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AN INFINITE DIMENSIONAL VERSION OF THE SCHUR CONVEXITY PROPERTY AND APPLICATIONS

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We extend to infinite dimensional separable Hilbert spaces the Schur convexity property of eigenvalues of a symmetric matrix with real entries. Our framework includes both the case of linear, selfadjoint, compact operators, and that of linear selfadjoint operators that can be approximated by operators of finite rank and having a countable family of eigenvalues. The abstract results of the present paper are illustrated by several examples from mechanics or quantum mechanics, including the Sturm–Liouville problem, the Schrödinger equation, and the harmonic oscillator.

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1. Introduction

An important notion in the finite dimensional theory of convex functions is that of the *Schur convexity*. Roughly speaking, Schur–convex functions are real-valued mappings which are monotone with respect to the majorization ordering. A rigorous definition is stated in what follows. Let $\mathbb{R}^n_{\geq}$ denote the cone of vectors with nonincreasing components, that is,

$$\mathbb{R}^n_{\geq} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n;\ x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

The dual cone of the cone $\mathbb{R}^n_{\geq}$ is defined by

$$(\mathbb{R}^n_{\geq})^+ = \left\{y \in \mathbb{R}^n;\ (x, y) \geq 0 \quad \text{for all } x \in \mathbb{R}^n_{\geq}\right\}.$$

A straightforward computation shows that

$$(\mathbb{R}^n_{\geq})^+ = \left\{y \in \mathbb{R}^n;\ \sum_{i=1}^{j} y_i \geq 0 \quad \text{for all } j = 1, \dots, n-1 \text{ and } \sum_{i=1}^{n} y_i = 0\right\}.$$

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We recall (see, e.g., [2, 16]) that a function $f : \mathbb{R}^n \to \mathbb{R}$ is a *Schur convex* if it is $(\mathbb{R}^n_{\geq})^+$-isotone, that is, a

$$x, y \in \mathbb{R}^n_{\geq},\ y - x \in (\mathbb{R}^n_{\geq})^+ \Rightarrow f(x) \leq f(y).$$

The Schur-convex functions were introduced by Schur [19] in 1923 and they have many important applications in analytic inequalities. Hardy, Littlewood and Pólya [8] were also interested in some inequalities that are related to the Schur-convex functions. The notion of the Schur-convexity has shown its importance in many domains. For instance, Merkle proved in [12] that if $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is differentiable, then f' is convex if and only if the mapping

$$F(x, y) = \begin{cases} \dfrac{f(y) - f(x)}{y - x} & \text{if } y \neq x, \\ f'(x) & \text{if } y = x, \end{cases}$$

is a Schur convex. This property is applied in order to obtain some inequalities for the ratio of Gamma functions. We also refer to Hwang and Rothblum [10], who study optimization problems over partitions of a finite set and obtain conditions that allow for simple constructions of partitions that are uniformly optimal for all Schur-convex functions. Stochastic Schur convexity properties have been established by Shaked, Shanthikumar and Tong [20]. Exciting results such as Schur's analytic criteria for Schur convexity, equivalence with Muirhead's inequality, majorization and stochastic matrix conditions in $\mathbb{R}^n$, and Schur's majorization inequality can be found in the excellent book by Steele [21]. Recently, Guan [7] has proved that the complete elementary symmetric function $c_r = c_r(x) = \sum_{i_1+\cdots+i_n=r} x_1^{i_1} \cdots x_n^{i_n}$ and the function $c_r(x)/c_{r-1}(x)$ are Schur-convex functions in $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n);\ x_i > 0\}$, where r is a positive integer and $i_1, \ldots, i_n$ are nonnegative integers.

Zhang [23] proved that every Schur-convex function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is a symmetric function, that is, $f(x) = f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$ for any permutation $\sigma \in \mathcal{P}_n$ and for all $x = (x_1, \ldots, x_n) \in D$. The converse is not true (see, e.g., [16, p. 258]). However, if I is an open interval and $f : I^n \to \mathbb{R}$ is symmetric and of class C^1, then f is Schur-convex if and only if

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \quad \text{on } I^n,$$

for all $i, j \in \{1, \ldots, n\}$ (see [16, p. 259]).

Eigenvalues of real symmetric matrices exhibit remarkable convexity properties. Let $\mathbf{S}^n$ denote the set of all symmetric matrices $X \in \mathcal{M}_{n,n}(\mathbb{R})$. In [2, p. 108], it is stated the following elementary property of eigenvalues of $X \in \mathbf{S}^n$.

The Schur Convexity Property. *Let $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ be the eigenvalues (counted by multiplicity) of an arbitrary matrix $X \in \mathbf{S}^n$. Assume that $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_{\geq}$. Then, the functional $\varphi(X) = \sum_{i=1}^n \mu_i \lambda_i(X)$ is sublinear.*

A direct consequence of this result is that the mapping $\varphi : \mathbf{S}^n \to \mathbb{R}$ is convex.

In the particular case, $\mu_1 = \cdots = \mu_k = 1$, $\mu_{k+1} = \cdots = \mu_n = 0$ $(1 \le k \le n)$, we deduce that the sum of the largest k eigenvalues of a matrix $X \in \mathbf{S}^n$ is a convex function. An alternative proof is based on the observation that, for any fixed $1 \le k \le n$,

$$\lambda_1(X) + \lambda_2(X) + \cdots + \lambda_k(X) = \sup_{A \in \mathcal{A}} \operatorname{tr}(AA^T X), c \tag{1.1}$$

where

$$\mathcal{A} = \left\{ A \in \mathcal{M}_{n,k}(\mathbb{R});\ A^T A = \mathrm{I}_k \right\}.$$

Since $\mathcal{A}$ is a compact set, the supremum in (1.1) is attained in $\mathcal{A}$. We deduce that the mapping $\mathbf{S}^n \ni X \mapsto \lambda_1(X) + \lambda_2(X) + \cdots + \lambda_k(X)$ is convex, as a supremum of linear functions on $\mathbf{S}^n$. The extreme situations $k = 1$ and $k = n$ show that both the largest eigenvalue of X and the trace of X are convex functions on $\mathbf{S}^n$. We also deduce, by taking differences, that $\sum_{j=k+1}^n \lambda_j(X)$ is a concave function, for all $1 \le k \le n-1$. In particular, the mapping $\mathbf{S}^n \ni X \mapsto \lambda_n(X)$ is concave.

A classical result (see, e.g., [2, 17]) asserts that Schur-convex functions are precisely restrictions to $\mathbb{R}^n_{\geq}$ of symmetric convex functions. This result is strictly related to the class of convex functions $f : \mathbf{S}^n \to \mathbb{R}$ (like the functions $\sum_{j=1}^k \lambda_j(X)$) depending only on the eigenvalues of X. In fact, if we write $\operatorname{diag}(\lambda)$ (where $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$) for the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, and define a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ by $\Phi(\lambda) = f(\operatorname{diag}(\lambda))$, then Φ is convex and symmetric: $\Phi(\lambda) = \Phi(\sigma \circ \lambda)$ for all permutation $\sigma \in \mathcal{P}_n$. The converse is also true: if $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a symmetric convex function, then the function $f : \mathbf{S}^n \to \mathbb{R}$ defined by $f(X) = \Phi(\lambda(X))$ (where $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))^T$) is convex and satisfies $f(U^* X U) = f(X)$ whenever $U \in \mathcal{M}_{n,n}(\mathbb{R})$ is a unitary matrix. The above result is due to Davis [5].

The above considerations show that it is natural to impose an adequate "symmetry" assumption in order to obtain a Schur convexity property for linear operators defined on arbitrary Hilbert spaces. That is why we consider throughout this paper linear selfadjoint operators defined on infinite dimensional Hilbert spaces.

2. A Schur Convexity Property in Hilbert Spaces

In the first part of this section, we establish an infinite dimensional version of the Schur convexity property for linear, selfadjoint and compact operators defined on separable Hilbert spaces. Next, we extend this property to the class of linear selfadjoint operators that can be approximated by operators of finite rank. Several examples from mechanics and quantum mechanics illustrate both cases.

2.1. *Schur convexity property for selfadjoint, compact operators*

Let H be a separable Hilbert space and assume that $S : H \to H$ is a linear, selfadjoint and compact operator. Since S is compact then, by the Riesz–Schauder theorem ([14, Theorem VI.15]), the spectrum $\sigma(S)$ of S is a discrete set having no limit points except perhaps the origin. Moreover, any $\lambda \in \sigma(S)\setminus\{0\}$ is an eigenvalue of finite multiplicity. Next, the classical spectral theory of compact selfadjoint operators (see, e.g., [3, Proposition VI.9]) ensures that $\sigma(S) \subset [m, M]$ and $m, M \in \sigma(S)$, where $m = \inf\{(Su, u);\ u \in H,\ \|u\| = 1\}$ and $M = \sup\{(Su, u);\ u \in H,\ \|u\| = 1\}$. In conclusion, the spectrum of S is discrete and it consists of a countable family of eigenvalues $(\lambda_n(S))_{n\geq 1}$ with the additional property that $\lambda_n(S) \to 0$ as $n \to \infty$. At this stage, the Hilbert–Schmidt theorem ([14, Theorem VI.16]) implies that there is a complete orthonormal basis $(e_n)_{n\geq 1}$ of H such that $Se_n = \lambda_n e_n$ for all $n \geq 1$, where $\lambda_n = \lambda_n(S)$. So, $Sx = \sum_{n=1}^{\infty} \lambda_n(x, e_n)e_n$, for all $x \in H$.

We observe that for any fixed positive integer n, the set

$$\left\{\lambda \in \sigma(S);\ |\lambda| \geq \frac{1}{n}\right\}$$

is either empty or finite. Thus, we can rearrange the eigenvalues of S such that

$$\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S) \geq \cdots > 0 > \cdots \geq \lambda_{-n}(S) \geq \cdots \geq \lambda_{-2}(S) \geq \lambda_{-1}(S). \tag{2.1}$$

Moreover, the unique limit point of the sequence $(\lambda_n(S))_{n\in\mathbb{Z}}$ is 0. If S has a finite number of negative eigenvalues (say, n), we denote them by $\lambda_{-1}(S) \leq \cdots \leq \lambda_{-n}(S)$ and we set $\lambda_{-k}(S)$ for all $k \leq n+1$. We make a similar convention if S has finitely many positive eigenvalues. If 0 is an eigenvalue of S, we denote $\lambda_0(S) = 0$.

Denote by $\mathcal{K}_1(H)$ the vector space of linear, selfadjoint and compact operators $S : H \to H$.

We prove the following infinite dimensional variant of Schur's convexity property.

Theorem 2.1. *Let H be a separable Hilbert space and assume that $S : H \to H$ is an arbitrary compact selfadjoint operator. Assume that the eigenvalues of S are arranged as in* (2.1) *and let $(\mu_n)_{n\in\mathbb{Z}}$ be real numbers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots \geq \mu_{-n} \geq \cdots \geq \mu_{-2} \geq \mu_{-1}$ and $\sum_{n=-\infty}^{\infty} \mu_n$ is an absolutely convergent series.*

Then, the functional $\psi : \mathcal{K}_1(H) \to \mathbb{R}$ defined by $\psi(S) = \sum_{n=-\infty}^{\infty} \mu_n\lambda_n(S)$ is convex and lower semicontinuous.

Proof. We first observe that since $S \in \mathcal{K}_1(H)$ is not assumed to be a nuclear operator, then the series $\sum_{n\in\mathbb{Z}} \lambda_n(S)$ is not necessarily convergent. However, our hypothesis that the series $\sum_{n=-\infty}^{\infty} \mu_n$ is absolutely convergent implies that the series $\sum_{n=-\infty}^{\infty} \mu_n\lambda_n(S)$ is absolutely convergent, too, so the mapping ψ is well-defined. Indeed, for all $S \in \mathcal{K}_1(H)$,

$$|\psi(S)| \leq \sum_{n=-\infty}^{\infty} |\mu_n| \cdot |\lambda_n(S)| \leq \max\{-\lambda_{-1}(S), \lambda_1(S)\} \sum_{n=-\infty}^{\infty} |\mu_n| < \infty.$$

Any operator $S \in \mathcal{K}_1(H)$ is the norm limit of a sequence of operators of finite rank. Indeed, if $(e_n)_{n\in\mathbb{Z}}$ is a complete orthonormal basis of H so that $Se_n = \lambda_n(S)e_n$ for all $n \in \mathbb{Z}$, with $\lambda_n(S)$ arranged as in (2.1), then $Sx = \sum_{n=-\infty}^{\infty} \lambda_n(S)(x, e_n)e_n$, for all $x \in H$. Set, for any $m \geq 1$, $S_m x = \sum_{j=-m}^{m} \lambda_j(S)(x, e_j)e_j$, for all $x \in H$. Then $S_m \to S$ in $L(H)$ as $m \to \infty$ and the (nontrivial) eigenvalues of S_m are $\lambda_1(S) \geq \cdots \geq \lambda_m(S) > 0 > \lambda_{-m}(S) \geq \cdots \geq \lambda_{-1}(S)$. So, by the finite dimensional Schur convexity property, the mapping

$$\psi_m : \mathcal{K}_1(H) \to \mathbb{R}, \quad \psi_m(S) = \sum_{j=-m}^{m} \mu_j \lambda_j(S)$$

is sublinear. So, for any S, $T \in \mathcal{K}_1(H)$ and all $\alpha \in \mathbb{R}$,

$$\psi_m(S+T) \leq \psi_m(S) + \psi_m(T) \quad \text{and} \quad \psi_m(\alpha S) = |\alpha|\, \psi_m(S). \tag{2.2}$$

On the other hand,

$$|\psi(S) - \psi_m(S)| = \left| \sum_{|j|\geq m+1} \mu_j \lambda_j(S) \right| \leq \max\{-\lambda_{-m-1}(S), \lambda_{m+1}(S)\} \sum_{|j|\geq m+1} |\mu_j|.$$

Therefore

$$\psi_m(S) \to \psi(S) \quad \text{as } m \to \infty. \tag{2.3}$$

Thus, by (2.2) and (2.3), ψ is a sublinear functional. In particular, ψ is convex.

It remains to argue that ψ is lower semicontinuous, that is, $\psi(S) \leq \liminf_{n\to\infty} \psi(S_n)$ for all $S \in \mathcal{K}_1(H)$, provided $S_n \in \mathcal{K}_1(H)$ and $\|S_n - S\| \to 0$ as $n \to \infty$. The key ingredient is [6, Theorem 4.2], which asserts that $\lambda_j(S) = \lim_{n\to\infty} \lambda_j(S_n)$. Fix an integer $m \geq 1$ and choose arbitrarily $0 < \varepsilon < \max\{-\lambda_{-m}(S), \lambda_m(S)\}$. It follows that there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq N_0$,

$$\begin{aligned}
\psi_m(S) &= \sum_{j=-m}^{m} \mu_j \lambda_j(S) = \sum_{j=-m}^{m} \mu_j^+ \lambda_j(S) - \sum_{j=-m}^{m} \mu_j^- \lambda_j(S) \\
&\leq \sum_{j=-m}^{m} \mu_j^+(\lambda_j(S_n) + \varepsilon) - \sum_{j=-m}^{m} \mu_j^- \,(\lambda_j(S_n) - \varepsilon) \\
&= \sum_{j=-m}^{m} \mu_j \lambda_j(S_n) + \varepsilon \sum_{j=-m}^{m} |\mu_j| = \psi_m(S_n) + \varepsilon \sum_{j=-m}^{m} |\mu_j|.
\end{aligned}$$

Taking $\varepsilon \to 0$, we obtain $\psi_m(S) \leq \psi_m(S_n)$, for all positive integers m and n. So, for all $n \geq 1$,

$$\psi(S) = \lim_{m\to\infty} \psi_m(S_n) \leq \lim_{m\to\infty} \psi_m(S_n) = \psi(S_n).$$

We deduce that $\psi(S) \leq \liminf_{n\to\infty} \psi(S_n)$ and the proof is concluded. □

Examples. (1) *Sturm–Liouville differential operators.* Many eigenvalue problems in quantum mechanics as well as classical physics are described by the Sturm–Liouville problem.

$$\begin{cases} -\dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) + q(x)y = \Lambda y \quad \text{in } (0, L), \\ y(0) = y(L) = 0, \end{cases} \tag{2.4}$$

where $y(x)$ is the quantum mechanical wave function or other physical quantity, while $p \in C^1[0, L]$ ($p > 0$ in $[0, L]$) and $q \in C[0, L]$ are given functions that are determined by the nature of the system of interest. We can assume, without loss of generality, that $q \geq 0$ in $[0, L]$. Indeed, if not, we choose $C \in \mathbb{R}$ sufficiently large such that $q + C \geq 0$ in $[0, L]$ (in such a case, Λ is replaced by $\Lambda + C$ in (2.4)). Fix $f \in L^2(0, L)$. Thus, by the Lax-Milgram lemma, there exists a unique $u \in H^2(0, L) \cap H_0^1(0, L)$ such that

$$\begin{cases} -\dfrac{d}{dx}\left(p(x)\dfrac{dy}{dx}\right) + q(x)y = f \quad \text{in } (0, L), \\ y(0) = y(L) = 0. \end{cases}$$

Let $S : L^2(0, L) \to L^2(0, L)$ be the operator defined by $Sf = u$. Then, by [3, Theorem VIII.20], S is linear, selfadjoint, compact, and nonnegative. Let $\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S) \geq \cdots > 0$ denote the eigenvalues of S. Then $\Lambda_n(S) = 1/\lambda_n(S)$ is an eigenvalue corresponding to the Sturm–Liouville problem (2.4). In the particular case $p \equiv 1$ and $q \equiv 0$, a straightforward computation shows that $\lambda_n(S) = L^2(n^2\pi^2)^{-1}$.

Let μ_n ($n \geq 1$) be real numbers such that $\mu_i \geq \mu_j$ if $i < j$ and such that the series $\sum_{n=1}^{\infty} \mu_n$ converges absolutely. So, by Theorem 2.1, the mapping

$$S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$$

is convex and lower semicontinuous.

(2) *The electron atom model.* On the Hilbert space $H = L^2(\mathbb{R}^3)$, let x, y, z be the components of the momentum of the electron and denote by $r = (x, y, z)$ its position. Consider on H the selfadjoint operator

$$S = \Delta + \frac{\alpha}{|r|}, \quad |r| = \sqrt{x^2 + y^2 + z^2}.$$

Notice that the potential $V(|r|) = \alpha/|r|$ is the energy of the electric field surrounding the electron, α depends on the electron's charge, and $|r|$ is its distance from the atom's nucleus. As established in [15], S has no eigenvalues for any $\alpha < 0$ and, if $\alpha > 0$, then all eigenvalues of S are

$$\lambda_n(S) = \frac{\alpha}{4n^2}, \quad n = 1, 2, \ldots.$$

Let $(\mu_n)_{n\geq 1}$ be a sequence of real numbers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots$ and the series $\sum_{n=1}^{\infty} \mu_n$ converges absolutely. So, by Theorem 2.1, the mapping

$$S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$$

is convex and lower semicontinuous.

(3) *Nonrelativistic model for* 2*-electron atom.* Set $H = L^2(\mathbb{R}^6)$ and define on H the selfadjoint operator

$$S = \Delta_1 + \frac{\alpha}{|r_1|} + \Delta_2 + \frac{\beta}{|r_2|},$$

where $\alpha, \beta > 0$, $r_k = (x_k, y_k, z_k)$, and

$$\Delta_k = \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} + \frac{\partial^2}{\partial z_k^2}, \quad \text{for all } k = 1, 2.$$

Compare with [15], the eigenvalues of S are precisely

$$\lambda_{n,m}(S) = \frac{\alpha}{4n^2} + \frac{\beta}{4m^2}, \quad n, m = 1, 2, \ldots.$$

The countable family of positive numbers $(\lambda_{n,m}(S))_{n,m\geq 1}$ can be rearranged in a sequence $(\gamma_p(S))_{p\geq 1}$ such that $\gamma_i(S) \geq \gamma_j(S)$, provided $i < j$. Let $(\mu_p)_{p\geq 1}$ be a sequence of real numbers such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq \cdots$ and the series $\sum_{p=1}^{\infty} \mu_p$ converges absolutely. Thus, by Theorem 2.1, the mapping

$$S \mapsto \sum_{p=1}^{\infty} \mu_p \gamma_p(S)$$

is convex and lower semicontinuous.

(4) *Schrödinger operators with periodic potential.* The basic equation of quantum mechanics is the Schrödinger equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi. \tag{2.5}$$

Schrödinger [18] studied the stationary equation

$$\lambda\varphi = -\frac{\hbar^2}{2m}\Delta\varphi + V(x)\varphi, \tag{2.6}$$

which follows from (2.5) through $\psi(x,t) = \varphi(x)e^{-i\lambda t/\hbar}$. From (2.6), Schrödinger derived the spectrum of the hydrogen atom. In this case, V is the potential of the electrostatic attracting force of the atomic nucleus, while from the eigenvalues λ of (2.6), one obtains the energy levels of the electron of the hydrogen atom.

Solutions of Schrödinger's equation have to fulfill strict conditions to be useful in describing the electron. Some of the solutions are associated with special values of the electron's energy level, known as eigenvalues. We consider in what follows the class of piecewise continuous potential functions $V : \mathbb{R} \to \mathbb{R}$

which are periodic of period 2π. Let S denote the one dimensional Schrödinger operator associated to V defined on $L^2_{\rm per}(\mathbb{R})$ with 2π-periodic conditions. This operator is defined as follows: for any $f \in L^2_{\rm per}(\mathbb{R})$ periodic of period 2π, let $u \in H^1_{\rm per}(\mathbb{R})$ be the unique solution of the problem

$$\begin{cases} -u'' + V(x)u = f & \text{in } (0, 2\pi) \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases}$$

Then S is defined by $L^2_{\rm per}(\mathbb{R}) \ni f \mapsto u = Sf \in L^2_{\rm per}(\mathbb{R})$. According to [15, Theorem XIII.89], S has a countable family of eigenvalues $\lambda_1(S) > \lambda_2(S) > \cdots > \lambda_n(S) > \cdots$ and $\lambda_n(S) \to 0$ as $n \to \infty$. Assume that μ_n $(n \geq 1)$ are real numbers such that $\mu_i \geq \mu_j$ if $i < j$ and such that the series $\sum_{n=1}^{\infty} \mu_n$ converges absolutely. So, by Theorem 2.1, the mapping

$$S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$$

is convex and lower semicontinuous.

(5) *Indefinite weight elliptic problems on the whole space*. Consider the class of measurable functions $V : \mathbb{R}^N \to \mathbb{R}$ $(N \geq 3)$ such that $V^+ \in L^{N/2}(\mathbb{R}^N)$, where $V = V^+ - V^-$. We observe that this class contains potentials V satisfying $V^+(x) \leq C(1 + |x|^2)^{-\alpha}$ for all $x \in \mathbb{R}^N$, where $\alpha > 1$ and C is a positive constant. For some fixed $\lambda > 0$, let E be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left[|\nabla u|^2 + \max\left(\lambda V^-, \omega\right) u^2\right] dx,$$

where $\omega(x) = K(1 + |x|^2)^{-1}$ with $K > 0$ sufficiently small. Then, by [1, Lemma 0], the operator $S : E \to E^* \hookrightarrow E$ defined by $S\varphi = V^+\varphi$ is compact and selfadjoint. Next, by [1, Theorem 1], there exist infinitely many eigenvalues $\lambda_1(S) > \lambda_2(S) \geq \cdots \geq \lambda_n(S) \geq \cdots \geq 0$ of S with $\lambda_n(S) \to 0$ as $n \to \infty$. So, if μ_n $(n \geq 1)$ are real numbers such that $\mu_i \geq \mu_j$ if $i < j$ and $\sum_{n=1}^{\infty} |\mu_n| < \infty$ then, by Theorem 2.1, the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous.

2.2. *A More general framework*

Consider the class $\mathcal{K}_2(H)$ of linear selfadjoint operators $S : H \to H$ having a countable family of eigenvalues and such that S can be approximated by operators of finite rank. For any operator $S \in \mathcal{K}_2(H)$, passing eventually at a rearrangement, let $\lambda_1(S) \geq \lambda_2(S) \geq \cdots \geq \lambda_n(S) \geq \cdots$ denote the eigenvalues of S.

Fix a family $\mu = (\mu_1, \mu_2, \ldots, \mu_n, \ldots)$ of real numbers such that $\mu_i \geq \mu_j$ if $i < j$. Consider the class $\mathcal{K}_{2,\mu}(H)$ of operators $S \in \mathcal{K}_2(H)$ such that the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges.

Under these hypotheses, we establish the following infinite dimensional version of the Schur convexity property.

Theorem 2.2. *The functional* $\psi : \mathcal{K}_{2,\mu}(H) \to \mathbb{R}$ *defined by* $\psi(S) = \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ *is convex and lower semicontinuous.*

Proof. By the definition of $\mathcal{K}_{2,\mu}(H)$, for any operator belonging to this class there exists a sequence $(S_n)_{n\geq 1}$ of operators of finite rank such that $\|S_n - S\| \to 0$ as $n \to \infty$. So, by [6, Theorem 4.2], we have $\lim_{n\to\infty} \lambda_j(S_n) = \lambda_j(S)$, for all positive integer j. Define, for all $m \geq 1$,

$$\psi_m : \mathcal{K}_{2,\mu}(H) \to \mathbb{R}, \quad \psi_m(S) = \sum_{j=1}^{m} \mu_j \lambda_j(S).$$

Therefore

$$\lim_{n\to\infty} \sum_{j=1}^{m} \mu_j \lambda_j(S_n) = \sum_{j=1}^{m} \mu_j \lambda_j(S) = \psi_m(S). \tag{2.7}$$

On the other hand, since $S \in \mathcal{K}_{2,\mu}(H)$,

$$\lim_{m\to\infty} \psi_m(S) = \sum_{j=1}^{\infty} \mu_j \lambda_j(S) = \psi(S). \tag{2.8}$$

Let $S, T \in \mathcal{K}_{2,\mu}(H)$ and assume that S_n, T_n are operators of finite rank such that $\|S_n - S\| \to 0$ and $\|T_n - T\| \to 0$ as $n \to \infty$. Applying the Schur convexity property, we obtain

$$\psi_m(S_n + T_n) \leq \psi_m(S_n) + \psi_m(T_n), \quad \text{for all } m, n \geq 1.$$

Taking $n \to \infty$ and using (2.7), we find

$$\psi_m(S + T) \leq \psi_m(S) + \psi_m(T), \quad \text{for all } m \geq 1.$$

Next, by (2.8), we deduce that

$$\psi(S + T) \leq \psi(S) + \psi(T), \quad \text{for all } S, T \in \mathcal{K}_{2,\mu}(H).$$

A similar argument shows that ψ is positive homogeneous.

The lower semicontinuity of ψ follows with the same arguments as in the proof of Theorem 2.1. □

Examples. (1) *Schrödinger operators with arbitrary potential.* Let H_0 denote the differential operator d^2/dx^2 on $L^2(0,1)$ with the boundary conditions $u(0) = u(1) = 0$ and assume that $V \in L^\infty(0,1)$ is an arbitrary potential. Let $\lambda_n(S)$ be the nth eigenvalue of the operator $S = H_0 + V$. Then, by, [15, Theorem XIII.82.5],

$$\lambda_n(S) = -n^2\pi^2 + \int_0^1 V(x)\, dx + o(1) \quad \text{as } n \to \infty. \tag{2.9}$$

Fix the real numbers μ_n $(n \geq 1)$ such that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges. Using the asymptotic estimate (2.9), we deduce that, for the last purpose, it is enough to choose μ_n so that $\mu_n = O\left(n^{-p}\right)$, for some $p > 3$. Then, by Theorem 2.2, the mapping

$$S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$$

is convex and lower semicontinuous.

(2) *Wave functions on infinite depth wells.* Fix arbitrarily the positive numbers a and b. Define the following discontinuous potential energy of a particle in the force field

$$V(x) = \begin{cases} -\infty & \text{if } x < -b, \\ 0 & \text{if } -b < x < a, \\ -\infty & \text{if } x > a. \end{cases}$$

Consider the Schrödinger equation,

$$\begin{cases} \dfrac{\hbar^2}{2m}\psi'' + V(x)\psi = \lambda\psi \\ \psi(-b) = \psi(a) = 0, \end{cases}$$

where m is the mass of the particle and $\hbar$ is Dirac's constant (reduced Planck's constant). Compare with [13, p. 102], the definition of V forces $\psi = 0$ outside $(-b, a)$. A straightforward computation shows that the eigenvalues of the associated operator S are given by

$$\lambda_n(S) = -\frac{\hbar^2\pi^2}{2m(a+b)^2}\, n^2.$$

Fix the real numbers μ_n $(n \geq 1)$ such that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges. The above expression of eigenvalues shows that it is enough to choose μ_n so that $\mu_n = O\left(n^{-p}\right)$, for some $p > 3$. Applying Theorem 2.2, we deduce that the mapping

$$S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$$

is convex and lower semicontinuous.

(3) *Linear harmonic oscillator.* Consider the Schrödinger equation on the whole real axis

$$\begin{cases} \dfrac{\hbar^2}{2m}\psi'' + V(x)\psi = \lambda\psi \\ \lim\limits_{|x| \to \infty} \psi(x) = 0 = 0. \end{cases} \tag{2.10}$$

In the particular case where $V(x) = -m\omega^2 x^2/2$, the above problem describes the linear harmonic oscillator. Compare with [13, p. 74], the energy levels of the corresponding linear operator S are given by $\lambda_n(S) = -\hbar\omega(n + 1/2)$. So, letting $(\mu_n)_{geq1}$ so that $\mu_i \geq \mu_j$ if $i < j$ and such that the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$

converges, Theorem 2.2 implies that the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous.

We point out that in the case of Morse potentials $V(x) = V_0(e^{-2x/a} - 2e^{-x/a})$ the number of eigenvalues of the problem (2.10) is finite.

(4) *Periodic standing waves of Schrödinger's equation.* In his Ph.D. thesis defended in 1923, de Broglie showed that an electron, or any other particle, has a wave associated with it. The second equation established by de Broglie establishes that the kinetic energy of a particle is directly proportional to its angular frequency. De Broglie's work resulted in the equation $\lambda = \hbar\omega$, where λ is the kinetic energy of the associated wave and ω is the angular frequency of the particle. With the same notations as in the previous example, we consider the Schrödinger equation with periodic boundary conditions

$$\begin{cases} \dfrac{\hbar^2}{2m}\psi'' + V(x)\psi = \lambda\psi & \text{in } (-b, a), \\ \psi(-b) = \psi(a), \\ \psi'(-b) = \psi'(a). \end{cases}$$

Outside the fundamental segment of length $L = a + b$, the standing wave ψ is prolonged by periodicity such that $\psi(x+L) = \psi(x)$, for all $x \in \mathbb{R}$. In [13, p. 108], it is provided a class of potentials V for which the associated bound state energies to the above problem are given by

$$\lambda_n(S) = -\frac{2\hbar\pi}{L}\, n.$$

Thus, by Theorem 2.2, the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous, provided $(\mu_n)_{n\geq 1}$ are chosen so that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges.

(5) *Generalized model of the helium atom.* Let S be the differential operator on $L^2(\mathbb{R}^{3n})$ given by

$$S = \sum_{i=1}^{3n} \left(-\frac{\Delta_i}{2m_i} - \frac{n}{m_i} \right) + \sum_{i<j} \left(\frac{\nabla_i \cdot \nabla_j}{M} + \frac{1}{|r_i - r_j|} \right),$$

where M and m_i ($1 \leq i \leq n$) are arbitrary positive numbers. Compare with [15], the above operator has been introduced by Zhislin and S can be viewed as the Hamiltonian of a system consisting of a nucleus of mass M and n electrons of masses $m_1, \ldots, m_n$, after the center of the mass motion has been removed. This model generalizes the elementary model of the helium atom which is described by the operator S on $L^2(\mathbb{R}^6)$ given by

$$S = -\Delta_1 - \Delta_2 - \frac{2}{|r_1|} - \frac{2}{|r_2|} + \frac{1}{|r_1 - r_2|}.$$

In both cases (see Kato's Theorem and Theorem XIII.7 in [15, p. 89]) the operator S has a countable family of eigenvalues which can be supposed to be arranged so that $\lambda_i(S) \geq \lambda_j(S)$ if $i < j$ (notice that $\lambda_1(S) < -1$ in the case of the elementary model of the helium atom). Fix the real numbers μ_n $(n \geq 1)$ such that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges. Thus, by Theorem 2.2, the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous.

(6) *Schrödinger operators with unbounded potential.* Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ belonging to the class of operators which are bounded from above and such that $V(x) \to -\infty$ as $|x| \to \infty$. Then, by [15, Theorem XIII.67], the Schrödinger operator $S = -\Delta + V$ has a countable family of eigenvalues such that

$$\lambda_1(S) \geq \cdots \geq \lambda_n(S) \geq \cdots \quad \text{and} \quad \lambda_n(S) \to -\infty \text{ as } n \to \infty.$$

Consider the real numbers μ_n $(n \geq 1)$ such that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges. Applying Theorem 2.2, we deduce that the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous.

(7) *Quasilinear anisotropic Sturm–Liouville problems.* Let $\alpha \geq 0$, $p > 1$, and $0 \leq a < b < \infty$. Assume that $q, s \in L^\infty(a,b)$ and $\operatorname{ess\,inf}_{x\in(a,b)} s(x) > 0$. Consider the quasilinear anisotropic eigenvalue problem

$$\begin{cases} r^{-\alpha}\left(r^\alpha |u'|^{p-2}u'\right)' + q(r)|u|^{p-2}u = \lambda s(r)|u|^{p-2}u \quad \text{in } (a,b), \\ \gamma_1\left(|u|^{p-2}u\right)(a) + \gamma_2\left(r^\alpha|u'|^{p-2}u'\right)(a) = 0, \\ \gamma_3\left(|u|^{p-2}u\right)(b) + \gamma_4\left(r^\alpha|u'|^{p-2}u'\right)(b) = 0, \end{cases} \tag{2.11}$$

where $\gamma_i \in \mathbb{R}$ $(i = 1, \ldots, 4)$ such that $\gamma_1^2 + \gamma_2^2 > 0$ and $\gamma_3^2 + \gamma_4^2 > 0$.

We distinguish two cases: the regular case where $a > 0$ or $a = 0$ and $0 \leq \alpha < p-1$, and the singular case defined by $a = 0$, $\alpha \geq p-1$. In the singular case the boundary condition at the origin is $u'(0) = 0$. In both cases Walter [22] proved that problem (2.11) has a countable number of simple eigenvalues $\lambda_1(S) > \cdots > \lambda_n(S) > \cdots$, $\lim_{n\to\infty} \lambda_n(S) = -\infty$ and the corresponding eigenfunction u_n has $n-1$ simple zeroes in (a,b). Consider the real numbers μ_n $(n \geq 1)$ such that $\mu_i \geq \mu_j$ if $i < j$ and the series $\sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ converges. So, by Theorem 2.2, the mapping $S \mapsto \sum_{n=1}^{\infty} \mu_n \lambda_n(S)$ is convex and lower semicontinuous.

3. Conclusions

In this paper, we have extended the Schur convexity property of the eigenvalues of a symmetric matrix with real entries in the framework of infinite dimensional Hilbert spaces. First, we have considered the case of linear, selfadjoint, and compact operators. Next, we have established a corresponding version of the Schur convexity property for linear selfadjoint operators that can be approximated by operators of finite rank and having a countable family of eigenvalues. Our abstract results have been illustrated by various examples, including Sturm–Liouville problems, Schrödinger operators with variable potential, the electron atom model, the linear harmonic oscillator, the generalized model of the helium atom, and wave functions on infinite depth wells. We have been concerned with linear operators with discrete spectrum and our results do not cover the case of operators with a continuous spectrum.

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