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Existence results for Kirchhoff–type superlinear problems involving the fractional Laplacian

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In this paper, we study the existence and multiplicity of solutions for Kirchhoff-type superlinear problems involving non-local integro-differential operators. As a particular case, we consider the following Kirchhoff-type fractional Laplace equation:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \dfrac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplace operator, $s \in (0,1)$, $N > 2s$, Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function satisfying certain assumptions, and $f(x,u)$ is superlinear at infinity. By computing the critical groups at zero and at infinity, we obtain the existence of non-trivial solutions for the above problem via Morse theory. To the best of our knowledge, our results are new in the study of Kirchhoff–type Laplacian problems.

Keywords: Fractional Laplacian; Kirchhoff–type problem; critical groups; Morse theory

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1. Introduction and main results

In this paper, we consider the following Kirchhoff-type problems with homogeneous Dirichlet boundary conditions involving the non-local integro-differential operator

$$\begin{cases} -M\left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right)\mathcal{L}_K u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N > 2s$, $s \in (0,1)$. We assume that $\mathcal{L}_K$ is the integro-differential operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \left(u(x+y)+u(x-y)-2u(x)\right) K(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^N, \tag{1.2}$$

with the kernel $K : \mathbb{R}^N\backslash\{0\} \to (0,+\infty)$ such that

(K_1) $mK \in L^1(\mathbb{R}^N)$, *where* $m(x) = \min\{|x|^2, 1\}$;

(K_2) *there exists* $\Lambda > 0$ *such that* $K(x) \geqslant \Lambda|x|^{-(N+2s)}$ *for any* $x \in \mathbb{R}^N\backslash\{0\}$.

A typical model for K is given by the singular kernel $K(x) = |x|^{-(N+2s)}$ that gives rise to the fractional Laplace operator $-(-\Delta)^s$, which, up to normalization factors, may be defined as

$$-(-\Delta)^s\varphi(x) := \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y, \quad x \in \mathbb{R}^N \tag{1.3}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Moreover, we suppose that the Kirchhoff function $M : [0,\infty) \to (0,\infty)$ is a continuous function satisfying the following conditions:

(M_1) *There exists* $m_0 > 0$ *such that* $M(t) \geqslant m_0$ *for all* $t \in [0,\infty)$;

(M_2) *There exists* $\theta \in [1, 2_s^*/2)$ *such that* $\theta\mathscr{M}(t) \geqslant M(t)t$ *for all* $t \in [0,\infty)$, *where* $\mathscr{M}(t) = \int_0^t M(\tau)\mathrm{d}\tau$.

(M_3) $\mathcal{M}(\omega t) \leqslant \mathcal{M}(t)$ *for all* $\omega \in [0,1]$, *where* $\mathcal{M}(t) = \theta\mathscr{M}(t) - M(t)t$ *for any* $t \in [0,\infty)$.

A typical example for M is given by $M(t) = a + b\theta t^{\theta-1}$ with $a > 0$, $b \geqslant 0$ for all $t \in [0,\infty)$. In fact, we can deal with a class of Kirchhoff functions satisfying (M_1)–(M_3). An elementary calculation shows that $M(t) = 1 + (1+t)^{-1}$ is exactly such a simple example. It is worth pointing out that if $M(0) = 0$, then Kirchhoff type problem is degenerate; if $M(0) > 0$, then Kirchhoff type problem is non-degenerate. So our problem (1.1) studied in this paper is non–degenerate.

In recent years, the problem (1.1) has received a great attention since it is involved in the study of fractional and non-local operators, which arise in both the pure mathematical research and concrete real-world applications. For example, these kinds of operators appear in many fields such as, among the others, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves, see [**11**] and the references therein. This is one of the reasons why, recently, non-local fractional problems are widely studied in the literature. For instance, see [**6**, **7**, **11**, **26**, **32**–**34**, **39**] and the references therein for non-local fractional Laplacian equations.

It is worth mentioning that *Fiscella* and *Valdinoci* in [**15**] proposed such a Kirchhoff–type variational equation (1.1) which models the *nonlocal* aspect of the tension arising from the fractional length of the string. In particular, we point out that problem (1.1) could be viewed as a fractional version of the well–known Kirchhoff equation. More precisely, *Kirchhoff* established a model given by

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.4}$$

where ρ is the mass density, p_0 is the initial tension, h is the area of cross-section, E is the Young modulus of the material, L is the length of the string, which extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. In particular, equation (1.4) received much attention only after *Lions* in [**19**] proposed an abstract framework to the problem. For some motivation in the physical background for the fractional kirchhoff problem, we refer to [**15**, Appendix A]. Recently, some interesting results in the degenerate and non–degenerate cases can be found, for instance, in [**2**, **5**, **14**, **24**, **25**, **28**–**30**, **36**–**38**, **40**].

In this paper, we will show the existence of nontrivial weak solutions for problem (1.1) using Morse theory. By a weak solution of problem (1.1), we mean a solution to the following problem

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}}|u(x)-u(y)|^2K(x-y)\mathrm{d}x\mathrm{d}y\right)\\ \quad\iint_{\mathbb{R}^{2N}}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\mathrm{d}x\mathrm{d}y\\ =\int_\Omega f(x,u(x))\varphi(x)\mathrm{d}x\ \forall\varphi\in X_0,\\ u\in X_0,\end{cases} \tag{1.5}$$

where space X_0 will be introduced in § 2. It is worthy noting that $C_0^\infty(\mathbb{R}^N)$ is dense in X_0, see [**16**] for more details. Also, $X_0\subset H^s(\mathbb{R}^N)$, see [**32**, Lemma 5].

Let $f\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ with $f(x,0)=0$ and $F(x,t)=\int_0^t f(x,\tau)\mathrm{d}\tau$. Furthermore, the following hypotheses are imposed on the nonlinearity $f(x,u)$:

(f_1) *There exist* $C>0$ *and* $q\in(2,2_s^*)$ *such that*

$$|f(x,t)|\leqslant C(1+|t|^{q-1}),\quad \textit{for all } (x,t)\in\Omega\times\mathbb{R},$$

where $2_s^*:=2N/(N-2s)$;

(f_2) *There exist* $\mu > 2\theta$ *and* $\beta > 0$ *such that*

$$\mu F(x,t) \leqslant tf(x,t) + \beta t^2, \quad \textit{for all } (x,t) \in \Omega \times \mathbb{R};$$

(f_3) $f(x,t)t \geqslant 0$ *for any* $x \in \Omega, t \in \mathbb{R}$ *and* $\lim_{|t|\to\infty} (f(x,t))/(t^{2\theta-1}) = +\infty$ *uniformly on* $x \in \Omega$, *where* θ *is given in assumption* (M_1)*;*

(f_4) *There exist* $\nu \geqslant 1$ *and* $\mathcal{C} > 0$ *such that*

$$\nu\mathcal{F}(x,t) \geqslant \mathcal{F}(x,\eta t) - \mathcal{C}, \quad \textit{for} \quad (x,t) \in \Omega \times \mathbb{R},\ \eta \in [0,1],$$

where $\mathcal{F}(x,t) = f(x,t)t - 2\theta F(x,t)$*;*

(f_5) *There exist* $\delta > 0$ *and* $\gamma \in (0, \lambda_1)$ *such that*

$$F(x,t) \leqslant \frac{1}{2}\gamma m_0 t^2, \quad \text{for } x \in \Omega,\ |t| \leqslant \delta;$$

where m_0 *is given in assumption* (M_1)*;*

(f_6) *There exists* $r > 0$ *small such that*

$$\frac{1}{2}m_1\lambda_k t^2 \leqslant F(x,t) \leqslant \frac{1}{2}\frac{m_0}{\theta}\lambda_{k+1}t^2, \quad \textit{for} \quad x \in \Omega,\ t \in \mathbb{R} \textit{ with } |t| \leqslant r,$$

where $m_1 = \max_{0\leqslant t\leqslant 1} M(t)$.

Here λ_k ($k = 1, 2\ldots$) is the eigenvalue of operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary datum, see § 2 for more details.

As $M \equiv 1, \theta = 1, s = 1$, then problem (1.1) reduces to the usual Laplacian boundary value problem. In this situation, our condition (f_2) is much weaker than (f_2) with $\beta = 0$. Note that the latter condition, which originally attributes to Ambrosetti and Rabinowitz in [**1**], implies that for some $L_1,\ L_2 > 0$

$$F(x,t) \geqslant L_1|t|^\mu - L_2 \quad \text{for } (x,t) \in \Omega \times \mathbb{R}. \tag{1.6}$$

Evidently, (1.6) means the condition (f_3), which characterizes problem (1.1) as superlinear at infinity. Since the seminal work of Ambrosetti and Rabinowitz [**1**], such superlinear elliptic boundary value problem has been investigated by many authors, see for example [**13**, **21**]. Although (f_2) is quite natural and important to guarantee the boundedness of Palais–Smale sequence, this condition is somewhat restrictive and removes many nonlinearities. For instance, the function

$$f(x,t) = t\log(1 + |t|) \tag{1.7}$$

does not satisfy (f_2), but it satisfies the conditions (f_3) and (f_4). It is easy to see that (f_4) without the constant $\mathcal{C}$, which was due to Jeanjean in [**18**], is slightly stronger than (f_4). In recent years, such condition was often applied to consider the existence of nontrivial solutions for the superlinear problems (1.1) without Ambrosetti–Rabinowitz condition, for example, see [**4**, **13**, **21**] and the references therein. In [**12**], Dinca, Jebelean and Mawhin applied the mountain pass theorem to obtain the existence of one positive solution and one negative solution to the problem (1.1) under the assumptions (f_2) and (f_5). The condition (f_6), which owes to Liu and Su in [**22**], was applied to get the existence of at least two non-trivial solutions to the problem (1.1), see [**22**, Theorem 1.2] for more details. Also, see [**9**, **23**, **31**] for more results related to the condition (f_6).

When $M(t) = a + bt$ and $s = 1$, Kirchhoff type problems (1.1) have captured some research interest in the past 10 years. For example, by using the Yang index and critical group, Perera and Zhang [**27**] investigated the existence of solutions for problem (1.1), where $f(x, \cdot)$ is asymptotically linear at 0 and asymptotically 4-linear at infinity. Recently, Sun and Liu [**21**] considered the existence of solutions for problem (1.1) in the cases where the nonlinearity is superlinear near zero but asymptotically 4−linear at infinity, and the nonlinearity is asymptotically linear near zero but 4−superlinear at infinity. Similar results can also be found in [**9**, **13**, **21**–**23**] and the reference therein.

When $M \equiv 1, \theta = 1$, the problem (1.1) becomes the following problem:

$$\begin{cases} -\mathcal{L}_K u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.8}$$

where the integro–differential operator $\mathcal{L}_K$ was introduced by Servadei and Valdinoci [**33**] to prove a Lewy–Stampacchia estimate. In [**32**], Servadei and Valdinoci obtained the existence of solutions for the problem (1.8) by using the mountain pass theorem under the conditions (f_1), (f_2) and $f(x, t) = o(t)$ as $t \to 0$. In [**14**], Ferrara *et al.* employed the Morse theory and critical groups to study the existence and multiplicity of solutions for problem (1.8) under some hypotheses used in this paper. In [**5**], Binlin *et al.* investigated the existence of multiple solutions for problem (1.1) by using Morse theory and local linking. However, the authors just considered the asymptotically linear case at infinity. By means of Morse theory and the spectral properties of fractional p–Laplacian operator, Iannizzotto *et al.* in [**17**] considered the existence of non–trivial solutions under the Ambrosetti–Rabinowitz condition. Very recently, by using Morse theory and critical groups, Chen and Su in [**10**] obtained the existence of nontrivial weak solutions for problem (1.8) when the problem is resonant at both infinity and zero.

Inspired by the above works, we would like to study the existence of solutions for superlinear problem (1.1) involving the Kirchhoff term and fractional Laplacian. There is no doubt that we encounter serious difficulties because of the presence of the Kirchhoff function and the nonlocal nature of fractional Laplacian. To this end, we need to make more delicate estimates to overcome the intrinsic difficulties induced by these new features.

Now we are in the position to state our main results as follows.

Theorem 1.1. *Let (M_1)–(M_2), (f_1), (f_2), $(f_3)$and (f_6) hold. Then the problem (1.1) has at least one nontrivial weak solution $u \in H^s(\mathbb{R}^N)$.*

Theorem 1.2. *Let (M_1)–(M_3), (f_1) and (f_3)–(f_5) hold. Then the problem (1.1) has at least one nontrivial weak solution $u \in H^s(\mathbb{R}^N)$.*

Theorem 1.3. *Let* (M_1)–(M_3), (f_1), (f_3), (f_4) *and* (f_6) *hold. Then the problem* (1.1) *has at least one nontrivial weak solution* $u \in H^s(\mathbb{R}^N)$.

Although (f_4) with $\mathcal{C} = 0$ is weaker than the assumption that

$$\frac{f(x,t)}{t^{2\theta-1}} \text{ is increasing in } t \geqslant 0, \text{ and is decreasing in } t \leqslant 0.$$

However, such condition and (f_4) are global conditions on $f(x,t)$, and hence is not very satisfactory. For this purpose, we replace the condition (f_4) with the following local condition (see [**21**]):

(f_7) *There exists* $\chi > 0$ *such that*

$$\frac{f(x,t)}{t^{2\theta-1}} \quad \text{is increasing in } t \geqslant \chi \text{ and decreasing in } t \leqslant -\chi.$$

Consequently, we obtain the following result.

Theorem 1.4. *Let* (M_1)–(M_3), (f_1), (f_3), (f_6) *and* (f_7) *hold. Then the problem* (1.1) *has at least one nontrivial weak solution* $u \in H^s(\mathbb{R}^N)$.

Finally, we point out that our results extend in several directions previous works. First, our conditions are much weaker than the corresponding classic conditions in the Laplacian setting. Second, because of the presence of the Kirchhoff function, the proofs of our results seem not to be an easy adaptation of techniques employed in [**4**, **9**, **21**, **23**]. In fact, theorem 1.1 is new in the literature even in the Laplacian case. Theorem 1.2 could be viewed as an extension of theorem 1.1 in [**23**] which just considered (f_2) and (f_5) in the Laplacian setting. Theorems 1.3–1.4 extend theorems 1.2–1.3 in [**14**] to the Kirchhoff context, respectively. To the best of our knowledge, there are very few results known for the Kirchhoff–type problems involving the fractional Laplacian by Morse theory. We just refer the readers to [**5**]. In this sense, our results are new even in the Kirchhoff–type Laplacian case. Moreover, our methods can also be used to deal with other elliptic problems, such as the Schrödinger–Kirchhoff problems. Naturally, we will focus on the multiplicity results for problem (1.1) in the near future.

The paper is organized as follows. In § 2, we present some necessary preliminary knowledge. In § 3, we justify the Cerami condition and the Palais–Smale condition for the energy functional under our assumptions. In § 4, we compute the critical groups at zero and at infinity. In § 5, we give the proofs of Theorems 1.1–1.4.

2. Preliminaries

In this section, we give some preliminary results which will be used in the sequel. Firstly, we briefly recall the definition of functional space X_0 introduced in [**33**]. See also [**32**, **34**, **35**] for more details.

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$

and

$$\text{the map } (x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)} \text{ is in } L^2\big((\mathbb{R}^N \times \mathbb{R}^N)\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega), \mathrm{d}x\mathrm{d}y\big)$$

(here $\mathcal{C}\Omega := \mathbb{R}^N\backslash\Omega$). Also, we denote by X_0 the following linear subspace of X

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\}.$$

We remark that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [**33**, Lemma 11]. Moreover, the space X is endowed with the norm defined as

$$\|g\|_X := \|g\|_{L^2(\Omega)} + \left(\iint_Q |g(x)-g(y)|^2 K(x-y)\mathrm{d}x\,\mathrm{d}y\right)^{1/2}, \tag{2.1}$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N)\backslash\mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^N \times \mathbb{R}^N$.

Now, we can take the function

$$X_0 \ni v \mapsto \|v\|_{X_0} = \left(\iint_Q |v(x)-v(y)|^2 K(x-y)\mathrm{d}x\,\mathrm{d}y\right)^{1/2}, \tag{2.2}$$

as the norm on X_0. Also, $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space, with scalar product

$$\langle u, v\rangle_{X_0} := \iint_Q (u(x)-u(y))(v(x)-v(y))\,K(x-y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2.3}$$

see [**32**] for more details.

Notice that in (2.3) (and in the related scalar product) the integral can be extended to all $\mathbb{R}^N \times \mathbb{R}^N$, since $v \in X_0$ (and so $v = 0$ a.e. in $\mathbb{R}^N\backslash\Omega$).

In the following, we define by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called *Gagliardo norm*)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\iint_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \tag{2.4}$$

We remark that, even in the model case in which $K(x) = |x|^{-(N+2s)}$, the norms in (2.1) and (2.4) are not the same, because $\Omega \times \Omega$ is strictly contained in Q (this makes the classical fractional Sobolev space approach not sufficient for studying the problem).

Lemma 2.1 (See [**32**, Lemma 5]). *If $K : \mathbb{R}^N\backslash\{0\} \to (0, +\infty)$ satisfies assumptions (K_1)–(K_2), then $X \subset H^s(\Omega)$ and $X_0 \subset H^s(\mathbb{R}^N)$.*

Lemma 2.2 (See [**32**, Lemma 8]). *Let $K : \mathbb{R}^N\backslash\{0\} \to (0, +\infty)$ satisfies assumptions (K_1)–(K_2). Then the embedding $j : X_0 \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [1, 2_s^*]$, while it is compact whenever $\nu \in [1, 2_s^*)$.*

In [34], Servadei and Valdinoci investigated the eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, namely the eigenvalue of the following problem

$$\begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{2.5}$$

More precisely, the following weak formulation of (2.5) was discussed:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)\mathrm{d}x\mathrm{d}y \\ \quad = \lambda \int_\Omega u(x)\varphi(x)\mathrm{d}x, \quad \forall \varphi \in X_0 \\ u \in X_0. \end{cases} \tag{2.6}$$

We recall that $\lambda \in \mathbb{R}$ is an eigenvalue of $-\mathcal{L}_K$ if there exists a non-trivial solution $u \in X_0$ of problem (2.5) or its weak formulation (2.6), and any solution will be called an eigenfunction corresponding to the eigenvalue λ.

Lemma 2.3 (See [34, Proposition 9]). *Let $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ satisfies assumptions (K_1)–(K_2). Then*
(i) problem (2.6) admits an eigenvalue $\lambda_1 > 0$ that is simple and has an associated eigenfunction which is non-negative function $e_1 \in X_0$ with $\|e_1\|_{L^2(\Omega)} = 1$;
(ii) problem (2.6) has a set of the eigenvalues which consists of a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with

$$0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots \quad \text{and} \quad \lambda_k \to +\infty \text{ as } k \to +\infty.$$

Moreover, for any $k \in \mathbb{N}$ the eigenvalues can be characterized as follows:

$$\lambda_{k+1} = \min_{u \in \mathbb{P}_{k+1} \backslash \{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y}{\int_\Omega |u(x)|^2 \mathrm{d}x},$$

where $\mathcal{P}_{k+1} := \big\{u \in X_0 : \langle u, e_j\rangle_{X_0} = 0, \ \forall j = 1, \ldots, k\big\}$;
(iii) the sequence $\{e_k\}_{k\in\mathbb{N}}$ of eigenfunctions corresponding to $\{\lambda_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0;
(iv) each eigenvalue λ_k has finite multiplicicy.

Remark 2.1. From lemma 2.3 it follows that X_0 has the following direct sum decomposition

$$X_0 = \mathrm{span}\{e_1, e_2, \ldots, e_k\} \oplus \big(\mathrm{span}\{e_1, e_2, \ldots, e_k\}\big)^{\perp} = \mathrm{span}\{e_1, e_2, \ldots, e_k\} \oplus \mathcal{P}_{k+1}.$$

where the orthogonal $\perp$ is implied with respect to the scalar product of X_0.

3. The compactness conditions

First, we observe that problem (1.1) has a variational structure, indeed it is the Euler–Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows

$$\mathcal{J}(u) = \frac{1}{2}\mathscr{M}\left(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right) - \int_{\Omega} F(x, u(x))\mathrm{d}x.$$

It is well known that the functional $\mathcal{J}$ is Frechét differentiable in X_0 and for any $\varphi \in X_0$

$$\langle \mathcal{J}'(u), \varphi\rangle = M\left(\|u\|^2_{X_0}\right)\iint_{\mathbb{R}^{2N}} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)\mathrm{d}x\mathrm{d}y$$
$$- \int_{\Omega} f(x, u(x))\varphi(x)\mathrm{d}x.$$

Therefore, critical points of the functional $\mathcal{J}$ are weak solutions of problem (1.1).

Definition 3.1. The functional $\mathcal{J}$ satisfies the Cerami condition (for short, the (C) condition) if for $c \in \mathbb{R}$, any sequence $\{u_j\}_j \subset X_0$ such that $\mathcal{J}(u_j) \to c, (1 + ||u_j||)||\mathcal{J}'(u_j)||_{X_0^*} \to 0$ has a convergent subsequence. The functional Φ satisfies the Palais–Smale condition (for short, the (PS) condition) if for $c \in \mathbb{R}$, any sequence $\{u_j\}_j$ such that $\mathcal{J}(u_j) \to c$ and $\mathcal{J}'(u_j) \to 0$ in X_0^* has a convergent subsequence. Here X_0^* denotes the dual space of X_0.

Note that the (C) condition introduced by Cerami is a weak version of the (PS) condition. If $\mathcal{J}$ satisfied the (PS) condition or the (C) condition, then $\mathcal{J}$ satisfied the deformation condition (see [**23**]).

Lemma 3.1 (See [**37**, Lemma 3.6]). *Let (M_1) and (f_1) hold. Then any bounded sequence $\{u_j\}_j$ in X_0 such that $\mathcal{J}'(u_j) \to 0$ in X_0^* has a convergent subsequence.*

Lemma 3.2. *Suppose that (M_1)–(M_2), (f_1), (f_2), (f_3) and (f_6) are satisfied. Let $\{u_j\}_j$ be a sequence in X_0 such that $\mathcal{J}(u_j) \to c$ and $\mathcal{J}'(u_j) \to 0$ in X_0^*. Then $\{u_j\}_j$ is bounded in X_0.*

Proof. We argue by contradiction. Assume that $\|u_n\| \to \infty$ as $n \to \infty$. Then by (M_1) and (f_2), we obtain

$$\begin{aligned} c + o(1) &\geqslant \mathcal{J}(u_n) - \frac{1}{\mu}\langle \mathcal{J}'(u_n), u_n\rangle \\ &\geqslant \left(\frac{1}{2} - \frac{\theta}{\mu}\right)\mathscr{M}(\|u_n\|^2_{X_0}) - \frac{1}{\mu}\int_{\Omega} (\mu F(x, u_n) - f(x, u_n)u_n)\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{\theta}{\mu}\right) m_0\|u_n\|^2_{X_0} - \frac{\beta}{\mu}\int_{\Omega} u_n^2\mathrm{d}x. \end{aligned} \tag{3.1}$$

Let $v_n = ||u_n||^{-1}_{X_0}u_n$, then $||v_n||_{X_0} = 1$. By lemma 2.2, up to a subsequence, we have

$$v_n \rightharpoonup v \text{ in } X_0, \quad v_n \to v \text{ in } L^q(\mathbb{R}^N), \quad v_n \to v \quad \text{a.e.} \quad x \in \mathbb{R}^N \tag{3.2}$$

Multiplying by $||u_n||_{X_0}^{-2}$ on both sides of (3.1) and letting $n \to \infty$, we have

$$\int_\Omega v^2 \mathrm{d}x \geqslant \frac{\mu}{\beta}\left(\frac{1}{2} - \frac{\theta}{\mu}\right). \tag{3.3}$$

Obviously, (3.3) implies that $v \not\equiv 0$ thanks to the fact that $\mu > 2\theta$.

By (f_1), (f_3) and (f_6), it is easy to see that there is some $\zeta > 0$ such that

$$F(x,t) \geqslant -\zeta t^{2\theta}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.4}$$

As a result, it follows that

$$\int_{\{v=0\}} \frac{F(x,u_n)}{||u_n||_{X_0}^{2\theta}} \mathrm{d}x = \int_{\{v=0\}} \frac{F(x,u_n)}{|u_n|^{2\theta}} |v_n|^{2\theta} \mathrm{d}x \geqslant -\zeta \int_{\{v=0\}} |v_n|^{2\theta} \mathrm{d}x > -\infty \tag{3.5}$$

thanks to (3.2) and $2\theta < 2_s^*$. Moreover, for $x \in \{x \in \Omega : v(x) \neq 0\}$, we know that $|u_n(x)| \to +\infty$ as $n \to \infty$. By (f_3) again, we deduce that

$$\frac{F(x,u_n)}{||u_n||_{X_0}^{2\theta}} = \frac{F(x,u_n)}{|u_n|^{2\theta}} |v_n|^{2\theta} \to +\infty, \quad \text{as} \quad n \to \infty. \tag{3.6}$$

By (M_2), we have

$$\mathscr{M}(t) \leqslant \mathscr{M}(1) t^\theta, \tag{3.7}$$

for any $t \geqslant 1$.

Consequently, combining (3.5)–(3.7), we have

$$\begin{aligned} \frac{1}{2}\mathscr{M}(1) - \frac{c+o(1)}{||u_n||_{X_0}^{2\theta}} &\geqslant \frac{1/2\mathscr{M}(\|u_n\|_{X_0}^2) - \mathcal{J}(u_n)}{||u_n||_{X_0}^{2\theta}} & (3.8)\\ &= \left(\int_{\{v=0\}} + \int_{\{v\neq 0\}}\right) \frac{F(x,u_n)}{||u_n||_{X_0}^{2\theta}} \mathrm{d}x \\ &\geqslant \int_{\{v\neq 0\}} \frac{F(x,u_n)}{||u_n||_{X_0}^{2\theta}} \mathrm{d}x - \zeta \int_{\{v=0\}} |v_n|^{2\theta} \mathrm{d}x \to +\infty & (3.9) \end{aligned}$$

This a contradiction. Thus, the proof is complete. □

THEOREM 3.1. *Suppose that* (M_1)–(M_3), (f_1), (f_3) *and* (f_4) *hold. Then* $\mathcal{J}$ *satisfies the* (C) *condition.*

Proof. Similarly to the proof of Lemma 2.2 in [**13**], and for the reader's convenience, we will give the key steps of the proof. From lemma 3.1, it suffices to show the boundedness of (C) sequences. Assume $\mathcal{J}$ has an unbounded (C) sequence $\{u_n\}_n$. Up to a subsequence, we may assume that for some $c \in \mathbb{R}$

$$\mathcal{J}(u_n) \to c, \quad ||u_n||_{X_0} \to \infty, \quad \langle \mathcal{J}'(u_n), \quad u_n\rangle \to 0 \quad \text{as} \quad n \to \infty. \tag{3.10}$$

Let $v_n = ||u_n||_{X_0}^{-1} u_n$, then $||v_n||_{X_0} = 1$. By lemma 2.2, up to a subsequence, we have

$$v_n \rightharpoonup v \text{ in } X_0, \; v_n \to v \text{ in } L^q(\mathbb{R}^N), \; v_n \to v \; \text{ a.e. } x \in \mathbb{R}^N \tag{3.11}$$

If $v \equiv 0$, as in [**18**], we choose a sequence $\{t_n\}_n \subset [0,1]$ such that $\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(t u_n)$. For any positive integer m, we can choose $r = \sqrt{2m}$ such that

$r||u_n||_{X_0}^{-1} \in (0,1)$ as n large enough. Since $v_n \to 0$ in $L^q(\Omega)$ and (f_1), by the continuity of the Nemiskii operator, we know that $F(\cdot, rv_n) \to 0$ in $L^1(\Omega)$. Thus

$$\lim_{n\to\infty} \int_\Omega F(x, rv_n)\mathrm{d}x = 0. \tag{3.12}$$

Hence, for n large enough, it follows from (3.12) and (M_1) that

$$\mathcal{J}(t_n u_n) \geqslant \mathcal{J}(r||u_n||_{X_0}^{-1} u_n) = \mathcal{J}(rv_n) \geqslant m_0 m - \int_\Omega F(x, rv_n)\mathrm{d}x.$$

From which we deduce that $\mathcal{J}(t_n u_n) \to +\infty$ as $n \to \infty$. But $\mathcal{J}(0) = 0, \mathcal{J}(u_n) \to c$, so $t_n \in (0,1)$ and

$$\langle \mathcal{J}'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} \mathcal{J}(tu_n) = 0.$$

Now using (f_4) and (M_3), we get

$$\begin{aligned}
\frac{1}{\nu}\mathcal{J}(t_n u_n) &= \frac{1}{\nu}\left[\mathcal{J}(t_n u_n) - \frac{1}{2\theta}\langle \mathcal{J}'(t_n u_n), t_n u_n\rangle\right] + o(1) \\
&= \frac{1}{2\nu\theta}\mathcal{M}(\|t_n u_n\|_{X_0}^2) + \frac{1}{2\nu\theta}\int_\Omega \mathcal{F}(x, t_n u_n)\mathrm{d}x + o(1) \\
&\leqslant \frac{1}{2\theta}\left[\mathcal{M}(\|u_n\|_{X_0}^2) + \int_\Omega \mathcal{F}(x, u_n)\mathrm{d}x\right] + \frac{\mathcal{C}}{2\theta\nu} + o(1) \\
&= \mathcal{J}(u_n) - \frac{1}{2\theta}\langle \mathcal{J}'(u_n), u_n\rangle + \frac{\mathcal{C}}{2\theta\nu} + o(1) \\
&\to \frac{\mathcal{C}}{2\theta\nu} + c.
\end{aligned} \tag{3.13}$$

This contradicts the fact that $\mathcal{J}(t_n u_n) \to +\infty$ as $n \to \infty$.

If $v \not\equiv 0$, then the set $\Omega' := \{x \in \Omega : v(x) \neq 0\}$ has positive Lebesgue measure and hence $|u_n(x)| \to +\infty$ as $n \to \infty$. Observe that the third limit in (3.10) gives

$$M(||u_n||_{X_0}^2)||u_n||_{X_0}^2 - \int_\Omega f(x, u_n)u_n \mathrm{d}x = \langle \mathcal{J}'(u_n), u_n\rangle = o(1).$$

Since $f(x,u)u \geqslant 0$ in (f_3), by (3.7) and (M_2), we obtain

$$\begin{aligned}
1 - o(1) &= \int_\Omega \frac{f(x, u_n)u_n}{M(||u_n||_{X_0}^2)||u_n||_{X_0}^2}\mathrm{d}x \\
&\geqslant \int_\Omega \frac{f(x, u_n)u_n}{\theta\mathscr{M}(||u_n||_{X_0}^2)}\mathrm{d}x \\
&\geqslant \frac{1}{\theta\mathscr{M}(1)}\left(\int_{\{v\neq 0\}} + \int_{\{v=0\}}\right)\frac{f(x, u_n)u_n}{|u_n|^{2\theta}}|v_n|^{2\theta}\mathrm{d}x \\
&\geqslant \frac{1}{\theta\mathscr{M}(1)}\int_{\{v\neq 0\}} \frac{f(x, u_n)u_n}{|u_n|^{2\theta}}|v_n|^{2\theta}\mathrm{d}x.
\end{aligned} \tag{3.14}$$

By (f_3) again, we deduce that

$$\frac{f(x,u_n)u_n}{|u_n|^{2\theta}}|v_n|^{2\theta} \to +\infty, \quad \text{as } n \to \infty.$$

Then the Fatou's lemma implies

$$\int_{\{v\neq 0\}} \frac{f(x,u_n)u_n}{|u_n|^{2\theta}}|v_n|^{2\theta}\mathrm{d}x \to +\infty, \quad \text{as } n \to \infty.$$

This contradicts (3.14). Thus, $\{u_n\}_n$ is bounded in X_0. □

LEMMA 3.3 (See [**21**, Lemma 2.3]). *If (f_7) holds, then for any $x \in \Omega$, $\mathcal{F}(x,t)$ is increasing in $t \geqslant \nu$ and decreasing in $t \leqslant -\nu$, where $\mathcal{F}(x,t) = f(x,t)t - 2\theta F(x,t)$. In particular, there exists $\widetilde{C} > 0$ such that for $x \in \Omega$ and $0 \leqslant m \leqslant t$ or $t \leqslant m \leqslant 0$*

$$\mathcal{F}(x,m) \leqslant \mathcal{F}(x,t) + \widetilde{C}.$$

THEOREM 3.2. *Suppose that (M_1)–(M_3), (f_1), (f_3) and (f_7) are fulfilled. Then $\mathcal{J}$ satisfies the (C) condition.*

Proof. From the proof of Theorem 3.1 it is easy to see that it suffices to verify the validity of (3.13). In view of lemma 3.3 and (M_3), we have

$$\begin{aligned}
\mathcal{J}(t_n u_n) &= \mathcal{J}(t_n u_n) - \frac{1}{2\theta}\langle \mathcal{J}'(t_n u_n), t_n u_n\rangle + o(1)\\
&= \frac{1}{2\theta}\mathcal{M}(\|t_n u_n\|^2_{X_0}) + \frac{1}{2\theta}\int_\Omega \mathcal{F}(x,t_n u_n)\mathrm{d}x + o(1)\\
&\leqslant \frac{1}{2\theta}\left[\mathcal{M}(\|u_n\|^2_{X_0}) + \int_\Omega \mathcal{F}(x,u_n)\mathrm{d}x\right] + \frac{1}{2\theta}\widetilde{C}|\Omega| + o(1)\\
&= \mathcal{J}(u_n) - \frac{1}{2\theta}\langle \mathcal{J}'(u_n), u_n\rangle + \frac{1}{2\theta}\widetilde{C}|\Omega| \to c + \frac{1}{2\theta}\widetilde{C}|\Omega|.
\end{aligned}$$

This is a contradiction. The proof is therefore complete. □

4. Critical groups

Since $f(x,0) = 0$, the problem (1.1) has a trivial solution $u \equiv 0$. Therefore, we are interested in the existence of non–trivial weak solutions for problem (1.1). It follows from Morse theory that comparing the critical groups of functional at zero and at infinity may lead to the existence of non–trivial weak solutions to problem (1.1) (see [**8**]).

Let Y be a real Banach space and $\mathcal{J} \in C^1(Y,\mathbb{R}), \mathcal{K} = \{u \in Y : \mathcal{J}'(u) = 0\}$. Then the qth critical group of $\mathcal{J}$ at an isolated critical point $u \in \mathcal{K}$ with $\mathcal{J}(u) = c$ is defined by

$$C_q(\mathcal{J},u) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U\backslash\{u\}), \quad q \in \mathbb{N} := \{0,1,2,\ldots\},$$

where $\mathcal{J}^c = \{u \in Y : \mathcal{J}(u) \leqslant c\}$, U is any neighbourhood of u, containing the unique critical point, H_q is the singular relative homology with coefficients in an Abelian group G.

We say that $u \in \mathcal{K}$ is a homological nontrivial critical point of $\mathcal{J}$ if at least one of its critical groups is nontrivial. Now we present the following result about the critical groups at zero which will be used later.

PROPOSITION 4.1 (See [**22**, Proposition 2.1]). *Assume that $\mathcal{I}$ has a critical point $u = 0$ with $\mathcal{I}(0) = 0$. Suppose that $\mathcal{I}$ has a local linking at 0 with respect to $Y = V \oplus W$, $k = \dim V < \infty$, that is, there exists $\rho > 0$ small such that*

$$\begin{cases} \mathcal{I}(u) \leqslant 0, \ \text{for } u \in V, \ ||u|| \leqslant \rho; \\ \mathcal{I}(u) > 0, \ \text{for } u \in W, \ 0 < ||u|| \leqslant \rho. \end{cases}$$

Then $C_k(\mathcal{I}, 0) \ncong 0$. Hence, 0 is a homological nontrivial critical point of $\mathcal{I}$.

If $\mathcal{J}$ satisfies the (C) condition and the critical values of $\mathcal{J}$ are bounded from below by some $a < \inf \mathcal{J}(\mathcal{K})$, then the critical groups of $\mathcal{J}$ at infinity were introduced by Bartsch and Li [**3**] as

$$C_q(\mathcal{J}, \infty) := H_q(X, \mathcal{J}^a), \quad q \in \mathbb{N}. \tag{4.1}$$

If $\mathcal{J}$ satisfies the (C) condition, then $\mathcal{J}$ satisfies the deformation condition. By the deformation lemma, the right–hand side of (4.1) does not depend on the choice of a.

With the help of the following result, we will prove the main results. For more details about Morse theory, we refer the interested readers to [**8**].

PROPOSITION 4.2 (See [**8**]). *Let $\mathcal{J} \in C^1(Y, \mathbb{R})$ satisfies the (C) condition and $\mathcal{J}$ has only finitely many critical points. Then*

(i) If there is some $q \in \mathbb{N}$ such that $C_q(\mathcal{J}, \infty) \ncong 0$, then $\mathcal{J}$ has a critical point u with $C_q(\mathcal{J}, u) \ncong 0$.

(ii) Let 0 be an isolated critical point of $\mathcal{J}$. If $\mathcal{K} = \{0\}$, then $C_q(\mathcal{J}, \infty) = C_q(\mathcal{J}, 0)$ for all $q \in \mathbb{N}$.

It follows from proposition 4.2 that if $C_q(\mathcal{J}, \infty) \neq C_q(\mathcal{J}, 0)$ for some $q \in \mathbb{N}$, then $\mathcal{J}$ must have a nontrivial critical point. Therefore, we must compute the critical groups at zero and at infinity.

For the proofs of our theorems, in what follows we may assume that Φ has only finitely many critical points. Since $\mathcal{J}$ satisfies the (C) condition, then the critical groups $C_q(\mathcal{J}, \infty)$ at infinity make sense.

THEOREM 4.1. *If (M_1), (f_1) and (f_5) are satisfied, then $C_q(\mathcal{J}, 0) \cong \delta_{q,0} G$ for each $q \in \mathbb{N}$.*

Proof. For any $u \in X_0$, it follows from (M_1), (f_1) and (f_5) that

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\mathscr{M}\left(\iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\mathrm{d}x\mathrm{d}y\right) - \int_\Omega F(x,u)\mathrm{d}x \\ &= \frac{1}{2}\mathscr{M}\left(\|u\|_{X_0}^2\right) - \int_{\{|u|\leqslant\delta\}} F(x,u)\mathrm{d}x - \int_{\{|u|>\delta\}} F(x,u)\mathrm{d}x \\ &\geqslant \frac{1}{2}m_0\|u\|_{X_0}^2 - \int_{\{|u|\leqslant\delta\}} \frac{\gamma}{2}m_0|u|^2\mathrm{d}x - \int_{\{|u|>\delta\}} |F(x,u)|\mathrm{d}x \\ &\geqslant \frac{1}{2}m_0\left(1-\frac{\gamma}{\lambda_1}\right)\|u\|_{X_0}^2 - C\|u\|_{L^q(\Omega)}^q. \end{aligned} \tag{4.2}$$

Thus $u=0$ is a local minimizer of $\mathcal{J}$ and hence $C_q(\mathcal{J},0) \cong \delta_{q,0}G$ for each $q \in \mathbb{N}$. This finishes the proof of Theorem 4.1. □

THEOREM 4.2. *If (M_1), (M_2), (f_1) and (f_6) are satisfied, then $C_k(\mathcal{J},0) \ncong 0$, where $k \in \mathbb{N}^* := \{1,2,\ldots\}$.*

Proof. Since $f(x,0)=0$, the functional $\mathcal{J}$ has a trivial critical point $u=0$. In view of lemma 4.1, it suffices to verify the fact that the functional $\mathcal{J}$ has a local linking at 0 with respect to $X_0 = V_k \oplus \mathcal{P}_{k+1}$, where $V_k := \operatorname{span}\{e_1,e_2,\ldots,e_k\}$ and $\mathcal{P}_{k+1}$ is as in remark 2.1.

(i) Since V_k is finite dimensional, we have that for given $r>0$, there exists $\rho \in (0,1)$ small such that for $u \in V_k$, $||u||_{X_0} \leqslant \rho$ implies $|u(x)| \leqslant r$ for a.e. $x \in \Omega$. Let $u \in V_k$. Then

$$u(x) = \sum_{i=1}^{k} c_i e_i(x),$$

with $c_i \in \mathbb{R}, i=1,\ldots,k$. Because $\operatorname{span}\{e_1,e_2,\ldots,e_k,\ldots\}$ is an orthogonal basis of $L^2(\Omega)$ and an orthogonal one of X_0 by lemma 2.3, we have

$$\|u\|_{L^2(\Omega)}^2 = \sum_{i=1}^{k} c_i^2,\ \ \|u\|_{X_0}^2 = \sum_{i=1}^{k} c_i^2\|e_i\|_{X_0}^2.$$

In terms of (f_6), we obtain that for $u \in V_k$ with $\|u\|_{X_0} \leqslant \rho$,

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\mathscr{M}\left(\|u\|_{X_0}^2\right) - \int_\Omega F(x,u)\mathrm{d}x \\ &= \frac{1}{2}\mathscr{M}\left(\|u\|_{X_0}^2\right) - \int_{\{|u(x)|<r\}} F(x,u(x))\mathrm{d}x \\ &\leqslant \frac{1}{2}m_1\|u\|_{X_0}^2 - \frac{\lambda_k}{2}m_1\|u\|_{L^2(\Omega)}^2 \\ &= \frac{1}{2}m_1\sum_{i=1}^{k} c_i^2(\|e_i\|_{X_0}^2 - \lambda_k) \leqslant 0, \end{aligned} \tag{4.3}$$

where $m_1 = \max\limits_{0\leqslant t\leqslant 1} M(t)$.

(ii) For $u \in \mathcal{P}_{k+1}$, we write $u = v + w$, where $v \in \operatorname{span}\{e_k\}$ and $w \in \mathcal{P}_{k+2}$. Obviously, v and w are orthogonal in $L^2(\Omega)$ and X_0. Therefore, we deduce from (M_1) and (M_2) that

$$\begin{aligned}
\mathcal{J}(u) &= \frac{1}{2}\mathscr{M}\left(\|u\|_{X_0}^2\right) - \frac{1}{2\theta}m_0\lambda_{k+1}\int_\Omega u^2 \mathrm{d}x + \int_\Omega \left(\frac{1}{2\theta}m_0\lambda_{k+1}u^2 - F(x,u(x))\right)\mathrm{d}x \\
&\geqslant \frac{1}{2\theta}M\left(\|u\|_{X_0}^2\right)\|u\|_{X_0}^2 - \frac{1}{2\theta}m_0\lambda_{k+1}\int_\Omega u^2\mathrm{d}x \\
&\quad + \int_\Omega \left(\frac{1}{2\theta}m_0\lambda_{k+1}u^2 - F(x,u(x))\right)\mathrm{d}x \\
&\geqslant \frac{1}{2\theta}m_0\left(1 - \frac{\lambda_{k+1}}{\lambda_{k+2}}\right)\|w\|_{X_0}^2 + \int_\Omega \left(\frac{1}{2\theta}m_0\lambda_{k+1}u^2 - F(x,u(x))\right)\mathrm{d}x. \qquad (4.4)
\end{aligned}$$

For $|u(x)| \leqslant r$, it follows from (f_6) that

$$\int_{\{|u(x)|\leqslant r\}} \left(\frac{1}{2\theta}m_0\lambda_{k+1}u^2 - F(x,u(x))\right)\mathrm{d}x \geqslant 0. \qquad (4.5)$$

For $|u(x)| > r$, $\|u\|_{X_0} < \rho$ means $\|v\|_{X_0} < \rho$ and then $|v(x)| < r/2$ when ρ is small enough. Hence, we obtain

$$|w(x)| = |u(x) - v(x)| \geqslant |u(x)| - |v(x)| \geqslant \frac{1}{2}|u(x)|.$$

Therefore, by means of (f_1), we have

$$\int_{\{|u(x)|>r\}} \left(F(x,u(x) - \frac{1}{2\theta}m_0\lambda_{k+1}u^2\right)\mathrm{d}x \leqslant C2^q\int_\Omega |w(x)|^q\mathrm{d}x. \qquad (4.6)$$

By lemma 2.2, we have

$$\int_{\{|u(x)|>r\}} \left(F(x,u(x) - \frac{1}{2\theta}m_0\lambda_{k+1}u^2\right)\mathrm{d}x \leqslant C\|w\|_{X_0}^q. \qquad (4.7)$$

Therefore, we obtain

$$\mathcal{J}(u) \geqslant \frac{1}{2\theta}m_0\left(1 - \frac{\lambda_{k+1}}{\lambda_{k+2}}\right)\|w\|_{X_0}^2 - C\|w\|_{X_0}^q. \qquad (4.8)$$

Now we claim that $\mathcal{J}(u)$ is positive as $u \in \mathcal{P}_{k+1}$ with $0 < \|u\|_{X_0} \leqslant \rho$ as ρ is small enough.

Case 1: If $w \not\equiv 0$, the assertion is clear because of $q > 2$ when ρ is small enough.

Case 2: If $w \equiv 0$, $v \not\equiv 0$, and $\|v\|_{X_0} \leqslant \rho$, then $|v| \leqslant r$ when ρ is small enough. Thus from (f_6) it follows that

$$\mathcal{J}(v) \geqslant \int_\Omega \left(\frac{1}{2\theta} m_0 \lambda_{k+1} v^2 - F(x, v(x))\right) \mathrm{d}x > 0. \tag{4.9}$$

So, we conclude that

$$\mathcal{J}(u) > 0, \text{ for any } u \in \mathcal{P}_{k+1},\ 0 < ||u|| \leqslant \rho. \tag{4.10}$$

Combining (4.3) and (4.10), we complete the proof of Theorem 4.2, thanks to proposition 4.1. □

THEOREM 4.3. *If (M_2), (f_1), (f_3) and (f_4) are satisfied, then $C_q(\mathcal{J}, \infty) \cong 0$ for all $q \in \mathbb{N}$.*

Proof. We follow the idea of lemma 2.3 in [8]. Let $S = \{u \in X_0 : \|u\|_{X_0} = 1\}$. For each $u \in S$, by the Fatou lemma and (f_3), we have

$$\lim_{t\to+\infty} \int_\Omega \frac{F(x, tu)}{|t|^{2\theta}} \mathrm{d}x \geqslant \int_\Omega \lim_{t\to+\infty} \frac{F(x, tu)}{|tu|^{2\theta}} |u|^{2\theta} \mathrm{d}x = +\infty. \tag{4.11}$$

Thus, from (3.7) it yields that

$$\begin{aligned} \mathcal{J}(tu) &= \frac{1}{2}\mathscr{M}(\|tu\|^2) - \int_\Omega F(x, tu)\mathrm{d}x \\ &\leqslant \frac{1}{2}\mathscr{M}(1)t^{2\theta} - \int_\Omega F(x, tu)\mathrm{d}x \\ &\leqslant t^{2\theta}\left(\frac{1}{2}\mathscr{M}(1) - \int_\Omega \frac{F(x, tu)}{|t|^{2\theta}}\mathrm{d}x\right) \to -\infty \quad \text{as } t \to +\infty. \end{aligned} \tag{4.12}$$

Next we claim: there exists $A > (1)/(2\nu\theta)\mathcal{C}|\Omega|$ such that for any $a > A$, if $\mathcal{J}(tu) \leqslant -a$, then $(d/dt)\mathcal{J}(tu) < 0$. In fact, from (f_4) it follows that

$$\mathcal{F}(x, t) \geqslant -\frac{\mathcal{C}}{\nu} \text{ for } (x, t) \in \Omega \times \mathbb{R}. \tag{4.13}$$

Thus, if $\mathcal{J}(tu) = \frac{1}{2}\mathscr{M}(\|tu\|^2_{X_0}) - \int_\Omega F(x, tu)\mathrm{d}x \leqslant -a$, then we have

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(tu) &= \langle \mathcal{J}'(tu), u\rangle \\ &\leqslant \frac{1}{t}\left[\theta\mathscr{M}(\|tu\|^2_{X_0}) - \int_\Omega f(x, tu)tu\mathrm{d}x\right] \\ &\leqslant \frac{1}{t}\left[-\int_\Omega \mathcal{F}(x, tu)\mathrm{d}x - 2\theta a\right], \\ &\leqslant \frac{1}{t}\left(\frac{\mathcal{C}}{\nu}|\Omega| - 2\theta a\right) < 0, \end{aligned}$$

thanks to (M_2) and (4.13). By the Implicit Function Theorem, there is a unique $T(u) \in C(S, \mathbb{R})$ such that $\mathcal{J}(T(u)u) = -a$. As in [20, Lemma 2.4], using the function T to construct a strong deformation retract from $X_0\backslash\{0\}$ to $\mathcal{J}^{-a}$. Therefore, it follows from $\dim X_0 = +\infty$ that

$$C_q(\mathcal{J}, \infty) = H_q(X_0, \mathcal{J}^{-a}) \cong H_q(X_0, X_0\backslash\{0\}) \cong 0, \ \forall q \in \mathbb{N}.$$

The proof is thus complete. □

THEOREM 4.4. *If (M_2), (f_1), (f_3) and (f_7) are satisfied, then $C_k(\mathcal{J}, \infty) \cong 0$, $k \in \mathbb{N}$.*

Proof. This proof is similar to that of theorem 4.3. Here we also give some key steps, just for the reader's convenience. Let $S = \{u \in X_0 : \|u\|_{X_0} = 1\}$. By (4.12), we have for each $u \in S$

$$\mathcal{J}(tu) \to -\infty \text{ as } t \to +\infty.$$

Let $m = 0$ in Lemma 3.3, we get

$$\mathcal{F}(x, t) \geqslant -\widetilde{C} \quad for \ (x, t) \in \Omega \times \mathbb{R}. \tag{4.14}$$

Choose $a < \min\left\{\inf_{\|u\|_{X_0} \leqslant 1} \mathcal{J}(u), -\frac{1}{2\theta}\widetilde{C}|\Omega|\right\}$, then for any $u \in S$, there exists $t > 1$ such that $\mathcal{J}(tu) \leqslant a$. Therefore, if $\mathcal{J}(tu) \leqslant a$, then by (4.14) and (M_2)

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(tu) &\leqslant \frac{1}{t}[\theta\mathscr{M}(\|tu\|^2_{X_0}) - \int_\Omega f(x, tu)tu\mathrm{d}x] \\ &\leqslant \frac{1}{t}[-\int_\Omega \mathcal{F}(x, tu)\mathrm{d}x + 2\theta a] \\ &\leqslant \frac{1}{t}(2\theta a + \widetilde{C}|\Omega|) < 0.\end{aligned}$$

The following proof is the same as the one of theorem 4.3, so we omit it. □

THEOREM 4.5. *If (M_2), (f_1), (f_2) and (f_3) are satisfied, then $C_k(\mathcal{J}, \infty) \cong 0$, $k \in \mathbb{N}$.*

Proof. We follow the argument in [9, Lemmas 2.4–2.5]. Here we give the details of the key steps, just for the reader's convenience. Let $S = \{u \in X_0 : \|u\|_{X_0} = 1\}$. By (4.12), we have for each $u \in S$

$$\mathcal{J}(tu) \to -\infty \text{ as } t \to +\infty.$$

Step 1: We first prove that there exists $A > 0$ such that for any $u \in X_0$ with $\mathcal{J}(u) \leqslant -A$, we have

$$\left.\frac{d}{dt}\right|_{t=1} \mathcal{J}(tu) < 0. \tag{4.15}$$

In fact, if this were not true, then there would exist $u_n \in X_0$ such that

$$\mathcal{J}(u_n) = \frac{1}{2}\mathscr{M}(\|u_n\|^2_{X_0}) - \int_\Omega F(x, u_n)\mathrm{d}x \leqslant -n \tag{4.16}$$

and

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=1}\mathcal{J}(tu_n) &= \langle \mathcal{J}'(u_n), u_n\rangle \\ &= M(\|u_n\|_{X_0}^2)\|u_n\|_{X_0}^2 - \int_\Omega f(x,u_n)u_n \mathrm{d}x \geqslant 0. \end{aligned} \tag{4.17}$$

Then from (4.17) and (f_3) it follows that $\|u_n\|_{X_0} \to +\infty$ as $n\to\infty$. Furthermore, by (M_1), (M_2) and (f_2), we obtain

$$\begin{aligned}-\mu n &\geqslant \mu\mathcal{J}(u_n) - \langle \mathcal{J}'(u_n), u_n\rangle \\ &= \frac{\mu}{2}\mathscr{M}(\|u_n\|_{X_0}^2) - M(\|u_n\|^2)\|u_n\|_{X_0}^2 + \int_\Omega \Big[f(x,u_n)u_n - \mu F(x,u_n)\Big]\mathrm{d}x \\ &\geqslant \left(\frac{\mu}{2}-\theta\right) m_0\|u_n\|_{X_0}^2 - \beta\int_\Omega u_n^2 \mathrm{d}x. \end{aligned} \tag{4.18}$$

Let $v_n = ||u_n||_{X_0}^{-1}u_n$, then $||v_n||_{X_0} = 1$. By lemma 2.2, up to a subsequence, we have

$$v_n \rightharpoonup v_0 \text{ in } X_0,\ v_n \to v_0 \text{ in } L^2(\mathbb{R}^N),\ v_n \to v_0 \ \text{ a.e. } x\in\mathbb{R}^N. \tag{4.19}$$

If $v_0 \equiv 0$, then we have

$$0 \geqslant \left(\frac{\mu}{2}-\theta\right) m_0\|v_n\|_{X_0}^2 - \beta\int_\Omega v_n^2\mathrm{d}x \to \left(\frac{\mu}{2}-\theta\right) m_0. \tag{4.20}$$

This is a contradiction. Therefore, $v_0 \not\equiv 0$.

Note that for $x\in\Omega' = \{x\in\Omega : v(x)\neq 0\}$, we have that $|u_n(x)|\to+\infty$ as $n\to\infty$. This, together with (4.17), (3.7), (f_3) and (M_2), implies that

$$\begin{aligned} +\infty \leftarrow \int_{\Omega'} \frac{f(x,u_n)u_n}{|u_n|^{2\theta}}|v_n|^{2\theta}\mathrm{d}x &\leqslant \frac{\int_\Omega f(x,u_n)u_n\mathrm{d}x}{\|u_n\|_{X_0}^{2\theta}} \\ &\leqslant \frac{M(\|u_n\|_{X_0}^2)\|u_n\|_{X_0}^2}{\|u_n\|_{X_0}^{2\theta}} \leqslant \theta\mathscr{M}(1). \end{aligned}$$

This contradiction yields the validness of *Step 1.*

Step 2: Let

$$a = \max\{A+1, \sup_{\|u\|_{X_0}\leqslant 1} |\mathcal{J}(u)|+1\}$$

Then for any $u\in S$, there is $\vartheta>0$ such that $\mathcal{J}(\vartheta u) = -a < -A$. According to *Step 1*, we have

$$\frac{d}{d\alpha}\bigg|_{\alpha=\vartheta}\mathcal{J}(\alpha u) = \frac{1}{\vartheta}\frac{d}{dt}\bigg|_{t=1}\mathcal{J}(t\vartheta u) < 0. \tag{4.21}$$

By using the Implicit Function Theorem, we obtain a unique continuous function $T: S\to(0,+\infty)$ given by $T(u)=\vartheta$ such that for any $u\in S$, we have that $\mathcal{J}(T(u)u) = -a$.

The remainder is the same as the one of theorem 4.3, and hence it is omitted here. Thus, we complete the proof of Theorem 4.5. □

5. Proof of Theorems 1.1–1.4

In this section, we give the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.1. From lemmas 3.1 and 3.2, we know that $\mathcal{J}$ satisfies the (PS) condition. By means of theorems 4.2 and 4.5, we have that $C_k(\mathcal{J},\infty) \neq C_k(\mathcal{J},0)$. Hence theorem 1.1 follows from proposition 4.2 and lemma 2.1. □

Proof of Theorem 1.2. From theorem 3.1, we know that $\mathcal{J}$ satisfies the (C) condition. In view of theorems 4.1 and 4.3, we obtain that $C_0(\mathcal{J},\infty) \neq C_0(\mathcal{J},0)$. Hence theorem 1.2 follows from proposition 4.2 and lemma 2.1. □

Proof of Theorem 1.3. From theorem 3.1, we know that $\mathcal{J}$ satisfies the (C) condition. In view of theorems 4.2 and 4.3, we obtain that $C_k(\mathcal{J},\infty) \neq C_k(\mathcal{J},0)$. Hence theorem 1.3 follows from proposition 4.2 and lemma 2.1. □

Proof of Theorem 1.4. In accordance with theorem 3.2, $\mathcal{J}$ satisfies the (C) condition. In terms of theorems 4.2 and 4.4, we deduce that $C_k(\mathcal{J},\infty) \neq C_k(\mathcal{J},0)$. Hence theorem 1.4 follows from proposition 4.2 and lemma 2.1. □

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