Least energy solutions for Choquard equations involving vanishing potentials and exponential growth

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Abstract. In this paper, we consider the existence of solutions for Choquard equation of the form

$$-\Delta u + V(|x|)u = [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u), \qquad x \in \mathbb{R}^2,$$

where the nonlinear term $f(s)$ has exponential growth, the radial potentials $V,\ Q : \mathbb{R}^+ \to \mathbb{R}$ are unbounded, singular at the origin or decaying to zero. By combining the variational methods, Trudinger-Moser inequality and some new approaches to estimate precisely the minimax level of the energy functional, we prove the existence of a nontrivial solution for the above problem under some weaker assumptions. Our study extends and improves the results of [Albuquerque-Ferreira-Severo, Milan J. Math. 89 (2021)] and [Alves-Shen, J. Differential Equations, 344 (2023)].

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1 Introduction and main results

In the present paper, we consider the existence of solutions for nonlinear Choquard equations of the form

$$-\Delta u + V(|x|)u = [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u), \qquad x \in \mathbb{R}^2, \tag{1.1}$$

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where $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $F(t) := \int_0^t f(s)\mathrm{d}s$, the radial potentials $V, Q : \mathbb{R}^+ \to \mathbb{R}$ are unbounded, singular at the origin or decaying to zero at infinity. The Riesz potential I_α with order $\alpha \in (0, 2)$ is defined for $x \in \mathbb{R}^2 \setminus \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{2-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{2-\alpha}{2})}{2^\alpha \pi \Gamma(\frac{\alpha}{2})}.$$

We recall here that Γ is the classical Gamma function and $*$ denotes the convolution on $\mathbb{R}^2$. Explicitly, we introduce the following assumptions on V and f:

(V1) $V \in \mathcal{C}((0, +\infty), (0, +\infty))$, and there exist $a_0, a > -2$ such that

$$\limsup_{r \to 0^+} \frac{V(r)}{r^{a_0}} < +\infty, \qquad \liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0;$$

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > 0;$$

or

(F1′) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exists $\beta_0 > 0$ such that

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > \beta_0$$

and

$$\lim_{|t| \to \infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{for all } \beta < \beta_0;$$

(F2) $\lim_{t \to 0} \left| \frac{f(t)}{t^{\alpha/2}} \right| = l \in [0, +\infty)$;

(F3) there exists $\mu > 1$ such that

$$f(t)t \geq \mu F(t) > 0, \quad \forall\, t \in \mathbb{R} \setminus \{0\};$$

(F4) there exists $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0 |f(t)|, \qquad \forall\, |t| \geq t_0.$$

To facilitate the subsequent analysis, we introduce the following notations. Let

$$E := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : u \text{ is radial, } |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|) u^2 \mathrm{d}x < +\infty \right\}$$

and define

$$(u, v) := \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + V(|x|) uv)\, \mathrm{d}x, \qquad \forall\, u, v \in E; \qquad \|u\|^2 := \sqrt{(u, u)} \quad \forall\, u \in E.$$

Then E is a Hilbert space with the above inner product $(\cdot,\cdot)$ (see [1, Proposition 2.1]). Furthermore, for $s \in [1,+\infty)$, we define

$$L^s(\mathbb{R}^2, Q) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is mensurable}, \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u|^s \mathrm{d}x < \infty \right\}.$$

It is widely known that the term $|x|^{\alpha-2} * (Q(x)F(u))$ can be regarded as the convolution between the Riesz potential $|x|^{\alpha-2}$ and $Q(x)F(u)$. Thus, problem (1.1) is closely related to the Choquard equation arising from the study of Bose-Einstein condensation and can be exploited to describe the finite-range many-body interactions between particles. For $N \geq 3$, the Choquard equation under the convolution of the Riesz potential is simply of the form

$$-\Delta u + u = \left(|x|^{\alpha-N} * |u|^p\right) u^{p-2}u, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

In the relevant physical case in which $N = 3, \alpha = 1$ and $p = 2$, Equation (1.2) turns into the Choquard-Pekar equation, which was used by Pekar [2] to describe a polaron at rest in the quantum field theory. It was also investigated by Choquard to characterize an electron trapped in its own hole as an approximation to the Hartree-Fock theory for a one component plasma [3]. Subsequently, Lieb [4] and Lions [5] obtained the existence and uniqueness of positive solutions to (1.2) by variational methods. It should be pointed out that Equation (1.2) was also proposed by Moroz-Penrose-Tod in [6] as a model for self-gravitating particles where it can be viewed as the classical Schrödinger-Newton equation, see e.g. [7–9].

Problem (1.2) has attracted a lot of interest in recent years and has been extensively investigated in the literature, such as, the existence and multiplicity of nontrivial solutions in [10–12], the sign-changing solutions in [13, 14] and the semiclassical solutions in [15, 16]. We also refer to reference [17] for a broad survey of the Choquard equations. We emphasize that all the results mentioned above require that the exponent $\frac{N+\alpha}{N} \leq p \leq \frac{N+\alpha}{N-2}$, which is deduced from the Hardy-Littlewood-Sobolev inequality (see Lemma 2.1) and the Sobolev embedding theorem.

Notably, the case $N = 2$ is very special, as the corresponding Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^s(\mathbb{R}^2)$ for all $s \in [2,+\infty)$, but $H^1(\mathbb{R}^2) \not\subseteq L^\infty(\mathbb{R}^2)$. In dimension $N = 2$, in order to address problems with exponential growth, one of the most important tools is the Trudinger-Moser inequality, which can be seen as a substitute of the Sobolev inequality. The first version of the Trundiger-Moser inequality in $\mathbb{R}^2$ was established by Cao in [18], see also [19, 20] and reads as follow:

i) if $\beta > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1\right) \mathrm{d}x < \infty;$$

ii) if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2 \leq M < \infty$, and $\beta < 4\pi$, then there exists a constant $\mathcal{C}(M, \beta)$, which depends only on M and β, such that

$$\int_{\mathbb{R}^2} \left(e^{\beta u^2} - 1\right) \mathrm{d}x \leq \mathcal{C}(M, \beta).$$

Based on Trundiger-Moser inequality, we say that f has subcritical growth on $\mathbb{R}^2$ at $t = \pm\infty$ if (F1) holds, and f has critical growth on $\mathbb{R}^2$ at infinity if (F1$'$) holds. Now let us recall some related works for (1.1) with subcritical or critical exponential growth.

When $Q(x) \equiv 1$, (1.1) reduces to the following Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u), \qquad x \in \mathbb{R}^2. \tag{1.3}$$

Consider the case $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^2$. In [21], Alves-Cassani-Tarsi-Yang studied (1.3) with the critical exponent $\beta_0 = 4\pi$ under (F3) and the following assumptions on f:

- (f_1) $f(t) = 0$ for $t \leq 0$, $0 \leq f(t) \leq Ce^{4\pi t^2}$, $\forall\, t > 0$;
- (f_2) there exists $p > \frac{\alpha}{2}$ and $C_p > 0$ such that $f(t) \sim C_p t^p$, as $t \to 0$;
- (f_3) there exist $t_1 > 0, M_1 > 0$ and $q \in (0, 1]$ such that $0 < t^q F(t) \leq M_1 f(t), \forall\, t \geq t_1$;
- (f_4) $\lim\limits_{t\to\infty} \dfrac{tf(t)F(t)}{e^{8\pi t^2}} \geq k$ with $k > \inf\limits_{\rho>0} \dfrac{\alpha(1+\alpha)(2+\alpha)^2}{16\pi^2\rho^{2+\alpha}} e^{\frac{2+\alpha}{4}V_\rho \rho^2}$, where $V_\rho := \sup\limits_{|x|\leq\rho} V(x)$.

They obtained nontrivial solutions for (1.3) by using the mountain pass lemma and showing that the mountain pass level shall be less than $\frac{2+\alpha}{8}$ provided that the potential V is periodic or satisfies the Rabinowitz type condition introduced in [22]. We point out that the threshold of the mountain pass level can be directly deduced from the condition (f_4). By considering a sequence of measures which have uniformly bounded total variation and using the Radon-Nicodym theorem, they [21] showed that the weak limit of a Palais-Smale sequence for the energy functional associated with (1.3) is a solution, moreover, the weak limit is proved to be nonzero by using the condition (f_4) which constitutes the core of the proof. For the case that V change sign, Qin-Tang [23] developed a direct approach to deal with the equation (1.3) with both critical exponential growth and strongly indefinite features when 0 lies in a gap of the spectrum of the operator $-\Delta + V$. They proved the existence of nontrivial solutions for Equation (1.3) under (F1$'$), (F2) with $l = 0$, (F3), (F4) and the following assumption:

- (f_5) $\liminf\limits_{t\to\infty} \dfrac{f(t)}{e^{\beta_0 t^2}} > \dfrac{\sqrt{\alpha(1+\alpha)}(2+\alpha)}{\sqrt{2\pi A_\alpha}\rho^{1+\alpha/2}} e^{4(2+\alpha)\pi(1+\rho)^2\mathcal{B}_0^2/(2+\rho)}$ where $\rho > 0$ satisfies $2(2+\alpha)\pi\rho^2\mathcal{B}_0^2 < 1$ and $\mathcal{B}_0 > 0$ is an embedding constant.

(f_5) in [23] plays the same role like (f_4) in [21]and enables taking advantage of the Moser functions to pull down the critical value to a particular threshold value $\frac{(2+\alpha)\pi}{2\beta_0}$ and then showing that the weak limit of a Palais-Smale sequence is nonvanishing. They also established the existence of ground states and geometrically distinct solutions for the equation (1.3) when the nonlinearity has subcritical exponential growth.

Compare with equation (1.3), the weight function Q will cause difficulties in estimating the threshold value of equation (1.1), and thus more efforts are required when studying the equation (1.1). Albuquerque-Ferreira-Severo[1] established a new version of the Trudinger-Moser inequality (see [1, Proposition 2.9]), which will plays an important role in their arguments. Based on the weighted Trudinger-Moser inequality, Albuquerque-Ferreira-Severo[1], by a standard argument, proved that (1.1) has a nontrivial weak solution provided that V, Q and f satisfy (V1), (F1′), (f_3) and the following assumptions:

(Q1′) $Q \in \mathcal{C}((0,+\infty),(0,+\infty))$, and there exists $b_0 > -\frac{2+\alpha}{2}$ and $b < \frac{a(2+\alpha)}{4}$ such that

$$\limsup_{r\to 0^+} \frac{Q(r)}{r^{b_0}} < +\infty, \qquad \limsup_{r\to+\infty} \frac{Q(r)}{r^{b}} < +\infty;$$

(Q2′) $\liminf_{r\to 0^+} \frac{Q(r)}{r^{b_0}} > 0;$

(f_6) $f \in \mathcal{C}(\mathbb{R}_+,\mathbb{R})$ and $\lim_{t\to 0^+} \frac{f(t)}{t^{\alpha/2}} = 0;$

(f_7) there exists $\theta > 1$ such that $\theta F(t) \le f(t)t, \quad \forall\, t \ge 0;$

(f_8) $\kappa_* := \liminf_{t\to+\infty} \frac{F(t)}{e^{\beta_0 t^2}} > 0.$

Later Shen-Rădulescu-Yang [24] studied the equation (1.1), where the potential V and the weight Q satisfy the following hypotheses:

(K) $V, Q \in \mathcal{C}(\mathbb{R}^2)$ and there exist some positive constants γ, τ, a, A and b such that

$$\frac{a}{1+|x|^\gamma} \le V(x) \le A \quad \text{and} \quad 0 < Q(x) \le \frac{b}{1+|x|^\tau},$$

where $V(x), Q(x) \sim |x|^{-\tau}$ as $|x| \to +\infty$ and (γ,τ) satisfies one of the following assumptions:

(i) $0 < \gamma < 2$ and $(2+\alpha)\gamma/4 \le \tau < +\infty$, or $0 < \gamma \le 4\tau/(2+\alpha) < 2$;

(ii) $\gamma = 2$ and $(2+\alpha)/2 \le \tau < +\infty$;

(iii) $\gamma > 2$ and $(2+\alpha)/2 \le \tau < +\infty$;

and f satisfies (F1′), (f_3), (f_8) as well as

(f_9) $f\in\mathcal{C}^1(\mathbb{R})$, $f(t)\equiv 0$ for all $t\leq 0$ and $\lim_{t\to 0^+}\frac{f(t)}{t^{\alpha/2}}=0$;

(f_{10}) there exists a constant $\delta\in[0,1)$ such that $\frac{F(t)f'(t)}{f^2(t)}\geq\delta,\ \forall\ t>0$.

They investigated the existence of nontrivial solutions of mountain-pass type for (1.1). Furthermore, they also proved that the nontrivial solution is a bound state, namely a solution belonging to $H^1(\mathbb{R}^2)$, for some particular (γ,τ).

We emphasize that hypothesis (f_8) is usually used to estimate the minimax level of the energy functional associated to the critical exponential growth problems. In [1, 24], hypothesis (f_8) is crucial to overcome the obstacles caused by the critical exponential term. Indeed, by (f_8), mountain-pass level c can be controlled by a fine threshold $\frac{\pi(2+\alpha+2b_0)}{2\beta_0}$ ($b_0=0$ in [24]) under which the compactness can be restored for the critical case (see [1, Proposition 6.2] and [24, Lemma 3.3]).

Inspired by the works mentioned above, two natural questions arise:

(Q1) Can we establish the existence of nontrival solution of (1.1) by using new assumptions on the weight function Q that are different from (Q1′) and (Q2′)?

(Q2) As we can see (f_8) is an essential technical condition for the critical exponential growth problems. When studying (1.1) in critical exponential case, can we weaken (f_8) to more general conditions?

The main purpose of this article is to address the above questions. Based on the above observations and inspired by the work [25], we shall further study the existence of nontrivial solutions for equation (1.1) under subcritical and critical exponential growth. Besides (V1), (F1), (F1′) and (F2)-(F4), we introduce the following assumptions:

(Q1) $Q\in\mathcal{C}((0,+\infty),(0,+\infty))$, and there exists $b_0>-\frac{2+\alpha}{2}$ such that

$$\limsup_{r\to 0^+}\frac{Q(r)}{r^{b_0}}<+\infty,\qquad \limsup_{r\to+\infty}\frac{Q(r)^{\frac{4}{2+\alpha}}}{V(r)}=0;$$

(Q2) there holds

$$\liminf_{r\to 0^+}\frac{\int_0^r sQ(s)\mathrm{d}s}{r^{2+b_0}}=:\zeta_0>0;$$

(F5) $\kappa:=\liminf_{t\to+\infty}\frac{tF(t)}{e^{\beta_0 t^2}}>0.$

Obviously, (F5) is much weaker than (f_8) used in [1, 24]. Our approach is based on delicate estimates for the upper bound for the mountain-pass minimax level c.

Remark 1.1. *It is clearly to see that (Q2′) implies (Q2). Since $b<\frac{a(2+\alpha)}{4}$ in (Q1′), it is also easy to verify that (V1) and (Q1′) imply (Q1). However, there are many functions $V(r)$ and $Q(r)$*

satisfying (V1), (Q1) and (Q2) but not (Q1′). For example, $V(r) = \log(1+r)$ *and* $Q(r) = r^\sigma$ *satisfy (V1), (Q1) and (Q2) when* $\sigma \in \left(-\frac{2+\alpha}{2}, 0\right]$ *but not (Q1′) when* $\sigma = 0$.

Definition 1.2. We say that u is a least energy solution to (1.1) if $u \in E$ such that $\Phi(u) = m := \inf\limits_{\mathcal{M}} \Phi$, where

$$\mathcal{M} := \{u \in E \setminus \{0\} : \Phi'(u) = 0\}. \tag{1.4}$$

Specifically, we are ready to state the main results in the present paper.

Theorem 1.3. *Assume that* V, Q *and* f *satisfy* (V1), (Q1) *and* (F1)-(F3). *Then* (1.1) *has a least energy solution* $u \in E \setminus \{0\}$.

Theorem 1.4. *Assume that* V, Q *and* f *satisfy* (V1), (Q1), (Q2), (F1′), (F2), (F3), (F4) *and* (F5). *Then* (1.1) *has a least energy solution* $u \in E \setminus \{0\}$.

Remark 1.5. *(F5) is a very mild condition involving the behavior of the nonlinearity* f *at infinity. Since (F5) is much weaker than* (f_8) *used in [1, 24], Theorem* 1.4 *seems to be an innovative result to some extent, which improves and extends the existing results in the direction concerning Choquard equations.*

Remark 1.6. *We give an explicit examples of nonlinear term satisfying our assumptions* (F1), (F2) *and* (F3) *as follows:*

$$f_1(t) = \frac{\alpha+2}{2}|t|^{\frac{\alpha-2}{2}} t e^{\beta t^2/2} + \beta |t|^{\frac{\alpha+2}{2}} t e^{\beta t^2/2} \ (\beta > 0) \text{ and } F_1(t) = \int_0^t f_1(s) \mathrm{d}s = |t|^{\frac{\alpha+2}{2}} e^{\beta t^2/2}.$$

The example of nonlinear term satisfying our assumptions (F1′), (F2), (F3), (F4) *and* (F5) *can be given as follows:*

$$f_2(t) = \frac{\alpha+2}{2}|t|^{\frac{\alpha-2}{2}} t e^{\beta_0 t^2} + 2\beta_0 |t|^{\frac{\alpha+2}{2}} t e^{\beta_0 t^2} \ (\beta_0 > 0) \text{ and } F_2(t) = \int_0^t f_2(s) \mathrm{d}s = |t|^{\frac{\alpha+2}{2}} e^{\beta_0 t^2}.$$

When $Q(|x|) = \frac{1}{\sqrt{A_\alpha}|x|^\mu}$ in (1.1), where $0 \le \mu < \frac{\alpha}{2}$, it transforms into the following Schrödinger equation with Stein-Weiss Potential

$$-\Delta u + V(|x|)u = \frac{1}{|x|^\mu} \left(\int_{\mathbb{R}^2} \frac{F(u(y))}{|x-y|^{2-\alpha}|y|^\mu} dy \right) f(u), \text{ in } \mathbb{R}^2. \tag{1.5}$$

There exists $b_0 = -\mu > -\frac{\alpha}{2} > -\frac{2+\alpha}{2}$ such that $(Q1)$ and $(Q2)$ hold.

Corollary 1.7. *Assume that* V *and* f *satisfy* (V1) *and* (F1)-(F3). *Then* (1.5) *has a least energy solution* $u \in E \setminus \{0\}$.

Corollary 1.8. *Assume that V and f satisfy* (V1), (F1′), (F2), (F3), (F4) *and* (F5). *Then* (1.5) *has a least energy solution $u \in E \setminus \{0\}$.*

Remark 1.9. *When $V(x) \equiv$constant, there exists $a_0 = 0$, $a = 0$ such that $(V1)$ holds. Thus Theorem 1.4 and Corollary 1.8 extend and cover the main result in [26].*

The paper is organized as follows. In Section 2, we give the variational setting and some preliminary lemmas. Section 3 is devoted to the subcritical exponential growth case where Theorem 1.3 is proved. In Section 4, we consider the critical exponential growth case, and complete the proof of Theorem 1.4.

Throughout the sequel, we denote the usual Lebesgue space with norm $\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p dx\right)^{\frac{1}{p}}$ by $L^p(\mathbb{R}^2)$, where $1 \le p < \infty$, $B_r := \{x \in \mathbb{R}^2 : |x| < r\}$ for all $r > 0$, and C_i denotes different positive constant in different place.

2 Variational framework and preliminaries

Under assumptions (F1′) and (F2), fix $\beta > \beta_0$, we know that for any $q \ge 0$, there exists $\theta_1 > 0$, $\theta_{2,q} > 0, \theta_{3,q} > 0$ such that

$$|f(t)| \le \theta_1 |t|^{\frac{\alpha}{2}} + \theta_{2,q}(e^{\beta t^2} - 1)|t|^q, \quad \forall\, t \in \mathbb{R}, \tag{2.1}$$

$$|F(t)| \le \theta_1 |t|^{\frac{2+\alpha}{2}} + \theta_{2,q}(e^{\beta t^2} - 1)|t|^{q+1}, \quad \forall\, t \in \mathbb{R} \tag{2.2}$$

and

$$|F(t)| \le \theta_{3,q}(e^{\beta t^2} - 1)|t|^{q+1}, \quad \forall\, |t| \ge 1. \tag{2.3}$$

Similarly, under assumptions (F1) and (F2), we know that (2.1), (2.2) and (2.3) hold for fixed $\beta > 0$.

Lemma 2.1. (Hardy-Littlewood-Sobolev inequality,[27]) Let $s, r > 1$ and $0 < \alpha < 2$ with $\frac{2-\alpha}{2} + \frac{1}{s} + \frac{1}{r} = 2$, $g \in L^s(\mathbb{R}^2)$, $h \in L^r(\mathbb{R}^2)$. There exists a sharp constant $\mathcal{C}(\alpha, s, r)$, independent of g, h, such that

$$\int_{\mathbb{R}^2} (I_\alpha * g) h \mathrm{d}x \le \mathcal{C}(\alpha, s, r)\|g\|_s \|h\|_r. \tag{2.4}$$

In particular,

$$\int_{\mathbb{R}^2} (I_\alpha * g) h \mathrm{d}x \le \mathcal{C}_0 \|g\|_{4/(2+\alpha)} \|h\|_{4/(2+\alpha)}, \tag{2.5}$$

where $\mathcal{C}_0 := \mathcal{C}(\alpha, 4/(2+\alpha), 4/(2+\alpha))$.

Lemma 2.2. (Cauchy-Schwarz type inequality,[28]) For $g, h \in L^1_{\text{loc}}(\mathbb{R}^2)$, there holds

$$\int_{\mathbb{R}^2}(I_\alpha * |g|)|h|\mathrm{d}x \le \left[\int_{\mathbb{R}^2}(I_\alpha * |g|)|g|\mathrm{d}x\int_{\mathbb{R}^2}(I_\alpha * |h|)|h|\mathrm{d}x\right]^{\frac{1}{2}}. \tag{2.6}$$

Lemma 2.3. ([29]) *Assume that* (V1) *holds. Then for any* $r_0 > 0$,

$$|u(x)| \le \mathcal{C}_0\|u\||x|^{-(a+2)/4}, \qquad \forall\ u \in E,\ |x| \ge r_0 > 0. \tag{2.7}$$

In the following lemma, we establish the embeddings $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ for all $s \ge 2$ under conditions (V1) and (Q1) and give a simple proof.

Lemma 2.4. *Assume that* (V1) *and* (Q1) *hold. Then the embeddings* $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ *are continuous and compact for all* $2 \le s < \infty$. *Therefore there exists* $\gamma_s > 0$ *such that*

$$\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u|^s\mathrm{d}x\right)^{\frac{1}{s}} \le \gamma_s\|u\|, \qquad \forall\ u \in E, \quad 2 \le s < \infty. \tag{2.8}$$

Proof. Fixed $s \in [2, +\infty)$. For any $u \in E$, let $\bar{u}(x) = u(x) - u(x/|x|) := u(x) - u_0$. Then $u_0 \in \mathbb{R}$ and $\bar{u} \in H^1_0(B_1)$. By Poincaré inequality, the Hölder inequality and Sobolev imbedding theorem, one has

$$\int_{B_1}|\bar{u}|^\varrho\mathrm{d}x \le C(\varrho)\left(\int_{B_1}|\nabla\bar{u}|^2\mathrm{d}x\right)^{\frac{\varrho}{2}}, \qquad \forall\ \varrho \ge 1. \tag{2.9}$$

Since $b_0 > -\frac{2+\alpha}{2}$, we can choose $p > 1$ such that $p\frac{4b_0}{2+\alpha} > -2$. Let $p' := p/(p-1)$. Then it follows from the Hölder inequality and (2.9) that

$$\begin{aligned}\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}|\bar{u}|^s\mathrm{d}x &\le \left(\int_{B_1}|x|^{\frac{4pb_0}{2+\alpha}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_1}|\bar{u}|^{p's}\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\le C_1\left(\frac{2\pi}{2+\frac{4pb_0}{2+\alpha}}\right)^{\frac{1}{p}}\|\nabla\bar{u}\|_2^s\\ &= C_1\left(\frac{2\pi}{2+\frac{4pb_0}{2+\alpha}}\right)^{\frac{1}{p}}\|\nabla u\|_2^s.\end{aligned} \tag{2.10}$$

Hence from (2.10) and Lemma 2.3, we have

$$\begin{aligned}\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}|u|^s\mathrm{d}x &\le 2^{s-1}\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}(|\bar{u}|^s + |u_0|^s)\,\mathrm{d}x\\ &= 2^{s-1}\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}|\bar{u}|^s\mathrm{d}x + \frac{2^s\pi}{2+\frac{4pb_0}{2+\alpha}}|u_0|^s \le C_2\|u\|^s.\end{aligned} \tag{2.11}$$

By using Lemma 2.3, one has

$$\int_{B_1^c}V(|x|)|u|^s\mathrm{d}x \le \mathcal{C}_0^{s-2}\|u\|^{s-2}\int_{B_1^c}V(|x|)u^2\mathrm{d}x \le C_3\|u\|^s. \tag{2.12}$$

Thus, it follows from (V1), (Q1), (2.11) and (2.12) that

$$\begin{aligned}\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u|^s\mathrm{d}x &= \int_{B_1} Q(|x|)^{\frac{4}{2+\alpha}}|u|^s\mathrm{d}x + \int_{B_1^c} Q(|x|)^{\frac{4}{2+\alpha}}|u|^s\mathrm{d}x \\ &\leq C_4\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}|u|^s\mathrm{d}x + C_5\int_{B_1^c}V(|x|)|u|^s\mathrm{d}x \\ &\leq C_6\|u\|^s, \quad \forall\, u\in E. \end{aligned}\tag{2.13}$$

This shows that (2.8) holds, i.e. the embeddings $E \hookrightarrow L^s(\mathbb{R}^2, Q)$ are continuous for all $2 \leq s < \infty$.

Next, we prove that the above embeddings are also compact. Let $\{u_n\} \subset E$ be such that $\|u_n\| \leq C_7$. Without loss of generality, we may assume $u_n \rightharpoonup 0$. We claim $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for all $2 \leq s < \infty$.

For any $\varepsilon > 0$, it follows from (Q1) and Lemma 2.3 that there exists $R_\varepsilon > 1$ such that

$$\begin{aligned}\int_{B_{R_\varepsilon}^c} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^s\mathrm{d}x &\leq \varepsilon\int_{B_{R_\varepsilon}^c}V(|x|)|u_n|^s\mathrm{d}x \\ &\leq \varepsilon\mathcal{C}_0^{s-2}\|u_n\|^{s-2}\int_{B_{R_\varepsilon}^c}V(|x|)u_n^2\mathrm{d}x \\ &\leq \varepsilon\mathcal{C}_0^{s-2}\|u_n\|^s \leq \varepsilon\mathcal{C}_0^{s-2}C_7^s. \end{aligned}\tag{2.14}$$

Since $u_n \rightharpoonup 0$, then $u_n \to 0$ in $L^\varrho_{\mathrm{loc}}(\mathbb{R}^2)$ for $\varrho \in [1, +\infty)$. It follows that

$$\begin{aligned}\int_{B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^s\mathrm{d}x &\leq C_8\int_{B_{R_\varepsilon}}|x|^{\frac{4b_0}{2+\alpha}}|u_n|^s\mathrm{d}x \\ &\leq C_8\left(\frac{2\pi R_\varepsilon^{2+\frac{4pb_0}{2+\alpha}}}{2+\frac{4pb_0}{2+\alpha}}\right)^{\frac{1}{p}}\left(\int_{B_{R_\varepsilon}}|u_n|^{p's}\mathrm{d}x\right)^{\frac{1}{p'}} \\ &= o(1). \end{aligned}\tag{2.15}$$

From (2.14) and (2.15), we can deduce that $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for all $2 \leq s < \infty$ due to the arbitrariness of $\varepsilon > 0$.

□

Lemma 2.5. ([25, Theorem 1.2]) *Assume that V and Q satisfy* (V1) *and* (Q1). *Then the following conclusions hold.*

i) *If $\beta > 0$ and $u \in E$, then*

$$\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\beta u^2} - 1\right)\mathrm{d}x < \infty.$$

ii) *If $0 < \beta < \frac{4\pi(2+\alpha+2b_0)}{2+\alpha}$, then there exists a constant $C > 0$ such that*

$$\sup_{u\in E, \|u\|\leq 1}\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\beta u^2} - 1\right)\mathrm{d}x \leq C.$$

In view of Lemma 2.5, it is easy to see that the energy functional associated to problem (1.1)

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u|^2+V(|x|)u^2\right]\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u))]Q(|x|)F(u)\mathrm{d}x,\ u\in E \tag{2.16}$$

is well defined under (V1), (Q1), (F1) (or (F1′)) and (F2), and by using standard arguments, we can show that $\Phi\in\mathcal{C}^1(E,\mathbb{R})$ with derivative given by

$$\langle\Phi'(u),\varphi\rangle=\int_{\mathbb{R}^2}[\nabla u\cdot\nabla\varphi+V(|x|)u\varphi]\,\mathrm{d}x-\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u))]Q(|x|)f(u)\varphi\mathrm{d}x \tag{2.17}$$

for all $\varphi\in E$. Now we say that $u\in E$ is a weak solution to problem (1.1) if, for all $\varphi\in C_0^\infty(\mathbb{R}^2)$ it holds that $\langle\Phi'(u),\varphi\rangle=0$.

Finally, let

$$X:=\left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^2):|\nabla u|\in L^2(\mathbb{R}^2)\ \text{ and }\ \int_{\mathbb{R}^2}V(|x|)u^2\mathrm{d}x<+\infty\right\}.$$

Then X is a Hilbert space when endowed with inner product

$$(u,v)_X=\int_{\mathbb{R}^2}[\nabla u\cdot\nabla v+V(|x|)uv]\mathrm{d}x.$$

Next, inspired by [25], we will show that E is a natural constraint to look for critical points of Φ, namely the critical points of the functional restricted to E are true critical points in X.

Lemma 2.6. *Assume that* (V1), (Q1), (F1) (or (F1′)) *hold. If $\bar{u}$ is a critical point of Φ restricted to E, then $\bar{u}$ is a critical point of Φ on X.*

Proof. We only proof the case where (F1′) holds, as the case where (F1) holds is similar. Set

$$T_{\bar{u}}(\varphi):=\int_{\mathbb{R}^2}[\nabla\bar{u}\cdot\nabla\varphi+V(|x|)\bar{u}\varphi]\,\mathrm{d}x-\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi\mathrm{d}x. \tag{2.18}$$

We first prove that $T_{\bar{u}}$ is a bounded linear functional on X. Let $\psi\in\mathcal{C}^\infty([0,+\infty),[0,1])$ be a cut-off function verifying

$$\psi(r)=1,\ \forall\ r\in[0,1];\quad \psi(r)=0,\ \forall\ r\in[2,+\infty);\quad\text{and}\quad |\psi'(r)|\le 2,\quad\forall\ r\in[0,+\infty). \tag{2.19}$$

By (V1), (2.19), Poincaré inequality, the Hölder inequality and Sobolev imbedding theorem, one has

$$\begin{aligned}\int_{B_1}|\varphi(x)|^s\mathrm{d}x &\le \int_{B_2}|\psi(|x|)\varphi(x)|^s\mathrm{d}x\\ &\le C_{13}\left(\int_{B_2}|\nabla(\psi\varphi)|^2\mathrm{d}x\right)^{\frac{s}{2}}\\ &\le C_{14}\left(\int_{B_2}\left(|\varphi\nabla\psi|^2+|\psi\nabla\varphi|^2\right)\mathrm{d}x\right)^{\frac{s}{2}}\end{aligned}$$

$$
\begin{aligned}
&\le C_{15}\left(\int_{B_2\setminus B_1}|\varphi|^2\mathrm{d}x+\int_{B_2}|\nabla\varphi|^2\mathrm{d}x\right)^{\frac{s}{2}}\\
&\le C_{16}\left(\int_{B_2\setminus B_1}V(|x|)|\varphi|^2\mathrm{d}x+\int_{B_2}|\nabla\varphi|^2\mathrm{d}x\right)^{\frac{s}{2}}\\
&\le C_{16}\|\varphi\|_X^s,\qquad \forall\ s\ge 1,\quad \varphi\in X.
\end{aligned}\tag{2.20}
$$

Since $b_0>-\frac{2+\alpha}{2}$, we can choose $p>1$ such that $p\frac{4b_0}{2+\alpha}>-2$. Let $p':=p/(p-1)$. Then it follows from (V1), (Q1) and the Hölder inequality that

$$
\begin{aligned}
\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|\varphi|^2\mathrm{d}x &= \int_{B_1}Q(|x|)^{\frac{4}{2+\alpha}}|\varphi|^2\mathrm{d}x+\int_{B_1^c}Q(|x|)^{\frac{4}{2+\alpha}}|\varphi|^2\mathrm{d}x\\
&\le C_{17}\int_{B_1}|x|^{\frac{4b_0}{2+\alpha}}|\varphi|^2\mathrm{d}x+C_{18}\int_{B_1^c}V(|x|)|\varphi|^2\mathrm{d}x\\
&\le C_{17}\left(\int_{B_1}|x|^{\frac{4pb_0}{2+\alpha}}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{B_1}|\varphi|^{2p'}\mathrm{d}x\right)^{\frac{1}{p'}}+C_{18}\int_{B_1^c}V(|x|)|\varphi|^2\mathrm{d}x\\
&\le C_{19}\|\varphi\|_X^2,\qquad \forall\ \varphi\in X.
\end{aligned}\tag{2.21}
$$

By (F1′), (F2), (2.20), (2.21), Lemma 2.4 and Theorem 2.5, one has

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi\mathrm{d}x\right|\\
\le\ & \mathcal{C}_0\left(\int_{\mathbb{R}^2}|Q(|x|)F(\bar{u})|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{\mathbb{R}^2}|Q(|x|)f(\bar{u})\varphi|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\le\ & C_{20}\left(\int_{\mathbb{R}^2}\left|Q(|x|)\left[\bar{u}^2+\left(e^{2\beta_0\bar{u}^2}-1\right)|\bar{u}|\right]\right|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
&\times\left(\int_{\mathbb{R}^2}\left|Q(|x|)\left[|\bar{u}|+\left(e^{2\beta_0\bar{u}^2}-1\right)\right]|\varphi|\right|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\le\ & C_{21}\left[\left(\int_{\mathbb{R}^2}|Q(|x|)\bar{u}^2|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}+\left(\int_{\mathbb{R}^2}\left|Q(|x|)\left(e^{2\beta_0\bar{u}^2}-1\right)|\bar{u}|\right|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\right]\\
&\times\left[\left(\int_{\mathbb{R}^2}|Q(|x|)\bar{u}\varphi|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}+\left(\int_{\mathbb{R}^2}\left|Q(|x|)\left(e^{2\beta_0\bar{u}^2}-1\right)|\varphi|\right|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\right]\\
\le\ & C_{22}\left[\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|\bar{u}|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}+\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{16\beta_0}{2+\alpha}\bar{u}^2}-1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{8}}\right.\\
&\left.\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|\bar{u}|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{8}}\right]\times\left[\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|\bar{u}|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{8}}\right.\\
&\left.+\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{16\beta_0}{2+\alpha}\bar{u}^2}-1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{8}}\right]\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|\varphi|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{8}}\\
\le\ & C_{23}\|\varphi\|_X,\qquad \forall\ \varphi\in X.
\end{aligned}\tag{2.22}
$$

From (2.18) and (2.22), we obtain

$$
|T_{\bar{u}}(\varphi)| \le |(\bar{u},\varphi)|+\left|\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi\mathrm{d}x\right|
$$

$$\leq \|\bar{u}\|_X\|\varphi\|_X + C_{23}\|\varphi\|_X \leq C_{24}\|\varphi\|_X, \quad \forall\, \varphi \in X. \tag{2.23}$$

The above shows that $T_{\bar{u}}$ is a bounded linear functional on X. The Riesz Representation Theorem in the Hilbert space X guarantees the existence of a unique $\bar{v} \in X$ such that

$$T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X, \quad \forall\, \varphi \in X. \tag{2.24}$$

Let $\mathcal{O}(2)$ the group of orthogonal transformations in $\mathbb{R}^2$. Then, by using a change of variables, we get

$$T_{\bar{u}}(g\varphi) = T_{\bar{u}}(\varphi) \text{ and } \|g\varphi\|_X = \|\varphi\|_X, \quad \forall\, \varphi \in X;\ \forall\, g \in \mathcal{O}(2),$$

which, together with (2.24), yields

$$(\varphi, g\bar{v})_X = T_{\bar{u}}(g^{-1}\varphi) = T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X,\ \forall\, \varphi \in X,\ \forall\, g \in \mathcal{O}(2). \tag{2.25}$$

By uniqueness, one has $g\bar{v} = \bar{v}, \forall\, g \in \mathcal{O}(2)$. This shows that $\bar{v} \in E$. Since $\bar{u}$ is a critical point of Φ restricted to E, it follows that $0 = T_{\bar{u}}(\bar{v}) = \|\bar{v}\|_X^2$. Hence,

$$T_{\bar{u}}(\varphi) = (\varphi, \bar{v})_X = (\varphi, 0)_X = 0,\ \forall\, \varphi \in X,$$

i.e. $\bar{u}$ is a critical point of Φ on X. □

3 The subcritical case

We establish the same conclusion in Lemma 4.1 below. So, to avoid repetition, we omit the proof of Lemma 3.1 here, which can be deduced obviously from Lemma 4.1.

Lemma 3.1. *Assume that* (V1), (Q1), (F1) *and* (F2) *hold. Then there exists a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{3.1}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Lemma 3.2. *Assume that* (Q1), (F1) *and* (F2) *hold. Let $u_n \rightharpoonup \bar{u}$ in E. Then for every $v \in E$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)v\mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})v\mathrm{d}x. \tag{3.2}$$

Proof. Since $u_n \rightharpoonup \bar{u}$ in E, we can choose $C_1 > 0$ such that $\|u_n\| \le C_1$. Set $\beta \in (0, \pi\alpha(2+\alpha+2b_0)/(2+\alpha)C_1^2)$. Then from (2.2), the Hölder inequality and Lemma 2.5-ii), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x \\
\le\ & \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2|u_n|^{\frac{2+\alpha}{2}} + C_2(e^{\beta u_n^2}-1)|u_n|\right)^{\frac{4}{2+\alpha}} \mathrm{d}x \\
\le\ & 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x + 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u_n^2}-1\right)^{\frac{4}{2+\alpha}} |u_n|^{\frac{4}{2+\alpha}} \mathrm{d}x \\
\le\ & 4C_2^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x + 4C_2^{\frac{4}{2+\alpha}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}4\beta u_n^2}-1\right) \mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}} \\
& \times \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x\right)^{\frac{2}{2+\alpha}} \\
\le\ & 4C_2^{\frac{4}{2+\alpha}} \gamma_2^2 \|u_n\|^2 + 4C_2^{\frac{4}{2+\alpha}} C_3 \gamma_2^2 \|u_n\|^{\frac{4}{2+\alpha}} \\
\le\ & 4C_2^{\frac{4}{2+\alpha}} \gamma_2^2 C_1^2 + 4C_2^{\frac{4}{2+\alpha}} C_3 \gamma_2^2 C_1^{\frac{4}{2+\alpha}} =: K_1^{\frac{4}{2+\alpha}}.
\end{aligned}
\tag{3.3}
$$

By Lemma 2.4, we know that $v \in L^s(\mathbb{R}^2, Q)$ for $s \ge 2$. And then for any $\varepsilon > 0$, we can choose $R_\varepsilon > 0$ such that

$$
\left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} v^2 \mathrm{d}x\right)^{\frac{1}{2}} < \varepsilon. \tag{3.4}
$$

From (2.5), (2.1), (3.3), (3.4), the Hölder inequality and Lemma 2.5-ii), one obtains

$$
\begin{aligned}
&\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)v| \mathrm{d}x \\
\le\ & \mathcal{C}_0 \left(\int_{\mathbb{R}^2} |Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & \mathcal{C}_0 K_1 \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2|u_n|^{\frac{\alpha}{2}} + C_2\left(e^{\beta u_n^2}-1\right)\right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}} \mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & C_4 \left[\left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2 \mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}} + \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}4\beta u_n^2}-1\right) \mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}}\right]^{\frac{2+\alpha}{4}} \\
& \times \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
\le\ & C_4 \left(\gamma_2^2 \|u_n\|^{\frac{2\alpha}{2+\alpha}} + C_3\right)^{\frac{2+\alpha}{4}} \left(\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
\le\ & C_5 \varepsilon.
\end{aligned}
\tag{3.5}
$$

Similarly, we can deduce that

$$
\int_{\mathbb{R}^2 \setminus B_{R_\varepsilon}} |[I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})v| \mathrm{d}x \le C_6 \varepsilon. \tag{3.6}
$$

Since $v \in L^2(\mathbb{R}^2, Q)$, it follows that there exists $\delta_\varepsilon > 0$ such that

$$\left(\int_A Q(|x|)^{\frac{4}{2+\alpha}} v^2 \mathrm{d}x\right)^{\frac{1}{2}} < \varepsilon \ \text{ if meas}(A) \le \delta_\varepsilon \tag{3.7}$$

for all measurable set $A \subset B_{R_\varepsilon}$. Hence it follows from $\|u_n\| \le C_1$ that there exists $M_\varepsilon > 0$ such that

$$meas(\{x \in B_{R_\varepsilon} : |u_n(x)| \ge M_\varepsilon\}) \le \delta_\varepsilon, \quad meas(\{x \in B_{R_\varepsilon} : |\bar{u}(x)| \ge M_\varepsilon\}) \le \delta_\varepsilon. \tag{3.8}$$

Let

$$A_n := \{x \in B_{R_\varepsilon} : |u_n(x)| \ge M_\varepsilon\}, \quad A_0 := \{x \in B_{R_\varepsilon} : |\bar{u}(x)| \ge M_\varepsilon\},$$

$$D_0 := \{x \in B_{R_\varepsilon} : |\bar{u}(x)| = M_\varepsilon\}.$$

Then it follows from (2.5), (2.1), (3.3), (3.7), (3.8), the Hölder inequality and Lemma 2.5-ii) that

$$\begin{aligned}
& \int_{A_n \cup D_0} |[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)v|\mathrm{d}x \\
\le\ & \mathcal{C}_0 \left(\int_{\mathbb{R}^2} |Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \left(\int_{A_n \cup D_0} |Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & \mathcal{C}_0 K_1 \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(C_2|u_n|^{\frac{\alpha}{2}} + C_2\left(e^{\beta u_n^2} - 1\right)\right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & C_7 \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^{\frac{2\alpha}{2+\alpha}} |v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right. \\
& \left. + \int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\beta u_n^2} - 1\right)^{\frac{4}{2+\alpha}} |v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & C_7 \left[\left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |u_n|^2\mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}} + \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} \left(e^{\alpha^{-1}(2+\alpha)\beta u_n^2} - 1\right)\mathrm{d}x\right)^{\frac{\alpha}{2+\alpha}}\right]^{\frac{2+\alpha}{4}} \\
& \times \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
\le\ & C_7 \left(\gamma_2^2 \|u_n\|^{\frac{2\alpha}{2+\alpha}} + C_3\right)^{\frac{2+\alpha}{4}} \left(\int_{A_n \cup D_0} Q(|x|)^{\frac{4}{2+\alpha}} |v|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
\le\ & C_8 \varepsilon.
\end{aligned} \tag{3.9}$$

Similarly, we can show that

$$\int_{A_0} |[I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})v|\mathrm{d}x \le C_6 \varepsilon. \tag{3.10}$$

Choose $K_\varepsilon > \max\{1, t_0\}$ such that

$$K_\varepsilon^{-1} \left(\int_{B_{R_\varepsilon}} |Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} < \varepsilon \tag{3.11}$$

and

$$\int_{B_{R_\varepsilon}\setminus A_0}[I_\alpha*(Q(|x|)F(\bar{u})\chi_{|\bar{u}|>K_\varepsilon})]|Q(|x|)f(\bar{u})v|\mathrm{d}x<\varepsilon. \tag{3.12}$$

Then from (2.5), (2.3) and (3.11) the Hölder inequality and Lemma 2.5-ii), one has

$$\begin{aligned}
&\int_{B_{R_\varepsilon}\setminus(A_n\cup D_0)}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\geq K_\varepsilon})]Q(|x|)f(u_n)v\Big|\mathrm{d}x\\
\leq\ & \mathcal{C}_0\left(\int_{|u_n|\geq K_\varepsilon}|Q(|x|)F(u_n)|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{B_{R_\varepsilon}\setminus(A_n\cup D_0)}|Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\leq\ & C_9K_\varepsilon^{-1}\left(\int_{|u_n|\geq K_\varepsilon}\left|Q(|x|)\left(e^{\beta u_n^2}-1\right)|u_n|^2\right|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{B_{R_\varepsilon}}|Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\leq\ & C_{10}K_\varepsilon^{-1}\left(\int_{|u_n|\geq K_\varepsilon}Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\alpha^{-1}4\beta u_n^2}-1\right)\mathrm{d}x\right)^{\frac{\alpha}{4}}\left(\int_{|u_n|\geq K_\varepsilon}Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^4\mathrm{d}x\right)^{\frac{1}{2}}\\
&\times\left(\int_{B_{R_\varepsilon}}|Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\leq\ & C_{11}K_\varepsilon^{-1}\left(\int_{B_{R_\varepsilon}}|Q(|x|)f(u_n)v|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\leq\ & C_{11}\varepsilon.
\end{aligned} \tag{3.13}$$

Let

$$\zeta_n(x):=\frac{1}{A_\alpha}(I_\alpha*(|Q(|x|)F(u_n)|\chi_{|u_n|\leq K_\varepsilon}))(x)=\int_{\mathbb{R}^2}\frac{|Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon}}{|x-y|^{2-\alpha}}\mathrm{d}y \tag{3.14}$$

and

$$\zeta(x):=\frac{1}{A_\alpha}(I_\alpha*(|Q(|x|)F(\bar{u})|\chi_{|\bar{u}|\leq K_\varepsilon}))(x)=\int_{\mathbb{R}^2}\frac{|Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)|\leq K_\varepsilon}}{|x-y|^{2-\alpha}}\mathrm{d}y. \tag{3.15}$$

Then from (2.2), (3.14) and (3.15), one has

$$\begin{aligned}
&|\zeta_n(x)-\zeta(x)|\\
&\leq\int_{\mathbb{R}^2}\frac{\Big||Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon}-|Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)|\leq K_\varepsilon}\Big|}{|x-y|^{2-\alpha}}\mathrm{d}y\\
&\leq\left(\int_{|x-y|\leq R}\Big||Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon}-|Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)|\leq K_\varepsilon}\Big|^{\frac{4-\alpha}{\alpha}}\mathrm{d}y\right)^{\frac{\alpha}{4-\alpha}}\\
&\quad\times\left(\int_{|x-y|\leq R}\frac{1}{|x-y|^{(4-\alpha)/2}}\mathrm{d}y\right)^{\frac{4-2\alpha}{4-\alpha}}\\
&\quad+\left(\int_{|x-y|>R}\Big||Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon}-|Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)|\leq K_\varepsilon}\Big|^{\frac{4}{2+\alpha}}\mathrm{d}y\right)^{\frac{2+\alpha}{4}}
\end{aligned}$$

$$\times \left(\int_{|x-y|>R} \frac{1}{|x-y|^4} \mathrm{d}y\right)^{\frac{2-\alpha}{4}}$$

$$\leq \left(\int_{|x-y|\leq R} \Big||Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon} - |Q(|y|)F(\bar{u}(y))|\chi_{|\bar{u}(y)|\leq K_\varepsilon}\Big|^{\frac{4-\alpha}{\alpha}} \mathrm{d}y\right)^{\frac{\alpha}{4-\alpha}} \left(\frac{4\pi}{\alpha} R^{\frac{\alpha}{2}}\right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ C_\varepsilon \left(\int_{|x-y|>R} Q(|y|)^{\frac{4}{2+\alpha}} |u_n|^2 + \int_{|x-y|>R} Q(|y|)^{\frac{4}{2+\alpha}} |\bar{u}|^2 \mathrm{d}y\right)^{\frac{2+\alpha}{4}} \left(\frac{\pi}{R^2}\right)^{\frac{2-\alpha}{4}}$$

$$\leq \left(\frac{4\pi}{\alpha} R^{\frac{\alpha}{2}}\right)^{\frac{4-2\alpha}{4-\alpha}} o(1) + C_{12} C_\varepsilon \left(\frac{\pi}{R^2}\right)^{\frac{2-\alpha}{4}}, \ \forall\, x \in \mathbb{R}^2, \tag{3.16}$$

which implies that

$$\zeta_n(x) \to \zeta(x), \ \forall\, x \in \mathbb{R}^2. \tag{3.17}$$

From (2.2) and (3.14), we have

$$|\zeta_n(x)| \leq \int_{\mathbb{R}^2} \frac{\Big||Q(|y|)F(u_n(y))|\chi_{|u_n(y)|\leq K_\varepsilon}\Big|}{|x-y|^{2-\alpha}} \mathrm{d}y$$

$$\leq \left(\int_{|x-y|\leq 1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4-\alpha}{\alpha}} \mathrm{d}y\right)^{\frac{\alpha}{4-\alpha}} \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{(4-\alpha)/2}} \mathrm{d}y\right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ \left(\int_{|x-y|>1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4}{2+\alpha}} \mathrm{d}y\right)^{\frac{2+\alpha}{4}} \left(\int_{|x-y|>1} \frac{1}{|x-y|^4} \mathrm{d}y\right)^{\frac{2-\alpha}{4}}$$

$$= \left(\int_{|x-y|\leq 1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4-\alpha}{\alpha}} \mathrm{d}y\right)^{\frac{\alpha}{4-\alpha}} \left(\frac{4\pi}{\alpha}\right)^{\frac{4-2\alpha}{4-\alpha}}$$

$$+ \left(\int_{|x-y|>1} |Q(|y|)F(u_n(y))\chi_{|u_n(y)|\leq K_\varepsilon}|^{\frac{4}{2+\alpha}} \mathrm{d}y\right)^{\frac{2+\alpha}{4}} \pi^{\frac{2-\alpha}{4}}$$

$$\leq \max_{B(x,1)} |Q(|y|)| \max_{|t|\leq K_\varepsilon} |F(t)| \pi^{\frac{\alpha}{4-\alpha}} \left(\frac{4\pi}{\alpha}\right)^{\frac{4-2\alpha}{4-\alpha}} + C_\varepsilon \left(\int_{|x-y|>1} Q(|y|)^{\frac{4}{2+\alpha}} |u_n|^2\right)^{\frac{2+\alpha}{4}} \pi^{\frac{2-\alpha}{4}}$$

$$\leq C_{13} \max_{B(x,1)} |Q(|y|)| + C_{14} C_\varepsilon, \ \forall\, x \in \mathbb{R}^2. \tag{3.18}$$

It follows that

$$\left|\zeta_n(x) Q(|x|) f(u_n(x)) \chi_{|u_n(x)|\leq M_\varepsilon} v(x)\right|$$
$$\leq \left[\left(C_{13} \max_{y\in B_{R_\varepsilon+1}} |Q(|y|)| + C_{14}\right) \max_{x\in B_{R_\varepsilon}} |Q(|x|)| \max_{|t|\leq M_\varepsilon} |f(t)|\right] |v(x)| \in L^1(B_{R_\varepsilon}), \ \forall\, x \in B_{R_\varepsilon}. \tag{3.19}$$

Since $u_n \rightharpoonup \bar{u}$ in E, we can deduce that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^2$. By (3.17), we can deduce that

$$\zeta_n(x) Q(|x|) f(u_n(x)) \chi_{|u_n(x)|\leq M_\varepsilon} v(x) \to \zeta(x) Q(|x|) f(\bar{u}(x)) \chi_{|\bar{u}(x)|\leq M_\varepsilon} v(x) \quad a.e.\ x \in B_{R_\varepsilon} \setminus D_0.$$

Therefore, (3.17), together with (3.19) and Lebesgue dominated convergence theorem lead to

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon} \setminus (A_n \cup D_0)} [I_\alpha * (Q(|x|)F(u_n)\chi_{|u_n|\leq K_\varepsilon})] Q(|x|) f(u_n) v \mathrm{d}x$$

$$= \int_{B_{R_\varepsilon}\setminus A_0} [I_\alpha * (Q(|x|)F(\bar{u})\chi_{|\bar{u}|\le K_\varepsilon})]Q(|x|)f(\bar{u})v\mathrm{d}x. \tag{3.20}$$

It follows from (3.5), (3.6), (3.9), (3.10), (3.12), (3.13) and (3.20) that (3.2) holds due to the arbitrariness of $\varepsilon > 0$.

□

By a standard argument, we can get the following lemma.

Lemma 3.3. *Assume that* (V1), (Q1) *and* (F1)-(F3) *hold. Then any sequence* $\{u_n\}$ *satisfying* (3.1) *is bounded in* E.

Lemma 3.4. *Assume that* (V1), (Q1) *and* (F1)-(F3) *hold. Then* Φ *satisfies* $(C)_c$ *condition.*

Proof. Applying Lemmas 3.1 and 3.3, we deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (3.1) and $\|u_n\| \le C_1$ for some constant $C_1 > 0$. Since $\|u_n\| \le C_1$, by Lemmas 2.4, we may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2, Q)$ for $s \in [2, \infty)$, and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. By (2.16) and (4.1), we can deduce that

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n) \le C_{15}. \tag{3.21}$$

Hence it follows from (3.21), (3.21), (2.3), Lemma 2.5-ii), $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for $s \ge 2$ and Hölder inequality that

$$\begin{aligned}
& \int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u})\Big|\mathrm{d}x \\
\le\ & \left(\int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n)\Big|\mathrm{d}x\right)^{\frac{1}{2}} \\
& \times \left(\int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)f(u_n)(u_n - \bar{u}))]Q(|x|)f(u_n)(u_n - \bar{u})\Big|\mathrm{d}x\right)^{\frac{1}{2}} \\
\le\ & \sqrt{C_{15}} \left(\int_{\mathbb{R}^2} \Big|Q(|x|)f(u_n)(u_n - \bar{u})\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\
\le\ & C_{16} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^2\mathrm{d}x\right)^{\frac{4\alpha}{(2+\alpha)^2}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n - \bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}} \\
& + C_{16} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{8\beta}{2+\alpha}u_n^2} - 1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{8}} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n - \bar{u}|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{8}} \\
\le\ & C_{17} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n - \bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}} + C_{17} \left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n - \bar{u}|^{\frac{8}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{8}} \\
=\ & o(1).
\end{aligned} \tag{3.22}$$

Which yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u})\mathrm{d}x = 0. \tag{3.23}$$

Similarly, we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n-\bar{u})\mathrm{d}x=0. \tag{3.24}$$

By (3.1), it is obvious that

$$\langle\Phi'(u_n)-\Phi'(\bar{u}),u_n-\bar{u}\rangle=o(1). \tag{3.25}$$

From (3.23), (3.24) and (3.25), we can get

$$\begin{aligned}\|u_n-\bar{u}\|^2 &= \langle\Phi'(u_n)-\Phi'(\bar{u}),u_n-\bar{u}\rangle-\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n-\bar{u})\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n-\bar{u})\mathrm{d}x\\ &= o(1).\end{aligned}$$

□

The following lemma allows us to find a least energy solution for (1.1) with positive level.

Lemma 3.5. *Assume that* (V1), (Q1), (F1), (F2) *and* (F3) *hold. Then*

i) $\sigma_0:=\inf\{\|u\|:u\in\mathcal{M}\}>0$;

ii) $m:=\inf\{\Phi(u):u\in\mathcal{M}\}>0$.

Proof. (i) Choose $\beta\in(0,\pi(2+\alpha+b_0)/2)$. For any $u\in\mathcal{M}$, if $\|u\|<1$, by (2.1) and (2.2), we have

$$\begin{aligned}\|u\|^2 &= \int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u))]Q(|x|)f(u)u\mathrm{d}x\\ &\le \left(\int_{\mathbb{R}^2}|Q(|x|)F(u)|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{\mathbb{R}^2}|Q(|x|)f(u)u|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\ &\le C_{18}\left[\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}\left(|u|^2+\left(e^{\beta u^2}-1\right)^{\frac{4}{2+\alpha}}|u|^2\right)\mathrm{d}x\right]^{\frac{2+\alpha}{2}}\\ &\le C_{19}\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|u|^2\mathrm{d}x\right)^{\frac{2+\alpha}{2}}+C_{19}\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{(2+\alpha)^{-1}8\beta u^2}-1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\ &\quad\times\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|u|^4\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\ &\le C_{20}\|u\|^{2+\alpha}+C_{21}\|u\|^{2+\alpha}=C_{22}\|u\|^{2+\alpha}.\end{aligned} \tag{3.26}$$

Thus there exists $C_{23}>0$ such that $\|u\|\ge C_{23},\ \forall\ u\in\mathcal{M},\ \|u\|<1$. Taking $\sigma_0=\min\{C_{23},\ 1\}$, then $\|u\|\ge\sigma_0,\ \forall\ u\in\mathcal{M}$.

(ii) For any $u\in\mathcal{M}$, by (F3) and item i), it is easy to see that

$$\Phi(u) = \Phi(u)-\frac{1}{2\mu}\langle\Phi'(u),u\rangle\ge\frac{\mu-1}{2\mu}\|u\|^2\ge\frac{\mu-1}{2\mu}\sigma_0^2.$$

Thus we can deduce that item ii) holds. □

Proof of Theorem 1.3. First, we prove that $\mathcal{M} \neq \emptyset$. Applying Lemmas 3.1 and 3.2, we deduce that there exists a sequence $\{v_n\} \subset E$ satisfying (3.1) and $\|v_n\| \leq C_{24}$ for some constant $C_{24} > 0$. Hence, there exists $v \in E$ such that, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$ and $v_n \to v$ a.e. in $\mathbb{R}^2$. By Lemma 3.4, we know that $v_n \to v$ in E. Thus

$$\Phi(v) = \lim_{n\to\infty} \Phi(v_n) = c > 0, \quad \Phi'(v) = \lim_{n\to\infty} \Phi'(v_n) = 0.$$

Which imply that $v \neq 0$ and $v \in \mathcal{M}$.

Next, we prove that $m := \inf_{\mathcal{M}} \Phi(u)$ is achieved. By Lemma 3.5, we have $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. It is easy to check that

$$\begin{aligned}
m + o(1) =& \Phi(u_n) = \Phi(u_n) - \frac{1}{2\mu}\langle \Phi'(u_n), u_n\rangle \\
=& \frac{\mu-1}{2\mu}\|u_n\|^2 + \frac{1}{2\mu}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)\,[f(u_n)u_n - \mu F(u_n)]\,\mathrm{d}x \\
\geq& \frac{\mu-1}{2\mu}\|u_n\|^2.
\end{aligned}$$

Which imply that $\{u_n\}$ is bounded in E. Hence, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. As in the proof of Lemma 3.4, we can deduce that $u_n \to u$ in E. Thus

$$\Phi(u) = \lim_{n\to\infty} \Phi(u_n) = m > 0, \quad \Phi'(u) = \lim_{n\to\infty} \Phi'(u_n) = 0.$$

□

4 The critical case

Lemma 4.1. *Assume that* (V1), (Q1), (F1′) *and* (F2) *hold. Then there exists a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \qquad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{4.1}$$

where c is given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Proof. Let γ_s be defined by (2.8). By (F1′) and (F2), there exists $C_1 > 0, C_2 > 0$ such that

$$|F(t)|^{\frac{4}{2+\alpha}} \leq C_1 t^2 + C_2|t|^2\left(e^{\frac{8\beta_0}{2+\alpha}t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{4.2}$$

In view of Lemma 2.5-ii), we have

$$\begin{aligned}\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{16\beta_0}{2+\alpha}u^2}-1\right)\mathrm{d}x &= \int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{16\beta_0}{2+\alpha}\|u\|^2(u/\|u\|)^2}-1\right)\mathrm{d}x \\ &\leq C_3, \quad \forall\ \|u\|\leq\sqrt{\pi(2+\alpha+2b_0)/5\beta_0}. \end{aligned} \tag{4.3}$$

From (2.8), (4.2) and (4.3), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u))]Q(|x|)F(u)\mathrm{d}x \\ \leq\ & \left(\int_{\mathbb{R}^2}|Q(|x|)F(u)|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{2}} \\ \leq\ & \left[C_1\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}u^2\mathrm{d}x + C_2\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|u|^2\left(e^{\frac{8\beta_0}{2+\alpha}u^2}-1\right)\mathrm{d}x\right]^{\frac{2+\alpha}{2}} \\ \leq\ & C_4\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}u^2\mathrm{d}x\right)^{\frac{2+\alpha}{2}} \\ & +C_5\left(\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}|u|^4\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left[\int_{\mathbb{R}^2}Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{16\beta_0}{2+\alpha}u^2}-1\right)\mathrm{d}x\right]^{\frac{2+\alpha}{4}} \\ \leq\ & C_6\|u\|^{2+\alpha}, \quad \forall\ \|u\|\leq\sqrt{\pi(2+\alpha+2b_0)/5\beta_0}. \end{aligned} \tag{4.4}$$

Hence, it follows from (2.16) and (4.4) that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2-\frac{1}{2}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u))]Q(|x|)F(u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2-C_6\|u\|^{2+\alpha}, \quad \forall\ \|u\|\leq\sqrt{\pi(2+\alpha+2b_0)/5\beta_0}. \end{aligned} \tag{4.5}$$

Therefore, there exist $\kappa_0>0$ and $0<\rho_0<\sqrt{\pi(2+\alpha+2b_0)/5\beta_0}$ such that

$$\Phi(u)\geq\kappa_0, \quad \forall\ u\in S:=\{u\in E:\|u\|=\rho_0\}. \tag{4.6}$$

Now we choose $w_0\in E\setminus\{0\}$, it is easy to show that $\lim_{t\to\infty}\Phi(tw_0)=-\infty$ due to (F3). Hence, we can choose $T>0$ such that $e:=Tw_0\in\{u\in E:\|u\|>\rho_0\}$ and $\Phi(e)<0$, then in view of the mountain-pass lemma, we deduce that there exists a sequence $\{u_n\}\subset E$ satisfying (4.1). □

Lemma 4.2. *Assume that* (V1), (Q1), (F1′), (F2) *and* (F3) *hold. Then any sequence* $\{u_n\}$ *satisfying* (4.1) *is bounded in* E.

Lemma 4.3. *Assume that* (Q1), (F1′), (F2), (F3), (F4) *hold. Let* $u_n\rightharpoonup\bar{u}$ *in* E *and*

$$\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n\leq K_0 \tag{4.7}$$

for some constant $K_0>0$. *Then there hold:*

i) *for every* $\varphi\in E\cap\mathcal{C}_0^\infty(\mathbb{R}^2)$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi\mathrm{d}x=\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi\mathrm{d}x; \tag{4.8}$$

ii)

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)F(u_n)\mathrm{d}x=\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x. \tag{4.9}$$

Proof. The proof of item ii) is similar to [1, Claim 6.3], so we omit the details here. Next, inspired by [23, Lemma 4.8], we prove i). By the Fatou's Lemma and (4.7), we have

$$\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\bar{u}\le K_0. \tag{4.10}$$

Let $\Omega=\mathrm{supp}\varphi$. For any given $\varepsilon>0$, let $M_\varepsilon:=K_0\|\varphi\|_\infty\varepsilon^{-1}$. Then it follows from (F4), (4.7) and (4.10) that for n large,

$$\begin{aligned}&\int_{|u_n|\ge M_\varepsilon}|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi|\mathrm{d}x\\ \le\ &\frac{\|\varphi\|_\infty}{M_\varepsilon}\int_{|u_n|\ge M_\varepsilon}|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n|\mathrm{d}x\\ \le\ &\frac{\|\varphi\|_\infty}{M_\varepsilon}K_0=\varepsilon\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&\int_{|\bar{u}|\ge M_\varepsilon}|[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi|\mathrm{d}x\\ \le\ &\frac{\|\varphi\|_\infty}{M_\varepsilon}\int_{|\bar{u}|\ge M_\varepsilon}|[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\bar{u}|\mathrm{d}x\\ \le\ &\frac{\|\varphi\|_\infty}{M_\varepsilon}K_0=\varepsilon.\end{aligned} \tag{4.12}$$

Let $D_\varepsilon=\{x\in\Omega:|\bar{u}(x)|=M_\varepsilon\}$. Since $|Q(|x|)f(u_n)\chi_{|u_n|\le M_\varepsilon}|\to|Q(|x|)f(\bar{u})\chi_{|\bar{u}|\le M_\varepsilon}|$ a.e. in $\Omega\setminus D_\varepsilon$ and

$$|Q(|x|)f(u_n)\chi_{|u_n|\le M_\varepsilon}|\le\max_{x\in\Omega}Q(|x|)\max_{|t|\le M_\varepsilon}|f(t)|<\infty,\quad\forall\,x\in\Omega,$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty}\int_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\le M_\varepsilon\}}|Q(|x|)f(u_n)|^{\frac{4}{2+\alpha}}\mathrm{d}x=\int_{(\Omega\setminus D_\varepsilon)\cap\{|\bar{u}|\le M_\varepsilon\}}|Q(|x|)f(\bar{u})|^{\frac{4}{2+\alpha}}\mathrm{d}x. \tag{4.13}$$

Choose $K_\varepsilon>t_0$ such that

$$\|\varphi\|_\infty\mathcal{C}_0\left(\frac{M_0K_0}{K_\varepsilon}\right)^{\frac{1}{2}}\left(\int_{(\Omega\setminus D_\varepsilon)\cap\{|\bar{u}|\le M_\varepsilon\}}|Q(|x|)f(\bar{u})|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}<\varepsilon \tag{4.14}$$

and

$$\int_{|\bar{u}|<M_\varepsilon}[I_\alpha*(Q(|x|)F(\bar{u})\chi_{|\bar{u}|>K_\varepsilon})]|Q(|x|)f(\bar{u})\varphi|\mathrm{d}x<\varepsilon. \tag{4.15}$$

Then from (F4), (4.13) and (4.14), one has

$$\begin{aligned}
&\int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|\ne M_\varepsilon)}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\ge K_\varepsilon})]Q(|x|)f(u_n)\varphi\Big|\mathrm{d}x\\
\le\ &\|\varphi\|_\infty\int_{\Omega\setminus D_\varepsilon}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\ge K_\varepsilon})]Q(|x|)f(u_n)\chi_{|u_n|\le M_\varepsilon}\Big|\mathrm{d}x\\
\le\ &\|\varphi\|_\infty\left(\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\ge K_\varepsilon})]Q(|x|)F(u_n)\chi_{|u_n|\ge K_\varepsilon}\right)^{\frac{1}{2}}\\
&\times\left(\int_{\mathbb{R}^2}[I_\alpha*(|Q(|x|)f(u_n)|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\le M_\varepsilon\}})]|Q(|x|)f(u_n)|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\le M_\varepsilon\}}\mathrm{d}x\right)^{\frac{1}{2}}\\
\le\ &\|\varphi\|_\infty\mathcal{C}_0\left(\int_{|u_n|\ge K_\varepsilon}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)F(u_n)\right)^{\frac{1}{2}}\\
&\times\left(\int_{\mathbb{R}^2}\Big|Q(|x|)f(u_n)\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\le M_\varepsilon\}}\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\le\ &\|\varphi\|_\infty\mathcal{C}_0\left(\frac{M_0}{K_\varepsilon}\int_{|u_n|\ge K_\varepsilon}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n\right)^{\frac{1}{2}}\\
&\times\left(\int_{(\Omega\setminus D_\varepsilon)\cap\{|\bar{u}|\le M_\varepsilon\}}|Q(|x|)f(u_n)|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\le\ &\|\varphi\|_\infty\mathcal{C}_0\left(\frac{M_0K_0}{K_\varepsilon}\right)^{\frac{1}{2}}\left(\int_{(\Omega\setminus D_\varepsilon)\cap\{|\bar{u}|\le M_\varepsilon\}}|Q(|x|)f(\bar{u})|^{\frac{4}{2+\alpha}}\mathrm{d}x+o(1)\right)^{\frac{2+\alpha}{4}}\\
<\ &\varepsilon+o(1).
\end{aligned}\tag{4.16}$$

For any $x\in\mathbb{R}^2$, we define $\zeta_n(x)$ and $\zeta(x)$ as in (3.14) and (3.15), then from (F1′), (F2), (3.16) and (3.18), we also have

$$\zeta_n(x)\to\zeta(x),\quad\forall\ x\in\mathbb{R}^2\tag{4.17}$$

and

$$|\zeta_n(x)|\le C_7\max_{B(x,1)}|Q(|y|)|+C_8,\ \forall\ x\in\mathbb{R}^2.\tag{4.18}$$

It follows that

$$\begin{aligned}
&\big|\zeta_n(x)Q(|x|)f(u_n(x))\chi_{|u_n|\le M_\varepsilon}(x)\varphi(x)\big|\\
\le&\left(C_7\max_{B(x,1)}|Q(|y|)|+C_8\right)\|\varphi\|_\infty\max_{x\in\Omega}|Q(|x|)|\max_{|t|\le M_\varepsilon}|f(t)|<\infty,\ \forall\ x\in\Omega.
\end{aligned}\tag{4.19}$$

Since $u_n\rightharpoonup\bar{u}$ in E, we can deduce that $u_n\to\bar{u}$ a.e. $x\in\mathbb{R}^2$. By (4.17), we can deduce that

$$\zeta_n(x)Q(|x|)f(u_n(x))\chi_{|u_n|\le M_\varepsilon}(x)\varphi(x)\to\zeta(x)Q(|x|)f(\bar{u}(x))\chi_{|\bar{u}|\le M_\varepsilon}(x)\varphi(x)\ \ a.e.\ x\in\Omega\setminus D_\varepsilon,$$

Therefore, (4.19) and Lebesgue dominated convergence theorem lead to

$$\lim_{n\to\infty}\int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|\ne M_\varepsilon)}[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\le K_\varepsilon})]Q(|x|)f(u_n)\varphi\mathrm{d}x$$

$$= \int_{|\bar{u}|<M_\varepsilon}[I_\alpha*(Q(|x|)F(\bar{u})\chi_{|\bar{u}|\le K_\varepsilon})]Q(|x|)f(\bar{u})\varphi\mathrm{d}x. \tag{4.20}$$

Since $u_n \rightharpoonup \bar{u}$ in E, we can deduce that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^2$ and then, by $\mathrm{E}_{\Gamma\mathrm{opo}\mathrm{B}}$ theorem, we have $u_n \rightrightarrows \bar{u},\ x \in \Omega\setminus A$, where $m(A)<\varepsilon$. For large n, one has

$$\{x\in\Omega\setminus A : |\bar{u}(x)| = M_\varepsilon\} \subset \left\{x\in\Omega\setminus A : |u_n(x)| > \frac{M_\varepsilon}{2}\right\}.$$

And then from Lemmas 2.1 and 2.2,

$$\begin{aligned}
& \int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)}\Big|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi\Big|\mathrm{d}x\\
= & \int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon))\cap(\Omega\setminus A)}\Big|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi\Big|\mathrm{d}x\\
& +\int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)\cap A}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|\le K_\varepsilon})]Q(|x|)f(u_n)\varphi\Big|\mathrm{d}x\\
& +\int_{(|u_n|\le M_\varepsilon)\cap(|\bar{u}|=M_\varepsilon)\cap A}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|> K_\varepsilon})]Q(|x|)f(u_n)\varphi\Big|\mathrm{d}x\\
\le & \|\varphi\|_\infty\int_{(|u_n|>M_\varepsilon/2)\cap(\Omega\setminus A)}|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)|\mathrm{d}x\\
& +\|\varphi\|_\infty\mathcal{C}_0\left(\int_{\mathbb{R}^2}\Big|Q(|x|)F(u_n)\chi_{|u_n|\le K_\varepsilon}\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{\mathbb{R}^2}\Big|Q(|x|)f(u_n)\chi_{(|u_n|\le M_\varepsilon)\cap A}\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
& +\|\varphi\|_\infty\left(\int_{\mathbb{R}^2}\Big|[I_\alpha*(Q(|x|)F(u_n)\chi_{|u_n|>K_\varepsilon})]Q(|x|)F(u_n)\chi_{|u_n|>K_\varepsilon}\Big|\mathrm{d}x\right)^{\frac{1}{2}}\\
& \times\left(\int_{\mathbb{R}^2}\Big|[I_\alpha*(Q(|x|)f(u_n)\chi_{(|u_n|\le M_\varepsilon)\cap A})]Q(|x|)f(u_n)\chi_{(|u_n|\le M_\varepsilon)\cap A}\Big|\mathrm{d}x\right)^{\frac{1}{2}}\\
\le & \frac{2\|\varphi\|_\infty}{M_\varepsilon}\int_{|u_n|>M_\varepsilon/2}|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n|\mathrm{d}x\\
& +C_\varepsilon\|\varphi\|_\infty\mathcal{C}_0\left(\int_{|u_n|\le K_\varepsilon}Q(|x|)^{\frac{4}{2+\alpha}}u_n^2\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\left(\int_{(|u_n|\le M_\varepsilon)\cap A}\Big|Q(|x|)f(u_n)\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
& +C_\varepsilon\sqrt{\frac{M_0}{K_\varepsilon}}\sqrt{\mathcal{C}_0}\left(\int_{|u_n|>K_\varepsilon}|[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n|\mathrm{d}x\right)^{\frac{1}{2}}\\
& \times\left(\int_{(|u_n|\le M_\varepsilon)\cap A}\Big|Q(|x|)f(u_n)\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}}\\
\le & C_9\varepsilon.
\end{aligned} \tag{4.21}$$

It follows from (4.11), (4.12), (4.20), (4.15), (4.16) and (4.21) that (4.8) holds due to the arbitrariness of $\varepsilon>0$. □

As in [30], we define Moser type functions $w_n(x)$ supported in $B_d := B_d(0)$ as follows:

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \le |x| \le d/n; \\ \frac{\log(d/|x|)}{\sqrt{\log n}}, & d/n \le |x| \le d; \\ 0, & |x| \ge d. \end{cases} \tag{4.22}$$

By an elemental computation, we have

$$\|\nabla w_n\|^2 = \int_{B_d \setminus B_{d/n}} |\nabla w_n|^2 \mathrm{d}x = 1 \tag{4.23}$$

and

$$\int_{\mathbb{R}^2} |x|^{a_0} w_n^2 \mathrm{d}x = \frac{2d^{2+a_0}}{(2+a_0)^2} \delta_n, \tag{4.24}$$

where

$$\delta_n := \frac{1}{(2+a_0)\log n} - \frac{1}{(2+a_0)n^{2+a_0}\log n} - \frac{1}{n^{2+a_0}} > 0. \tag{4.25}$$

Lemma 4.4. *Assume that* (V1), (Q1), (Q2) *and* (F1′)-(F5) *hold. Then there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$c \le \max_{t \ge 0} \Phi(tw_{\bar{n}}) < \frac{(2+\alpha+2b_0)\pi}{2\beta_0}. \tag{4.26}$$

Proof. By (V1), we can deduce that there exists $d > 0$ such that

$$V(r) \le (\tau+1) r^{a_0}, \quad \forall\ 0 < r \le d \tag{4.27}$$

and

$$\frac{(2+\alpha+2b_0)(\tau+1)d^{2+a_0}}{(2+a_0)^3} < \frac{1}{3}. \tag{4.28}$$

By (F5), we know that there exists $t_\kappa > 0$ such that

$$tF(t) \ge \frac{\kappa}{2} e^{\beta_0 t^2}, \quad \forall\ t \ge t_\kappa. \tag{4.29}$$

From (V1), (F5), (2.16), (4.23), (4.24) and (4.27), we have

$$\begin{aligned} \Phi(tw_n) &= \frac{t^2}{2}\left(\|\nabla w_n\|_2^2 + \int_{\mathbb{R}^2} V(|x|) w_n^2 \mathrm{d}x\right) - \frac{1}{2}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(tw_n))] Q(|x|)F(tw_n) \mathrm{d}x \\ &\le \frac{t^2}{2}\left(\|\nabla w_n\|_2^2 + \int_{B_d} (\tau+1)|x|^{a_0} w_n^2 \mathrm{d}x\right) - \frac{1}{2}\int_{B_{d/n}} [I_\alpha * (Q(|x|)F(tw_n))] Q(|x|)F(tw_n) \mathrm{d}x \\ &\le \frac{t^2}{2}\left[1 + \frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] - \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}} \left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2 \left(\int_0^{d/n} rQ(r) \mathrm{d}r\right)^2. \end{aligned} \tag{4.30}$$

To show (4.26) we have three cases to distinguish. From now on, in the sequel, all inequalities hold for large $n \in \mathbb{N}$.

Case i). $t \in \left[0, \sqrt{\frac{(2+\alpha+2b_0)\pi}{2\beta_0}}\right]$. Then it follows (F5), (Q2) and (4.30) that

$$\Phi(tw_n) \le \frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] \le \frac{(2+\alpha+2b_0)\pi}{3\beta_0}. \tag{4.31}$$

Clearly, there exists $\bar{n} \in \mathbb{N}$ such that (4.26) holds.

Case ii). $t \in \left[\sqrt{\frac{(2+\alpha+2b_0)\pi}{2\beta_0}}, \sqrt{\frac{(2+\alpha+2b_0)(1+\kappa)\pi}{\beta_0}}\right]$. Then $t\sqrt{\log n}/\sqrt{2\pi} \ge t_\kappa$ for large $n \in \mathbb{N}$, it follows (Q2) and (4.29) that

$$\begin{aligned}
& \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2 \\
\ge\ & \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\frac{2\pi}{t^2\log n}\frac{\kappa^2}{4}e^{\pi^{-1}\beta_0 t^2\log n}\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2 \\
\ge\ & \frac{\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\frac{\kappa^2\beta_0}{(2+\alpha+2b_0)(1+\kappa)\log n}e^{\pi^{-1}\beta_0 t^2\log n}\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2 \\
\ge\ & \frac{\pi^2\kappa^2\beta_0\zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}\log n}e^{\pi^{-1}\beta_0 t^2\log n}.
\end{aligned} \tag{4.32}$$

It follows from (4.25), (4.30) and (4.32) that

$$\begin{aligned}
\Phi(tw_n) \le\ & \frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right] - \frac{2\pi^2 n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n} rQ(r)\mathrm{d}r\right)^2 \\
\le\ & \frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right] - \frac{\pi^2\kappa^2\beta_0\zeta_0^2 d^{2+\alpha+2b_0}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}\log n}e^{\pi^{-1}\beta_0 t^2\log n} \\
=:\ & \psi_n(t).
\end{aligned} \tag{4.33}$$

Let $t_n > 0$ such that $\psi_n'(t_n) = 0$. Then

$$1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n} = \frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)n^{2+\alpha+2b_0}}e^{\pi^{-1}\beta_0 t^2\log n}. \tag{4.34}$$

It follows that

$$t_n^2 = \frac{(2+\alpha+2b_0)\pi}{\beta_0} - \frac{\pi}{\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)} + O\left(\frac{1}{\log^2 n}\right). \tag{4.35}$$

It follows from (4.28), (4.33) and (4.35), we have

$$\begin{aligned}
\psi_n(t) \le\ & \psi_n(t_n) = \frac{t_n^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right] - \frac{\pi}{2\beta_0\log n}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right] \\
=\ & \frac{1}{2}\left[\frac{(2+\alpha+2b_0)\pi}{\beta_0} - \frac{\pi}{\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2 d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)}\right]\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right] \\
& -\frac{\pi}{2\beta_0\log n} + O\left(\frac{1}{\log^2 n}\right) \\
=\ & \frac{(2+\alpha+2b_0)\pi}{2\beta_0} + \frac{\pi}{\beta_0\log n}\left[\frac{(2+\alpha+2b_0)(\tau+1)d^{2+a_0}}{(2+a_0)^3} - \frac{1}{2}\right]
\end{aligned}$$

$$-\frac{\pi}{2\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)}+O\left(\frac{1}{\log^2 n}\right)$$

$$<\frac{(2+\alpha+2b_0)\pi}{2\beta_0}-\frac{\pi}{2\beta_0\log n}\log\frac{\pi\kappa^2\beta_0^2\zeta_0^2d^{2+\alpha+2b_0}}{2^{3-\alpha}(2+\alpha+2b_0)(1+\kappa)}+O\left(\frac{1}{\log^2 n}\right). \tag{4.36}$$

Hence, combining (4.33) with (4.36), one has

$$\Phi(tw_n)\le\psi_n(t)\le\psi_n(t_n)<\frac{(2+\alpha+2b_0)\pi}{2\beta_0}. \tag{4.37}$$

Case iii). $t\in\left(\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}(1+\kappa)},+\infty\right)$. Then $t\sqrt{\log n}/\sqrt{2\pi}\ge t_\kappa$ for large $n\in\mathbb{N}$, it follows (4.25), (4.29) and (4.30) that

$$\begin{aligned}\Phi(tw_n)&\le\frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right]-\frac{2\pi^2n^{2-\alpha}}{(2d)^{2-\alpha}}\left[F\left(\frac{t\sqrt{\log n}}{\sqrt{2\pi}}\right)\right]^2\left(\int_0^{d/n}rQ(r)\mathrm{d}r\right)^2\\&\le\frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^2}\delta_n\right]-\frac{2\pi^2n^{2-\alpha}}{(2d)^{2-\alpha}}\left(\frac{t^2\log n}{2\pi}\right)^{-1}\frac{\kappa^2}{4}e^{\pi^{-1}\beta_0t^2\log n}\left(\int_0^{d/n}rQ(r)\mathrm{d}r\right)^2\\&\le\frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right]-\frac{\pi^3\kappa^2\zeta_0^2d^{2+\alpha+2b_0}}{2^{4-\alpha}n^{2+\alpha+2b_0}t^2\log n}e^{\pi^{-1}\beta_0t^2\log n}\\&\le\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right]\frac{(2+\alpha+2b_0)\pi(1+\kappa)}{2\beta_0}-\frac{\pi^2\kappa^2\zeta_0^2d^{2+\alpha+2b_0}\beta_0n^{(2+\alpha+2b_0)\kappa}}{2^{4-\alpha}(2+\alpha+2b_0)(1+\kappa)\log n}\\&\le\frac{(2+\alpha+2b_0)\pi}{3\beta_0},\end{aligned} \tag{4.38}$$

which implies that there exists $\bar{n}\in\mathbb{N}$ such that (4.26) holds. In the above derivation process, we use the fact that the function

$$h_n(t):=\frac{t^2}{2}\left[1+\frac{2(\tau+1)d^{2+a_0}}{(2+a_0)^3\log n}\right]-\frac{\pi^3\kappa^2\zeta_0^2d^{2+\alpha+2b_0}}{2^{4-\alpha}n^{2+\alpha+2b_0}t^2\log n}e^{\pi^{-1}\beta_0t^2\log n}$$

is decreasing on $t\in\left(\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}(1+\kappa)},+\infty\right)$, since its stagnation points tend to $\sqrt{\frac{(2+\alpha+2b_0)\pi}{\beta_0}}$ as $n\to\infty$.

□

Lemma 4.5. *Assume that* (V1), (Q1), (Q2), (F1′), (F2), (F3) *and* (F5) *hold. Then* Φ *satisfies* $(C)_c$ *condition.*

Proof. Applying Lemmas 4.1 and 4.2, we deduce that there exists a sequence $\{u_n\}\subset E$ satisfying (4.1) and $\|u_n\|\le C_{10}$ for some constant $C_{10}>0$. Since $\|u_n\|\le C_{10}$, by Lemma 2.4, we may thus assume, passing to a subsequence if necessary, that $u_n\rightharpoonup\bar{u}$ in E, $u_n\to\bar{u}$ in $L^s(\mathbb{R}^2,Q)$ for $s\in[2,\infty)$ and $u_n\to\bar{u}$ a.e. on $\mathbb{R}^2$. It follows from (2.17) and (4.1) that

$$\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)u_n\le C_{11}. \tag{4.39}$$

In view of Lemma 4.3, for any $\phi \in E \cap \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi \mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})\varphi \mathrm{d}x. \tag{4.40}$$

By (4.1) and (4.40), it is easy to deduce that $\Phi'(\bar{u}) = 0$. Then by (F3), one has

$$\Phi(\bar{u}) = \Phi(\bar{u}) - \frac{1}{\mu}\langle \Phi'(\bar{u}), \bar{u}\rangle \geq 0.$$

By (4.39) and Lemma 4.3, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x. \tag{4.41}$$

Thus

$$\begin{aligned} c + o(1) &= \Phi(u_n) = \frac{1}{2}\|u_n - \bar{u}\|^2 + \frac{1}{2}\|\bar{u}\|^2 - \frac{1}{2}\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(\bar{u}))]Q(|x|)F(\bar{u})\mathrm{d}x + o(1) \\ &= \frac{1}{2}\|u_n - u\|^2 + \Phi(\bar{u}) + o(1) \\ &\geq \frac{1}{2}\|u_n - u\|^2 + o(1). \end{aligned}$$

Hence, it follows from (2.16), (4.1), (4.26) and (4.41) that

$$\|u_n - u\|^2 \leq 2c + o(1) := \frac{(2+\alpha+2b_0)\pi}{\beta_0}(1 - 3\bar{\varepsilon}) + o(1). \tag{4.42}$$

Lemma 4.4 implies that $\bar{\varepsilon} > 0$. Now we choose $q \in (1, 2)$ such that

$$\frac{(1+\bar{\varepsilon})^2(1-3\bar{\varepsilon})q^2}{1-\bar{\varepsilon}} < 1. \tag{4.43}$$

By (2.16) and (4.1), we can deduce that

$$\int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n) \leq C_{12}. \tag{4.44}$$

Let $q' = q/(q-1)$. Then $q' > 2$. Hence it follows from (4.42), (4.43), (4.44), (2.1), Lemma 2.5-ii), $u_n \to 0$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$, Hölder inequality and Young inequality that

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n - \bar{u})\Big|\mathrm{d}x \\ \leq\ & \left(\int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)F(u_n))]Q(|x|)F(u_n)\Big|\mathrm{d}x\right)^{\frac{1}{2}} \\ &\times \left(\int_{\mathbb{R}^2} \Big|[I_\alpha * (Q(|x|)f(u_n)(u_n - \bar{u}))]Q(|x|)f(u_n)(u_n - \bar{u})\Big|\mathrm{d}x\right)^{\frac{1}{2}} \\ \leq\ & C_{13}\left(\int_{\mathbb{R}^2} \Big|Q(|x|)f(u_n)(u_n - \bar{u})\Big|^{\frac{4}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4}} \\ \leq\ & C_{14}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^2\mathrm{d}x\right)^{\frac{4\alpha}{(2+\alpha)^2}}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n - \bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}} \end{aligned}$$

$$
\begin{aligned}
&+C_{14}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{4q}{2+\alpha}\beta_0(1+\bar{\varepsilon})u_n^2}-1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{4q}}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^{\frac{4q'}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4q'}}\\
\leq\ & C_{14}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^2\mathrm{d}x\right)^{\frac{4\alpha}{(2+\alpha)^2}}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}}\\
&+C_{14}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{4q}{2+\alpha}\beta_0(1+\bar{\varepsilon})^2(u_n-\bar{u})^2}e^{\frac{4q}{2+\alpha}\beta_0(1+\bar{\varepsilon})^2\varepsilon^{-1}\bar{u}^2}-1\right)\mathrm{d}x\right)^{\frac{2+\alpha}{4q}}\\
&\times\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^{\frac{4q'}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4q'}}\\
\leq\ & C_{14}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n|^2\mathrm{d}x\right)^{\frac{4\alpha}{(2+\alpha)^2}}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}}\\
&+C_{14}\left[\frac{1}{q}\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{4q^2}{2+\alpha}\beta_0(1+\bar{\varepsilon})^2\|u_n-\bar{u}\|^2[(u_n-\bar{u})/\|u_n-\bar{u}\|]^2}-1\right)\mathrm{d}x\right.\\
&\left.+\frac{1}{q'}\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}\left(e^{\frac{4qq'}{2+\alpha}\beta_0(1+\bar{\varepsilon})^2\varepsilon^{-1}\bar{u}^2}-1\right)\mathrm{d}x\right]^{\frac{2+\alpha}{4q}}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^{\frac{4q'}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4q'}}\\
\leq\ & C_{15}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^2\mathrm{d}x\right)^{\frac{8}{(2+\alpha)^2}}+C_{16}\left(\int_{\mathbb{R}^2} Q(|x|)^{\frac{4}{2+\alpha}}|u_n-\bar{u}|^{\frac{4q'}{2+\alpha}}\mathrm{d}x\right)^{\frac{2+\alpha}{4q'}}\\
=\ & o(1). \qquad (4.45)
\end{aligned}
$$

Which yields that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n-\bar{u})\mathrm{d}x=0. \qquad (4.46)$$

Similarly, we can deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n-\bar{u})\mathrm{d}x=0. \qquad (4.47)$$

By (4.1), it is obvious that

$$\langle\Phi'(u_n)-\Phi'(\bar{u}),u_n-\bar{u}\rangle=o(1). \qquad (4.48)$$

From (4.46), (4.47) and (4.48), we can get

$$
\begin{aligned}
\|u_n-\bar{u}\|^2 &= \langle\Phi'(u_n)-\Phi'(\bar{u}),u_n-\bar{u}\rangle-\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(u_n))]Q(|x|)f(u_n)(u_n-\bar{u})\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}[I_\alpha*(Q(|x|)F(\bar{u}))]Q(|x|)f(\bar{u})(u_n-\bar{u})\mathrm{d}x\\
&= o(1).
\end{aligned}
$$

□

Lemma 4.6. *Assume that* (V1), (Q1), (F1′), (F2) *and* (F3) *hold. Then*

i) $\sigma_0:=\inf\{\|u\|:u\in\mathcal{M}\}>0$;

ii) $m:=\inf\{\Phi(u):u\in\mathcal{M}\}>0$.

The proof is similar to Lemma 3.5, so we omit it here.

Proof of Theorem 1.4. First, we prove that $\mathcal{M} \neq \emptyset$. Applying Lemmas 4.1 and 4.2, we deduce that there exists a sequence $\{v_n\} \subset E$ satisfying (4.1) and $\|v_n\| \leq C_{17}$ for some constant $C_{17} > 0$. Hence, there exists $v \in E$ such that, up to a subsequence, $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $v_n \to v$ a.e. in $\mathbb{R}^2$. By Lemma 4.5, we know that $v_n \to v$ in E. Thus

$$\Phi(v) = \lim_{n\to\infty} \Phi(v_n) = c > 0, \quad \Phi'(v) = \lim_{n\to\infty} \Phi'(v_n) = 0.$$

Which imply that $v \neq 0$ and $v \in \mathcal{M}$.

Next, we prove that $m := \inf_{\mathcal{M}} \Phi(u)$ is achieved. By Lemma 4.6, we have $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. As in the proof of Lemma 4.2, we can deduce that $\{u_n\}$ is bounded in E. Hence, there exists $u \in E$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^s(\mathbb{R}^2, Q)$ for $s \geq 2$ and $u_n \to u$ a.e. in $\mathbb{R}^2$. In view of Lemma 4.3, for any $\phi \in E \cap \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|)f(u_n)\varphi \mathrm{d}x = \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|)f(u)\varphi \mathrm{d}x. \tag{4.49}$$

By (4.1) and (4.49), it is easy to deduce that $\Phi'(u) = 0$. From the weak lower semicontinuous of norm and Fatou's Lemma, we can deduce that

$$\begin{aligned}
m = &\lim_{n\to\infty} \Phi(u_n) = \lim_{n\to\infty} \left(\Phi(u_n) - \frac{1}{\mu} \langle \Phi'(u_n), u_n \rangle \right) \\
= &\lim_{n\to\infty} \left(\frac{\mu-1}{2\mu} \|u_n\|^2 + \frac{1}{2\mu} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u_n))]Q(|x|) \left[f(u_n)u_n - \mu F(u_n) \right] \mathrm{d}x \right) \\
\geq &\frac{\mu-1}{2\mu} \|u\|^2 + \frac{1}{2\mu} \int_{\mathbb{R}^2} [I_\alpha * (Q(|x|)F(u))]Q(|x|) \left[f(u)u - \mu F(u) \right] \mathrm{d}x \\
= &\Phi(u) - \frac{1}{\mu} \langle \Phi'(u), u \rangle = \Phi(u) \geq m.
\end{aligned}$$

This shows that $u \in \mathcal{M}$ and $\Phi(u) = m$. □

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References

[1] F.B.S. Albuquerque, M.C. Ferreira, U.B. Severo, Ground State Solutions for a Nonlocal Equation in $\mathbb{R}^2$ Involving Vanishing Potentials and Exponential Critical Growth, Milan J. Math. 89 (2021) 263-294.

[2] S.I. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.

[3] E.H. Lieb, B. Simon, The Hartree-Fock theory for Coulomb systems, Commun. Math. Phys. 53 (1977) 185-194.

[4] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1977) 93-105.

[5] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980) 1063-1073.

[6] I.M. Moroz, R. Penrose, P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, Class. Quantum Gravity 15 (1998) 2733-2742.

[7] P. Choquard, J. Stubbe, M. Vuffray, Stationary solutions of the Schrödinger-Newton model-an ODE approach, Differ. Integral Equ. 21 (2008) 665-679.

[8] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativ. Gravit. 28 (1996) 581-600.

[9] P. Tod, I.M. Moroz, An analytical approach to the Schrödinger-Newton equations, Nonlinearity 12 (1999) 201-216.

[10] L. Ma, L. Zhao, Classification of positive solitary solutions of the nonlinear Choquard equation, Arch. Ration. Mech. Anal. 195 (2010) 455-467.

[11] V. Moroz, J.V. Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Am. Math. Soc. 367 (2015) 6557-6579.

[12] D.D. Qin, V.D. Rădulescu, X.H. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, J. Differential Equations, 275 (2021) 652-683.

[13] M. Ghimenti, J.V. Schaftingen, Nodal solutions for the Choquard equation, J. Funct. Anal. 271 (2016) 107-135.

[14] J.K. Xia, Z.-Q. Wang, Saddle solutions for the Choquard equation, Calc. Var. Partial Differential Equations, 58 (2019) 85.

[15] V. Moroz, J.V. Schaftingen, Semi-classical states for the Choquard equation, Calc. Var. Partial Differential Equations, 52 (2015) 199-235.

[16] J. Zhang, Q.F. Wu, D.D. Qin, Semiclassical solutions for Choquard equations with Berestycki-Lionstype conditions, Nonlinear Anal. 188 (2019) 22-49.

[17] V. Moroz, J. van Schaftingen, A guide to the Choquard equation, J. Fixed Point Theory Appl. 19 (2017) 773-813.

[18] D.M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, Comm. Partial Differ. Equ. 17 (1992) 407-435.

[19] S. Adachi, K. Tanaka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, Proc. Am. Math. Soc. 128 (2000) 2051-2057.

[20] D. Cassani, F. Sani, C. Tarsi, Equivalent Moser type inequalities in $\mathbb{R}^2$ and the zero mass case, J. Funct. Anal. 267 (2014) 4236-4263.

[21] C.O. Alves, D. Cassani, C. Tarsi, M.B. Yang, Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$, J. Differential Equations, 261 (2016) 1933-1972.

[22] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270-291.

[23] D.D. Qin, X.H. Tang, On the planar Choquard equation with indefinite potential and critical exponential growth, J. Differential Equations, 285 (2021) 40-98.

[24] L.J. Shen, V.D. Rădulescu, M.B. Yang, Planar Schrödinger-Choquard equations with potentials vanishing at infinity: The critical case, J. Differential Equations, 329 (2022) 206-254.

[25] X.Y. Lin, X.H. Tang, Schrödinger equations in $\mathbb{R}^2$ with critical exponential growth and concave nonlinearities, J. Math Anal Appl. 514 (2022) 126252.

[26] C.O. Alves, L.J. Shen, Critical Schrödinger equations with Stein-Weiss convolution parts in $\mathbb{R}^2$, J. Differential Equations, 344 (2023) 352-404.

[27] E. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997.

[28] L. Mattner, Strict definiteness of integrals via complete monotonicity of derivatives, Trans. Am. Math. Soc. 349 (1997) 3321-3342.

[29] J. Su, Z.-Q. Wang, M. Willem, Nonlinear Schrödinger equations with unbounded and decaying radial potentials, Commun. Contemp. Math. 9 (2007) 571-583.

[30] D.G. de Figueiredo, O.H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. Partial Differential Equations, 3 (1995) 139-153.