Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–17, *2025*
DOI:10.1017/prm.2025.10071

Blow-up of cylindrically symmetric solutions for fractional NLS

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(Received 11 October 2024; revised 24 June 2025; accepted 24 June 2025)

In this paper, we consider blow-up of solutions to the Cauchy problem for the following fractional Nonlinear Schrödinger Equation (NLS),

$$\mathrm{i}\,\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u \quad \text{in } \mathbb{R} \times \mathbb{R}^N,$$

where $N \geq 2$, $1/2 < s < 1$, and $0 < \sigma < 2s/(N-2s)$. In the mass critical and supercritical cases, we establish a criterion for blow-up of solutions to the problem for cylindrically symmetric data. The results extend the known ones with respect to blow-up of solutions to the problem for radially symmetric data.

Keywords: blow-up; cylindrical symmetry solutions; fractional NLS; mass critical case; mass supercritical case

2020 *Mathematics Subject Classification:* Primary: 35R11; Secondary: 35B44

1. Introduction

In this paper, we are concerned with blow-up of cylindrically symmetric solutions to the following fractional NLS,

$$\begin{cases} \mathrm{i}\,\partial_t u = (-\Delta)^s u - |u|^{2\sigma} u, \\ u(0) = u_0 \in H^s(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$, $1/2 < s < 1$, and $0 < \sigma < 2s/(N-2s)$. The fractional Laplacian $(-\Delta)^s$ is characterized as $\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi)$ for $\xi \in \mathbb{R}^N$, where $\mathcal{F}$ denotes the Fourier transform. Eq. (1.1) was introduced by Laskin in [18, 19] and can be seen as a canonical model for a nonlocal dispersive Partial Differential Equation (PDE) with focusing nonlinearity. Evolution equations with nonlocal dispersion as (1.1) naturally arise in various physical settings, such as in the continuum limit of discrete models with long-range interactions [17] and in the description of Boson stars, as well as in water wave dynamics [4, 9, 20].

When $s=1$, the early study of the existence of finite time blow-up solutions to (1.1) for initial data with finite variance is due to Glassey in [13]. The result was later extended by Ogawa and Tsutsumi in [22] and Holmer and Roudenko in [15] to radially symmetric initial data with infinite variance. While $0 < s < 1$, despite that (1.1) bears a strong resemblance to the classical NLS, the existence of blow-up solutions to (1.1) was open for a long time until the work of Boulenger et al. [3]. In [3], they proved a general criterion for blow-up of solutions to (1.1) with radially symmetric data. Nevertheless, the consideration of blow-up solutions to (1.1) with non-radially symmetric data left open so far. Inspired by the aforementioned works, the first aim of the present paper is to investigate blow-up of solutions to (1.1) with initial data belonging to Σ_N defined by

$$\Sigma_N := \left\{ u \in H^s(\mathbb{R}^N) : u(y, x_N) = u(|y|, x_N), x_N u \in L^2(\mathbb{R}^N) \right\},$$

where $x = (y, x_N) \in \mathbb{R}^N$ and $y = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. We derive the existence of finite time blowing-up of solutions to (1.1) with initial data belonging to Σ_N in the mass supercritical case $\sigma > 2s/N$, see theorem 1.1. The second aim of the paper is to establish blow-up of solutions to (1.1) with initial data belonging to Σ defined by

$$\Sigma := \left\{ u \in H^s(\mathbb{R}^N) : u(y, x_N) = u(|y|, x_N) \right\}.$$

We obtain the existence of finite time blow-up of solutions to (1.1) with initial data belonging to Σ for $\sigma > 2s/(N-1)$, see theorem 1.3. It is worth quoting [1, 2, 7, 8, 10, 14], where blow-up of cylindrically symmetric solutions to the local NLS with initial data in Σ_N was considered. The latter papers can be regarded as extensions of the ones in the seminal work due to Martel [21]. Since the problem under our consideration is nonlocal, the essential arguments we adapt here are greatly different from the ones used to deal with the local NLS.

Let us now mention the work of Hong and Sire [16], where the local well-posedness of solutions to (1.1) in $H^s(\mathbb{R}^N)$ was investigated. Problem (1.1) satisfies the conservation of the mass and the energy given respectively by

$$M[u] := \int_{\mathbb{R}^N} |u|^2 \, dx,$$

$$E[u] := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx.$$

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For further clarifications, we shall fix some notations. Let us define

$$s_c := \frac{N}{2} - \frac{s}{\sigma}.$$

We refer to the cases $s_c < 0$, $s_c = 0$, and $s_c > 0$ as mass subcritical, critical, and supercritical, respectively. The end case $s_c = s$ is energy critical. Note that the cases $s_c = 0$ and $s_c = s$ correspond to the exponents $\sigma = 2s/N$ and $\sigma = 2s/(N-2s)$, respectively. For $1 \leq p < \infty$ and $N \geq 1$, we denote by $L^q(\mathbb{R}^N)$ the usual Lebesgue space with the norm

$$\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|u\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} |u|.$$

The Sobolev space $H^s(\mathbb{R}^N)$ is equipped with the standard norm

$$\|u\| := \|(-\Delta)^{s/2} u\|_2 + \|u\|_2.$$

In addition, we denote by $Q \in H^s(\mathbb{R}^N)$ the ground state to the following fractional nonlinear elliptic equation,

$$(-\Delta)^s Q + Q - |Q|^{2\sigma} Q = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The uniqueness of ground states was recently revealed in [11, 12]. Throughout the paper, we shall write $X \lesssim Y$ to denote that $X \leq CY$ for some irrelevant constant $C > 0$.

The main results of the present paper read as follows and they provide blow-up criteria for solutions of problem (1.1) with cylindrically symmetric data.

THEOREM 1.1. *(Blow-up for Mass-Supercritical Case) Let $N \geq 3$, $1/2 < s < 1$, and $0 < s_c \leq s$ with $0 < \sigma \leq s$. Suppose that $u_0 \in \Sigma_N$ satisfies that either $E[u_0] < 0$ or $E[u_0] \geq 0$ and*

$$E[u_0]^{s_c} M[u_0]^{s-s_c} < E[Q]^{s_c} M[Q]^{s-s_c}, \tag{1.3}$$

$$\|(-\Delta)^{s/2} u_0\|_2^{s_c} \|u_0\|_2^{s-s_c} > \|(-\Delta)^{s/2} Q\|_2^{s_c} \|Q\|_2^{s-s_c}. \tag{1.4}$$

Then the solution $u \in C([0, T_{max}), H^s(\mathbb{R}^N))$ to (1.1) with initial datum u_0 blows up in finite time, i.e. $0 < T_{max} < +\infty$.

REMARK 1.1. When $s_c = s$, by [3, Appendix B], then $Q \notin L^2(\mathbb{R}^N)$. In this situation, the conditions (1.3) and (1.4) become the following ones,

$$E[u_0] < E[Q], \quad \|(-\Delta)^{s/2} u_0\|_2 > \|(-\Delta)^{s/2} Q\|_2.$$

THEOREM 1.2. *(Blow-up for Mass-Critical Case) Let $N \geq 3$, $1/2 < s < 1$, and $s_c = 0$. Suppose that $u_0 \in \Sigma_N$ satisfies that $E[u_0] < 0$. Then the solution $u \in$*

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$C([0,T_{max}),H^s(\mathbb{R}^N))$ to (1.1) either blows up in finite time, i.e.$T_{max}<+\infty$ or blows up in infinite time, i.e.$T_{max}=+\infty$ and

$$\|(-\Delta)^{s/2}u(t)\|_2 \geq Ct^s \quad \textit{for any } t \geq t_0,$$

where $\mathrm{C}>0$ *and* $t_0=t_0>0$ *are constants depending only on* u_0, s *and* N.

REMARK 1.2. The assumption that $0<\sigma\leq s$ is technical. It is unknown whether theorems 1.1 and 1.2 remain hold for $\sigma>s$.

To prove theorems 1.1 and 1.2, the crucial arguments lie in establishing localized virial estimates (2.8) and (2.36) for cylindrically symmetric solutions to (1.1), where (2.36) is a refined version of (2.8) used to discuss blow-up of the solutions to (1.1) for $s_c=0$. First we need to introduce a localized virial quantity $\mathcal{M}_{\varphi_R}[u]$ defined by (2.4), where φ_R defined by (2.3) is a cylindrically symmetric function. Then, adapting [3, Lemma 2.1], we can derive the virial identity (2.9). At this stage, to get the desired conclusions, we need to properly estimate each term in (2.9). One of the key arguments is actually to estimate the following term,

$$\int_{\mathbb{R}}\int_{|y|\geq R}|u|^{2\sigma+2}\,dydx_N. \tag{1.5}$$

To do this, when $s=1$, one can make use of the following two crucial ingredients jointly with the classical Gagliardo–Nirenberg's inequality in $H^1(\mathbb{R})$ and the radial Sobolev's inequality in $H^1(\mathbb{R}^{N-1})$ to get the desired conclusion,

$$\sup_{x_N\in\mathbb{R}}\int_{\mathbb{R}^{N-1}}|u|^2\,dy \lesssim \left(\int_{\mathbb{R}^{N-1}}|\partial_{x_N}u|^2\,dy\right)^{\frac{1}{2}}, \tag{1.6}$$

$$\partial_{x_N}\left(\int_{\mathbb{R}^{N-1}}|u|^2\,dy\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^{N-1}}|\partial_{x_N}u|^2\,dy\right)^{\frac{1}{2}}, \tag{1.7}$$

see for example [1, 7, 8, 21]. However, to our knowledge, it seems rather difficult to generalize the estimates (1.6) and (1.7) to the nonlocal cases. Therefore, we cannot follow the strategies in [1, 8, 21] to handle the term (1.5). In fact, when $0<s<1$, by tactfully employing Sobolev's inequality in $W^{s,1}(\mathbb{R})$ and certain fractional chain rule, we then have that

$$\int_{\mathbb{R}}\|u\|_{L^2_y}^{\frac{2}{1-\sigma}}\,dx_N \lesssim \left(\int_{\mathbb{R}^{N-1}}\left\|(-\partial_{x_Nx_N})^{s/2}u\right\|_{L^2_{x_N}}^2\,dy\right)^{\frac{\sigma}{2s(1-\sigma)}}, \quad 0<\sigma\leq s.$$

With this at hand, we are now able to estimate the term (1.5). This then completes the proofs.

It would also be interesting to investigate blow-up of solutions to (1.1) for cylindrically symmetric initial data belonging to Σ without the restriction that $x_Nu_0\in L^2(\mathbb{R}^N)$. In this respect, we have the following result.

THEOREM 1.3. *(Blowup for Mass-Supercritical Case Revisited) Let $N\geq 4$, $1/2<s<1$, and $0<s_c\leq s$ with $2s/(N-1)<\sigma\leq s$. Suppose that $u_0\in\Sigma$ satisfies*

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$E[u_0] < 0$. *Then the solution* $u \in C([0, T_{max}), H^s(\mathbb{R}^N))$ *to (1.1) with initial datum* u_0 *blows up in finite time, i.e.* $0 < T_{max} < +\infty$.

To establish theorem 1.3, we need to introduce a new localized virial quantity $\mathcal{M}_{\psi_R}[u]$ defined by (2.39). Following closely the proof of lemma 2.1, one can get localized virial estimate (2.40) for cylindrically symmetric solutions to (1.1). This then implies the desired conclusion.

2. Proofs of main results

In this section, we are going to prove theorems 1.1 and 1.2. Let us first introduce a localized virial quantity. Let $\psi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a radially symmetric and smooth function such that

$$\psi(r) := \begin{cases} \dfrac{r^2}{2} & r \le 1, \\ const. & r \ge 10, \end{cases} \qquad \psi''(r) \le 1, \quad r = |y|.$$

Let $\psi_R : \mathbb{R}^{N-1} \to \mathbb{R}$ be the radially symmetric function defined by the scaling

$$\psi_R(r) := R^2 \psi\left(\frac{r}{R}\right), \quad R > 0. \tag{2.1}$$

It is straightforward to verify that

$$1 - \psi_R''(r) \ge 0, \quad 1 - \frac{\psi_R'(r)}{r} \ge 0, \quad N - 1 - \Delta\psi_R(r) \ge 0, \quad r \ge 0. \tag{2.2}$$

Let $\varphi : \mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ be a smooth function defined by

$$\varphi_R(x) := \psi_R(r) + \frac{x_N^2}{2}, \quad x = (y, x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}. \tag{2.3}$$

Now we introduce the localized virial quantity as

$$\mathcal{M}_{\varphi_R}[u] := 2\mathrm{Im} \int_{\mathbb{R}^N} \overline{u} \left(\nabla\varphi_R \cdot \nabla u\right) dx. \tag{2.4}$$

For convenience, we shall give the well-known fractional radial Sobolev's inequality in [5]. For every radial function $f \in H^s(\mathbb{R}^{N-1})$ with $N \ge 3$, then

$$|y|^{\frac{N-2}{2}} |f(y)| \le C(N, s) \|(-\Delta)^{s/2} f\|_2^{\frac{1}{2s}} \|f\|_2^{1-\frac{1}{2s}}, \quad y \ne 0. \tag{2.5}$$

Also we shall present the well-known Gagliardo–Nirenberg's inequality in [23]. For any $f \in H^1(\mathbb{R})$ and $p > 2$, then

$$\|f\|_p \le C(s, p) \|(-\Delta)^{s/2} f\|_2^{\alpha} \|f\|_2^{1-\alpha}, \quad \alpha = \frac{p-2}{2ps}. \tag{2.6}$$

Let $f : \mathbb{R}^{N-1} \to \mathbb{C}$ be a radially symmetric and smooth function, then

$$\partial_{kl}^2 f = \left(\delta_{kl} - \frac{x_k x_l}{r^2}\right) \frac{\partial_r f}{r} + \frac{x_k x_l}{r^2} \partial_{rr}^2 f, \tag{2.7}$$

where δ_{kl} is the Kronecker symbol and $1 \leq k, l \leq N-1$.

In the following, we are going to estimate the evolution of $\mathcal{M}_{\varphi_R}[u(t)]$ along time, which is the key to establish theorems 1.1 and 1.2.

Lemma 2.1. *Let $N \geq 3$, $1/2 < s < 1$, and $0 < \sigma \leq s$. Suppose that $u \in C([0, T_{max}); H^s(\mathbb{R}^N))$ is the solution to (1.1) with initial datum $u_0 \in \Sigma_N$. Then, for any $t \in [0, T_{max})$, there holds that*

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] &\leq 4s \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(t)|^2 \, dx - \frac{2\sigma N}{\sigma+1} \int_{\mathbb{R}^N} |u(t)|^{2\sigma+2} \, dx \\ &\quad + C\left(R^{-2s} + R^{-\sigma(N-2)}\left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(t)|^2 \, dx\right)\right) \\ &= 4\sigma N E[u_0] - 2(\sigma N - 2s) \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(t)|^2 \, dx \\ &\quad + C\left(R^{-2s} + R^{-\sigma(N-2)}\left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(t)|^2 \, dx\right)\right). \end{aligned} \tag{2.8}$$

Proof. Define

$$u_m(t,x) := c_s \frac{1}{-\Delta + m} u(t,x) = c_s \mathcal{F}^{-1}\left(\frac{\mathcal{F}(u)(t,\xi)}{|\xi|^2 + m}\right), \quad m > 0,$$

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

It follows from [3, Lemma 2.1] that

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] = \int_0^\infty \int_{\mathbb{R}^N} m^s \left(4 \sum_{k,l=1}^N \overline{\partial_k u_m} \left(\partial_{kl}^2 \varphi_R\right) \partial_l u_m - \left(\Delta^2 \varphi_R\right) |u_m|^2\right) dx dm \\ - \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^N} (\Delta \varphi_R) |u|^{2\sigma+2} \, dx. \end{aligned} \tag{2.9}$$

Let us start with treating the first term on the right-hand side of (2.9). Observe that

$$
\begin{aligned}
&4\sum_{k,l=1}^{N}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kl}\varphi_R\right)\partial_l u_m\,dxdm\\
&=4\sum_{k,l=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kl}\varphi_R\right)\partial_l u_m\,dxdm\\
&\quad+4\sum_{k=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kN}\varphi_R\right)\partial_N u_m\,dxdm\\
&\quad+4\sum_{l=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_N u_m}\left(\partial^2_{Nl}\varphi_R\right)\partial_l u_m\,dxdm\\
&\quad+4\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_N u_m}\left(\partial^2_{NN}\varphi_R\right)\partial_N u_m\,dxdm. \qquad (2.10)
\end{aligned}
$$

Using (2.3) and (2.7), we have that

$$
\begin{aligned}
&\sum_{k,l=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kl}\varphi_R\right)\partial_l u_m\,dxdm\\
&=\sum_{k,l=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kl}\psi_R\right)\partial_l u_m\,dxdm\\
&=\int_0^\infty\int_{\mathbb{R}^N} m^s\left(\psi_R\partial^2_{rr}\psi\right)|\nabla_y u_m|^2\,dxdm.
\end{aligned}
$$

It is clear to see from (2.3) that $\partial^2_{jN}\varphi_R = 0$ for $1 \le j \le N-1$. Therefore, there holds that

$$
\begin{aligned}
&\sum_{k=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_k u_m}\left(\partial^2_{kN}\varphi_R\right)\partial_N u_m\,dxdm\\
&=\sum_{l=1}^{N-1}\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_N u_m}\left(\partial^2_{Nl}\varphi_R\right)\partial_l u_m\,dxdm=0.
\end{aligned}
$$

In addition, since $\partial^2_{NN}\varphi_R = 1$, then

$$
\int_0^\infty\int_{\mathbb{R}^N} m^s\overline{\partial_N u_m}\left(\partial^2_{NN}\varphi_R\right)\partial_N u_m\,dxdm=\int_0^\infty\int_{\mathbb{R}^N} m^s|\partial_N u_m|^2\,dx.
$$

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As an application of Plancherel's identity and Fubini's theorem (see [3, (2.12)]), we know that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^N} m^s |\nabla u_m|^2 \,dxdm &= \frac{\sin \pi s}{\pi} \int_{\mathbb{R}^N} \int_0^\infty \frac{m^s}{(|\xi|^2+m)^2} |\xi|^2 |\mathcal{F}(u)|^2 \,dmd\xi \\ &= s \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}(u)|^2 \,d\xi = s \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \,dx.\end{aligned} \tag{2.11}$$

Consequently, going back to (2.10) and utilizing (2.2), we get that

$$\begin{aligned}&4 \sum_{k,l=1}^N \int_0^\infty \int_{\mathbb{R}^N} m^s \overline{\partial_k u_m} \left(\partial^2_{kl} \varphi_R\right) \partial_l u_m \,dxdm \\ &= 4 \int_0^\infty \int_{\mathbb{R}^N} m^s |\nabla u_m|^2 \,dxdm \\ &\quad - 4 \int_0^\infty \int_{\mathbb{R}^N} m^s \left(1 - \partial^2_{rr} \psi_R\right) |\nabla_y u_m|^2 \,dxdm \\ &\leq 4s \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \,dx.\end{aligned} \tag{2.12}$$

Furthermore, applying [3, Lemma A.2], we get that

$$\left| \int_0^\infty \int_{\mathbb{R}^N} m^s \left(\Delta^2 \varphi_R\right) |u_m|^2 \,dxdm \right| \lesssim \|\Delta^2 \varphi_R\|_\infty^s \|\Delta \varphi_R\|_\infty^{1-s} \|u\|_2^2 \lesssim R^{-2s}. \tag{2.13}$$

Now we are going to deal with the second term on the right-hand side of (2.9). Noting (2.3), one readily finds that

$$\begin{aligned}\int_{\mathbb{R}^N} (\Delta \varphi_R) |u|^{2\sigma+2} \,dx &= \int_{\mathbb{R}^N} (\Delta \psi_R) |u|^{2\sigma+2} \,dx + \int_{\mathbb{R}^N} |u|^{2\sigma+2} \,dx \\ &= N \int_{\mathbb{R}^N} |u|^{2\sigma+2} \,dx + \int_{\mathbb{R}^N} (\Delta \psi_R - N + 1) |u|^{2\sigma+2} \,dx.\end{aligned} \tag{2.14}$$

Obviously, from (2.1), there holds that $\Delta \psi_R(r) - N + 1 = 0$ for $0 \leq r \leq R$. Therefore, by (2.14), we conclude that

$$\int_{\mathbb{R}^N} (\Delta \varphi_R) |u|^{2\sigma+2} \,dx = N \int_{\mathbb{R}^N} |u|^{2\sigma+2} \,dx + \int_{\mathbb{R}} \int_{|y| \geq R} (\Delta \psi_R - N + 1) |u|^{2\sigma+2} \,dydx_N. \tag{2.15}$$

It is worth mentioning that $\Delta \psi_R - N + 1 \in L^\infty(\mathbb{R}^{N-1})$. In what follows, the aim is to estimate the second term on the right-hand side of (2.37). It is simple to notice that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^{2\sigma+2} \, dy dx_N \leq \int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^{2\sigma} \|u\|_{L^2_y}^2 \, dx_N. \tag{2.16}$$

To estimate the term on the right-hand side of (2.16), we first consider the case that $\sigma = s$. In this case, applying Hölder's inequality, we get that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^{2s+2} \, dx \leq \left(\int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^2 \, dx_N \right)^s \left(\int_{\mathbb{R}} \|u\|_{L^2_y}^{\frac{2}{1-s}} \, dx_N \right)^{1-s}. \tag{2.17}$$

In virtue of (2.5), Hölder's inequality and the conservation of mass, we have that

$$\begin{aligned} \int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^2 \, dx_N &\lesssim R^{-(N-2)} \int_{\mathbb{R}} \|(-\Delta_y)^{s/2} u\|_{L^2_y}^{\frac{1}{s}} \|u\|_{L^2_y}^{2-\frac{1}{s}} \, dx_N \\ &\leq R^{-(N-2)} \left(\int_{\mathbb{R}} \|(-\Delta_y)^{s/2} u\|_{L^2_y}^2 \, dx_N \right)^{\frac{1}{2s}} \left(\int_{\mathbb{R}} \|u\|_{L^2_y}^2 \, dx_N \right)^{\frac{2s-1}{2s}} \\ &\lesssim R^{-(N-2)} \left(\int_{\mathbb{R}} \|(-\Delta_y)^{s/2} u\|_{L^2_y}^2 \, dx_N \right)^{\frac{1}{2s}}. \end{aligned} \tag{2.18}$$

It follows from [6] that $(-\Delta)^{\frac{s}{2}}$ can be equivalently represented as

$$(-\Delta)^{\frac{s}{2}} u(x) := C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|x-z|^{N+s}} \, dz, \tag{2.19}$$

where $C_{N,s} \in \mathbb{R}$ is a constant given by

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \dots, \xi_N).$$

Furthermore, Gagliardo's semi-norm in $H^s(\mathbb{R}^N)$ is represented by

$$\left\| (-\Delta)^{s/2} u \right\|_2^2 = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx dy.$$

By (2.19), we are able to calculate that

$$(-\partial_{x_N x_N}^2)^{\frac{s}{2}} |u|^2 = 2|u| (-\partial_{x_N x_N}^2)^{\frac{s}{2}} |u| - I_s(|u|, |u|), \tag{2.20}$$

where

$$I_s(|u|, |u|) := C_{1,s} \int_{\mathbb{R}} \frac{\left(|u|(y, x_N) - |u|(y, x_N') \right)^2}{|x_N - x_N'|^{1+s}} \, dx_N'.$$

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From the definition of Gagliardo's semi-norm in $H^{\frac{s}{2}}(\mathbb{R})$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \left\|(-\partial_{x_N x_N})^{s/4}|u|\right\|^2_{L^2_{x_N}}\, dy \\
&= \frac{C_{1,s}}{2}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left(|u|(y,x_N)-|u|(y,x'_N)\right)^2}{|x_N-x'_N|^{1+s}}\, dx'_N dx_N dy \\
&= \frac{1}{2}\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} I_s(|u|,|u|)\, dx_N dy.
\end{aligned} \tag{2.21}$$

In addition, from interpolation inequality, we find that

$$\left\|(-\partial_{x_N x_N})^{s/4}|u|\right\|_{L^2_{x_N}} \lesssim \left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|_{L^2_{x_N}} \|u\|_{L^2_{x_N}}. \tag{2.22}$$

Taking into account (2.20), (2.21), and (2.22), Sobolev's inequality, Minkowski's inequality, Hölder's inequality, and the conservation of mass, we then obtain that

$$\begin{aligned}
\int_{\mathbb{R}} \|u\|^{\frac{2}{1-s}}_{L^2_y}\, dx_d &\lesssim \left(\int_{\mathbb{R}} \left|(-\partial_{x_N x_N})^{s/2}\left(\|u\|^2_{L^2_y}\right)\right|\, dx_N\right)^{\frac{1}{1-s}} \\
&= \left(\int_{\mathbb{R}} \left|\int_{\mathbb{R}^{N-1}} (-\partial_{x_N x_N})^{s/2}\left(|u|^2\right)\, dy\right|\, dx_N\right)^{\frac{1}{1-s}} \\
&\le \left(\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}} \left|(-\partial_{x_N x_N})^{s/2}\left(|u|^2\right)\right|\, dx_N dy\right)^{\frac{1}{1-s}} \\
&\lesssim \left(\int_{\mathbb{R}^{N-1}} \|u\|_{L^2_{x_N}} \left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|_{L^2_{x_N}}\, dy\right)^{\frac{1}{1-s}} \\
&\quad + \left(\int_{\mathbb{R}^{N-1}} \left\|(-\partial_{x_N x_N})^{s/4}|u|\right\|^2_{L^2_{x_N}}\, dy\right)^{\frac{1}{1-s}} \\
&\lesssim \left(\int_{\mathbb{R}^{N-1}} \left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|^2_{L^2_{x_N}}\, dy\right)^{\frac{1}{2(1-s)}} \\
&\quad + \left(\int_{\mathbb{R}^{N-1}} \left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|_{L^2_{x_N}} \|u\|_{L^2_{x_N}}\, dy\right)^{\frac{1}{1-s}} \\
&\lesssim \left(\int_{\mathbb{R}^{N-1}} \left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|^2_{L^2_{x_N}}\, dy\right)^{\frac{1}{2(1-s)}},
\end{aligned} \tag{2.23}$$

where Sobolev's inequality we used is from the fact that $L^{\frac{1}{1-s}}(\mathbb{R})$ is continuously embedded into $W^{s,1}(\mathbb{R})$. Moreover, by the definition of Gagliardo's semi-norm in $H^s(\mathbb{R})$, we know that

$$\left\|(-\partial_{x_N x_N})^{s/2}|u|\right\|^2_{L^2_{x_N}} \le \left\|(-\partial_{x_N x_N})^{s/2}u\right\|^2_{L^2_{x_N}}, \tag{2.24}$$

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because $||u|(y, x_N) - |u|(y, x'_N)| \le |u(y, x_N) - u(y, x'_N)|$ for any $x_N, x'_N \in \mathbb{R}$. Now going back to (2.17) and applying (2.18), (2.23), and (2.24), we then derive that

$$\int_{\mathbb{R}} \int_{|y| \ge R} |u|^{2s+2} \, dx \lesssim R^{-(N-2)s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx. \tag{2.25}$$

Next we consider the case that $0 < \sigma < s$. In this case, taking into account (2.16) and Hölder's inequality, we know that

$$\int_{\mathbb{R}} \int_{|y| \ge R} |u|^{2\sigma+2} \, dx \le \left(\int_{\mathbb{R}} \|u\|^2_{L^\infty(|y| \ge R)} \, dx_N \right)^{\sigma} \left(\int_{\mathbb{R}} \|u\|^{\frac{2}{1-\sigma}}_{L^2_y} \, dx_N \right)^{1-\sigma}. \tag{2.26}$$

In view of (2.20), Sobolev's inequality, Hölder's inequality, interpolation inequality, and the conservation of mass, we are able to similarly show that

$$\begin{aligned} \int_{\mathbb{R}} \|u\|^{\frac{2}{1-\sigma}}_{L^2_y} \, dx_N &\le \left(\int_{\mathbb{R}} \left| (-\partial_{x_N x_N})^{\sigma/2} \left(\|u\|^2_{L^2_y} \right) \right| \, dx_N \right)^{\frac{1}{1-\sigma}} \\ &\lesssim \left(\int_{\mathbb{R}^{N-1}} \left\| (-\partial_{x_N x_N})^{\sigma/2} |u| \right\|^2_{L^2_{x_N}} \, dy \right)^{\frac{1}{2(1-\sigma)}} \\ &\lesssim \left(\int_{\mathbb{R}^{N-1}} \left\| (-\partial_{x_N x_N})^{s/2} |u| \right\|^{\frac{2\sigma}{s}}_{L^2_{x_N}} \|u\|^{\frac{2(s-\sigma)}{s}}_{L^2_{x_N}} \, dy \right)^{\frac{1}{2(1-\sigma)}} \\ &\lesssim \left(\int_{\mathbb{R}^{N-1}} \left\| (-\partial_{x_N x_N})^{s/2} |u| \right\|^2_{L^2_{x_N}} \, dy \right)^{\frac{\sigma}{2s(1-\sigma)}}. \end{aligned} \tag{2.27}$$

Making use of (2.18) and (2.27), we then obtain from (2.26) that

$$\begin{aligned} \int_{\mathbb{R}} \int_{|y| \ge R} |u|^{2\sigma+2} \, dx &\lesssim R^{-\sigma(N-2)} \left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right)^{\frac{\sigma}{s}} \\ &\lesssim R^{-\sigma(N-2)} + R^{-\sigma(N-2)} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx. \end{aligned} \tag{2.28}$$

Combining (2.25) and (2.28), we then have that

$$\int_{\mathbb{R}} \int_{|y| \ge R} |u|^{2\sigma+2} \, dx \lesssim R^{-\sigma(N-2)} + R^{-\sigma(N-2)} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx, \quad 0 < \sigma \le s.$$

It then follows from (2.37) that

$$\int_{\mathbb{R}^N} (\Delta \varphi_R) \, |u|^{2\sigma+2} \, dx \le N \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx + C R^{-\sigma(N-2)} \left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right),$$

$0 < \sigma \le s$.

This together with (2.12) and (2.13) then clearly leads to the desired conclusion and the proof is completed. □

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Proof of theorem 1.1. Using lemma 2.1 and following the proof of [3, Theorem 1.1] for the case $\sigma > 2s/N$, we are able to conclude the proof. Let us now sketch the proof. We shall suppose by contradiction that $u(t)$ exists globally in time. First we consider the case that $E[u_0] < 0$. Since $\sigma > 2s/N$, from lemma 2.1, then

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] \leq 2\sigma N E[u_0] < 0, \quad R \gg 1.$$

Then we conclude that there exists $t_1 \gg 1$ such that $\mathcal{M}_{\varphi_R}[u(t)] < 0$ for any $t \geq t_1$. Arguing as the proof of [3, Theorem 1.1], we are now able to derive that

$$\mathcal{M}_{\varphi_R}[u(t)] \lesssim -\int_{t_1}^{t} |\mathcal{M}_{\varphi_R}[u(\tau)]|^{2s}\, d\tau.$$

Solving this inequality gives that there exists $t_* < +\infty$ such that

$$\mathcal{M}_{\varphi_R}[u(t)] \lesssim -|t - t_*|^{1-2s}.$$

Therefore, we know that $\mathcal{M}_{\varphi_R}[u(t)] \to -\infty$ as $t \to t_*$ due to $s > \frac{1}{2}$. This shows that $u(t)$ cannot exist globally in time, namely $u(t)$ blows up in finite time.

Next we are going to treat the case that $E[u_0] \geq 0$ and (1.3)–(1.4) hold. It follows from lemma 2.1 that

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] \lesssim 4P(u(t)) + C\left(R^{-2s} + R^{-\sigma(N-2)}\left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(t)|^2\, dx\right)\right), \tag{2.29}$$

where

$$P[u] := s\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\, dx - \frac{N\sigma}{2\sigma+2}\int_{\mathbb{R}^N} |u|^{2\sigma+2}\, dx.$$

In addition, one can observe that there exists $\epsilon > 0$ small enough that

$$4P[u(t)] + \epsilon\left\|(-\Delta)^{s/2}u(t)\right\|_2^2 \leq -\delta, \tag{2.30}$$

where $\delta > 0$ is a constant. Indeed, since (1.3) and (1.4) hold, then there exists $\eta > 0$ such that

$$E[u_0]M[u_0]^{\frac{s-s_c}{s_c}} \leq (1-\eta)E[Q]M[Q]^{\frac{s-s_c}{s_c}}. \tag{2.31}$$

Further, we are able to show that there exists $\eta' > 0$ depending on η such that

$$\left\|(-\Delta)^{s/2}u(t)\right\|_2 \|u(t)\|_2^{\frac{s-s_c}{s_c}} \geq (1+\eta')\left\|(-\Delta)^{s/2}Q\right\|_2 \|Q\|_2^{\frac{s-s_c}{s_c}}. \tag{2.32}$$

Observe that

$$E[Q]M[Q]^{\frac{s-s_c}{s_c}} = \frac{N\sigma - 2s}{2N\sigma}\left\|(-\Delta)^{s/2}Q\right\|_2^2 \|Q\|_2^{\frac{2(s-s_c)}{s_c}}, \tag{2.33}$$

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because $Q \in H^s(\mathbb{R}^N)$ is the ground state to (1.2). Combining (2.31)–(2.33) and the conservation laws, we then derive that

$$\begin{aligned}
&\left(4P[u(t)] + \epsilon \left\|(-\Delta)^{s/2}u(t)\right\|_2^2\right) M[u_0]^{\frac{s-s_c}{s_c}}\\
&= \left(4P[u(t)] + \epsilon \left\|(-\Delta)^{s/2}u(t)\right\|_2^2\right) M[u(t)]^{\frac{s-s_c}{s_c}}\\
&= \left(4N\sigma E[u(t)] + (4s+\epsilon-2N\sigma)\left\|(-\Delta)^{s/2}u(t)\right\|_2^2\right) M[u(t)]^{\frac{s-s_c}{s_c}}\\
&\le \gamma \left\|(-\Delta)^{s/2}Q\right\|_2^2 \|Q\|_2^{\frac{2(s-s_c)}{s_c}},
\end{aligned}$$

where

$$\gamma := 2(N\sigma - 2s)\left((1-\eta) - (1+\eta')^2\right) + \epsilon(1+\eta')^2.$$

Noting that $\gamma < 0$ by taking $\epsilon > 0$ small enough, we then get that (2.30) holds true. As an application of (2.29) and (2.30), we then have that

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] \lesssim -\frac{\delta}{2}, \quad R \gg 1.$$

Similarly, we have that $u(t)$ blows up in finite time. This completes the proof. □

Proof of theorem 1.2. To prove theorem 1.2, we need a refined version of lemma 2.1. Define

$$\widetilde{\psi}_{1,R}(r) := 1 - \partial_{rr}^2\psi_R, \quad \widetilde{\psi}_{2,R}(r) := N - 1 - \Delta\psi_R \ge 0.$$

Taking advantage of (2.11), (2.12), (2.13), and (2.37), we know from (2.9) that

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] =& 8sE[u_0] - 4\int_0^\infty \int_{\mathbb{R}^N} m^s \widetilde{\psi}_{1,R}|\nabla_y u_m|^2\, dxdm\\
&+ \frac{4s}{N+2s}\int_{\mathbb{R}}\int_{|y|\ge R} \widetilde{\psi}_{2,R}|u|^{\frac{4s}{N}+2}\, dydx_N + \mathcal{O}(R^{-2s}).
\end{aligned} \tag{2.34}$$

According to Hölder's inequality, we see that

$$\begin{aligned}
\int_{\mathbb{R}}\int_{|y|\ge R} \widetilde{\psi}_{2,R}|u|^{\frac{4s}{N}+2}\, dydx_N &\le \int_{\mathbb{R}} \left\|\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}}|u|\right\|_{L^\infty(|y|\ge R)}^{\frac{4s}{N}} \left\|\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}}|u|\right\|_{L_y^2}^2\, dx_N\\
&\le \left(\int_{\mathbb{R}} \left\|\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}}|u|\right\|_{L^\infty(|y|\ge R)}^2 dx_N\right)^{\frac{2s}{N}}\\
&\quad\times \left(\int_{\mathbb{R}} \left\|\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}}|u|\right\|_{L_y^2}^{\frac{2N}{N-2s}} dx_N\right)^{\frac{N-2s}{N}}.
\end{aligned}$$

It follows from (2.18) and (2.27) with $\sigma = 2s/N$ that

$$\int_{\mathbb{R}} \left\| \widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} |u| \right\|_{L^\infty(|y|\geq R)}^2 dx_N \lesssim R^{-(N-2)} \left(\int_{\mathbb{R}} \left\| (-\Delta_y)^{s/2} \left(\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} |u| \right) \right\|_{L^2_y}^2 dx_N \right)^{\frac{1}{2s}},$$

$$\int_{\mathbb{R}} \left\| \widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} |u| \right\|_{L^2_y}^{\frac{2N}{N-2s}} dx_N \lesssim \left(\int_{\mathbb{R}^{N-1}} \left\| (-\partial_{x_N x_N})^{s/2} \left(\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} |u| \right) \right\|_{L^2_{x_N}}^2 dy \right)^{\frac{1}{N-2s}}.$$

Consequently, we have that

$$\begin{aligned} \int_{\mathbb{R}} \int_{|y|\geq R} \widetilde{\psi}_{2,R} |u|^{\frac{4s}{N}+2} \, dy dx_N &\lesssim R^{-\frac{2s(N-2)}{N}} \left(\int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \left(\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} u \right) \right|^2 dx \right)^{\frac{1}{N}} \\ &\leq \eta \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \left(\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} u \right) \right|^2 dx + \mathcal{O}\left(\eta^{-\frac{1}{N-1}} R^{-\frac{2s(N-2)}{N-1}} \right), \end{aligned} \tag{2.35}$$

where we also used Young's inequality with $\eta > 0$. Moreover, adapting the elements presented in the proof of [3, Lemma 2.3], we are able to derive that

$$s \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \left(\widetilde{\psi}_{2,R}^{\frac{N}{2N+4s}} u \right) \right|^2 dx = \int_0^\infty \int_{\mathbb{R}^N} m^s \widetilde{\psi}_{2,R}^{\frac{N}{N+2s}} |\nabla u_m|^2 \, dx dm + \mathcal{O}\left(1 + R^{-2} + R^{-4}\right).$$

It then yields from (2.35) that

$$\begin{aligned} \int_{\mathbb{R}} \int_{|y|\geq R} \widetilde{\psi}_{2,R} |u|^{\frac{4s}{N}+2} \, dy dx_N &\leq \frac{\eta}{s} \int_0^\infty \int_{\mathbb{R}^N} m^s \widetilde{\psi}_{2,R}^{\frac{N}{N+2s}} |\nabla u_m|^2 \, dx dm \\ &\quad + \mathcal{O}\left(\eta^{-\frac{1}{N-1}} R^{-\frac{2s(N-2)}{N-1}} + \eta \left(1 + R^{-2} + R^{-4}\right) \right). \end{aligned}$$

Inserting into (2.34), we then conclude that

$$\begin{aligned} \frac{d}{dt} \mathcal{M}_{\varphi_R}[u(t)] &= 8sE[u_0] - 4 \int_0^\infty \int_{\mathbb{R}^N} m^s \left(\widetilde{\psi}_{1,R} - \frac{\eta}{N+2s} \widetilde{\psi}_{2,R}^{\frac{N}{N+2s}} \right) \widetilde{\varphi}_R |\nabla_y u_m|^2 \, dx dm \\ &\quad + \mathcal{O}\left(\eta^{-\frac{1}{N-1}} R^{-\frac{2s(N-2)}{N-1}} + R^{-2s} + \eta \left(1 + R^{-2} + R^{-4}\right) \right). \end{aligned} \tag{2.36}$$

At this point, using the refined version of lemma 2.1 given by (2.36) and following the proof of [3, Theorem 1.1] for the case $\sigma = 2s/N$, we are able to conclude the proof. Let us now sketch the proof. We shall assume that $u(t)$ exists globally in time. Utilizing (2.36) and arguing as the proof of [3, Theorem 1.1], we have that

$$\frac{d}{dt} \mathcal{M}_{\varphi_R}[u(t)] \leq 4sE[u_0] < 0, \quad R \gg 1.$$

It follows that there exist $t_0 \gg 1$ and $c > 0$ depending on s and $E[u_0]$ such that

$$\mathcal{M}_{\varphi_R}[u(t)] \leq -ct, \quad \forall\, t \geq t_0. \tag{2.37}$$

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On the other hand, reasoning as the proof of [3, Theorem 1.1], we also know that

$$|\mathcal{M}_{\varphi_R}[u(t)]| \lesssim \left\|(-\Delta)^{s/2}u\right\|_2^{1/s} + 1. \tag{2.38}$$

Combining (2.37) and (2.38), we then get the desired conclusion. This completes the proof. □

To discuss blow-up of solutions to (1.1) with initial data belonging to Σ, we shall introduce a new localized virial quantity. Let ψ_R be defined by (2.1). The localized virial quantity is indeed defined by

$$\mathcal{M}_{\psi_R}[u] := 2\mathrm{Im}\int_{\mathbb{R}^N} \overline{u}\,(\nabla\psi_R \cdot \nabla_y u)\, dx. \tag{2.39}$$

LEMMA 2.2. *Let $N \geq 3$, $1/2 < s < 1$, and $0 < \sigma \leq s$. Suppose that $u \in C([0, T_{max}); H^s(\mathbb{R}^N))$ is the solution to (1.1) with initial datum $u_0 \in \Sigma$. Then, for any $t \in [0, T_{max})$, there holds that*

$$\begin{aligned} \frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] &\leq 4s\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(t)|^2\, dx - \frac{2\sigma(N-1)}{\sigma+1}\int_{\mathbb{R}^N} |u(t)|^{2\sigma+2}\, dx \\ &\quad + C\left(R^{-2s} + R^{-\sigma(N-2)}\left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(t)|^2\, dx\right)\right) \\ &= 4\sigma(N-1)E[u_0] - 2(\sigma(N-1) - 2s)\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(t)|^2\, dx \\ &\quad + C\left(R^{-2s} + R^{-\sigma(N-2)}\left(1 + \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(t)|^2\, dx\right)\right). \end{aligned} \tag{2.40}$$

Proof. Replacing the roles of φ_R in the proof of lemma 2.1 by ψ_R and repeating the proof of lemma 2.1, we then obtain the desirable conclusion. This completes the proof. □

Proof of theorem 1.3. Since $E[u_0] < 0$ and $\sigma(N-1) > 2s$, by applying lemma 2.2, then we are able to get that

$$\frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] \leq 2\sigma(N-1)E[u_0] < 0, \quad R \gg 1.$$

This then immediately implies the desired conclusion by following the proof of [3, Theorem 1.1]. Hence the proof is completed. □

Acknowledgements
T. Gou warmly thanks Dr. Luigi Forcella for fruitful discussions on the proof of theorem 1.3.

Conflict of interests
The authors declare that there are no conflict of interests.

Funding

T. Gou was supported by the National Natural Science Foundation of China (No. 12101483) and the Postdoctoral Science Foundation of China. The research of V.D. Rădulescu was supported by the grant 'Nonlinear Differential Systems in Applied Sciences' of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8/22. He also acknowledges the support of the AGH University of Kraków under grant no. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education. Z. Zhang was supported by the National Key R & D Program of China (2022YFA1005601) and the National Natural Science Foundation of China (No. 12031015).

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