 that the functional  $J_\varepsilon$  verifies the Palais-Smale condition. Hence, passing to a subsequence we can find some  $u \in S_\varepsilon^+$  such that  $m_\varepsilon(u_n) \rightarrow m_\varepsilon(u)$  in  $X_\varepsilon$  as  $n \rightarrow \infty$ . Combining this fact with Lemma 2-(c) we conclude that  $u_n \rightarrow u$  in  $S_\varepsilon^+$  as  $n \rightarrow \infty$ .  $\square$

We end this section by showing the following result:

**Lemma 10.** *Let  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  be a sequence such that  $u_n \xrightarrow{w} u$  in  $X_\varepsilon$  and let  $\mathcal{A}_t : \mathbb{R} \mapsto \mathbb{R}$  defined by  $\mathcal{A}_t(\tau) = |\tau|^{t-2}\tau$  ( $t \in \{p, q\}$ ). Setting  $w_n = u_n - u$ , then for all  $\varepsilon \geq 0$  and  $n \in \mathbb{N}$  large enough we have*

$$\begin{aligned} \text{(a)} & \left( [u_n]_{s,p}^p + [u_n]_{s,q}^q \right) - \left( [w_n]_{s,p}^p + [w_n]_{s,q}^q \right) - \left( [u]_{s,p}^p + [u]_{s,q}^q \right) = o_n(1); \\ \text{(b)} & \left( \left| V(\varepsilon \cdot)^{1/p} u_n \right|_p^p + \left| V(\varepsilon \cdot)^{1/q} u_n \right|_q^q \right) - \left( \left| V(\varepsilon \cdot)^{1/p} w_n \right|_p^p + \left| V(\varepsilon \cdot)^{1/q} w_n \right|_q^q \right) \\ & - \left( \left| V(\varepsilon \cdot)^{1/p} u \right|_p^p + \left| V(\varepsilon \cdot)^{1/q} u \right|_q^q \right) = o_n(1); \\ \text{(c)} & |\mathcal{A}_p(u_n) - \mathcal{A}_p(w_n) - \mathcal{A}_p(u)|^{\frac{p-1}{p}} + |\mathcal{A}_q(u_n) - \mathcal{A}_q(w_n) - \mathcal{A}_q(u)|^{\frac{q-1}{q}} = o_n(1); \\ \text{(d)} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{\mathcal{A}_p(u_n(x) - u_n(y))}{|x - y|^{\frac{N+sp}{p(p-1)}}} - \frac{\mathcal{A}_p(w_n(x) - w_n(y))}{|x - y|^{\frac{N+sp}{p(p-1)}}} - \frac{\mathcal{A}_p(u(x) - u(y))}{|x - y|^{\frac{N+sp}{p(p-1)}}} \right|^{\frac{p}{p-1}} dx dy \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{\mathcal{A}_q(u_n(x) - u_n(y))}{|x - y|^{\frac{N+sq}{q(q-1)}}} - \frac{\mathcal{A}_q(w_n(x) - w_n(y))}{|x - y|^{\frac{N+sq}{q(q-1)}}} - \frac{\mathcal{A}_q(u(x) - u(y))}{|x - y|^{\frac{N+sq}{q(q-1)}}} \right|^{\frac{q}{q-1}} dx dy \\ & = o_n(1); \\ \text{(e)} & \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) F(u_n) dx - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) F(w_n) dx \\ & - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u) \right) F(u) dx = o_n(1) \end{aligned}$$

and for any  $\varphi \in X_\varepsilon$  with  $\|\varphi\|_{X_\varepsilon} \leq 1$  it holds

$$(f) \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) f(u_n) \varphi dx - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) f(w_n) \varphi dx \\ - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \varphi dx = o_n(1).$$

*Proof.* We first point out that the proofs of (a) and (b) follow from the Brezis-Lieb Lemma (see [17]). Moreover, arguing as in the proof Lemma 3.2 of Mercuri & Willem [43], we see that (c) holds true.

Next, we give the details of the proofs of (d), (e) and (f) for the convenience of readers.

(d) **Case 1:**  $2 \leq t \in \{p, q\}$ .

We deduce from the Mean Value Theorem, the Young inequality and  $t \geq 2$  that for any fixed  $\sigma > 0$  there exists  $C_\sigma > 0$  such that

$$(25) \quad \left| |a+b|^{t-2}(a+b) - |a|^{t-2}a \right| \leq \sigma |a|^{t-1} + C_\sigma |b|^{t-1} \quad \text{for all } a, b \in \mathbb{R}.$$

In (25) we take

$$a = \frac{w_n(x) - w_n(y)}{|x-y|^{\frac{N+st}{t}}} \quad \text{and} \quad b = \frac{u(x) - u(y)}{|x-y|^{\frac{N+st}{t}}}$$

and we have

$$\left| \frac{\mathcal{A}_t(u_n(x) - u_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(w_n(x) - w_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} \right| \leq \sigma \frac{|w_n(x) - w_n(y)|^{t-1}}{|x-y|^{\frac{N+st}{t/(t-1)}}} + C_\sigma \frac{|u(x) - u(y)|^{t-1}}{|x-y|^{\frac{N+st}{t/(t-1)}}}.$$

Now we introduce the mapping  $H_{\sigma,n}^1 : \mathbb{R}^{2N} \mapsto \mathbb{R}_+$  given by

$$H_{\sigma,n}^1(x, y) := \max \left\{ \left| \frac{\mathcal{A}_t(u_n(x) - u_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(w_n(x) - w_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(u(x) - u(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} \right| \right. \\ \left. - \sigma \frac{|w_n(x) - w_n(y)|^{t-1}}{|x-y|^{\frac{N+st}{t/(t-1)}}}, 0 \right\}.$$

Then  $H_{\sigma,n}^1(x, y) \rightarrow 0$  a.e. in  $\mathbb{R}^{2N}$  as  $n \rightarrow \infty$  and

$$0 \leq H_{\sigma,n}^1(x, y) \leq C \frac{|u(x) - u(y)|^{t-1}}{|x-y|^{\frac{N+st}{t/(t-1)}}} \in L^{\frac{t}{t-1}}(\mathbb{R}^{2N})$$

for some constant  $C > 0$ . Hence from the Dominated Convergence Theorem we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |H_{\sigma,n}^1|^{\frac{t}{t-1}} dx dy \rightarrow 0 \text{ in } \mathbb{R} \text{ as } n \rightarrow \infty.$$

In addition, according to the definition of  $H_{\sigma,n}^1$  we have

$$\left| \frac{\mathcal{A}_t(u_n(x) - u_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(w_n(x) - w_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(u(x) - u(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} \right|^{\frac{t}{t-1}} \\ \leq C \left( \sigma^{\frac{t}{t-1}} \frac{|w_n(x) - w_n(y)|^t}{|x-y|^{N+st}} + (H_{\sigma,n}^1(x, y))^{\frac{t}{t-1}} \right)$$

for some constant  $C > 0$ . This implies that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{\mathcal{A}_t(u_n(x) - u_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(w_n(x) - w_n(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(u(x) - u(y))}{|x-y|^{\frac{N+st}{t/(t-1)}}} \right|^{\frac{t}{t-1}} dx dy = o_n(1)$$

holds true for  $n \in \mathbb{N}$  large enough.

**Case 2:**  $1 < t < 2$ .

Invoking Lemma 3.1 in Mercuri & Willem [43] and applying the Dominated Convergence Theorem, for  $n \in \mathbb{N}$  large enough we obtain that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{\mathcal{A}_t(u_n(x) - u_n(y))}{|x - y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(w_n(x) - w_n(y))}{|x - y|^{\frac{N+st}{t/(t-1)}}} - \frac{\mathcal{A}_t(u(x) - u(y))}{|x - y|^{\frac{N+st}{t/(t-1)}}} \right|^{\frac{t}{t-1}} dx dy = o_n(1)$$

holds true.

Now, combining Case 1 and Case 2 we complete the proof of (d).

(e) Using the Mean Value Theorem, hypotheses  $(f_1)$ – $(f_2)$  and Young's inequality, we can infer that for any  $\sigma > 0$  there exists  $C_\sigma > 0$  such that

$$|F(u_n) - F(w_n) - F(u)|^{\frac{2N}{2N-\mu}} \leq \sigma \left( |u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\nu}{2N-\mu}} \right) + C_\sigma \left( |u|^{\frac{2Np}{2N-\mu}} + |u|^{\frac{2N\nu}{2N-\mu}} \right).$$

Define the following mapping

$$H_{\sigma,n}^2 := \max \left\{ |F(u_n) - F(w_n) - F(u)|^{\frac{2N}{2N-\mu}} - \sigma \left( |u_n|^{\frac{2Np}{2N-\mu}} + |u_n|^{\frac{2N\nu}{2N-\mu}} \right), 0 \right\}.$$

Then, arguing as in the proof of Lemma 10-(d) and using the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$ , for  $n \in \mathbb{N}$  sufficiently large we can derive that

$$(26) \quad \int_{\mathbb{R}^N} |F(u_n) - F(w_n) - F(u)|^{\frac{2N}{2N-\mu}} dx = o_n(1).$$

On the other hand, we can deduce from Theorem 4.3 of Lieb & Loss [38] that

$$(27) \quad \int_{\mathbb{R}^N} \left| \frac{1}{|x|^\mu} * (F(u_n) - F(w_n) - F(u)) \right|^{\frac{2N}{\mu}} dx = o_n(1)$$

for  $n \in \mathbb{N}$  large enough.

We note that

$$\begin{aligned} & \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) F(u_n) dx - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) F(w_n) dx \\ &= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * (F(u_n) - F(w_n)) \right) (F(u_n) - F(w_n)) dx \\ &+ 2 \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * (F(u_n) - F(w_n)) \right) F(w_n) dx. \end{aligned}$$

Using the above equality and taking into account that  $F(w_n) \xrightarrow{w} 0$  in  $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ , together with (26)–(27), we can show that (e) holds true.

(f) We first prove that

$$(28) \quad \sup_{\|\varphi\|_{X_\varepsilon} \leq 1} \int_{\mathbb{R}^N} |(f(u_n) - f(w_n) - f(u)) \varphi|^{\frac{2N}{2N-\mu}} dx = o_n(1)$$

for  $n \in \mathbb{N}$  large enough.

For any fixed  $0 < \sigma < 1$ , we deduce from hypothesis  $(f_1)$  that there exists  $0 < \lambda_0 := \lambda_0(\sigma) < 1$  such that

$$(29) \quad f(t) \leq \sigma |t|^{p-1} \quad \text{for all } |t| \leq 2\lambda_0.$$

In addition, by hypothesis  $(f_2)$  we can find  $\lambda_1 := \lambda_1(\sigma) > 2$  such that

$$(30) \quad |f(t)| \leq \sigma |t|^{\nu-1} \quad \text{for all } |t| \geq \lambda_1 - 1.$$

Next, by the continuity of  $f$ , there is a positive constant  $\gamma := \gamma(\sigma) < \lambda_0$  such that

$$(31) \quad |f(t_1) - f(t_2)| \leq \lambda_0^{p-1} \sigma \quad \text{for all } |t_1 - t_2| \leq \gamma, |t_1|, |t_2| \leq \lambda_1 + 1.$$

Additionally, combining  $(f_1)$  and  $(f_2)$  we can conclude that there exists  $C(\sigma) > 0$  such that

$$(32) \quad f(t) \leq C(\sigma) |t|^{p-1} + \sigma |t|^{\nu-1} \quad \text{for all } t \in \mathbb{R}.$$

We now estimate the following term:

$$\int_{\mathbb{R}^N \setminus B_R} |(f(u_n) - f(w_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx.$$

Since  $u \in X_\varepsilon$ , together with (32), the Hölder inequality (8) and  $(V_1)$ , we know that there exists  $R = R(\sigma) > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_R} |f(u)\varphi|^{\frac{2N}{2N-\mu}} dx &\leq C \left( \int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{2N(p-1)}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx + \int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{2N(\nu-1)}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx \right) \\ &\leq C \left( \left( \int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{2Np}{2N-\mu}} dx \right)^{\frac{p-1}{p}} + \left( \int_{\mathbb{R}^N \setminus B_R} |u|^{\frac{2N\nu}{2N-\mu}} dx \right)^{\frac{\nu-1}{\nu}} \right) \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} \\ (33) \quad &\leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} \end{aligned}$$

for some constant  $C > 0$ .

Let us define

$$D_{u_n}^1 := \{x \in \mathbb{R}^N \setminus B_R : |u_n(x)| \leq \lambda_0\}.$$

According to (29), the Hölder inequality, (8),  $(V_1)$  and the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$ , we have

$$\begin{aligned} &\int_{D_{u_n}^1 \cap |u| \leq \gamma} |(f(u_n) - f(w_n))\varphi|^{\frac{2N}{2N-\mu}} dx \\ &\leq \int_{D_{u_n}^1 \cap |u| \leq \gamma} (\sigma|u_n|^{p-1} + \sigma|w_n|^{p-1})^{\frac{2N}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx \\ (34) \quad &\leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} \end{aligned}$$

for some constant  $C > 0$ .

Set

$$D_{u_n}^2 := \{x \in \mathbb{R}^N \setminus B_R : |u_n(x)| \geq \lambda_1\}.$$

Arguing as in (34), together with (30), we can obtain

$$(35) \quad \int_{D_{u_n}^2 \cap |u| \leq \gamma} |(f(u_n) - f(w_n))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}.$$

for some positive constant  $C > 0$ .

Now, we introduce the following set:

$$D_{u_n}^3 := \{x \in \mathbb{R}^N \setminus B_R : \lambda_0 \leq |u_n(x)| \leq \lambda_1\}.$$

It is easy to check that  $|D_{u_n}^3| < +\infty$ , since  $u_n \in X_\varepsilon$ . So, from (31) it follows that

$$(36) \quad \int_{D_{u_n}^3 \cap |u| \leq \gamma} |(f(u_n) - f(w_n))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}$$

for some constant  $C > 0$ .

Thus, we deduce from (34)–(36) that

$$(37) \quad \int_{(\mathbb{R}^N \setminus B_R) \cap |u| \leq \gamma} |(f(u_n) - f(w_n))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} \quad \text{for all } n \in \mathbb{N}.$$

Thanks to  $u \in X_\varepsilon$ , we have  $|(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}| \rightarrow 0$  as  $R \rightarrow \infty$ . Using (32) again, we deduce that there exists  $R := R(\sigma) > 0$  such that

$$\begin{aligned}
& \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}} |(f(u_n) - f(w_n))\varphi|^{\frac{2N}{2N-\mu}} dx \\
& \leq \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}} (\sigma(|u_n|^{\nu-1} + |w_n|^{\nu-1})|\varphi| + C(\sigma)(|u_n|^{p-1} + |w_n|^{p-1})|\varphi|)^{\frac{2N}{2N-\mu}} dx \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + C \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}} (C(\sigma)(|u_n|^{p-1} + |w_n|^{p-1})|\varphi|)^{\frac{2N}{2N-\mu}} dx \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + (2C(\sigma))^{\frac{2N}{2N-\mu}} \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}} \left( |u_n|^{\frac{2N(p-1)}{2N-\mu}} + |w_n|^{\frac{2N(p-1)}{2N-\mu}} \right) |\varphi|^{\frac{2N}{2N-\mu}} dx \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + C(C(\sigma))^{\frac{2N}{2N-\mu}} \left( \int_{(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}} |\varphi|^{\frac{2Np}{2N-\mu}} dx \right)^{\frac{1}{p}} \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + C(C(\sigma))^{\frac{2N}{2N-\mu}} \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} |(\mathbb{R}^N \setminus B_R) \cap \{|u| \geq \gamma\}|^{\frac{2sp-\mu}{p(2N-\mu)}} \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}},
\end{aligned}$$

where  $C$  is a positive constant. Combining this inequality with (33) and (37), we deduce that there is a constant  $C > 0$  such that for all  $n \in \mathbb{N}$

$$(38) \quad \int_{\mathbb{R}^N \setminus B_R} |(f(u_n) - f(w_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}.$$

Recalling that  $u_n \xrightarrow{w} u$  in  $X_\varepsilon$  as  $n \rightarrow \infty$  and passing to a subsequence (still denoted by  $\{u_n\}_{n \in \mathbb{N}}$ ), we can assume that  $u_n \rightarrow u$  strongly in  $L^{\frac{2Np}{2N-\mu}}(B_R)$  (since  $0 < \mu < 2sp$ ) and there is a function  $d \in L^{\frac{2Np}{2N-\mu}}(B_R)$  such that  $|u_n(x)|, |u(x)| \leq d(x)$  a.e.  $x \in B_R$ .

For  $n \in \mathbb{N}$  large enough we have

$$\begin{aligned}
& \int_{B_R} |f(w_n)\varphi|^{\frac{2N}{2N-\mu}} dx \leq \int_{B_R} (C(\sigma)|w_n|^{p-1} + \sigma|w_n|^{\nu-1})^{\frac{2N}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + (2C(\sigma))^{\frac{2N}{2N-\mu}} \left( \int_{B_R} |w_n|^{\frac{2Np}{2N-\mu}} \right)^{\frac{p-1}{p}} |\varphi|^{\frac{2N}{2N-\mu}} \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}},
\end{aligned}$$

where  $C > 0$  is some constant.

Set  $D_{u_n}^4 := \{x \in B_R : |u_n(x) - u(x)| \geq 1\}$ . So, we get

$$\begin{aligned}
& \int_{D_{u_n}^4} |(f(u_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \\
& \leq \int_{D_{u_n}^4} (C(\sigma)(|u_n|^{p-1} + |u|^{p-1}) + \sigma(|u_n|^{\nu-1} + |u|^{\nu-1}))^{\frac{2N}{2N-\mu}} |\varphi|^{\frac{2N}{2N-\mu}} dx \\
& \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + C(2C(\sigma))^{\frac{2N}{2N-\mu}} \left( \int_{D_{u_n}^4} d^{\frac{2Np}{2N-\mu}} dx \right)^{\frac{p-1}{p}} \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}},
\end{aligned}$$

where  $C > 0$  is some constant. We observe that  $|D_{u_n}^4| \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  and then we can deduce that there exists some constant  $C > 0$  such that

$$(40) \quad \int_{D_{u_n}^4} |(f(u_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma\|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}},$$

provided that  $n \in \mathbb{N}$  is sufficiently large.

In addition, from  $u \in X_\varepsilon$  it follows that

$$\left| \{x \in \mathbb{R}^N : |u(x)| \geq L\} \right| \rightarrow 0 \text{ in } \mathbb{R}^N \text{ as } L \rightarrow +\infty.$$

Using the above fact and invoking (32) again, we can infer that there exists  $L := L(\sigma) > 0$  such that

$$\begin{aligned} & \int_{(B_R \setminus D_{u_n}^4) \cap \{x \in \mathbb{R}^N : |u(x)| \geq L\}} |(f(u_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \\ & \leq \int_{(B_R \setminus D_{u_n}^4) \cap \{x \in \mathbb{R}^N : |u(x)| \geq L\}} \left| (C(\sigma)(|u_n|^{p-1} + |u|^{p-1}) + \sigma(|u_n|^{\nu-1} + |u|^{\nu-1})) \varphi \right|^{\frac{2N}{2N-\mu}} dx \\ & \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} + C(C(\sigma))^{\frac{2N}{2N-\mu}} \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}} |(B_R \setminus D_{u_n}^4) \cap \{x \in \mathbb{R}^N : |u(x)| \geq L\}|^{\frac{2sp-\mu}{p(2N-\mu)}} \\ (41) \quad & \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}, \end{aligned}$$

where  $C > 0$  is a constant. Additionally, we can deduce from the Dominated Convergence Theorem that

$$\int_{(B_R \setminus D_{u_n}^4) \cap \{x \in \mathbb{R}^N : |u(x)| \leq L\}} |f(u_n) - f(u)|^{\frac{2Np}{(2N-\mu)(p-1)}} dx = o_n(1)$$

for  $n \in \mathbb{N}$  large enough.

It follows that

$$(42) \quad \int_{(B_R \setminus D_{u_n}^4) \cap \{x \in \mathbb{R}^N : |u(x)| \leq L\}} |(f(u_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}$$

for some constant  $C > 0$ .

Hence we deduce from (39), (40), (41) and (42) that there exists some constant  $C > 0$  such that

$$(43) \quad \int_{B_R} |(f(u_n) - f(w_n) - f(u))\varphi|^{\frac{2N}{2N-\mu}} dx \leq C\sigma \|\varphi\|_{X_\varepsilon}^{\frac{2N}{2N-\mu}}$$

for large enough  $n \in \mathbb{N}$ .

Putting together (38) and (43), we conclude that (28) holds true.

On account of the fact that  $\frac{1}{|x|^\mu} * F(u) \in L^{\frac{2N}{\mu}}(\mathbb{R}^N)$ , for any  $0 < \sigma < 1$  there exists  $R_1 := R_1(\sigma) > 0$  such that

$$\left( \int_{\mathbb{R}^N \setminus B_{R_1}} \left| \frac{1}{|x|^\mu} * F(u) \right|^{\frac{2N}{\mu}} dx \right)^{\frac{\mu}{2N}} < \sigma.$$

Using the above all information and by a straightforward computation, for  $n \in \mathbb{N}$  large enough we can conclude that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(w_n) \varphi dx \right| \\ & \leq \int_{B_{R_1}} \left| \left( \frac{1}{|x|^\mu} * F(u) \right) f(w_n) \varphi \right| dx + \int_{\mathbb{R}^N \setminus B_{R_1}} \left| \left( \frac{1}{|x|^\mu} * F(u) \right) f(w_n) \varphi \right| dx \\ & \leq C_u \left( \int_{B_{R_1}} |f(w_n) \varphi|^{\frac{2N}{2N-\mu}} dx \right)^{\frac{2N-\mu}{2N}} \\ & \quad + \left( \int_{\mathbb{R}^N \setminus B_{R_1}} \left| \frac{1}{|x|^\mu} * F(u) \right|^{\frac{2N}{\mu}} dx \right)^{\frac{\mu}{2N}} \left( \int_{\mathbb{R}^N} |f(w_n) \varphi|^{\frac{2N}{2N-\mu}} dx \right)^{\frac{2N-\mu}{2N}} \\ (44) \quad & \leq C\sigma \|\varphi\|_{X_\varepsilon}, \end{aligned}$$

where  $C_u$  and  $C$  are positive constants.

Next, we prove the following relation:

$$(45) \quad \left| \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) f(u)\varphi dx \right| \leq C\sigma \|\varphi\|_{X_\varepsilon}$$

for  $n \in \mathbb{N}$  sufficiently large.

Indeed, it is easy to check that, for any  $\sigma \in (0, 1)$  and some  $R_2 := R(\sigma) > 0$ , there exists some constant  $C > 0$  such that

$$(46) \quad \int_{B_{R_2}} |F(w_n)| \left( \int_{B_{R_2}} \frac{|f(u(x))\varphi(x)|}{|x-y|^\mu} dx \right) dy \leq C\sigma \|\varphi\|_{X_\varepsilon},$$

$$(47) \quad \int_{\mathbb{R}^N} |F(w_n)| \left( \int_{\mathbb{R}^N \setminus B_{R_2}} \frac{|f(u(x))\varphi(x)|}{|x-y|^\mu} dx \right) dy \leq C\sigma \|\varphi\|_{X_\varepsilon}$$

for  $n \in \mathbb{N}$  large enough.

To prove (45), now we need to estimate the following part:

$$\Phi_n := \int_{\mathbb{R}^N \setminus B_{R_2}} |F(w_n)| \left( \int_{B_{R_2}} \frac{|f(u(x))\varphi(x)|}{|x-y|^\mu} dx \right) dy.$$

To this end, we are divided into two parts for discussion.

(\*) If  $f(u(x))\varphi(x) = 0$  a.e. on  $B_{R_2}$ , so, we have

$$\Phi_n \leq C\sigma \|\varphi\|_{X_\varepsilon} \text{ for all } \sigma > 0 \text{ and for some constant } C > 0.$$

(\*\*) If  $|\{x \in B_{R_2} : f(u(x))\varphi(x) \neq 0\}| > 0$ , that is,

$$\int_{B_{R_2}} |f(u)\varphi|^{\frac{6N}{6N-\mu}} dx > 0.$$

In addition, we can easily derive that

$$\int_{B_{R_2}} |f(u)\varphi|^{\frac{6N}{6N-\mu}} dx \leq C \|\varphi\|_{X_\varepsilon}^{\frac{6N}{6N-\mu}} |B_{R_2}|^{\frac{2\mu}{6N-\mu}} \text{ for some constant } C > 0.$$

Let us define

$$d_\sigma := \left( \frac{\|f(u)\varphi\|_{L^{\frac{6N}{6N-\mu}}(B_{R_2})}}{\sigma \|\varphi\|_{X_\varepsilon}} \right)^{\frac{3}{\mu}} \text{ and } \widehat{R}_2 := R_2 + \sigma^{-\frac{3}{\mu}} |B_{R_2}|^{\frac{6}{6N-\mu}} C^{\frac{6N-\mu}{2N\mu}}.$$

In the case (\*\*), we can apply the above relations, Theorem 4.3 of Lieb & Loss [38], the Sobolev continuous embedding and the local compactness Sobolev embedding to infer that,



for  $n \in \mathbb{N}$  sufficiently large,

$$\begin{aligned}
\Phi_n &= \int_{B_{R_2}} |f(u)\varphi| \left( \int_{\mathbb{R}^N \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&= \int_{B_{R_2}} |f(u)\varphi| \left( \int_{\mathbb{R}^N \setminus B_{R_2+d\sigma}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&+ \int_{B_{R_2}} |f(u)\varphi| \left( \int_{B_{R_2+d\sigma} \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&\leq C \frac{\sigma \|\varphi\|_{X_\varepsilon}}{|f(u)\varphi|_{L^{\frac{6N}{6N-\mu}}(B_{R_2})}} \int_{B_{R_2}} |f(u)\varphi| \left( \int_{\mathbb{R}^N \setminus B_{R_2+d\sigma}} \frac{|F(w_n(y))|}{|x-y|^{2\mu/3}} dy \right) dx \\
&+ \int_{B_{R_2}} |f(u)\varphi| \left( \int_{B_{R_2+d\sigma} \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&\leq C \frac{\sigma \|\varphi\|_{X_\varepsilon}}{|f(u)\varphi|_{L^{\frac{6N}{6N-\mu}}(B_{R_2})}} \int_{B_{R_2}} |f(u)\varphi| \left( \int_{\mathbb{R}^N} \frac{|F(w_n(y))|}{|x-y|^{2\mu/3}} dy \right) dx \\
&+ \int_{B_{R_2}} |f(u)\varphi| \left( \int_{B_{R_2+d\sigma} \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&\leq C \frac{\sigma \|\varphi\|_{X_\varepsilon}}{|f(u)\varphi|_{L^{\frac{6N}{6N-\mu}}(B_{R_2})}} |f(u)\varphi|_{L^{\frac{6N}{6N-\mu}}(B_{R_2})} \\
&+ \int_{B_{R_2}} |f(u)\varphi| \left( \int_{B_{\hat{R}_2} \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
&\leq C\sigma \|\varphi\|_{X_\varepsilon} + \int_{B_{R_2}} |f(u)\varphi| \left( \int_{B_{\hat{R}_2} \setminus B_{R_2}} \frac{|F(w_n(y))|}{|x-y|^\mu} dy \right) dx \\
(48) \quad &\leq C\sigma \|\varphi\|_{X_\varepsilon}.
\end{aligned}$$

Thus, relations (46), (47), (48) and (\*) imply that (45) holds true. Finally, we point out that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) f(u_n) \varphi dx - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) f(w_n) \varphi dx \\
&= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * (F(u_n) - F(w_n)) \right) (f(u_n) - f(w_n)) \varphi dx \\
&+ \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * (F(u_n) - F(w_n)) \right) f(w_n) \varphi dx \\
&+ \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(w_n) \right) (f(u_n) - f(w_n)) \varphi dx.
\end{aligned}$$

Then the above equality combined with (26), (27), (28), (44), (45), Theorem 4.3 of Lieb & Loss [38], the Hölder inequality, the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_\varepsilon$  and the fact that  $\|\varphi\|_{X_\varepsilon} \leq 1$  yields that (f) is true. This proof is now complete.  $\square$

### 3. THE LIMIT PROBLEM

To the best of our knowledge, there is no result about the nonlinear fractional  $(p, q)$ -Choquard problem. That is why we need to consider the following limit problem associated

with problem (3):

$$(49) \quad \left\{ \begin{array}{l} (-\Delta)_p^s u + (-\Delta)_q^s u + V_0(|u|^{p-2}u + |u|^{q-2}u) = \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N. \end{array} \right\}$$

It is worth pointing out that we denote by  $X_{V_0}$  the space  $X_\varepsilon$  when  $\varepsilon = 0$ . For all  $u \in X_{V_0}$ , we introduce the corresponding energy functional associated with problem (49) defined by

$$I_{V_0}(u) := \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q + V_0 \left( \frac{1}{p}|u|_p^p + \frac{1}{q}|u|_q^q \right) - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u) \right) F(u) dx.$$

It is obvious that the functional  $I_{V_0}$  is well-defined and belongs to  $C^1$ , with its differential given by

$$\begin{aligned} \langle I'_{V_0}(u), v \rangle &= \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + \int_{\mathbb{R}^N} V_0 (|u|^{p-2}u + |u|^{q-2}u) v dx \\ &\quad - \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u) \right) f(u) v dx, \end{aligned}$$

for all  $u, v \in X_{V_0}$ . So, it is easy to check that all the solutions of problem (49) correspond to critical points of the functional  $I_{V_0}$ .

Next, we denote by  $\mathcal{M}_{V_0}$  the Nehari manifold associated with the functional  $I_{V_0}$ , that is,

$$\mathcal{M}_{V_0} := \{u \in X_{V_0} \setminus \{0\} : \langle I'_{V_0}(u), u \rangle = 0\}.$$

Moreover, we define  $c_{V_0} := \inf_{u \in \mathcal{M}_{V_0}} I_{V_0}(u)$ .

Now, we define the following sets

$$X_{V_0}^+ := \{u \in X_{V_0} : |\text{supp}(u^+)| > 0\} \quad \text{and} \quad S_{V_0}^+ = S_{V_0} \cap X_{V_0}^+,$$

where  $S_{V_0}$  is the unit sphere of  $X_{V_0}$ . We point out that  $S_{V_0}^+$  is also an incomplete  $C^{1,1}$ -manifold of codimension one and contained in  $X_{V_0}^+$ . Hence,  $X_{V_0} = T_u(S_{V_0}^+) \oplus \mathbb{R}u$  for each  $u \in S_{V_0}^+$ , where

$$T_u(S_{V_0}^+) := \left\{ v \in X_{V_0} : \langle u, v \rangle_{s,p} + \langle u, v \rangle_{s,q} + \int_{\mathbb{R}^N} V_0 (|u|^{p-2}u + |u|^{q-2}u) v dx = 0 \right\}.$$

With arguments as in the proof of Lemma 2, we can deduce that the following property holds true.

**Lemma 11.** *Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$  are fulfilled, then we have the following properties:*

- for all  $u \in X_{V_0}^+$ , there exists a unique  $t_u > 0$  such that  $t_u u \in \mathcal{M}_{V_0}$ . Furthermore,  $\hat{m}_{V_0}(u) = t_u u$  is the unique maximum of  $I_{V_0}(tu)$ ;
- there exists  $\tau > 0$  independent of  $u$  such that  $t_u \geq \tau$  for all  $u \in S_{V_0}^+$ . Moreover, for each compact set  $\mathcal{W} \subset S_{V_0}^+$ , there exists a constant  $C_{\mathcal{W}} > 0$  such that  $t_u \leq C_{\mathcal{W}}$  for all  $u \in \mathcal{W}$ ;
- the mapping  $\hat{m}_{V_0} : X_{V_0}^+ \mapsto \mathcal{M}_{V_0}$  is continuous and  $m_{V_0} := \hat{m}_{V_0}|_{S_{V_0}^+}$  is a homeomorphism between  $S_{V_0}^+$  and  $\mathcal{M}_{V_0}$ , and the inverse of  $m_{V_0}$  is given by  $m_{V_0}^{-1}(u) := u/\|u\|_{X_{V_0}}$ ;
- $c_{V_0} := \inf_{u \in \mathcal{M}_{V_0}} I_{V_0}(u) > 0$  and  $I_{V_0}$  is bounded below on  $\mathcal{M}_{V_0}$  by some positive constant;
- let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . If there exists a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset S_{V_0}^+$  such that  $\text{dist}(u_n, \partial S_\varepsilon^+) \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\|m_{V_0}(u_n)\|_{X_{V_0}} \rightarrow +\infty$  and  $J_\varepsilon(m_\varepsilon(u_n)) \rightarrow +\infty$  as  $n \rightarrow \infty$ ;
- $I_{V_0}$  is coercive on  $\mathcal{M}_{V_0}$ .

Now, let us define the mappings

$$\hat{\psi}_{V_0} : X_{V_0}^+ \mapsto \mathbb{R} \quad \text{and} \quad \psi_{V_0} : S_{V_0}^+ \mapsto \mathbb{R}$$

by  $\hat{\psi}_{V_0}(u) := I_{V_0}(\hat{m}_{V_0}(u))$  for all  $u \in X_{V_0}^+$  and  $\psi_{V_0} := \hat{\psi}_{V_0}|_{S_{V_0}^+}$ .

**Lemma 12.** *Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$  hold true. Then,*

- (a)  $\hat{\psi}_{V_0} \in C^1(X_{V_0}^+, \mathbb{R})$  and  $\langle \hat{\psi}'_{V_0}(u), v \rangle = \frac{\|\hat{m}_{V_0}(u)\|_{X_{V_0}}}{\|u\|_{X_{V_0}}} \langle I'_{V_0}(\hat{m}_{V_0}(u)), v \rangle$  for all  $u \in X_{V_0}^+$ , all  $v \in X_{V_0}$ ;
- (b)  $\psi_{V_0} \in C^1(S_{V_0}^+, \mathbb{R})$  and  $\langle \psi'_{V_0}(u), v \rangle = \|m_{V_0}(u)\|_{X_{V_0}} \langle I'_{V_0}(m_{V_0}(u)), v \rangle$  for all  $u \in S_{V_0}^+$ , all  $v \in T_u(S_{V_0}^+)$ ;
- (c) if  $\{u_n\}_{n \in \mathbb{N}}$  is a Palais-Smale sequence for  $\psi_{V_0}$ , then  $\{m_{V_0}(u_n)\}_{n \in \mathbb{N}}$  is a Palais-Smale sequence for  $I_{V_0}$ . If  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$  is bounded Palais-Smale sequence for  $I_{V_0}$ , then  $\{m_{V_0}^{-1}(u_n)\}_{n \in \mathbb{N}} \subset S_{V_0}^+$  is a Palais-Smale sequence for  $\psi_{V_0}$ ;
- (d)  $u \in S_{V_0}^+$  is a critical point of  $\psi_{V_0}$  if and only if  $m_{V_0}(u) \in \mathcal{M}_{V_0}$  is a critical point of  $I_{V_0}$ . Moreover, the corresponding critical values coincide and

$$\inf_{u \in S_{V_0}^+} \psi_{V_0}(u) = \inf_{u \in \mathcal{M}_{V_0}} I_{V_0}(u) = c_{V_0}.$$

**Remark 3.** *The following variational characterization for  $c_{V_0}$  is fulfilled:*

$$c_{V_0} = \inf_{u \in \mathcal{M}_{V_0}} I_{V_0}(u) = \inf_{u \in X_{V_0}^+} \max_{t > 0} I_{V_0}(tu) = \inf_{u \in S_{V_0}^+} \max_{t > 0} I_{V_0}(tu).$$

**Lemma 13.** *Assume that  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is a Palais-Smale sequence of the functional  $I_{V_0}$  at the level  $c_{V_0}$ . Then the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded and there exist a sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  and some constants  $R, \alpha > 0$  such that*

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n|^q dx \geq \alpha.$$

*Proof.* Arguing as in Lemma 5, it is obvious to see that  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded. Now, arguing by contradiction suppose that for any  $R > 0$  the following relation

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^q dx = 0$$

holds true. Then, using Lemma 2.2 in Alves, Ambrosio & Isernia [1], we know that for all  $r \in (p, q_s^*)$

$$(50) \quad u_n \rightarrow 0 \text{ in } L^r(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

So, we deduce from Theorem 4.3 of Lieb & Loss [38], hypotheses  $(f_1)$ – $(f_2)$ , the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  and (50) that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) f(u_n) u_n dx = 0.$$

Moreover, by the boundedness of the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$ , we see that  $\langle I'_{V_0}(u_n), u_n \rangle = o_n(1)$  as  $n \rightarrow \infty$ , that is,

$$[u_n]_{s,p}^p + [u_n]_{s,q}^q + V_0 (|u_n|_p^p + |u_n|_q^q) = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(u_n) \right) f(u_n) u_n dx + o_n(1) \text{ as } n \rightarrow \infty.$$

This implies that  $\|u_n\|_{X_{V_0}} = o_n(1)$  as  $n \rightarrow \infty$ . We get a contradiction since  $I_{V_0}(u_n) \rightarrow c_{V_0} > 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . The proof is now complete.  $\square$

**Theorem 14.** *Problem (49) has a positive ground state solution.*

*Proof.* We deduce from a variant of the Mountain Pass Theorem without the Palais-Smale condition that there is a Palais-Smale sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  of the functional  $I_{V_0}$  at the level  $c_{V_0}$ . As in the proof of Lemma 5, we can conclude that the sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is

bounded. Then we are able to deduce from Lemma 13 that there exist a sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  and some constants  $R, \alpha > 0$

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |u_n|^q dx \geq \alpha.$$

Set  $v_n(x) := u_n(x + y_n)$ . Then we have

$$\int_{B_R(0)} |v_n|^q dx \geq \frac{\alpha}{2}.$$

It is easy to check that  $I_{V_0}(v_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  and  $I'_{V_0}(v_n) \rightarrow 0$  in  $X_{V_0}^*$  as  $n \rightarrow \infty$ . Clearly, the sequence  $\{v_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is also bounded. Thus, we may suppose that there exists  $0 \neq v \in X_{V_0}$  such that  $v_n \xrightarrow{w} v$  (up to a subsequence) in  $X_{V_0}$  as  $n \rightarrow \infty$ . Moreover, arguing as in Lemma 7, we have  $I'_{V_0}(v) = 0$ . Due to  $v \neq 0$ , we can derive that  $v \in \mathcal{M}_{V_0}$  and so  $I_{V_0}(v) \geq c_{V_0}$ . On the other hand, from hypothesis  $(f_3)$  and Fatou's Lemma we deduce that

$$I_{V_0}(v) = I_{V_0}(v) - \frac{1}{q} \langle I'_{V_0}(v), v \rangle \leq \liminf_{n \rightarrow \infty} \left( I_{V_0}(v_n) - \frac{1}{q} \langle I'_{V_0}(v_n), v_n \rangle \right) = c_{V_0}.$$

So, we obtain  $I_{V_0}(v) = c_{V_0}$ .

We can also show that this ground state solution  $v$  is positive. Set  $v^- := \min\{v, 0\}$ , then  $v^- \in X_{V_0}$ . Recalling that  $f(t) = 0$  for  $t \leq 0$  and using  $\langle I'_{V_0}(v), v^- \rangle = 0$ , we obtain

$$\|v^-\|_{V_0, p}^p + \|v^-\|_{V_0, q}^q \leq 0.$$

This leads to  $v^- = 0$  and so  $v \geq 0$  on  $\mathbb{R}^N$ . Thus  $v \geq 0$  and  $v \neq 0$ . Similar to the proof of Lemma 21, we see that  $v \in L^\infty(\mathbb{R}^N)$ . In addition, from Corollary 2.1 in Ambrosio & Rădulescu [11] it follows that  $v \in C^\sigma(\mathbb{R}^N)$  for some  $\sigma \in (0, 1)$ . Now, it follows from the proof of Theorem 1.1-(ii) in Jaroš [33] that  $v > 0$  on  $\mathbb{R}^N$ . This ends the proof.  $\square$

Next, we introduce a compactness result for the autonomous problem, which will be very useful in the sequel.

**Lemma 15.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Assume that  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$  is a sequence satisfying  $I_{V_0}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , then the sequence  $\{u_n(\cdot + y_n)\}_{n \in \mathbb{N}} \subset X_{V_0}$  possesses a convergent subsequence for some sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ .*

*Proof.* On account of the fact that  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$  and  $I_{V_0}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , Lemma 11-(c), Lemma 12-(d) and the definition of  $c_{V_0}$  imply that

$$w_n := m_{V_0}^{-1}(u_n) \in S_{V_0}^+ \text{ for all } n \in \mathbb{N}$$

and

$$\psi_{V_0}(w_n) = I_{V_0}(u_n) \rightarrow c_{V_0} = \inf_{w \in S_{V_0}^+} \psi_{V_0}(w) \text{ in } \mathbb{R} \text{ as } n \rightarrow \infty.$$

Now, we introduce the following map

$$\mathcal{I}(u) = \begin{cases} \psi_{V_0}(u) & \text{if } u \in S_{V_0}^+, \\ +\infty & \text{if } u \in \partial S_{V_0}^+. \end{cases}$$

It is worth to pointing out that the following essential facts:

- (i)  $(\overline{S_{V_0}^+}, \delta_{V_0})$ , where  $\delta_{V_0}(u, v) = \|u - v\|_{X_{V_0}}$ , is a complete metric space;
- (ii)  $\mathcal{I} \in C(\overline{S_{V_0}^+}, \mathbb{R} \cup \{+\infty\})$ , by Lemma 11-(e);
- (iii)  $\mathcal{I}$  is bounded below, by Lemma 12-(d).

We deduce from the Ekeland variational principle (see Ekeland [28]) that there exists a Palais-Smale sequence  $\{\hat{w}_n\}_{n \in \mathbb{N}} \subset S_{V_0}^+$  of the functional  $\psi_{V_0}$  at the level  $c_{V_0}$  such that  $\|\hat{w}_n - w_n\|_{X_{V_0}} = o_n(1)$  as  $n \rightarrow \infty$ , that is,  $\psi_{V_0}(\hat{w}_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  and  $\psi'_{V_0}(\hat{w}_n) \rightarrow 0$  in  $T_{\hat{w}_n}(S_{V_0}^+)^*$  as  $n \rightarrow \infty$ . Therefore, from Lemma 12-(c) it follows that the sequence  $\{m_{V_0}(\hat{w}_n)\}_{n \in \mathbb{N}} \subset \mathcal{M}_{V_0}$  is Palais-Smale sequence of the functional  $I_{V_0}$  at the level  $c_{V_0}$ . Then, we can conclude that the sequence  $\{m_{V_0}(\hat{w}_n)\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded Palais-Smale sequence at the level  $c_{V_0}$ . Now, let  $v_n := m_{V_0}(\hat{w}_n)$ . According to Theorem 14, we see that there exist a sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  and  $0 \neq \hat{v} \in X_{V_0}$  such that

$$(51) \quad \begin{cases} \hat{v}_n(\cdot) := v_n(\cdot + y_n) \xrightarrow{w} \hat{v} & \text{in } X_{V_0} \text{ as } n \rightarrow \infty, \\ I_{V_0}(\hat{v}_n) = c_{V_0} + o_n(1), \quad I'_{V_0}(\hat{v}_n) = o_n(1) & \text{as } n \rightarrow \infty, \\ I_{V_0}(\hat{v}) = c_{V_0}, \quad I'_{V_0}(\hat{v}) = 0. \end{cases}$$

Set  $\tilde{v}_n := \hat{v}_n - \hat{v}$ . Applying Lemma 10, the Hölder inequality and (51), we can deduce that

$$I_{V_0}(\tilde{v}_n) = o_n(1) \quad \text{and} \quad I'_{V_0}(\tilde{v}_n) = o_n(1) \text{ as } n \rightarrow \infty.$$

Consequently,  $\tilde{v}_n \rightarrow 0$  in  $X_{V_0}$  as  $n \rightarrow \infty$ , that is,  $\hat{v}_n \rightarrow \hat{v}$  in  $X_{V_0}$  as  $n \rightarrow \infty$ . From Lemma 11-(c) and (51), we infer that  $u_n(\cdot + y_n) \rightarrow \hat{v} \in \mathcal{M}_{V_0}$  as  $n \rightarrow \infty$ . So, we now complete the proof.  $\square$

#### 4. THE BARYCENTER MAP

Next, we need to establish a relationship between the topology of  $M$  and the number of positive solutions for problem (9). To this end, we first choose  $\delta > 0$  such that

$$(52) \quad M_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\} \subset \Lambda$$

and take  $\eta \in C^\infty([0, +\infty), [0, 1])$  non-increasing satisfying  $\eta(t) = 1$  for  $t \in (0, \delta/2)$ ,  $\eta(t) = 0$  for  $t \in [\delta, +\infty)$  and  $|\eta'(t)| \leq C$  for some constant  $C > 0$ .

Let  $\omega$  be a positive ground state solution to the autonomous problem (49). For any  $y \in M$  we define the following function

$$\Psi_{\varepsilon, y}(x) := \eta(|\varepsilon x - y|) \omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)$$

with the unique number  $t_\varepsilon > 0$  satisfying

$$\max_{t \geq 0} J_\varepsilon(t\Psi_{\varepsilon, y}) = J_\varepsilon(t_\varepsilon\Psi_{\varepsilon, y}),$$

and let us consider the mapping  $\Phi_\varepsilon : M \mapsto \mathcal{N}_\varepsilon$  defined by

$$\Phi_\varepsilon(y) := t_\varepsilon\Psi_{\varepsilon, y}.$$

**Lemma 16.** *The mapping  $\Phi_\varepsilon$  verifies the following property:*

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M.$$

*Proof.* Arguing by contradiction, we may assume that there exist  $\delta_0 > 0$ ,  $\{y_n\}_{n \in \mathbb{N}} \subset M$  and  $\varepsilon_n \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  such that

$$(53) \quad |J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0}| \geq \delta_0.$$

We first note that for each  $n \in \mathbb{N}$  and for all  $z \in B_{\frac{\delta}{\varepsilon_n}}$ ,  $\varepsilon_n z \in B_\delta$ , hence  $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$ .

Applying the change of variable  $z = (\varepsilon_n x - y_n)/\varepsilon_n$  and recalling that  $G = F$  in  $\Lambda$  and  $\eta(t) = 0$  for  $t \geq \delta$ , we have

$$\begin{aligned}
J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, p}}^p + \frac{t_{\varepsilon_n}^q}{q} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, q}}^q \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon_n x, t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}) \right) G(\varepsilon_n x, t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}) dx \\
&= \frac{t_{\varepsilon_n}^p}{p} \left( [\eta(|\varepsilon_n \cdot|) \omega]_{s, p}^p + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n) (\eta(|\varepsilon_n z|) \omega(z))^p dz \right) \\
&\quad + \frac{t_{\varepsilon_n}^q}{q} \left( [\eta(|\varepsilon_n \cdot|) \omega]_{s, q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n) (\eta(|\varepsilon_n z|) \omega(z))^q dz \right) \\
(54) \quad &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|z|^\mu} * F(t_{\varepsilon_n} \eta(|\varepsilon_n z|) \omega(z)) \right) F(t_{\varepsilon_n} \eta(|\varepsilon_n z|) \omega(z)) dz.
\end{aligned}$$

Next, we prove that the sequence  $\{t_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset \mathbb{R}$  verifies  $t_{\varepsilon_n} \rightarrow 1$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Using the definition of  $t_{\varepsilon_n}$ , we see that  $t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n} \in \mathcal{N}_{\varepsilon_n}$ , that is,

$$\begin{aligned}
&t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, p}}^p + \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, q}}^q \\
&= \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}) \right) \frac{f(t_{\varepsilon_n} \Psi_{\varepsilon_n, y_n}) \Psi_{\varepsilon_n, y_n}}{t_{\varepsilon_n}^{q-1}} dx \\
&\quad \text{(on account of the fact that } g = f, G = F \text{ on } \Lambda) \\
(55) \quad &= \int_{\mathbb{R}^N} \left( \frac{1}{|z|^\mu} * F(t_{\varepsilon_n} \eta(|\varepsilon_n z|) \omega(z)) \right) \frac{f(t_{\varepsilon_n} \eta(|\varepsilon_n z|) \omega(z)) \eta(|\varepsilon_n z|) \omega(z)}{t_{\varepsilon_n}^{q-1}} dz.
\end{aligned}$$

Taking into account  $\eta(|x|) = 1$  for  $x \in B_{\frac{\delta}{2}}$  and  $B_{\frac{\delta}{2}} \subset B_{\frac{\delta}{\varepsilon_n}}$  for  $n \in \mathbb{N}$  large enough, we deduce from (55) that

$$t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, p}}^p + \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, q}}^q \geq \int_{B_{\frac{\delta}{2}}} \int_{B_{\frac{\delta}{2}}} \frac{F(t_{\varepsilon_n} \omega(x)) f(t_{\varepsilon_n} \omega(z)) \omega(z)}{t_{\varepsilon_n}^{q-1} |z-x|^\mu} dx dz.$$

On the other hand,  $\omega$  is a continuous and positive function in  $\mathbb{R}^N$ , so there exists  $\bar{z} \in \mathbb{R}^N$  such that

$$\omega(\bar{z}) = \min_{z \in \bar{B}_{\frac{\delta}{2}}} \omega(z) > 0.$$

Thus, we deduce from hypothesis  $(f_4)$  that

$$(56) \quad t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, p}}^p + \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, q}}^q \geq \delta^{-\mu} \left| B_{\frac{\delta}{2}} \right|^2 \frac{F(t_{\varepsilon_n} \omega(\bar{z})) f(t_{\varepsilon_n} \omega(\bar{z}))}{(t_{\varepsilon_n} \omega(\bar{z}))^{q-1}} \omega(\bar{z})^q.$$

According to the Dominated Convergence Theorem and Lemma 2.2 in Ambrosio [7], we can deduce that

$$(57) \quad \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, t}} \rightarrow \|\omega\|_{V_{0, t}} \in (0, +\infty) \text{ in } \mathbb{R} \text{ as } n \rightarrow \infty \quad \text{for } t \in \{p, q\}$$

and

$$(58) \quad \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\Psi_{\varepsilon_n, y_n}) \right) F(\Psi_{\varepsilon_n, y_n}) dx \rightarrow \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\omega) \right) F(\omega) dx \text{ in } \mathbb{R} \text{ as } n \rightarrow \infty.$$

Therefore, if  $t_{\varepsilon_n} \rightarrow +\infty$ , we can conclude that the left-hand side of (56) satisfies the following property:

$$(59) \quad \lim_{n \rightarrow \infty} \left( t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, p}}^p + \|\Psi_{\varepsilon_n, y_n}\|_{V_{\varepsilon_n, q}}^q \right) = \|\omega\|_{V_{0, q}}^q,$$

since  $q > p$ . On the other hand, we deduce from hypothesis  $(f_3)$  that

$$(60) \quad \lim_{n \rightarrow \infty} \frac{F(t_{\varepsilon_n} \omega(\bar{z}))}{(t_{\varepsilon_n} \omega(\bar{z}))^{\frac{q}{2}}} = \lim_{n \rightarrow \infty} \frac{f(t_{\varepsilon_n} \omega(\bar{z}))}{(t_{\varepsilon_n} \omega(\bar{z}))^{\frac{q}{2}-1}} = +\infty.$$

From (56), (59) and (60) we get a contradiction. Then, we pass to a subsequence and assume that there exists  $t_0$  such that  $t_{\varepsilon_n} \rightarrow t_0 \geq 0$ . In addition, applying (55), (57) and combining with  $(f_1)$ – $(f_2)$ , we deduce that  $t_0 > 0$ .

In (55) we pass to the limit as  $n \rightarrow \infty$ , then we can use (57) and the Dominated Convergence Theorem to infer that

$$(61) \quad t_0^{p-q} \|\omega\|_{V_0,p}^p + \|\omega\|_{V_0,q}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(t_0 \omega) \right) \frac{f(t_0 \omega)}{t_0^{q-1}} \omega dx.$$

Due to  $\omega \in \mathcal{M}_{V_0}$ , we derive that

$$(62) \quad \|\omega\|_{V_0,p}^p + \|\omega\|_{V_0,q}^q = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\omega) \right) \frac{f(\omega)}{t_0^{q-1}} \omega dx.$$

From the fact that the functions  $\frac{F(t)}{t^{\frac{q}{2}}}$  and  $\frac{f(t)}{t^{\frac{q}{2}-1}}$  are increasing for  $t > 0$ , together with (61) and (62), it follows that  $t_0 = 1$ .

In (54) we pass to the limit as  $n \rightarrow \infty$ , together with (58), we have

$$\lim_{n \rightarrow \infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_{V_0}(\omega) = c_{V_0}.$$

This is a contradiction, since (53). So, we complete the proof of the lemma.  $\square$

Let us consider the function  $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$  such that  $h(\varepsilon) := \sup_{y \in M} |J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}|$  for all  $\varepsilon > 0$ . Then, we define the following subset of  $\mathcal{N}_\varepsilon$ :

$$\hat{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\}.$$

From Lemma 16 it follows that  $h(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Additionally, we deduce from the definition of  $h$  that  $\Phi_\varepsilon(y) \in \hat{\mathcal{N}}_\varepsilon$  for any  $y \in M$  and  $\varepsilon > 0$ , and so  $\hat{\mathcal{N}}_\varepsilon \neq \emptyset$ .

For any  $\delta > 0$  given by (52), let us choose  $\rho := \rho(\delta) > 0$  such that  $M_\delta \subset B_\rho$ . Then we introduce the map  $\zeta : \mathbb{R}^N \mapsto \mathbb{R}^N$  defined by

$$\zeta(x) := \begin{cases} x & \text{if } |x| < \rho, \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

Now, we introduce the following barycenter map  $\beta_\varepsilon : \mathcal{N}_\varepsilon \mapsto \mathbb{R}^N$  defined by

$$\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^N} \zeta(\varepsilon x) (|u|^p + |u|^q) dx}{\int_{\mathbb{R}^N} (|u|^p + |u|^q) dx}$$

for all  $u \in \mathcal{N}_\varepsilon$ .

Then from the above information we can give the following lemma.

**Lemma 17.** *The function  $\beta_\varepsilon$  has the following property:*

$$\lim_{\varepsilon \rightarrow 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \text{ uniformly in } y \in M.$$

*Proof.* Arguing by contradiction assume that there exist  $\delta_0 > 0$ ,  $\{y_n\}_{n \in \mathbb{N}} \subset M$  and  $\varepsilon_n \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  such that

$$(63) \quad |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0.$$

Employing the definitions of  $\Phi_{\varepsilon_n}$ ,  $\beta_{\varepsilon_n}$ ,  $\zeta$  and using the change of variable  $z = (\varepsilon_n x - y_n)/\varepsilon_n$ , we can conclude that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} (\zeta(\varepsilon_n z + y_n) - y_n) (|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q) dz}{\int_{\mathbb{R}^N} (|\eta(|\varepsilon_n z|)\omega(z)|^p + |\eta(|\varepsilon_n z|)\omega(z)|^q) dz}.$$

On account of  $\{y_n\}_{n \in \mathbb{N}} \subset M \subset M_\delta$ , combining the Dominated Convergence Theorem we can derive that

$$\lim_{n \rightarrow \infty} |\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = 0,$$

which contradicts relation (63). This proof is now complete.  $\square$

**Lemma 18.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Assume that the sequences  $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$  and  $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_{\varepsilon_n}$  satisfy  $\varepsilon_n \rightarrow 0$  in  $\mathbb{R}$  and  $J_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , then there is a sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  such that the sequence  $\{\hat{u}_n(x) := u_n(x + \hat{y}_n)\}_{n \in \mathbb{N}}$  admits a subsequence which converges in  $X_{V_0}$ . Furthermore, the sequence  $\{y_n := \varepsilon_n \hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  has a subsequence  $\{y_n\}_{n \in \mathbb{N}}$  (still denoted by itself) such that  $y_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ .*

*Proof.* It is easy to verify that  $\{u_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded. Then, using Lemma 13, we know that there exist a sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  and some constants  $R, \alpha > 0$  such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(\hat{y}_n)} |u_n|^q dx \geq \alpha.$$

Let  $\hat{u}_n(x) := u_n(x + \hat{y}_n)$ . Consequently,  $\{\hat{u}_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded, and so passing to a subsequence we can suppose that there exists some  $0 \neq \hat{u} \in X_{V_0}$  such that  $\hat{u}_n \xrightarrow{w} \hat{u}$  in  $X_{V_0}$  as  $n \rightarrow \infty$ . Let  $t_n > 0$  be such that  $\hat{v}_n := t_n \hat{u}_n \in \mathcal{M}_{V_0}$  and let  $y_n := \varepsilon_n \hat{y}_n$ . Thus, we deduce that

$$\begin{aligned} c_{V_0} &\leq I_{V_0}(\hat{v}_n) \text{ (from the definition of } c_{V_0}\text{)} \\ &\leq \frac{1}{p}[\hat{v}_n]_{s,p}^p + \frac{1}{q}[\hat{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left( \frac{1}{p}|\hat{v}_n|^p + \frac{1}{q}|\hat{v}_n|^q \right) dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\hat{v}_n) \right) F(\hat{v}_n) dx \\ &\leq \frac{t_n^p}{p}[u_n]_{s,p}^p + \frac{t_n^q}{q}[u_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x) \left( \frac{t_n^p}{p}|u_n|^p + \frac{t_n^q}{q}|u_n|^q \right) dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon_n x, t_n u_n) \right) G(\varepsilon_n x, t_n u_n) dx \text{ (by (g}_2\text{))} \\ &= J_{\varepsilon_n}(t_n u_n) \leq J_{\varepsilon_n}(u_n) \text{ (since } u_n \in \mathcal{N}_{\varepsilon_n}\text{)} \\ &= c_{V_0} + o_n(1) \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that  $I_{V_0}(\hat{v}_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  and  $\hat{v}_n \in \mathcal{M}_{V_0}$ . Clearly, the sequence  $\{\hat{v}_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  is bounded. Therefore, up to a subsequence if necessary, still denoted by itself, we may assume that there exists  $\hat{v} \in X_{V_0}$  such that  $\hat{v}_n \xrightarrow{w} \hat{v}$  in  $X_{V_0}$  as  $n \rightarrow \infty$ . It is easy to see that the sequence  $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$  is bounded and it holds that  $t_n \rightarrow t_0 \geq 0$  as  $n \rightarrow \infty$ . Indeed,  $t_0 > 0$ . Otherwise,  $t_0 = 0$ , so, we infer from the boundedness of the sequence  $\{\hat{v}_n\}_{n \in \mathbb{N}} \subset X_{V_0}$  that  $\|\hat{v}_n\|_{X_{V_0}} = t_n \|\hat{u}_n\|_{X_{V_0}} \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , and so  $I_{V_0}(\hat{v}_n) \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , but this is impossible, since  $c_{V_0} > 0$ . Thus,  $t_0 > 0$ . We deduce from the uniqueness of the weak limit that  $\hat{v} = t_0 \hat{u}$  and  $\hat{u} \neq 0$ . Then, from Lemma 15 it follows that

$$(64) \quad \hat{v}_n \rightarrow \hat{v} \text{ in } X_{V_0} \text{ as } n \rightarrow \infty,$$

and so  $\hat{u}_n \rightarrow \hat{u}$  in  $X_{V_0}$  as  $n \rightarrow \infty$ . Moreover,  $I_{V_0}(\hat{v}) = c_{V_0}$  and  $\langle I'_{V_0}(\hat{v}), \hat{v} \rangle = 0$ .

Next, we shall prove that the sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  has a subsequence, still denoted by itself, such that  $y_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ . We first show the boundedness of the sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ . Otherwise, the sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  is not bounded. So, we may assume that there exists a subsequence, still denoted by itself, such that  $|y_n| \rightarrow +\infty$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Then, we choose  $R > 0$  large enough such that  $\Lambda \subset B_R$  and we may suppose that  $|y_n| > 2R$  for  $n \in \mathbb{N}$  sufficiently large, and so for all  $x \in B_{R/\varepsilon_n}$  we have

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R.$$



On account of the facts that  $u_n \in \mathcal{N}_{\varepsilon_n}$  and  $J_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , thus, for  $n \in \mathbb{N}$  large enough, we have that  $u_n \in \mathcal{B}$ . Consequently, from Lemma 6 it follows that

$$\left| \frac{1}{|x|^\mu} * G(\varepsilon_n x, u_n) \right|_{L^\infty(\mathbb{R}^N)} < \frac{K}{2} \text{ for } n \in \mathbb{N} \text{ large enough.}$$

Therefore, for  $n \in \mathbb{N}$  large enough we obtain

$$\begin{aligned} \|\hat{u}_n\|_{V_{0,p}}^p + \|\hat{u}_n\|_{V_{0,q}}^q &\leq \frac{K}{2} \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \hat{u}_n) \hat{u}_n dx \\ &\leq \frac{K}{2} \int_{B_{R/\varepsilon_n}} \tilde{f}(\hat{u}_n) \hat{u}_n dx + \frac{K}{2} \int_{B_{R/\varepsilon_n}^c} f(\hat{u}_n) \hat{u}_n dx \\ &\leq \frac{K}{2} \int_{B_{R/\varepsilon_n}} \frac{V_0}{K} (|\hat{u}_n|^p + |\hat{u}_n|^q) dx + \frac{K}{2} \int_{B_{R/\varepsilon_n}^c} f(\hat{u}_n) \hat{u}_n dx, \\ &\quad (\text{on account of the fact that } \tilde{f}(\hat{u}_n) \hat{u}_n \leq \frac{V_0}{K} (|\hat{u}_n|^p + |\hat{u}_n|^q) \text{ on } B_{R/\varepsilon_n}). \end{aligned}$$

Recalling that  $\hat{u}_n \rightarrow \hat{u}$  in  $X_{V_0}$  as  $n \rightarrow \infty$  and using the Dominated Convergence Theorem we obtain

$$\int_{B_{R/\varepsilon_n}^c} f(\hat{u}_n) \hat{u}_n dx = o_n(1) \text{ as } n \rightarrow \infty.$$

So, we have

$$\frac{1}{2} \left( \|\hat{u}_n\|_{V_{0,p}}^p + \|\hat{u}_n\|_{V_{0,q}}^q \right) \leq o_n(1) \text{ as } n \rightarrow \infty.$$

Using  $\hat{u}_n \rightarrow \hat{u} \neq 0$  in  $X_{V_0}$  as  $n \rightarrow \infty$  again, we see that this is a contradiction.

Now we get the boundedness of the sequence  $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$ . Passing to a subsequence (still denoted by  $\{y_n\}_{n \in \mathbb{N}}$ ), we may assume that there exists  $y_0 \in \mathbb{R}^N$  such that  $y_n \rightarrow y_0 \in \bar{\Lambda}$  as  $n \rightarrow \infty$ . In fact, if  $y_0 \notin \bar{\Lambda}$ , then we can find some constant  $r > 0$  such that  $y_n \in B_{r/2}(y_0) \subset \bar{\Lambda}^c$ . Arguing as before, we can reach a contradiction. So,  $y_0 \in \bar{\Lambda}$ .

It remains to show that  $V(y_0) = V_0$ . Arguing by contradiction again we may assume that  $V(y_0) > V_0$ . From (64), together with Fatou's Lemma and the invariance of  $\mathbb{R}^N$  by translations it follows that

$$\begin{aligned} c_{V_0} &= I_{V_0}(\hat{v}) \\ &< \liminf_{n \rightarrow \infty} \left( \frac{1}{p} [\hat{v}_n]_{s,p}^p + \frac{1}{q} [\hat{v}_n]_{s,q}^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left( \frac{1}{p} |\hat{v}_n|^p + \frac{1}{q} |\hat{v}_n|^q \right) dx \right. \\ &\quad \left. - \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * F(\hat{v}_n) \right) F(\hat{v}_n) dx \right) \\ &\leq \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(t_n u_n) \leq \liminf_{n \rightarrow \infty} J_{\varepsilon_n}(u_n) = c_{V_0}. \end{aligned}$$

This is impossible. So, from hypothesis  $(V_2)$  it follows that  $y_0 \in M$ . This proof is now complete.  $\square$

**Lemma 19.** *For any  $\delta > 0$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \sup_{u \in \hat{\mathcal{N}}_\varepsilon} \text{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

*Proof.* Let  $\varepsilon_n \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , then we can find a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n}$  such that

$$\sup_{u \in \hat{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1) \text{ as } n \rightarrow \infty.$$

Noting that  $\{u_n\}_{n \in \mathbb{N}} \subset \hat{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$ , we infer that

$$c_{V_0} \leq c_{\varepsilon_n} \leq J_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n)$$

and so

$$\lim_{n \rightarrow \infty} J_{\varepsilon_n}(u_n) = c_{V_0}.$$

Then, we deduce from Lemma 18 that for  $n \in \mathbb{N}$  large enough there exists some sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  such that  $y_n = \varepsilon_n \hat{y}_n \in M_\delta$ . So, we have

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} (\zeta(\varepsilon_n z + y_n) - y_n) (|u_n(z + \hat{y}_n)|^p + |u_n(z + \hat{y}_n)|^q) dz}{\int_{\mathbb{R}^N} (|u_n(z + \hat{y}_n)|^p + |u_n(z + \hat{y}_n)|^q) dz}.$$

Taking into account the facts that  $\{\hat{u}_n(\cdot + \hat{y}_n)\}_{n \in \mathbb{N}} \subset X_{V_0}$  has a convergent subsequence and  $\varepsilon_n z + y_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ , we derive that  $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$  in  $\mathbb{R}^N$  as  $n \rightarrow \infty$ . So, there exists a sequence  $\{y_n\}_{n \in \mathbb{N}} \subset M_\delta$  such that

$$\lim_{n \rightarrow \infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

This ends the proof of the lemma.  $\square$

## 5. MULTIPLE SOLUTIONS FOR PROBLEM (9)

In this section we shall establish a relationship between the topology of  $M$  and the number of solutions for problem (9). Since  $\mathcal{N}_\varepsilon$  is not a  $C^1$  submanifold of the space  $X_\varepsilon$ , we cannot use directly the standard Ljusternik-Schnirelmann theory, but we can bypass this difficulty by applying the abstract results in Szulkin & Weth [56].

**Theorem 20.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$ . Assume that  $(f_1)$ – $(f_4)$  and  $(V_1)$ – $(V_2)$  are fulfilled, then for any  $\delta > 0$  satisfying  $M_\delta \subset \Lambda$ , there exists  $\hat{\varepsilon}_\delta > 0$  such that for any  $\varepsilon \in (0, \hat{\varepsilon}_\delta)$  problem (9) possesses at least  $\text{cat}_{M_\delta}(M)$  positive solutions.*

*Proof.* For any fixed  $\varepsilon > 0$  we introduce the mapping  $\alpha_\varepsilon : M \mapsto S_\varepsilon^+$  defined by

$$\alpha_\varepsilon(y) := m_\varepsilon^{-1}(\Phi_\varepsilon(y)) \text{ for all } y \in M.$$

Consequently, we deduce from Lemma 16 that

$$(65) \quad \lim_{\varepsilon \rightarrow 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon \rightarrow 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \text{ uniformly in } y \in M.$$

Let us introduce the following function

$$h'(\varepsilon) := \sup_{y \in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) - c_{V_0}|.$$

According to (65), we see that  $h'(\varepsilon) \rightarrow 0$  in  $\mathbb{R}$  as  $\varepsilon \rightarrow 0$ . Also, we define the following set:

$$\hat{S}_\varepsilon^+ := \{\omega \in S_\varepsilon^+ : \psi_\varepsilon(\omega) \leq c_{V_0} + h'(\varepsilon)\}.$$

Clearly, for all  $y \in M$  and  $\varepsilon > 0$ ,  $\psi_\varepsilon(\alpha_\varepsilon(y)) \in \hat{S}_\varepsilon^+$ , and so  $\hat{S}_\varepsilon^+ \neq \emptyset$ .

From Lemma 16, Lemma 2-(c), Lemma 19 and Lemma 17 it follows that there exists  $\hat{\varepsilon} = \hat{\varepsilon}_\delta > 0$  such that the following diagram is well-defined:

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta \text{ for any } \varepsilon \in (0, \hat{\varepsilon}).$$

Using Lemma 17 and decreasing  $\hat{\varepsilon}$  if necessary, for all  $y \in M$  we have  $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + l(\varepsilon, y)$ , where  $|l(\varepsilon, y)| \leq \delta/2$  uniformly in  $y \in M$  and for all  $\varepsilon \in (0, \hat{\varepsilon})$ . Hence,  $H(t, y) := y + (1 - t)l(\varepsilon, y)$  for  $(t, y) \in [0, 1] \times M$  is homotopy between  $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$  and the inclusion map  $id : M \mapsto M_\delta$ . This means that

$$(66) \quad \text{cat}_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geq \text{cat}_{M_\delta}(M).$$

Additionally, let us choose a function  $h'(\varepsilon) > 0$  such that  $h'(\varepsilon) \rightarrow 0$  in  $\mathbb{R}$  as  $\varepsilon \rightarrow 0$  and such that  $c_{V_0} + h'(\varepsilon)$  is not a critical level for the functional  $J_\varepsilon$ . Using Corollary 9 and Theorem 27 in Szulkin & Weth [56], for  $\varepsilon > 0$  sufficiently small we can deduce that  $\psi_\varepsilon$  possesses at least  $\text{cat}_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M)$  critical points on  $\hat{S}_\varepsilon^+$ . So, Lemma 3 and (66) imply that the functional  $J_\varepsilon$  has at least  $\text{cat}_{M_\delta}(M)$  critical points in  $\hat{\mathcal{N}}_\varepsilon$ . This proof is now complete.  $\square$

## 6. PROOF OF THEOREM 1

The main idea is to show that the solutions obtained in Theorem 20 verify the following estimate:

$$\text{for } \varepsilon > 0 \text{ sufficiently small, } u_\varepsilon(x) \leq a \text{ for all } x \in \Lambda_\varepsilon^c.$$

Then we can deduce from this fact that these solutions are indeed solutions of the original problem (3). To this end, we shall treat the regularity of nonnegative solutions to problem (9). More precisely, we first establish the following result inspired by Moser [48] and Ambrosio & Rădulescu [11].

**Lemma 21.** *Let  $0 < \mu < sp$  and  $\nu < (N - \mu)q/(N - sq)$  and let  $\varepsilon_n \rightarrow 0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$  and  $u_n \in \hat{\mathcal{N}}_{\varepsilon_n}$  be a solution to problem (9). Then,  $J_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ , and there exists a sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  such that  $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^\infty(\mathbb{R}^N)$  and  $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leq C$  for all  $n \in \mathbb{N}$ , for some constant  $C > 0$ . Furthermore,*

$$(67) \quad \hat{u}_n(x) \rightarrow 0 \text{ as } |x| \rightarrow +\infty \text{ uniformly in } n \in \mathbb{N}.$$

*Proof.* On account of the fact that  $u_n \in \hat{\mathcal{N}}_{\varepsilon_n}$ , arguing as in the proof of Lemma 19, we see that  $J_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Then, from Lemma 18 it follows that there exists a sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  such that  $\hat{u}_n(\cdot) := u_n(\cdot + \hat{y}_n) \rightarrow \hat{u}(\cdot) \in X_{V_0}$  and  $y_n := \varepsilon_n \hat{y}_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ . For any  $L > 0$  and  $\beta > 1$  we introduce the function

$$\psi(\hat{u}_n) := \hat{u}_n \hat{u}_{n,L}^{q(\beta-1)} \in X_{\varepsilon_n}, \text{ where } \hat{u}_{n,L} := \min\{\hat{u}_n, L\}.$$

Taking  $\psi(\hat{u}_n)$  as test function, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{p-2} (\hat{u}_n(x) - \hat{u}_n(y)) (\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x - y|^{N+sp}} dx dy \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{q-2} (\hat{u}_n(x) - \hat{u}_n(y)) (\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x - y|^{N+sq}} dx dy \\ & + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |\hat{u}_n|^{p-2} \hat{u}_n \psi(\hat{u}_n) dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |\hat{u}_n|^{q-2} \hat{u}_n \psi(\hat{u}_n) dx \\ & = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, \hat{u}_n) \right) g(\varepsilon_n x + y_n, \hat{u}_n) \psi(\hat{u}_n) dx. \end{aligned}$$

Additionally, applying the boundedness of the sequence  $\{\hat{u}_n\}_{n \in \mathbb{N}} \subset X_{\varepsilon_n}$  and arguing as in the proof of Lemma 6, we can deduce that there exists  $C_0 > 0$  such that

$$\sup_{n \in \mathbb{N}} \left| \frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, \hat{u}_n) \right|_{L^\infty(\mathbb{R}^N)} \leq C_0.$$

According to the hypotheses on  $g$ , we see that for any  $\sigma > 0$  there exists  $C_\sigma > 0$  such that

$$|g(x, t)| \leq \sigma |t|^{p-1} + C_\sigma |t|^{q_s^* - 1} \text{ for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Choosing  $\sigma \in (0, V_0/C_0)$ , together with the above inequalities, we can infer that

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{p-2} (\hat{u}_n(x) - \hat{u}_n(y)) (\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x - y|^{N+sp}} dx dy \\ & + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\hat{u}_n(x) - \hat{u}_n(y)|^{q-2} (\hat{u}_n(x) - \hat{u}_n(y)) (\psi(\hat{u}_n(x)) - \psi(\hat{u}_n(y)))}{|x - y|^{N+sq}} dx dy \\ (68) \quad & \leq C \int_{\mathbb{R}^N} |\hat{u}_n|^{q_s^*} \hat{u}_{n,L}^{q(\beta-1)} dx \end{aligned}$$

for some constant  $C > 0$ .

Let us define the following functions

$$\varphi(t) := \frac{|t|^q}{q} \quad \text{and} \quad \Upsilon(t) := \int_0^t (\psi'(\tau))^{\frac{1}{q}} d\tau.$$

We first observe that  $\psi$  is an increasing function, hence

$$(69) \quad (a - b)(\psi(a) - \psi(b)) \geq 0 \quad \text{for all } a, b \in \mathbb{R}.$$

Then we can infer from (69) and the Jensen inequality that

$$(70) \quad \varphi'(a - b)(\psi(a) - \psi(b)) \geq |\Upsilon(a) - \Upsilon(b)|^q \quad \text{for all } a, b \in \mathbb{R}.$$

We also point out that

$$(71) \quad \Upsilon(\hat{u}_n) \geq \frac{1}{\beta} \hat{u}_n \hat{u}_{n,L}^{\beta-1}.$$

So, putting together with (68), (69), (70) and (71) and using the Sobolev embedding, we infer that there exists some constant  $C > 0$  such that

$$(72) \quad |\hat{u}_n \hat{u}_{n,L}^{\beta-1}|_{q_s^*}^q \leq C \beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q_s^*} \hat{u}_{n,L}^{q(\beta-1)} dx.$$

Choose  $\beta = \frac{q_s^*}{q}$  and let  $R > 0$  large enough. Combining  $\hat{u}_n \rightarrow \hat{u}$  in  $X_{V_0}$  as  $n \rightarrow \infty$  with the Hölder inequality, we can conclude that there exists some constant  $C > 0$  such that

$$\left( \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^{q_s^*} dx \right)^{\frac{q}{q_s^*}} \leq C \beta^q \int_{\mathbb{R}^N} R^{q_s^*-q} \hat{u}_n^{q_s^*} dx + C \epsilon \left( \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^{q_s^*} dx \right)^{\frac{q}{q_s^*}}.$$

Then, we choose a fixed  $\epsilon \in (0, 1/C)$  and infer that

$$\left( \int_{\mathbb{R}^N} \left( \hat{u}_n \hat{u}_{n,L}^{\frac{q_s^*-q}{q}} \right)^{q_s^*} dx \right)^{\frac{q}{q_s^*}} \leq C \beta^q \int_{\mathbb{R}^N} R^{q_s^*-q} \hat{u}_n^{q_s^*} dx < +\infty.$$

In the above inequality we pass to the limit as  $L \rightarrow +\infty$  and we obtain  $\hat{u}_n \in L^{\frac{q_s^*2}{q}}(\mathbb{R}^N)$ .

Thanks to  $0 \leq \hat{u}_{n,L} \leq \hat{u}_n$ , then in (72) we pass to the limit as  $L \rightarrow +\infty$  and we have

$$|\hat{u}_n|_{\beta q_s^*}^{\beta q} \leq C \beta^q \int_{\mathbb{R}^N} \hat{u}_n^{q_s^*+q(\beta-1)} dx.$$

This means that

$$\left( \int_{\mathbb{R}^N} \hat{u}_n^{\beta q_s^*} dx \right)^{\frac{1}{q_s^*(\beta-1)}} \leq (C^{1/q} \beta)^{\frac{1}{\beta-1}} \left( \int_{\mathbb{R}^N} \hat{u}_n^{q_s^*+q(\beta-1)} dx \right)^{\frac{1}{q(\beta-1)}}.$$

For  $1 \leq m \in \mathbb{N}$ , let us define

$$q_s^* + q(\beta_{m+1} - 1) = \beta_m q_s^* \quad \text{and} \quad \beta_1 = \frac{q_s^*}{q}.$$

It follows that

$$\beta_{m+1} = \beta_1^m (\beta_1 - 1) + 1$$

and so

$$\lim_{m \rightarrow \infty} \beta_m = +\infty.$$

Let us define

$$T_m := \left( \int_{\mathbb{R}^N} \hat{u}_n^{\beta_m q_s^*} dx \right)^{\frac{1}{q_s^*(\beta_m-1)}}.$$

Then, we have

$$T_{m+1} \leq (C^{1/q} \beta_{m+1})^{\frac{1}{\beta_{m+1}-1}} T_m.$$

Consequently, using a standard iteration argument we have

$$T_{m+1} \leq \prod_{k=1}^m (C^{1/q} \beta_{k+1})^{\frac{1}{\beta_{k+1}-1}} T_1 \leq \bar{C} T_1, \quad \text{where } \bar{C} \text{ is independent of } m.$$

In the above inequality we pass to the limit as  $m \rightarrow \infty$  and then we infer that  $|\hat{u}_n|_{L^\infty(\mathbb{R}^N)} \leq C$  uniformly in  $n \in \mathbb{N}$ .

Next, let us define

$$\kappa_n := -V(\varepsilon_n x + y_n) (\hat{u}_n^{p-1} + \hat{u}_n^{q-1}) + \left( \frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, \hat{u}_n) \right) g(\varepsilon_n x + y_n, \hat{u}_n).$$

We point out that  $\hat{u}_n$  satisfies the following equation:

$$(-\Delta)_p^s \hat{u}_n + (-\Delta)_q^s \hat{u}_n = \kappa_n \quad \text{in } \mathbb{R}^N.$$

From the growth hypotheses on  $g$ , Corollary 2.1 in Ambrosio & Rădulescu [11],  $\hat{u}_n \rightarrow \hat{u}$  in  $X_{V_0}$  as  $n \rightarrow \infty$  and the uniform boundedness of the sequence  $\{\hat{u}_n\}_{n \in \mathbb{N}}$  in  $L^\infty(\mathbb{R}^N) \cap X_{V_0}$ , we can conclude that  $\hat{u}_n(x) \rightarrow 0$  in  $\mathbb{R}$  as  $|x| \rightarrow +\infty$  uniformly with respect to  $n \in \mathbb{N}$ . This ends the proof of the lemma.  $\square$

**Proof of Theorem 1 completed.** We first choose  $\delta > 0$  small enough such that  $M_\delta \subset \Lambda$ . Then, we claim that there exists  $\bar{\varepsilon}_\delta > 0$  such that for any  $\varepsilon \in (0, \bar{\varepsilon}_\delta)$  and any solution  $u_\varepsilon \in \hat{\mathcal{N}}_\varepsilon$  of problem (9), we have

$$(73) \quad |u_\varepsilon|_{L^\infty(\Lambda_\varepsilon^c)} < a.$$

Otherwise, we may assume that there exists a subsequence  $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$  such that  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $u_{\varepsilon_n} \in \hat{\mathcal{N}}_{\varepsilon_n}$  such that  $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$  and

$$(74) \quad |u_{\varepsilon_n}|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geq a.$$

But we see that  $J_{\varepsilon_n}(u_{\varepsilon_n}) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Thus, from Lemma 18 it follows that there exists a sequence  $\{\hat{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^N$  such that  $\hat{u}_n(\cdot) := u_{\varepsilon_n}(\cdot + \hat{y}_n) \rightarrow \hat{u}(\cdot)$  in  $X_{V_0}$  and  $\varepsilon_n \hat{y}_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ .

Next, we choose  $r > 0$  such that  $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ , and so  $B_{\frac{r}{\varepsilon_n}}\left(\frac{y_0}{\varepsilon_n}\right) \subset \Lambda_{\varepsilon_n}$ . Furthermore, for  $n$  large enough we can deduce that  $\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(\hat{y}_n)$ . In addition, from (67), we see that  $\hat{u}_n(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$  uniformly in  $n \in \mathbb{N}$ . Therefore, we can find  $R > 0$  such that  $\hat{u}_n(x) < a$  for any  $|x| \geq R$ ,  $n \in \mathbb{N}$ . Consequently,  $u_{\varepsilon_n}(x) < a$  for any  $x \in B_R^c(\hat{y}_n)$ ,  $n \in \mathbb{N}$ . Moreover, for  $n \in \mathbb{N}$  sufficiently large, we know that

$$\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(\hat{y}_n) \subset B_R^c(\hat{y}_n).$$

Thus, we infer that  $u_{\varepsilon_n}(x) < a$  for any  $x \in \Lambda_{\varepsilon_n}^c$  and for all  $n \in \mathbb{N}$  large enough, which contradicts relation (74).

Fix  $\varepsilon \in (0, \varepsilon_\delta)$ , where  $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$ . From Theorem 20, we can see that problem (9) has at least  $\text{cat}_{M_\delta}(M)$  nontrivial solutions. Now, we use  $u_\varepsilon$  to denote one of these solutions, and so  $u_\varepsilon \in \hat{\mathcal{N}}_\varepsilon$ . Then, using (73) and recalling that the definitions of  $g$  and  $G$ , we can also infer that  $u_\varepsilon$  is a solution of problem (3). So, problem (3) possesses at least  $\text{cat}_{M_\delta}(M)$  nontrivial solutions.

Finally, we establish the behavior of the maximum points of solutions to problem (3). Let us choose  $\varepsilon_n \rightarrow 0$  and consider a sequence  $\{u_n\}_{n \in \mathbb{N}} \subset X_{\varepsilon_n}$  of solutions for problem (3) as before. From  $(g_1)$  it follows that there exists a positive constant  $\iota < a$  such that

$$(75) \quad g(\varepsilon x, t) \leq \frac{V_0}{K} (t^p + t^q) \quad \text{for any } x \in \mathbb{R}^N, t \in [0, \iota].$$

Since  $u_n \in \mathcal{N}_{\varepsilon_n}$  and  $J_{\varepsilon_n}(u_n) \rightarrow c_{V_0}$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . Then, for  $n \in \mathbb{N}$  sufficiently large, we have that  $u_n \in \mathcal{B}$ . Consequently, from Lemma 6 it follows that

$$(76) \quad \left| \frac{1}{|x|^\mu} * G(\varepsilon_n x, u_n) \right|_{L^\infty(\mathbb{R}^N)} < \frac{K}{2} \quad \text{for } n \in \mathbb{N} \text{ large enough.}$$

Arguing as before, there exists  $R > 0$  such that

$$(77) \quad |u_n|_{L^\infty(B_R^c(\hat{y}_n))} < \iota.$$

Furthermore, up to a subsequence, we may assume that

$$(78) \quad |u_n|_{L^\infty(B_R(\hat{y}_n))} \geq \iota.$$

Otherwise, if relation (78) does not hold, we deduce from (77) that  $|u_n|_{L^\infty(\mathbb{R}^N)} < \iota$ . Using  $u_n \in \mathcal{N}_{\varepsilon_n}$  again and (75)–(76), we have

$$\begin{aligned} \|u_n\|_{V_{\varepsilon_n,p}}^p + \|u_n\|_{V_{\varepsilon_n,q}}^q &\leq \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\mu} * G(\varepsilon_n x, u_n) \right) g(\varepsilon_n x, u_n) u_n dx, \\ &\leq \frac{V_0}{2} \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q) dx \quad \text{for all } n \in \mathbb{N} \text{ large enough.} \end{aligned}$$

This implies that  $\|u_n\|_{X_{\varepsilon_n}} = 0$  for all  $n \in \mathbb{N}$  sufficiently large, which is a contradiction. Consequently, relation (78) holds true. On account of (77) and (78), we can infer that if  $p_n$  is a global maximum point of  $u_n$  and  $p_n = \hat{y}_n + q_n$  for some  $q_n \in B_R$ . So,  $\varepsilon_n p_n \rightarrow y_0 \in M$  as  $n \rightarrow \infty$ , then using the continuity of the potential  $V$  we see that  $V(\varepsilon_n p_n) \rightarrow V(y_0) = V_0$  in  $\mathbb{R}$  as  $n \rightarrow \infty$ . This proof is now complete.  $\square$

#### DECLARATIONS

**Conflict of interest.** The authors declare that they have no competing interests.

**Data availability.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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(Y.P. Zhang) DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF DEFENSE TECHNOLOGY, CHANGSHA, HUNAN 410073, PR CHINA

*Email address:* zhangypzn@163.com

(V.D. Rădulescu) FACULTY OF APPLIED MATHEMATICS, AGH UNIVERSITY OF KRAKÓW, AL. MICKIEWICZA 30, 30-059 KRAKÓW, POLAND & BRNO UNIVERSITY OF TECHNOLOGY, FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION, TECHNICKÁ 3058/10, BRNO 61600, CZECH REPUBLIC & DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRAIOVA, 200585 CRAIOVA, ROMANIA

*Email address:* radulescu@inf.ucv.ro (Corresponding author)

(J. Chen) SCHOOL OF MATHEMATICS AND COMPUTING SCIENCES, HUNAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, XIANGTAN, HUNAN 411201, PR CHINA

*Email address:* cjhust@hnust.edu.cn

(D.D. Qin) SCHOOL OF MATHEMATICS AND STATISTICS, HNP-LAMA, CENTRAL SOUTH UNIVERSITY, CHANGSHA, HUNAN 410083, PR CHINA

*Email address:* qindd132@163.com