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Ground states for quasilinear equations of N-Laplacian type with critical exponential growth and lack of compactness

Sitong Chen[1], Dongdong Qin[1,*], Vicenţiu D. Rădulescu[2,3] & Xianhua Tang[1]

[1] *School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, China;*
[2] *Faculty of Applied Mathematics, AGH University of Kraków, Kraków 30-059, Poland;*
[3] *Department of Mathematics, University of Craiova, Craiova 200585, Romania*

Email: mathsitongchen@mail.csu.edu.cn, qindd132@163.com, radulescu@inf.ucv.ro, tangxh@mail.csu.edu.cn

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Abstract In this paper, (i) we present **unified approaches** to study the existence of ground state solutions and mountain-pass type solutions for the following quasilinear equation

$$-\Delta_N u + V(x)|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad N \geqslant 2$$

in three different cases allowing the potential $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ to be periodic, radially symmetric, or asymptotically constant, where $\Delta_N u := \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ and f has critical exponential growth. (ii) **Two new compactness lemmas** in $W^{1,N}(\mathbb{R}^N)$ for general nonlinear functionals are established which generalize the ones obtained in the radially symmetric space $W^{1,N}_{\text{rad}}(\mathbb{R}^N)$. (iii) Based on some key observations, we construct **a special path** allowing us to control the Mountain-pass minimax level by a fine threshold under which the compactness can be restored for the critical case. In particular, some delicate analyses are developed to overcome non-standard difficulties due to both the quasilinear characteristic of the equation and the lack of compactness aroused by the critical exponential growth of f. Our results extend and improve the ones of Alves et al. (2012), Ibrahim et al. (2015) ($N = 2$), and Masmoudi and Sani (2015) ($N \geqslant 3$) for the constant potential case; of Alves and Figueiredo (2009) for the periodic potential case; of Lam and Lu (2012) and Yang (2012) for the coercive potential case; of Chen et al. (Sci China Math, 2021) for the degenerate potential case, which are totally new even for the simpler semilinear case of $N = 2$. We believe that our approaches and strategies may be adapted and modified to attack more variational problems with critical exponential growth.

Keywords N-Laplacian equations, critical exponential growth, Trudinger-Moser inequality, ground state solution, Nehari manifold

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* Corresponding author

1 Introduction and main Results

This paper is concerned with the following quasilinear equation:

$$-\Delta_N u+V(x)|u|^{N-2}u=f(u)\quad \text{in } \mathbb{R}^N,\quad N\geqslant 2, \tag{1.1}$$

where Δ_N is the N-Laplacian operator, i.e., $\Delta_N u:=\operatorname{div}(|\nabla u|^{N-2}\nabla u)$, and V and f satisfies the following basic assumptions:

(V0) $V\in\mathcal{C}(\mathbb{R}^N,\mathbb{R})$ and $0<V_*:=\inf_{x\in\mathbb{R}^N}V(x)\leqslant\sup_{x\in\mathbb{R}^N}V(x):=V^*<\infty$;

(F1) $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$, and there exists an $\alpha_0>0$ such that

$$\lim_{|t|\to\infty}\frac{|f(t)|}{\mathrm{e}^{\alpha t^{N/(N-1)}}}=0\quad\text{for all } \alpha>\alpha_0 \tag{1.2}$$

and

$$\lim_{|t|\to\infty}\frac{|f(t)|}{\mathrm{e}^{\alpha t^{N/(N-1)}}}=+\infty\quad\text{for all } \alpha<\alpha_0; \tag{1.3}$$

(F2) $f(t)=o(|t|^{N-1})$ as $t\to 0$;

in particular, V is allowed to be periodic, radially symmetric, or asymptotical constant, i.e., V satisfies (V1), (V2) or (V3) below:

(V1) $V(x)$ is 1-periodic in $x_1,x_2,\dots,x_N$;

(V2) $V(x)$ is a radially symmetric function;

(V3) $V_*=\inf_{x\in\mathbb{R}^N}V(x)<V^*=\lim_{|x|\to\infty}V(x)$.

Problem (1.1) and its related quasilinear equation

$$-\Delta_p u+V(x)|u|^{p-2}u=f(u)\quad\text{in } \mathbb{R}^N \tag{1.4}$$

arise in several branches of mathematical physics, such as non-Newtonian fluids, pseudo-plastic fluids, turbulent fluids in porus media, and image processing [4,14], and find their applications in the fields of electromagnetism and astronomy where they are used to describe the behavior of electric and gravitational potentials [30]. There are fruitful results on the existence, nonexistence, and multiplicity of nontrivial solutions, ground states, and semiclassical solutions for these problems. When $p=2$, (1.4) becomes the well-known Schrödinger equation which has been widely investigated in the literature (see, for example, [8,33,40,44,46] and the references therein). For the case $p>2$, we refer readers to [3,16] for the existence and multiplicity of nontrivial solutions, [7] for the sign-changing solutions, and [32] for the semiclassical solutions. All these results are based on the Sobolev embedding theorem and critical point theory, and thus the nonlinearity is only allowed to have polynomial growth at infinity, precisely, assuming $F(t)=|t|^q$, the exponent q must satisfy $p\leqslant q\leqslant Np/(N-p)$. For the case $p=N$, the growth range of the exponent q can be further extended since the Sobolev embedding theorem shows that $W_0^{1,N}(\Omega)\subset L^q(\Omega)$ for all $q\geqslant N$; however, $W^{1,N}(\Omega)\not\subseteq L^\infty(\Omega)$. To find the maximal growth in this situation, Yudovich [49], Pohozaev [38], and Trudinger [45] showed independently that $W_0^{1,N}(\Omega)$ is embedded in the Orlicz space determined by the Young function $\mathrm{e}^{\alpha|t|^{N/(N-1)}}-1$ for some $\alpha>0$, precisely, the following refined result established by Moser [36] holds when $\Omega\subset\mathbb{R}^N$ is a bounded domain.

Lemma 1.1 (See [36]). *There exist a sharp constant $\alpha_N:=N\omega_{N-1}^{1/(N-1)}$ (where ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$) and a constant $C(N,\alpha)>0$ such that*

$$\sup_{\substack{u\in W_0^{1,N}(\Omega)\\ \|\nabla u\|_N=1}}\int_\Omega \mathrm{e}^{\alpha|u|^{N/(N-1)}}dx\leqslant C(N,\alpha)|\Omega|,\quad\forall\,\alpha\leqslant\alpha_N. \tag{1.5}$$

Moreover, the supremum in the above inequality is $+\infty$ if $\alpha>\alpha_N$.

Since the problem (1.1) studied in this paper is defined on the whole space $\mathbb{R}^N$, the above inequality (1.5) fails while it can be recovered either by weakening the exponent $\alpha_N=N\omega_{N-1}^{1/(N-1)}$ or by strengthening the Dirichlet norm $\|\nabla u\|_N$. Precisely, for the nonlinearity f having critical exponential growth at $\pm\infty$ as in (F1), we use the following version of the Trudinger-Moser inequality.

Lemma 1.2 (See [23, 31]). (i) *If $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$ $(N \geqslant 2)$, then $\int_{\mathbb{R}^N} \phi_N(\alpha|u|^{\frac{N}{N-1}})dx < \infty$, where*

$$\phi_N(t) := \mathrm{e}^t - \sum_{k=0}^{N-2} \frac{t^k}{k!} = \sum_{k=N-1}^{\infty} \frac{t^k}{k!}. \tag{1.6}$$

(ii) *If $\|\nabla u\|_N \leqslant 1$, $\|u\|_N \leqslant M < \infty$, and $\alpha < \alpha_N$, then there exists a constant $C(N, M, \alpha) > 0$, which depends only on N, M, and α such that*

$$\int_{\mathbb{R}^N} \phi_N(\alpha|u|^{\frac{N}{N-1}})dx \leqslant C(N, M, \alpha).$$

Moreover,

$$\sup_{\substack{u \in W^{1,N}(\mathbb{R}^N) \\ \|\nabla u\|_N^N + \|u\|_N^N \leqslant 1}} \int_{\mathbb{R}^N} \phi_N(\alpha|u|^{\frac{N}{N-1}})dx \begin{cases} < +\infty, & \text{if } \alpha \leqslant \alpha_N, \\ = +\infty, & \text{if } \alpha > \alpha_N. \end{cases} \tag{1.7}$$

This lemma was first established by Cao [9] in $\mathbb{R}^2$ and generalized to any dimension $N \geqslant 2$ by do Ó [23], Panda [37], and Alves and Figueiredo [4]. If we use the full Sobolev norm $\|u\|$ instead of the Dirichlet norm $\|\nabla u\|_N$, then the Moser's result (1.5) can be fully extended to the whole space $\mathbb{R}^N$, i.e., the above inequality (1.7) holds which means that the constant α_N is the best exponent. Such an inequality was shown by Ruf [41] for the case $N = 2$ via the symmetrization method and by Li and Ruf [31] for the case $N \geqslant 2$ by using the method of blow-up analysis. For recent results on the Trudinger-Moser type inequalities and their applications in the study of nonlinear elliptic equations and systems, we bring the reader's attention to the papers [1, 2, 11, 13, 24, 28, 34, 35] and the review article [18].

When $V \equiv c > 0$, we derive the following autonomous problem from (1.1):

$$-\Delta_N u + c|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad N \geqslant 2. \tag{1.8}$$

For the case $N = 2$, the existence of the ground state solution which has the least energy among all the solutions of (1.8) was studied in [6, 29, 42] via the variational method for the critical nonlinearity, and in [27] by the ordinary differential equation (ODE) technique including the supercritical nonlinearity. For a more general quasilinear equation involving indefinite nonlinearity, we refer readers to [21], where the existence of a nontrivial solution was proved via variational arguments in an Orlicz-Sobolev space with a version of the Trudinger-Moser inequality. Using the concentration-compactness principle introduced by Lions [33], do Ó et al. studied recently the case $N \geqslant 2$ in [24], and established an improved Trudinger-Moser type inequality by virtue of which the Palais-Smale compactness condition was obtained. Moreover, they showed that (1.8) has a radial ground state solution under (F1), (F2), the well-known Ambrosetti-Rabinowitz type condition (AR), and the following assumptions (F4), (F5′), and (F7′) (see [24, Theorem 3.1]).

(AR) There exists a $\mu > N$ such that $f(t)t \geqslant \mu F(t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$, where $F(t) = \int_0^t f(s)ds$.

(F4) There exist $M_0 > 0$ and $\bar{t}_0 > 0$ such that $F(t) \leqslant M_0|f(t)|$, $\forall\, |t| \geqslant \bar{t}_0$.

(F5′) $f(t) = 0$ for all $t \leqslant 0$, and $\frac{f(t)}{t^{N-1}}$ is increasing for $t > 0$.

(F6′) $\liminf_{t\to+\infty} \frac{tf(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} \geqslant \beta_0 > \frac{(N-2)!N\mathrm{e}c}{\alpha_0^{N-1}}$.

(F7′) There exists a $p > N$ such that $|f(t)| \geqslant C_p|t|^{p-1}$ for all $t \in \mathbb{R}$, where

$$C_p > \left(\frac{\alpha_0}{\alpha_N}\right)^{\frac{(p-N)(N-1)}{N}} \left(\frac{p-N}{p}\right)^{\frac{p-N}{N}} \mathcal{S}_{p,N;c}^p$$

and

$$\mathcal{S}_{p,N;c} := \inf_{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}} \frac{(\|\nabla u\|_N^N + c\|u\|_N^N)^{1/N}}{\|u\|_p}. \tag{1.9}$$

In recent papers [29, 35], a precise version of the Trudinger-Moser inequality in the whole space $\mathbb{R}^N$ was established by Ibrahim et al. [29] for the case $N = 2$ and by Masmoudi and Sani [35] for the case $N \geqslant 3$

(see Lemma 2.1); moreover, necessary and sufficient conditions for the boundedness and the compactness of general nonlinear functionals in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ were obtained by them (see [35, Theorems 1.5 and 1.6]). To find a ground state solution for the problem (1.8), they studied the constrained minimization problem $A_c := \inf\{\frac{1}{N}\|\nabla u\|_N^N : u \in \mathcal{P}_c\}$, where $\mathcal{P}_c$ is the set consisting of all the nonzero functions satisfying the Pohozaev identity for equation (1.8) defined by

$$\mathcal{P}_c := \left\{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\},\, c\int_{\mathbb{R}^N} |u|^N dx - N\int_{\mathbb{R}^N} F(u)dx = 0\right\}.$$

Since (1.8) is autonomous, it is easy to find a minimizing sequence $\{u_n\} \subset \mathcal{P}_c$ for A_c satisfying

$$\{u_n\} \subset W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N), \quad \frac{1}{N}\|\nabla u_n\|_N^N \to A_c, \quad \|u_n\|_N = 1$$

by Schwarz symmetrization and rescaling (see [35, Remark 7.2]) and easy to see that a minimizer of the infimum A_c is, up to a suitable change of scale, a ground state solution of (1.8) (see [35, Remark 7.3]). To investigate the attainability of A_c, Masmoudi and Sani [35] introduced the Trudinger-Moser ratio $\mathcal{C}^*_{\mathrm{TM}}$ and showed that $c < \mathcal{C}^*_{\mathrm{TM}}$ is equivalent to $A_c < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$, from which the attainability of A_c can be deduced provided that (F1), (F2), (AR), and (F4) are satisfied. It holds that

$$\mathcal{C}^*_{\mathrm{TM}} := \sup\left\{\frac{N}{\|u\|_N^N}\int_{\mathbb{R}^N} F(u)dx \;\middle|\; u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\},\, \|\nabla u\|_N^N \leqslant \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}\right\}. \tag{1.10}$$

Moreover, the following result was established in [29, Theorem 5.1] and [35, Theorem 7.4] for the case $N = 2$ and $N \geqslant 3$, respectively.

Theorem 1.3. *Let $N \geqslant 2$ and assume that f satisfies* (F1), (F2), (AR), *and* (F4). *Then there exists a $c_* \in (0, +\infty]$ such that for each $c \in (0, c_*)$,* (1.8) *admits a positive radial ground state solution. Moreover, $c_* = \mathcal{C}^*_{\mathrm{TM}}$ when $\mathcal{C}^*_{\mathrm{TM}} < +\infty$, while $c_* = +\infty$ is equivalent to*

$$\lim_{t \to +\infty} \frac{t^{N/(N-1)}F(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} = +\infty. \tag{1.11}$$

From the above Theorem, we see that a crucial condition to guarantee the existence of ground states is $c < \mathcal{C}^*_{\mathrm{TM}}$, which holds when either (F6′) or (F7′) is satisfied (see [35, Section 8]). As pointed out by Masmoudi and Sani [35, Remark 8.2], it seems to be difficult to compare the growth condition (F6′) with (F7′) as they prescribe the growth of f at infinity and near the origin respectively; however, a comparison between them can be seen in terms of the Trudinger-Moser ratio $\mathcal{C}^*_{\mathrm{TM}}$. Thus, the condition $c < \mathcal{C}^*_{\mathrm{TM}}$ unifies (F6′) and (F7′) in some extend. For a similar result about the case $N = 2$, we refer readers to [6, 42] and to [50, 51] for fractional problems.

Before recalling some related works for the non-autonomous problem (1.1) with critical exponential growth, we first introduce the following assumptions used in the references.

(V4) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$, and $\mathrm{meas}\{x \in \mathbb{R}^N : V(x) \leqslant M\} < +\infty$, $\forall\, M > 0$.

(V5) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$, and $V^{-\frac{1}{N-1}} \in L^1(\mathbb{R}^N)$.

(V6) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$, and $V^{-1} \in L^1(\mathbb{R}^N)$.

(F6″) Let $V_r := \max_{|x| \leqslant r} V(x)$. It holds that

$$\liminf_{t \to +\infty} \frac{tf(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} \geqslant \beta_0 > \inf_{r>0} \frac{N^N}{\alpha_0^{N-1} r^N} \mathrm{e}^{NV_r(N-2)!r^N/N^N}.$$

(F7″) There exists a $p > N$ such that $|f(t)| \geqslant C_p|t|^{p-1}$ for all $t \in \mathbb{R}$, where $C_p > [\frac{8^N\mu(p-N)}{p(\mu-N)}]^{\frac{p-N}{N}}\mathcal{S}^p_{p,N;V^*}$, $\mu > N$ is the constant in (AR), and $\mathcal{S}_{p,N;V^*}$ is defined by (1.9).

When $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$ satisfying (F1), (F2), (AR), (F5′), and the restrictively technical condition (F7″), Alves and Figueiredo [4] proved the existence of a positive solution for (1.1) if V satisfies the periodic condition (V1) and also studied the existence, multiplicity, and concentration of positive solutions if the operator $-\Delta_N$ is replaced by $-\epsilon^N\Delta_N$ for a sufficiently small parameter $\epsilon>0$, and V satisfies the Rabinowitz type condition (V0) with $V_* < \lim_{|x|\to\infty} V(x) = V^*$. As well known, the condition (V4) or (V5) or (V6) guarantees that the embedding from the space $\{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty\}$ into $L^q(\mathbb{R}^N)$ $(q \geqslant N)$ is compact; moreover, the exponent q can be extend to the range $[1,\infty)$ if (V5) is satisfied (see [48, Lemma 2.4]). Taking advantage of this and using the Mountain-pass theorem, Ekeland's variational principle, and Trudinger-Moser type inequality established in [2], Yang [48] studied a singular quasilinear elliptic equation including (1.1) and showed the existence of a nontrivial mountain-pass type solution for (1.1) under (V5), (F1), (F2), (AR), (F4), and (F6$''$) (see also [2, 25] for similar results where the nonhomogeneous case was considered and the important condition (V4) or (V6) was used). We emphasize that the small nonzero perturbation $\varepsilon h(x)$ in their equation plays a crucial role in showing the nontriviality of the solutions [2, 25, 30, 48]. In recent paper [30], using the assumptions (V4) (or (V6)), (F1), (F2), (AR), (F4), and a restrictive condition involving the behavior of f at infinity, Lam and Lu [30, Theorem 2.3] showed the existence of nontrivial solutions for (1.1) by the Mountain-pass approach. We emphasize that all the arguments of their proof in the above references depend crucially on the *compact embedding* $\{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < \infty\} \subset L^q(\mathbb{R}^N)$ deduced from (V4) or (V5) or (V6), and in particular, on the compact embedding of the space into $L^N(\mathbb{R}^N)$.

As pointed out by Masmoudi and Sani [35], there is a long way to go yet in the case where the potential V is a constant since the working space is $W^{1,N}(\mathbb{R}^N)$ and the embedding $W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ $(q \geqslant N)$ is continuous but not compact, even if we restrict attention to the radial case. In this paper, we study the non-autonomous problem (1.1) without assuming any coercive condition on the potential and are interested in the following three cases: the periodic case, radially symmetric case, and the asymptotical constant case, i.e., V satisfies (V1) or (V2) or (V3). Although we may define the following constraint set involving the Pohozaev identity for (1.1) as in the constant potential case [35]:

$$\mathcal{P}_V := \left\{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} \left(V(x) + \frac{1}{N}\nabla V \cdot x\right)|u|^N dx - N\int_{\mathbb{R}^N} F(u)dx = 0\right\},$$

and consider the similar minimization problem $A_V := \inf\{\frac{1}{N}\|\nabla u\|_N^N : u \in \mathcal{P}_V\}$, we cannot deduce a minimizing sequence $\{u_n\} \subset \mathcal{P}_V$, as in [35, Remark 7.2], such that

$$\{u_n\} \subset W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N), \quad \frac{1}{N}\|\nabla u_n\|_N^N \to A_V, \quad \|u_n\|_N = 1,$$

which lies at the heart of the proof in [35, Section 7], since the Schwarz symmetrization method and rescaling skill are no longer valid for the non-autonomous problem (1.1). Thus, the constrained minimization approach used in [29,35] is not adoptable in our case, it requires exploring some new methods and techniques in order to find nontrivial solutions and ground state solutions for (1.1). Precisely, the following question will be addressed in this paper.

Question 1. *Study the non-autonomous problem* (1.1) *with the potential satisfying the above three conditions* (V1)–(V3) *respectively, and establish the existence of Mountain-pass type solutions and ground state solutions under more general hypotheses than* (AR), (F5$'$)–(F7$'$), (F6$''$)–(F7$''$)*, and* (1.11)*. Due to the difficulty to compare the growth conditions like* (F6$'$) (or (F6$''$), (1.11)) *and* (F7$'$) (or (F7$''$)) *involving the behavior of the nonlinearity f at infinity and near the origin respectively, whether one can find a proper condition as in* [35] *unifying all the above conditions to guarantee the existence of ground states* (*see Remarks* 1.8 *and* 1.9)*.*

Recently, the following open question was proposed by Yang [48, Lines 6–7 from the bottom, p, 1682].

Question 2. *The lower bound* $\inf_{r>0} \frac{N^N}{\alpha_0^{N-1} r^N} e^{NV_r(N-2)!r^N/N^N}$ *of* β_0 *is not the best constant in* (F6$''$)*, and it would be interesting if one can find an explicit smaller number replacing it* (*see Remark* 1.8)*.*

In this paper, some new methods and techniques will be developed to solve the above two questions, precisely, we introduce the following *unified condition* (VF) involving supremum of the potential V and the Trudinger-Moser ratio $\mathcal{C}^*_{\mathrm{TM}}$, and use the following more general conditions (F3), (F5), (F6), and (F7) instead of the classical condition (AR), the strict monotonicity condition (F5′) and the stronger growth conditions (F6′) (or (F6″), (1.11)), and (F7′) (or (F7″)), respectively (see Remark 1.8).

(VF) $V^* = \sup_{x\in\mathbb{R}^N} V(x) < \mathcal{C}^*_{\rm TM}$, where $\mathcal{C}^*_{\rm TM}$ is defined by (1.10).

(F3) $f(t)t \geqslant NF(t) \geqslant 0$ for all $t \in \mathbb{R}$, and

$$\frac{f(t)}{t|t|^{N-2}} \geqslant \frac{V_*}{2} \Rightarrow f(t)t - NF(t) > 0.$$

(F3′) $f(t)t \geqslant 0$ for all $t \in \mathbb{R}$.

(F5) $t \mapsto \frac{f(t)}{|t|^{N-1}}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

(F6) $\liminf_{t\to+\infty} \frac{t^{N/(N-1)}F(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} \geqslant \kappa > \frac{(N-1)!V^*}{2\alpha_0^N}$.

(F7) There exists a $p > N$ such that $F(t) \geqslant \beta|t|^p$ for all $t \in \mathbb{R}$, where $\beta > \frac{V^*}{N}(\frac{\alpha_0}{\alpha_N})^{\frac{(p-N)(N-1)}{N}}\mathcal{C}^p_{p,N}$ and

$$\mathcal{C}_{p,N} := \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_N^{1-N/p}\|u\|_N^{N/p}}{\|u\|_p}.$$

The main difficulties of the non-autonomous problem (1.1) are the lack of compactness, quasilinear characteristic of the equation, and the difficulties aroused by the critical exponential growth of f at infinity. Motivated by the works [2, 24, 29, 35, 48], we first use the unified condition (VF) to construct *a special path* (defined later by (3.10)) based on some key observations, and this allows us to control the Mountain-pass minimax level by a fine threshold $\frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$ under which the compactness can be restored for the critical case (see Lemma 3.1). It will be shown later in Lemmas 3.2 and 3.3 that such a condition (VF) can be deduced both from the explicit and general hypotheses (F6) and (F7). Secondly, to achieve our goal of finding a nontrivial solution from the Cerami sequence obtained by the Mountain-pass theorem, we establish *two compactness lemmas* for general nonlinear functionals in $W^{1,N}(\mathbb{R}^N)$ which allow us to verify that the Cerami sequence is nonvanishing and generalize the related ones obtained by Ibrahim et al. [29, Theorem 1.5] for the case $N = 2$ and by Masmoudi and Sani [35, Theorem 1.6] for the case $N \geqslant 3$ in the radially symmetric space $W^{1,N}_{\rm rad}(\mathbb{R}^N)$ (see Lemmas 2.2 and 2.3).

Let us define the critical points set by

$$\mathcal{K} := \{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \Phi'(u) = 0\} \tag{1.12}$$

and the classic Nehari manifold

$$\mathcal{N} := \{u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\} : \langle\Phi'(u), u\rangle = 0\}, \tag{1.13}$$

where Φ is the energy functional associated with (1.1) defined later by (2.40). Now we are ready to state our first results for the periodic potential case.

Theorem 1.4. *Assume that V and f satisfy* (V0), (V1), *and* (F1)–(F4).

(i) *If* (VF) *holds, then* (1.1) *has a ground state solution $\tilde{u} \neq 0$, i.e., $\Phi(\tilde{u}) = \bar{m} := \inf_{\mathcal{K}} \Phi$.*

(ii) *If* (F6) *or* (F7) *holds, then* (1.1) *has a ground state solution $\tilde{u} \neq 0$.*

Using the monotonicity condition (F5) instead of the technical condition (F3), we find the ground state solution by minimizing the functional Φ on the Nehari manifold $\mathcal{N}$.

Theorem 1.5. *Assume that V and f satisfy* (V0), (V1), (F1), (F2), (F4), *and* (F5).

(i) *If* (VF) *holds, then* (1.1) *has a ground state solution $\bar{u} \in \mathcal{N}$ such that $\Phi(\bar{u}) = b := \inf_{\mathcal{N}} \Phi$.*

(ii) *If* (F6) *or* (F7) *holds, then* (1.1) *has a ground state solution $\bar{u} \in \mathcal{N}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi$.*

Set $\mathcal{N}_{\rm rad} := \{u \in W^{1,N}_{\rm rad}(\mathbb{R}^N)\setminus\{0\} : \langle\Phi'(u), u\rangle = 0\}$. Our main results concerning the radially symmetric potential case can be stated as follows.

Theorem 1.6. *Assume that V and f satisfy* (V0), (V2), (F1), (F2), *and* (F4).

(i) *If* (VF) *and* (F3) *hold, then* (1.1) *has a nontrivial Mountain-pass type solution.*

(ii) *If* (F6) (*or* (F7)) *and* (F3) *hold, then* (1.1) *has a nontrivial Mountain-pass type solution.*

(iii) *If* (VF) *and* (F5) *hold, then* (1.1) *has a Mountain-pass type solution* $\bar{u} \in \mathcal{N}_{\rm rad}$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{N}_{\rm rad}} \Phi$.

(iv) *If* (F6) (*or* (F7)) *and* (F5) *hold, then* (1.1) *has a Mountain-pass type solution* $\bar{u} \in \mathcal{N}_{\rm rad}$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{N}_{\rm rad}} \Phi$.

For the asymptotical constant potential case, i.e., the Rabinowitz type assumptions (V0) and (V3) are satisfied, we obtain the following existence results.

Theorem 1.7. *Assume that* V *and* f *satisfy* (V0), (V3), (F1), (F2), (F4), *and* (F5).

(i) *If* (VF) *holds, then* (1.1) *has a ground state solution* $\bar{u} \in \mathcal{N}$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi$.

(ii) *If* (F6) *or* (F7) *holds, then* (1.1) *has a ground state solution* $\bar{u} \in \mathcal{N}$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi$.

Remark 1.8. Conditions (F6) and (F7) are much weaker than (F6′) (or (F6″), (1.11)) and (F7′) (or (F7″)), respectively. Indeed, by noting that

$$\lim_{t\to+\infty} \frac{t^{N/(N-1)}F(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} = \lim_{t\to+\infty} \frac{NF(t)+(N-1)tf(t)}{N\alpha_0 \mathrm{e}^{\alpha_0 t^{N/(N-1)}}},$$

we see from (F4) that (F6) with $V^* = c$ can be deduced from the following assumption (H6) which is much weaker than (F6′).

(H6) $\liminf_{t\to+\infty} \frac{tf(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} \geqslant \kappa_1 > \frac{(N-2)!Nc}{2\alpha_0^{N-1}}$.

When (V0) holds, it is easy to see that (F6″) can be simplified to (F6′). By Lemma 3.4, it is easy to verify that (F7′) yields (F7) with $V^* = c$. Moreover, (F7) is much weaker than (F7″). Although it is difficult to compare the growth conditions (F6) and (F7), we succeed in finding a unified condition (VF) to guarantee existence of ground states for the non-autonomous problem (1.1) which unifies all the conditions (F6)–(F7), (F6′)–(F7′), (F6″)–(F7″), and (1.11) (see Lemmas 3.2 and 3.3). Thus the open questions 1–2 raised above are well solved in this paper.

Remark 1.9. Using the following condition (V0′) instead of (V0) and replacing V_* by V_0 in places where V_* appears, we see that the above Theorems 1.6 and 1.7 still hold.

(V0′) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ and for some $V_0 > 0$,

$$\int_{\mathbb{R}^N} [|\nabla u|^N + V(x)|u|^N]dx \geqslant V_0 \int_{\mathbb{R}^N} |u|^N dx, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N).$$

Then the above condition (V0′) allows the case where the potential $V(x)$ is nonnegative and vanishes on an open bounded domain in $\mathbb{R}^N$ (see [22]). Thus our argument and methods developed in this paper are valid for the degenerate potential case. Recently, Chen et al. [10] considered such a case where $N = 2$ and showed the existence of the ground state solution for (1.1) by establishing an associated Trudinger-Moser inequality and using the conditions (V3), (F1), (F2), (AR), (F4), (F5′), and (1.11). Our assumptions on the potential V and nonlinearity f are satisfied by a larger class of nonlinear functions. In this direction, our results improve and complement the related ones of [6, Corollary 1.5], [9], [24, Theorem 3.1], [29, Theorem 5.1], [35, Theorems 7.4 and 8.2], and [42, Theorem 2.2] for the constant potential case, and of [4, Theorem 1.1], [10, Theorem 1.3], [30, Theorem 2.3], and [48, Theorem 1.1], where the potential is large at infinity or satisfies the condition (V3).

There are many functions satisfying the conditions (F1)–(F6) such as

$$F(t) = |t|\phi_N(\alpha_0 t^{N/(N-1)}), \quad F(t) = \ln(1+|t|)\phi_N(\alpha_0 t^{N/(N-1)}), \tag{1.14}$$

and the following functions satisfy conditions (F1)–(F7):

$$F(t) = |t|\phi_N(\alpha_0 t^{N/(N-1)}) + \beta|t|^p, \quad F(t) = \ln(1+|t|)\phi_N(\alpha_0 t^{N/(N-1)}) + \beta|t|^p, \tag{1.15}$$

where $p > N$, $\phi_N(t)$ is defined by (1.6), and β is given by (F7).

Before completing this section, we sketch our proof. Using (F1), (F2), and Lemma 1.2, we apply the Mountain-pass theorem to get a Cerami sequence $\{u_n\} \subset W^{1,N}(\mathbb{R}^N)$ at the minimax level c^* defined later by (2.43). By virtue of the technical condition (F3) or the monotonicity condition (F5), we can show the boundedness of such a Cerami sequence. In order to derive a nontrivial solution from the weak limit $\bar{u} \in W^{1,N}(\mathbb{R}^N)$ of the sequence (i.e., $u_n \rightharpoonup \bar{u}$ in $W^{1,N}(\mathbb{R}^N)$), there are some preparations shall be made which can be summarized as follows:

(i) We first establish two compactness lemmas for general nonlinear functionals in $W^{1,N}(\mathbb{R}^N)$ to verify that the sequence $\{u_n\}$ is nonvanishing which generalize the related ones of [29, Theorem 1.5] and [35, Theorem 1.6] established in the radially symmetric space $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ (see Lemmas 2.2 and 2.3).

(ii) Since (1.1) is quasilinear, different from the case $N = 2$, the functional $B(u, v) := \int_{\mathbb{R}^N} |\nabla u|^{N-2}\nabla u \cdot \nabla v dx$ is not bilinear and it requires to verify that $B(u_n, v) \to B(\bar{u}, v)$ for any $v \in W^{1,N}(\mathbb{R}^N)$. This will be done via some analytical techniques and applying the result of Cherrier [15] (see Lemma 2.17).

(iii) Using the unified condition (VF), a special path shall be constructed to control the minimax level c^* such that $c^* < \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}$ which allows us to restore the compactness for the critical case. Based on the key observation that there exists a $\tilde{u} \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$ with $\|\nabla \tilde{u}\|_N^N < (\frac{\alpha_N}{\alpha_0})^{N-1}$ and $\varepsilon_0 > 0$ such that (see (3.4)) $(V^* + \varepsilon_0)\|\tilde{u}\|_N^N = N \int_{\mathbb{R}^N} F(\tilde{u})dx$, and in view of the monotonicity of $\Phi(\theta\tilde{u}_{t_0})$ with respect to $\theta \in [0, 1]$ for some $t_0 > 0$ (here $\tilde{u}_{t_0}(\cdot) := \tilde{u}(\cdot/t_0)$), we find a proper path involving $\tilde{u}_{t_0}$ (see (3.10)) to achieve our goal, see Lemma 3.1.

(iv) For the periodic potential case, i.e., (V1) holds, using the technical condition (F3) which is much weaker than (AR) and employing some analytic techniques, we show the existence of ground state solutions for the non-autonomous problem (1.1). Such existing results, to the best of our knowledge, seem to be new. In order to find a ground state solution constrained on the Nehari manifold, following Rabinowitz [40], we use the monotonicity condition (F5) and give the minimax characterization of $\inf_{\mathcal{N}} \Phi$ in Lemmas 2.12 and 2.13, i.e., $c* = \inf_{\mathcal{N}} \Phi = \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}} \max_{t\geqslant 0} \Phi(tu)$.

For the semilinear case $N = 2$, it is worth pointing out that (1.1) with sign-changing potential V and critical exponential nonlinearity f has been considered in [5, 12, 26], where the existence of nontrivial solutions was obtained via the generalized Nehari manifold method [5], the non-Nehari manifold method combined with an approximation scheme [43], and the Schwarz symmetrization method [26], respectively.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary results, establish two compactness lemmas for general nonlinear functionals in $W^{1,N}(\mathbb{R}^N)$, and show the boundedness of the Cerami sequence obtained by the Mountain-pass theorem. In Section 3, using a direct method and some delicate estimates, we show that the Mountain-pass minimax level can be controlled by a fine threshold under either the unified condition (VF) or the concrete condition (F6) (or (F7)). We study the periodic potential case in Section 4 and show Theorems 1.4 and 1.5. Section 5 is devoted to the radially symmetric potential case where Theorem 1.6 is proved. In Section 6, we consider the asymptotical constant potential case and complete the proof of Theorem 1.7.

2 Variational framework and preliminaries

Define

$$\|u\|^N := \int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)dx, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N).$$

Then by (V0), $\|\cdot\|$ is an equivalent norm with the standard one in $W^{1,N}(\mathbb{R}^N)$. Throughout the paper, we use $C_1, C_2, \ldots$ to denote positive constants possibly different in different places.

The following improved inequality yields Lemma 1.2 (see [29, 34] and [35, Section 6]), it was first established by Ibrahim et al. [29] for the case $N = 2$ and generalized to all the dimensions $N \geqslant 2$ by Masmoudi and Sani [35].

Lemma 2.1 (See [35, Theorem 1.4]). *Let $N \geqslant 2$. Then there exists a constant $C_N > 0$ such that*

$$\sup_{\substack{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\} \\ \|\nabla u\|_N^N \leqslant 1}} \frac{1}{\|u\|_N^N} \int_{\mathbb{R}^N} \frac{\phi_N(\alpha_N |u|^{\frac{N}{N-1}})}{(1 + |u|)^{\frac{N}{N-1}}} dx \leqslant C_N.$$

Moreover, this inequality fails if the power $\frac{N}{N-1}$ *in the denominator is replaced by any* $p < \frac{N}{N-1}$.

Applying Lemmas 1.2 and 2.1, we establish the following two compactness lemmas which allow us to verify that the Cerami sequence obtained later by Lemma 2.8 is nonvanishing. A similar version for functionals in the radially symmetric space $W^{1,N}_{\rm rad}(\mathbb{R}^N)$ was established in [29, Theorem 1.5] for the case $N = 2$ and [35, Theorem 1.6] for the case $N \geqslant 3$. Here, we consider more general nonlinear functionals in $W^{1,N}(\mathbb{R}^N)$ the results of which seem to be new and shall be of independent interest.

Lemma 2.2. *Let* $G : \mathbb{R} \to [0, +\infty)$ *be any continuous function and* $K > 0$ *such that*

$$\lim_{|t|\to+\infty} \frac{|t|^{N/(N-1)}G(t)}{\mathrm{e}^{K|t|^{N/(N-1)}}} = 0 \quad \text{and} \quad \lim_{|t|\to 0} \frac{G(t)}{|t|^N} = 0. \tag{2.1}$$

Suppose that there exists a sequence $\{u_n\} \subset W^{1,N}(\mathbb{R}^N)$ *satisfying*

(i) $u_n \rightharpoonup \bar{u}$ *in* $W^{1,N}(\mathbb{R}^N)$;

(ii) $\|\nabla u_n\|_N^N \leqslant (\frac{\alpha_N}{K})^{N-1}$;

(iii) $u_n \to \bar{u}$ *in* $L^p(\mathbb{R}^N)$ *for some* $p \geqslant N$.

Then it holds that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |G(u_n) - G(\bar{u})| dx = 0. \tag{2.2}$$

Proof. By (2.1), we have

$$\lim_{|t|\to+\infty} \frac{(1+|t|)^{N/(N-1)}G(t)}{\phi_N(K|t|^{N/(N-1)})} = \lim_{|t|\to+\infty} \frac{|t|^{N/(N-1)}G(t)}{\mathrm{e}^{K|t|^{N/(N-1)}}} = 0. \tag{2.3}$$

For any given $\varepsilon > 0$, it follows from (2.1) and (2.3) that there exist $\delta = \delta(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that

$$0 \leqslant G(t) \leqslant \varepsilon|t|^N, \quad \forall\, |t| \leqslant \delta \tag{2.4}$$

and

$$0 \leqslant G(t) \leqslant \varepsilon \frac{\phi_N(K|t|^{N/(N-1)})}{(1+|t|)^{N/(N-1)}}, \quad \forall\, |t| \geqslant M. \tag{2.5}$$

There are two possible cases to distinguish.

Case (1) $\|\nabla u_n\|_N^N \leqslant (\frac{\alpha_N}{2K})^{N-1}$. Then $K\|\nabla u_n\|_N^{N/(N-1)} \leqslant \frac{\alpha_N}{2}$. Hence, by (2.5) and Lemma 1.2-ii), we have

$$\begin{aligned}\int_{|u_n|\geqslant M} G(u_n)dx &\leqslant \varepsilon \int_{|u_n|\geqslant M} \frac{\phi_N(K|u_n|^{N/(N-1)})}{(1+|u_n|)^{N/(N-1)}} dx \\ &\leqslant \frac{\varepsilon}{(1+M)^{N/(N-1)}} \int_{|u_n|\geqslant M} \phi_N(K|u_n|^{N/(N-1)})dx \\ &\leqslant C_1\varepsilon. \end{aligned}\tag{2.6}$$

Case (2) $(\frac{\alpha_N}{2K})^{N-1} < \|\nabla u_n\|_N^N \leqslant (\frac{\alpha_N}{K})^{N-1}$. Then $\frac{\alpha_N}{2} < K\|\nabla u_n\|_N^{N/(N-1)} \leqslant \alpha_N$. Hence, by (2.5) and Lemma 2.1, we have

$$\begin{aligned}\int_{|u_n|\geqslant M} G(u_n)dx &\leqslant \varepsilon \int_{|u_n|\geqslant M} \frac{\phi_N(K|u_n|^{N/(N-1)})}{(1+|u_n|)^{N/(N-1)}} dx \\ &\leqslant \varepsilon \int_{|u_n|\geqslant M} \frac{\phi_N(\alpha_N(\frac{|u_n|}{\|\nabla u_n\|_N})^{\frac{N}{N-1}})}{[1+(\frac{\alpha_N}{2K})^{(N-1)/N}\frac{|u_n|}{\|\nabla u_n\|_N}]^{\frac{N}{N-1}}} dx \\ &\leqslant \frac{\varepsilon}{\min\{1, \frac{\alpha_N}{2K}\}} \int_{|u_n|\geqslant M} \frac{\phi_N(\alpha_N(\frac{|u_n|}{\|\nabla u_n\|_N})^{\frac{N}{N-1}})}{(1+\frac{|u_n|}{\|\nabla u_n\|_N})^{\frac{N}{N-1}}} dx\end{aligned}$$

$$\leqslant C_2\varepsilon. \tag{2.7}$$

The above two cases show that

$$\int_{|u_n|\geqslant M} G(u_n)dx \leqslant C_3\varepsilon. \tag{2.8}$$

On the other hand, from (2.4) and the boundedness of $\{\|u_n\|_N\}$, we obtain

$$\int_{|u_n|\leqslant\delta} G(u_n)dx \leqslant \varepsilon\int_{|u_n|\leqslant\delta} |u_n|^N dx \leqslant C_4\varepsilon. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\begin{aligned}\int_{(|u_n|\leqslant\delta)\cup(|u_n|\geqslant M)} G(u_n)dx &\leqslant \int_{|u_n|\leqslant\delta} G(u_n)dx + \int_{|u_n|\geqslant M} G(u_n)dx\\ &\leqslant (C_3+C_4)\varepsilon.\end{aligned} \tag{2.10}$$

By continuity of the function G, we can choose a constant $C_\varepsilon>0$ such that

$$0\leqslant G(t)\leqslant C_\varepsilon|t|^p, \quad \forall\,\text{a.e.}\ |t|\in[\delta,M]. \tag{2.11}$$

Noting that $u_n\to\bar{u}$ in $L^p(\mathbb{R}^N)$, by Lemma [46, Lemma A.1], we see that there exists a $w_0\in L^p(\mathbb{R}^N)$ such that

$$|u_n(x)|\leqslant w_0(x), \quad |\bar{u}(x)|\leqslant w_0(x) \quad \text{a.e. } x\in\mathbb{R}^N. \tag{2.12}$$

Now we can choose $R_\varepsilon>0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} G(\bar{u})dx<\varepsilon, \quad C_\varepsilon\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} |w_0|^p dx<\varepsilon. \tag{2.13}$$

Hence, it follows from (2.10)–(2.13) that

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} G(u_n)dx\\ &\quad\leqslant \int_{(\mathbb{R}^N\setminus B_{R_\varepsilon})\cap(\delta\leqslant|u_n|\leqslant M)} G(u_n)dx + \int_{(|u_n|\leqslant\delta)\cup(|u_n|\geqslant M)} G(u_n)dx\\ &\quad\leqslant C_\varepsilon\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} |w_0|^p dx + (C_3+C_4)\varepsilon\\ &\quad< (1+C_3+C_4)\varepsilon.\end{aligned} \tag{2.14}$$

Let $A_n^M:=\{x\in B_{R_\varepsilon}:|u_n(x)|\geqslant M\}$. Then it follows from (2.8) that

$$\int_{A_n^M} G(u_n)dx\leqslant C_3\varepsilon. \tag{2.15}$$

Since $\{\|u_n\|\}$ is bounded, it implies that $\text{meas}(A_n^M)\to 0$ as $M\to+\infty$ uniformly on $n\in\mathbb{N}$, so

$$\int_{A_n^M} G(\bar{u})dx = o_M(1) \quad \text{uniformly on } n\in\mathbb{N}. \tag{2.16}$$

By (2.1), one has

$$|G(u_n(x))-G(\bar{u}(x))|\chi_{B_{R_\varepsilon}\setminus A_n^M}(x)\leqslant M^N + C_5 \mathrm{e}^{KM^{N/(N-1)}} + G(\bar{u}(x)), \quad \forall\, x\in B_{R_\varepsilon}. \tag{2.17}$$

Since $u_n\to\bar{u}$ a.e. on $\mathbb{R}^N$, it is easy to verify that

$$|G(u_n(x))-G(\bar{u}(x))|\chi_{B_{R_\varepsilon}\setminus A_n^M}(x)\to 0 \quad \text{a.e. } x\in B_{R_\varepsilon}. \tag{2.18}$$

Hence it follows from (2.17), (2.18), and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_{B_{R_\varepsilon}\setminus A_n^M}|G(u_n)-G(\bar{u})|dx=0. \tag{2.19}$$

Hence, from (2.13)–(2.16) and (2.19), we derive

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^N}|G(u_n)-G(\bar{u})|dx\\&\leqslant\lim_{n\to\infty}\left[\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}}|G(u_n)-G(\bar{u})|dx+\int_{A_n^M}G(u_n)dx+\int_{A_n^M}G(\bar{u})dx\right]\\&\leqslant(3+2C_3+C_4)\varepsilon. \end{aligned}\tag{2.20}$$

Due to the arbitrariness of $\varepsilon>0$, we can deduce (2.2) from (2.20). □

Lemma 2.3. *Let $G,H:\mathbb{R}\to[0,+\infty)$ be two continuous functions satisfying*

$$\lim_{|t|\to\infty}\frac{G(t)}{H(t)}=0\quad\text{and}\quad\lim_{|t|\to0}\frac{G(t)}{|t|^N}=0. \tag{2.21}$$

Suppose that $u_n\rightharpoonup\bar{u}$ in $W^{1,N}(\mathbb{R}^N)$ and $u_n\to\bar{u}$ in $L^p(\mathbb{R}^N)$ for some $p\geqslant N$. If there exists a constant $K_1>0$ such that

$$\int_{\mathbb{R}^N}H(u_n)dx\leqslant K_1, \tag{2.22}$$

then it holds that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|G(u_n)-G(\bar{u})|dx=0. \tag{2.23}$$

Proof. For any given $\varepsilon>0$, it follows from (2.21) that there exist $\delta=\delta(\varepsilon)>0$ and $M=M(\varepsilon)>0$ such that

$$0\leqslant G(t)\leqslant\varepsilon|t|^N,\quad\forall\,|t|\leqslant\delta \tag{2.24}$$

and

$$0\leqslant G(t)\leqslant\varepsilon H(t),\quad\forall\,|t|\geqslant M. \tag{2.25}$$

Hence from (2.22) and (2.25), we have

$$\int_{|u_n|\geqslant M}G(u_n)dx\leqslant\varepsilon\int_{|u_n|\geqslant M}H(u_n)dx\leqslant K_1\varepsilon.$$

On the other hand, from (2.24) and the boundedness of $\{\|u_n\|_N\}$, we obtain

$$\int_{|u_n|\leqslant\delta}G(u_n)dx\leqslant\varepsilon\int_{|u_n|\leqslant\delta}|u_n|^Ndx\leqslant C_6\varepsilon.$$

By similar arguments as in the proof of Lemma 2.2, we can get (2.23). □

Lemma 2.4. *Assume that* (F1), (F2), (F3′), *and* (F4) *hold. Let $u_n\rightharpoonup\bar{u}$ in $W^{1,N}(\mathbb{R}^N)$ and $u_n\to\bar{u}$ in $L^p(\mathbb{R}^N)$ for some $p\geqslant N$. If there exists a constant $K_0>0$ such that*

$$\int_{\mathbb{R}^N}f(u_n)u_ndx\leqslant K_0, \tag{2.26}$$

then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}F(u_n)dx=\int_{\mathbb{R}^N}F(\bar{u})dx. \tag{2.27}$$

This Lemma is a direct corollary of Lemma 2.3 by setting $G(t) = F(t)$ and $H(t) = f(t)t$.

As well known, the Fréchet derivative of the functional $\int_{\mathbb{R}^N} F(u)dx$ is no longer weakly sequentially continuous; here we verify it with the help of the above auxiliary condition (2.26).

Lemma 2.5. *Assume that* (F1), (F2), *and* (F3′) *hold. Let* $u_n \rightharpoonup \bar{u}$ *in* $W^{1,N}(\mathbb{R}^N)$*. If* (2.26) *is satisfied, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(u_n)\phi dx = \int_{\mathbb{R}^N} f(\bar{u})\phi dx, \quad \forall\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{2.28}$$

Proof. Let $\Omega = \text{supp}\ \phi$. For any given $\varepsilon > 0$, we have

$$\int_{|u_n|\geqslant K_0\|\phi\|_\infty\varepsilon^{-1}} |f(u_n)\phi|dx \leqslant \frac{\varepsilon}{K_0}\int_{|u_n|\geqslant K_0\|\phi\|_\infty\varepsilon^{-1}} f(u_n)u_n dx < \varepsilon. \tag{2.29}$$

Since $f(\bar{u})\phi \in L^1(\Omega)$, there exists a $\delta > 0$ such that

$$\int_A |f(\bar{u})\phi|dx < \varepsilon, \quad \text{if meas}(A) \leqslant \delta \tag{2.30}$$

for all measurable set $A \subset \Omega$. Since $\{\|u_n\|\}$ is bounded, there exists a constant $C > 0$ such that $\|\nabla u_n\|_N + \|u_n\|_N \leqslant C$. Then for the fixed $\delta > 0$, we can find $M_1 > 0$ such that

$$\text{meas}(\{x \in \Omega : |u_n(x)| \geqslant M_1\}) \leqslant \delta, \quad \forall\, n \in \mathbb{N}. \tag{2.31}$$

Set $M_\varepsilon = \max\{M_1, K_0\|\phi\|_\infty\varepsilon^{-1}\}$ and let $A_n := \{x \in \Omega : |u_n(x)| \geqslant M_\varepsilon\}$. Then one has

$$\int_{A_n} |f(u_n)\phi|dx < \varepsilon, \quad \int_{A_n} |f(\bar{u})\phi|dx < \varepsilon. \tag{2.32}$$

Since $u_n \to \bar{u}$ a.e. Ω, it is easy to verify that

$$|[f(u_n(x)) - f(\bar{u}(x))]\phi(x)|\chi_{|u_n|\leqslant M_\varepsilon}(x) \to 0 \quad \text{a.e. } x \in \Omega. \tag{2.33}$$

Moreover, one has

$$|[f(u_n(x)) - f(\bar{u}(x))]\phi(x)|\chi_{|u_n|\leqslant M_\varepsilon}(x) \leqslant \|\phi\|_\infty \max_{|t|\leqslant M_\varepsilon} |f(t)| + |f(\bar{u}(x))\phi(x)| \quad \text{a.e. } x \in \Omega. \tag{2.34}$$

So it follows from (2.33), (2.34), and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_{\Omega\setminus A_n} |[f(u_n) - f(\bar{u})]\phi|dx = 0. \tag{2.35}$$

Hence from (2.32) and (2.35), we see that

$$\begin{aligned}
\lim_{n\to\infty}\left|\int_{\mathbb{R}^N}[f(u_n) - f(\bar{u})\phi]dx\right| &= \lim_{n\to\infty}\left|\int_{\Omega}[f(u_n) - f(\bar{u})\phi]dx\right| \\
&\leqslant \lim_{n\to\infty}\left[\int_{A_n} |f(u_n)\phi|dx + \int_{A_n} |f(\bar{u})\phi|dx\right] \\
&\leqslant 2\varepsilon,
\end{aligned}$$

which implies that (2.28) holds due to the arbitrariness of $\varepsilon > 0$. □

In the following lemma, we show the boundedness of general nonlinear functionals in $W^{1,N}(\mathbb{R}^N)$.

Lemma 2.6. *Assume that* (F1), (F2), (F3′), *and* (F4) *hold. Then for any* $L \in (0, (\frac{\alpha_N}{\alpha_0})^{N-1})$*, there exists a constant* $C_{L,F} > 0$ *such that*

$$\sup_{\substack{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla u\|_N^N\leqslant L}} \frac{1}{\|u\|_N^N}\int_{\mathbb{R}^N} F(u)dx \leqslant C_{L,F}. \tag{2.36}$$

Proof. Set $\alpha_* = \alpha_N/L^{1/(N-1)}$. Then $\alpha_* > \alpha_0$. By virtue of (F1), (F2), (F3′), and (F4), we deduce

$$\lim_{|t|\to\infty} \frac{|t|^{N/(N-1)}F(t)}{\mathrm{e}^{\alpha|t|^{N/(N-1)}}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0, \end{cases} \tag{2.37}$$

which, together with (F1) and (F2), implies that there exists a $C_1 > 0$ such that

$$|F(t)| \leqslant \frac{C_1\phi_N(\alpha_*|t|^{N/(N-1)})}{(1+|t|)^{N/(N-1)}}, \quad \forall\, t \in \mathbb{R}. \tag{2.38}$$

Let $v = (\frac{\alpha_*}{\alpha_N})^{(N-1)/N}u$. Then it follows from (2.38) and Lemma 2.1 that

$$\begin{aligned}
&\sup_{\substack{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla u\|_N^N\leqslant L}} \frac{1}{\|u\|_N^N}\int_{\mathbb{R}^N} F(u)dx \\
&= \left(\frac{\alpha_*}{\alpha_N}\right)^{N-1} \sup_{\substack{v\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla v\|_N^N\leqslant 1}} \frac{1}{\|v\|_N^N}\int_{\mathbb{R}^N} F\left(\left(\frac{\alpha_N}{\alpha_*}\right)^{(N-1)/N} v\right)dx \\
&\leqslant C_1 \sup_{\substack{v\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla v\|_N^N\leqslant 1}} \frac{1}{\|v\|_N^N}\int_{\mathbb{R}^N} \frac{\phi_N(\alpha_N|v|^{N/(N-1)})}{[1+(\alpha_N/\alpha_*)^{(N-1)/N}|v|]^{N/(N-1)}}dx \\
&\leqslant C_2 \sup_{\substack{v\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}\\ \|\nabla v\|_N^N\leqslant 1}} \frac{1}{\|v\|_N^N}\int_{\mathbb{R}^N} \frac{\phi_N(\alpha_N|v|^{N/(N-1)})}{(1+|v|)^{N/(N-1)}}dx \\
&\leqslant C_{L,F}.
\end{aligned}$$

This shows that (2.36) holds. □

Next, we introduce the following useful lemma established in [48].

Lemma 2.7 (See [48, Lemmas 2.1 and 2.2])**.** (i) *It holds that*

$$[\phi_N(s)]^p \leqslant \phi_N(ps), \quad \forall\, s \geqslant 0, \quad p \geqslant 1.$$

(ii) *For any $q, q' > 1$ with $\frac{1}{q} + \frac{1}{q'} = 1$, it holds that*

$$\phi_N(s+t) \leqslant \frac{1}{q}\phi_N(qs) + \frac{1}{q'}\phi_N(q't), \quad \forall\, s,t \geqslant 0.$$

For any $\varepsilon > 0, \alpha > \alpha_0$, and $q > 0$, it follows from (F1) and (F2) that there exists a $C = C(\varepsilon, \alpha, q) > 0$ such that

$$|F(t)| \leqslant \varepsilon|t|^N + C|t|^q\phi_N(\alpha|t|^{N/(N-1)}), \quad \forall\, t \in \mathbb{R}. \tag{2.39}$$

To apply the critical point theory to (1.1), we define the functional $\Phi : W^{1,N}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{N}\int_{\mathbb{R}^N} [|\nabla u|^N + V(x)|u|^N]dx - \int_{\mathbb{R}^N} F(u)dx, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N). \tag{2.40}$$

By (2.39) and a standard argument, one has $\Phi \in \mathcal{C}^1(W^{1,N}(\mathbb{R}^N), \mathbb{R})$ with

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^N} [|\nabla u|^{N-2}\nabla u\cdot\nabla v + V(x)|u|^{N-2}uv]dx - \int_{\mathbb{R}^N} f(u)vdx, \quad \forall\, u,v \in W^{1,N}(\mathbb{R}^N). \tag{2.41}$$

Hence, solutions of (1.1) are critical points of the functional (2.40).

Now we verify the Mountain-pass geometry for the functional Φ and find a corresponding Cerami sequence via the Mountain-pass theorem.

Lemma 2.8. *Assume that* (V0), (F1), (F2), *and* (F3′) *hold. Then there exists a sequence* $\{u_n\} \subset W^{1,N}(\mathbb{R}^N)$ *satisfying*

$$\Phi(u_n) \to c^*, \quad \|\Phi'(u_n)\|(1+\|u_n\|) \to 0, \tag{2.42}$$

where $c^* > 0$ *is given by*

$$c^* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \tag{2.43}$$

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], W^{1,N}(\mathbb{R}^N)) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}. \tag{2.44}$$

Proof. Using (2.39), we deduce that there exist constants $\alpha > \alpha_0$ and $C_1 > 0$ such that

$$F(t) \leqslant \frac{V_*}{2N}|t|^N + C_1|t|^{N+1}\phi_N(\alpha|t|^{N/(N-1)}), \quad \forall\, t \in \mathbb{R}. \tag{2.45}$$

From (2.45) and Lemmas 1.2(ii) and 2.7, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} F(u)dx &\leqslant \frac{V_*}{2N}\|u\|_N^N + C_1\int_{\mathbb{R}^N} \phi_N(\alpha|u|^{N/(N-1)})|u|^{N+1}dx \\ &\leqslant \frac{V_*}{2N}\|u\|_N^N + C_1\left[\int_{\mathbb{R}^N} \phi_N(2\alpha|u|^{N/(N-1)})dx\right]^{1/2}\|u\|_{2N+2}^{N+1} \\ &\leqslant \frac{1}{2N}\|u\|^N + C_2\|u\|^{N+1}, \quad \forall\, \|u\| \leqslant \left(\frac{\alpha_N}{4\alpha}\right)^{\frac{N-1}{N}}.\end{aligned} \tag{2.46}$$

Thus, (V0), together with (2.40) and (2.46), implies that

$$\begin{aligned}\Phi(u) &= \frac{1}{N}\|u\|^N - \int_{\mathbb{R}^N} F(u)dx \\ &\geqslant \frac{1}{N}\|u\|^N - \frac{1}{2N}\|u\|^N - C_2\|u\|^{N+1} \\ &\geqslant \frac{1}{2N}\|u\|^N - C_2\|u\|^{N+1}, \quad \forall\, \|u\| \leqslant \left(\frac{\alpha_N}{4\alpha}\right)^{\frac{N-1}{N}}.\end{aligned} \tag{2.47}$$

Consequently, there exist $\kappa_0 > 0$ and $0 < \rho_0 < (\frac{\alpha_N}{4\alpha})^{\frac{N-1}{N}}$ such that

$$\Phi(u) \geqslant \kappa_0, \quad \forall\, u \in S_{\rho_0} := \{u \in W^{1,N}(\mathbb{R}^N) : \|u\| = \rho_0\}. \tag{2.48}$$

For any $w_0 \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, it is easy, by (F1), to verify that $\lim_{t\to\infty} \Phi(tw_0) = -\infty$, so we can take a sufficiently large $T > 0$ such that $e := Tw_0 \in \{u \in W^{1,N}(\mathbb{R}^N) : \|u\| > \rho_0\}$ and $\Phi(e) < 0$. Then in view of the Mountain pass theorem, we conclude that there is a sequence $\{u_n\} \subset W^{1,N}(\mathbb{R}^N)$ satisfying (2.42). □

Lemma 2.9. *Assume that* (V0), (F1), (F2), *and* (F5) *hold. Then*

$$\Phi(u) \geqslant \Phi(tu) + \frac{1-t^N}{N}\langle\Phi'(u), u\rangle, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N), \quad t \geqslant 0. \tag{2.49}$$

Proof. Obviously, (F5) implies the following inequality:

$$\begin{aligned}\frac{1-t^N}{N}f(s)s + F(ts) - F(s) &= \int_t^1 \left(\frac{f(s)}{|s|^{N-2}s} - \frac{f(\tau s)}{|\tau s|^{N-2}\tau s}\right)|\tau s|^{N-2}\tau s^2 d\tau \\ &\geqslant 0, \quad \forall\, s \in \mathbb{R}, \quad t \geqslant 0.\end{aligned} \tag{2.50}$$

By (2.40), (2.41), and (2.50), we obtain

$$\Phi(u) - \Phi(tu) = \frac{1-t^N}{N}\|u\|^N - \int_{\mathbb{R}^N} [F(u) - F(tu)]dx$$

$$= \frac{1-t^N}{N}\langle \Phi'(u), u\rangle + \int_{\mathbb{R}^N}\left[\frac{1-t^N}{N}f(u)u - F(u) + F(tu)\right]dx$$
$$\geqslant \frac{1-t^N}{N}\langle \Phi'(u), u\rangle, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N), \quad t \geqslant 0.$$

□

From Lemma 2.9, we have the following corollary.

Corollary 2.10. *Assume that* (V0), (F1), (F2), *and* (F5) *hold. Then*

$$\Phi(u) \geqslant \max_{t\geqslant 0}\Phi(tu), \quad \forall\, u \in \mathcal{N}. \tag{2.51}$$

Lemma 2.11. *Assume that* (V0), (F1), (F2), *and* (F5) *hold. Then for any* $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$, *there exists a* $t_u > 0$ *such that* $t_u u \in \mathcal{N}$.

Proof. Let $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ be fixed and define a function $\zeta(t) := \Phi(tu)$ on $[0,\infty)$. Clearly, it follows from (2.40) and (2.41) that for $t \neq 0$,

$$\zeta'(t) = 0 \Leftrightarrow t^N\|u\|^N - \int_{\mathbb{R}^N} f(tu)tudx = 0 \Leftrightarrow \langle \Phi'(tu), tu\rangle = 0 \Leftrightarrow tu \in \mathcal{N}.$$

By (2.47), (F1), (F2), and the fact that $f(t)t \geqslant 0$, one has $\zeta(0) = 0$ and $\zeta(t) > 0$ for $t > 0$ sufficiently small and $\zeta(t) < 0$ for t large enough. Hence, $\max_{t\in(0,\infty)}\zeta(t)$ is achieved at some $t_u > 0$ so that $\zeta'(t_u) = 0$ and $t_u u \in \mathcal{N}$. □

Applying Corollary 2.10 and Lemma 2.11, we show the following lemma 2.12 and obtain the minimax characterization of $\inf_{\mathcal{N}}\Phi$ in following two lemmas.

Lemma 2.12. *Assume that* (V0), (F1), (F2), *and* (F5) *hold. Then*

$$b := \inf_{\mathcal{N}}\Phi = \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}}\max_{t\geqslant 0}\Phi(tu). \tag{2.52}$$

Lemma 2.13. *Assume that* (V0), (F1), (F2), *and* (F5) *hold. Then* $b = c^*$.

Proof. By Lemma 2.12, we choose a sequence $\{v_n\} \subset W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ such that

$$b \leqslant \max_{t\geqslant 0}\Phi(tv_n) < b + \frac{1}{n}, \quad n \in \mathbb{N}. \tag{2.53}$$

For any fixed $u \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$, it holds that $\Phi(tu) < 0$ for t large. Then there exist $t_n = t(v_n) > 0$ and $s_n > t_n$ such that

$$\Phi(t_n v_n) = \max_{t\geqslant 0}\Phi(tv_n), \quad \Phi(s_n v_n) < 0, \quad n \in \mathbb{N}. \tag{2.54}$$

Let $\gamma_n(t) = ts_n v_n$ for $t \in [0,1]$. Then $\gamma_n \in \Gamma$ defined by (2.44), and it follows from (2.53) and (2.54) that

$$\max_{t\in[0,1]}\Phi(\gamma_n(t)) = \max_{t\geqslant 0}\Phi(tv_n) < b + \frac{1}{n}, \quad n \in \mathbb{N},$$

which implies that $c^* \leqslant b$. On the other hand, the manifold $\mathcal{N}$ separates $W^{1,N}(\mathbb{R}^N)$ into two components $E^+ := \{u \in W^{1,N}(\mathbb{R}^N) : \langle \Phi'(u), u\rangle > 0\} \cup \{0\}$ and $E^- := \{u \in W^{1,N}(\mathbb{R}^N) : \langle \Phi'(u), u\rangle < 0\}$. Hence it follows from (2.49) that $\Phi(u) \geqslant 0$ for $u \in E^+$. By (F1) and (F2), E^+ contains a small ball around the origin. Thus every $\gamma \in \Gamma$ has to cross $\mathcal{N}$, because $\gamma(0) \in E^+$ and $\gamma(1) \in E^-$, so $b \leqslant c^*$. Therefore, $b = c^*$. □

As mentioned in Section 1, the technical condition (F3) and the monotonicity condition (F5) are weaker than the classic condition (AR) and the strictly monotonic condition (F5′), respectively, by virtue of which we show the boundedness of the Cerami sequence obtained in Lemma 2.8 in the following two lemmas.

Lemma 2.14. *Assume that* (V0) *and* (F1)–(F4) *hold. Then any sequence* $\{u_n\}$ *satisfying* (2.42) *is bounded in* $W^{1,N}(\mathbb{R}^N)$.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, we may suppose that $\|u_n\| \to \infty$ as $n \to \infty$. By (F3) and (F4), there exists an $R > \bar{t}_0$ such that

$$f(t)t \geqslant 2NF(t), \quad \forall\, |t| \geqslant R. \tag{2.55}$$

Let

$$A := \left\{ t \in [-R, R] : \frac{f(t)}{|t|^{N-2}t} < \frac{V_*}{2} \right\}, \quad B := \left\{ t \in [-R, R] : \frac{f(t)}{|t|^{N-2}t} \geqslant \frac{V_*}{2} \right\}, \tag{2.56}$$

and

$$\mathcal{F}(t) := \frac{1}{N} f(t)t - F(t). \tag{2.57}$$

Then it follows from (F1), (F3), and (2.57) that there exists a $c_0 > 0$ such that

$$|f(t)|^{N/(N-1)} \leqslant c_0 |t|^{1/(N-1)} \mathcal{F}(t), \quad \forall\, t \in B. \tag{2.58}$$

From (2.40)–(2.42) and (2.55), we have

$$\begin{aligned} c^* + o(1) &= \Phi(u_n) - \frac{1}{N}\langle \Phi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} \left(\frac{1}{N} f(u_n)u_n - F(u_n) \right) dx \\ &\geqslant \int_{|u_n| \leqslant R} \mathcal{F}(u_n) dx + \frac{1}{2N} \int_{|u_n| > R} f(u_n)u_n dx. \end{aligned} \tag{2.59}$$

Then it follows from (2.41), (2.42), and (2.56)–(2.59) that

$$\begin{aligned} 1 &= \frac{1}{\|u_n\|^N} \int_{\mathbb{R}^N} f(u_n)u_n dx + o(1) \\ &\leqslant \frac{1}{\|u_n\|^N} \left(\frac{V_*}{2} \int_A |u_n|^N dx + \int_B f(u_n)u_n dx \right) + o(1) \\ &\leqslant \frac{1}{2} + \frac{c_0^{(N-1)/N}}{\|u_n\|^N} \left(\int_{|u_n| \leqslant R} \mathcal{F}(u_n) dx \right)^{(N-1)/N} \|u_n\|_{N+1}^{(N+1)/N} + o(1) \\ &= \frac{1}{2} + o(1). \end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$. □

Lemma 2.15. *Assume that* (V0), (F1), (F2), (F4), *and* (F5) *hold. Then any sequence* $\{u_n\}$ *satisfying* (2.42) *is bounded in* $W^{1,N}(\mathbb{R}^N)$.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. By (F2) and (F5), we have

$$\frac{f(\theta t)\theta t}{\theta^N} \geqslant f(t)t \geqslant NF(t) \geqslant 0, \quad \forall\, t \in \mathbb{R}, \quad \theta \geqslant 1. \tag{2.60}$$

It follows from (F4) and (2.60) that there exists an $R > \bar{t}_0$ such that

$$f(t)t \geqslant 2NF(t), \quad \forall\, |t| \geqslant R. \tag{2.61}$$

From (2.40)–(2.42), (2.60), and (2.61), we have

$$\begin{aligned} c^* + o(1) &= \Phi(u_n) - \frac{1}{N}\langle \Phi'(u_n), u_n \rangle = \int_{\mathbb{R}^N} \left(\frac{1}{N} f(u_n)u_n - F(u_n) \right) dx \\ &\geqslant \frac{1}{2N} \int_{|u_n| > R} f(u_n)u_n dx. \end{aligned} \tag{2.62}$$

Let $\varrho := [2N(c^*+1)]^{1/N}$ and $t_n = \varrho/\|u_n\|$. Then $t_n \to 0$ as $n \to \infty$. It follows from (F2), (2.60), and (2.62) that

$$\begin{aligned}\int_{\mathbb{R}^N} F(t_nu_n)dx &= \int_{|u_n|\leqslant R} F(t_nu_n)dx + \int_{|u_n|>R} F(t_nu_n)dx \\ &\leqslant \int_{|u_n|\leqslant R} F(t_nu_n)dx + \frac{t_n^N}{N}\int_{|u_n|>R} f(u_n)u_ndx \\ &\leqslant \frac{V_*}{2N}\int_{|u_n|\leqslant R} |t_nu_n|^N dx + \frac{2\varrho^N(c^*+1)}{\|u_n\|^N} + o(1) \leqslant \frac{\varrho^N}{2N} + o(1). \end{aligned} \tag{2.63}$$

Hence, from (2.42), (2.49), and (2.63), one has

$$\begin{aligned} c^* + o(1) = \Phi(u_n) &\geqslant \Phi(t_nu_n) + \frac{1-t_n^N}{N}\langle \Phi'(u_n), u_n\rangle \\ &= \frac{t_n^N}{N}\|u_n\|^N - \int_{\mathbb{R}^N} F(t_nu_n)dx + o(1) \\ &= \frac{\varrho^N}{N} - \int_{\mathbb{R}^N} F(t_nu_n)dx + o(1) \\ &\geqslant \frac{\varrho^N}{2N} + o(1) \geqslant c^* + 1 + o(1). \end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$. □

Lemma 2.16 (See [15])**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $\tau > 0$ be a given number. Then it holds that*

$$\frac{\alpha_N^{N-1}}{2} = \sup\Big\{\beta^N : \sup_{\substack{u\in W^{1,N}(\Omega)\\ \int_\Omega(|\nabla u|^N+\tau|u|^N)dx\leqslant 1}} \int_\Omega \exp(\beta^{N/(N-1)}|u|^{N/(N-1)})dx < +\infty\Big\}.$$

With the help of the above lemma, inspired by [2], we verify the weak to weak* continuity of the Fréchet derivative of the functional $\int_{\mathbb{R}^N}|\nabla u|^Ndx$.

Lemma 2.17. *Assume that* (V0), (F1), *and* (F2) *hold. Let $\{u_n\}$ be a sequence satisfying $u_n \rightharpoonup \bar{u}$ in $W^{1,N}(\mathbb{R}^N)$ and* (2.42). *Then, up to a subsequence,*

$$\nabla u_n \to \nabla \bar{u} \quad a.e.\ in\ \mathbb{R}^N \tag{2.64}$$

and

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla \bar{u}|^{N-2}\nabla \bar{u} \quad in\ (L^{N/(N-1)}(\mathbb{R}^N))^N. \tag{2.65}$$

Proof. Since $u_n \rightharpoonup \bar{u}$ in $W^{1,N}(\mathbb{R}^N)$, we assume, up to a subsequence, that $\{u_n\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$, $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [2,\infty)$, and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^N$. By virtue of [46, Theorem 1.39-a)], without loss of generality, we may assume that

$$|\nabla u_n|^N + |u_n|^N \to \mu \quad \text{a.e. in } \mathcal{M}(\mathbb{R}^N) \tag{2.66}$$

and

$$|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup U \quad \text{in } (L^{N/(N-1)}(\mathbb{R}^N))^N, \tag{2.67}$$

where μ is a nonnegative regular measure and $U \in (L^{N/(N-1)}(\mathbb{R}^N))^N$.

Now, we can define an energy concentration set for any fixed $\delta > 0$, i.e.,

$$S_\delta := \{x \in \mathbb{R}^N : \forall\, r > 0,\ \mu(B_r(x)) \geqslant \delta\}. \tag{2.68}$$

Since $\{u_n\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$, S_δ must be a finite set, i.e., $S_\delta = \{x_1, x_2, \dots, x_m\}$. For any $x^* \in \mathbb{R}^N \setminus S_\delta$, there exists $0 < r < \min_{1\leqslant i\leqslant m}\|x^* - x_i\|$ such that $\mu(B_{2r}(x^*)) < \delta$. Choose $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N)$

such that $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(x) \equiv 1$ in $B_r(x^*)$, and $\varphi(x) \equiv 0$ in $\mathbb{R}^N \setminus B_{2r}(x^*)$. Thus it follows from (2.66) that

$$\begin{aligned}\limsup_{n\to\infty} \int_{B_r(x^*)} (|\nabla u_n|^N + |u_n|^N)dx &\leqslant \lim_{n\to\infty} \int_{B_{2r}(x^*)} (|\nabla u_n|^N + |u_n|^N)\varphi dx \\ &= \int_{B_{2r}(x^*)} \varphi d\mu \leqslant \mu(B_{2r}(x^*)) < \delta. \end{aligned} \tag{2.69}$$

Consequently, one has that for n large,

$$\int_{B_r(x^*)} (|\nabla u_n|^N + |u_n|^N)dx < 2\delta. \tag{2.70}$$

Thanks to Lemma 2.16, for sufficiently small $\delta > 0$, there exists some $q > 1$ such that

$$\int_{B_r(x^*)} |f(u_n)|^q dx < C_1. \tag{2.71}$$

Hence, from (2.71), Hölder's inequality, and the fact that $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [2, \infty)$, we can deduce that

$$\begin{aligned}\int_{B_r(x^*)} |f(u_n)(u_n - \bar{u})|dx &\leqslant \left(\int_{B_r(x^*)} |f(u_n)|^q dx\right)^{\frac{1}{q}} \left(\int_{B_r(x^*)} |u_n - \bar{u}|^{q'} dx\right)^{\frac{1}{q'}} \\ &= o(1), \end{aligned} \tag{2.72}$$

where $q' = q/(q-1)$. Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) \equiv 1$ in $B_{r/2}(x^*)$, and $\psi(x) \equiv 0$ in $\mathbb{R}^N \setminus B_r(x^*)$. Then by (V0), (2.41), (2.42), and (2.72), we obtain

$$\begin{aligned} o(1) &= \langle \Phi'(u_n) - \Phi'(\bar{u}), \psi(u_n - \bar{u})\rangle \\ &= \int_{\mathbb{R}^N} [(|\nabla u_n|^{N-2}\nabla u_n - |\nabla \bar{u}|^{N-2}\nabla \bar{u}) \cdot \nabla(\psi(u_n - \bar{u})) \\ &\quad + V(x)(|u_n|^{N-2}u_n - |\bar{u}|^{N-2}\bar{u})\psi(u_n - \bar{u})]dx - \int_{\mathbb{R}^N} [f(u_n) - f(\bar{u})]\psi(u_n - \bar{u})dx \\ &= \int_{\mathbb{R}^N} [\psi(|\nabla u_n|^{N-2}\nabla u_n - |\nabla \bar{u}|^{N-2}\nabla \bar{u}) \cdot (\nabla u_n - \nabla \bar{u}) \\ &\quad + (u_n - \bar{u})(|\nabla u_n|^{N-2}\nabla u_n - |\nabla \bar{u}|^{N-2}\nabla \bar{u}) \cdot \nabla \psi \\ &\quad + V(x)\psi(|u_n|^{N-2}u_n - |\bar{u}|^{N-2}\bar{u})(u_n - \bar{u})]dx - \int_{\mathbb{R}^N} f(u_n)\psi(u_n - \bar{u})dx + o(1) \\ &\geqslant \int_{B_{r/2}(x^*)} [(|\nabla u_n|^{N-2}\nabla u_n - |\nabla \bar{u}|^{N-2}\nabla \bar{u}) \cdot (\nabla u_n - \nabla \bar{u}) \\ &\quad + V_*(|u_n|^{N-2}u_n - |\bar{u}|^{N-2}\bar{u})(u_n - \bar{u})]dx \\ &\quad - \|\nabla \psi\|_\infty (\|\nabla u_n\|_N^{N-1} + \|\nabla \bar{u}\|_N^{N-1}) \left(\int_{B_r(x^*)} |u_n - \bar{u}|^N dx\right)^{\frac{1}{N}} + o(1) \\ &\geqslant 2^{2-N} \int_{B_{r/2}(x^*)} (|\nabla u_n - \nabla \bar{u}|^N + V_*|u_n - \bar{u}|^N)dx + o(1). \end{aligned} \tag{2.73}$$

In the above derivation process, we have used the following elementary inequality:

$$(|x|^{N-2}x - |y|^{N-2}y) \cdot (x - y) \geqslant 2^{2-N}|x - y|^N, \quad \forall\, x, y \in \mathbb{R}^N. \tag{2.74}$$

(2.73) shows that

$$\lim_{n\to\infty} \int_{B_{r/2}(x^*)} (|\nabla u_n - \nabla \bar{u}|^N + V_*|u_n - \bar{u}|^N)dx = 0. \tag{2.75}$$

Since S_δ is a finite set and $x^* \in \mathbb{R}^N \setminus S_\delta$ is arbitrary, it follows from (2.75) that (2.64) holds. This implies immediately (2.65) by [47, Proposition 5.4.7] and the boundedness of $\|\nabla u_n\|_N^N$. □

3 Estimates for the mountain-pass minimax level

In this section, we employ some new strategies and delicate analyses to determine a fine upper bound for the minimax level c^* given by Lemmas 2.8 and establish the relationships between the Trudinger-Moser ratio $\mathcal{C}^*_{\rm TM}$ and the numbers κ and β in the conditions (F6) and (F7), respectively.

Based on some key observations, we make use of the unified condition (VF) to find a fine threshold for the minimax level c^* by constructing a special path defined later by (3.10).

Lemma 3.1. *Assume that* (V0), (VF), (F1), (F2), (F3′), *and* (F4) *hold. Then it holds that*

$$\kappa_0 \leqslant c^* < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \tag{3.1}$$

where κ_0 *is defined by* (2.48).

Proof. Since $V^* < \mathcal{C}^*_{\rm TM}$, one has $V^* + 2\varepsilon_0 < \mathcal{C}^*_{\rm TM}$ for some $\varepsilon_0 > 0$. There are two possible cases: (i) $\mathcal{C}^*_{\rm TM} < +\infty$ and (ii) $\mathcal{C}^*_{\rm TM} = +\infty$.

Case (i) It follows from the definition of $\mathcal{C}^*_{\rm TM}$ that there exists a $\hat{u} \in W^{1,N}(\mathbb{R}^N)$ with $\|\nabla \hat{u}\|_N^N \leqslant (\frac{\alpha_N}{\alpha_0})^{N-1}$ satisfying

$$(V^* + \varepsilon_0)\|\hat{u}\|_N^N < N\int_{\mathbb{R}^N} F(\hat{u})dx. \tag{3.2}$$

This shows that $\Psi(\hat{u}) < 0$, where

$$\Psi(u) := (V^* + \varepsilon_0)\|u\|_N^N - N\int_{\mathbb{R}^N} F(u)dx, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N). \tag{3.3}$$

Let $h(s) := \Psi(s\hat{u})$ for $s > 0$. Since $h(1) < 0$ and $h(s) > 0$ for $s > 0$ small enough by (F2), there exists an $s_0 \in (0,1)$ satisfying $h(s_0) = 0$. Therefore, for $\tilde{u} := s_0\hat{u}$, we have

$$(V^* + \varepsilon_0)\|\tilde{u}\|_N^N = N\int_{\mathbb{R}^N} F(\tilde{u})dx. \tag{3.4}$$

Case (ii) It follows from the definition of $\mathcal{C}^*_{\rm TM}$ and Lemma 2.6 that there exists a $\hat{u} \in W^{1,N}(\mathbb{R}^N)$ with $\|\nabla \hat{u}\|_N^N < (\frac{\alpha_N}{\alpha_0})^{N-1}$ satisfying (3.2). Hence we can repeat the same arguments as Case (i) to get $\tilde{u} := s_0\hat{u}$ satisfying (3.4).

By (F1) and (F2), there exists a $C_1 > 0$ such that

$$|f(t)| \leqslant |t|^{N-1}[1 + C_1\phi_N(2\alpha_0|t|^{N/(N-1)})], \quad \forall\, t \in \mathbb{R}, \tag{3.5}$$

which, together with (F3′) and Lemmas 1.2(i) and 2.7, implies

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{f(\theta\tilde{u})}{|\theta\tilde{u}|^{N-2}\theta\tilde{u}}|\tilde{u}|^N dx &\leqslant \int_{\mathbb{R}^N} [1 + C_1\phi_N(2\alpha_0|\tilde{u}|^{N/(N-1)})]|\tilde{u}|^N dx \\ &\leqslant \|\tilde{u}\|_N^N + C_1\left[\int_{\mathbb{R}^N} \phi_N(4\alpha_0|\tilde{u}|^{N/(N-1)})dx\right]^{1/2}\|\tilde{u}\|_{2N}^N \\ &\leqslant C_2, \quad \forall\, \theta \in (0,1]\end{aligned} \tag{3.6}$$

for some constant $C_2 > 0$ independent of $\theta \in (0,1]$. Let $\tilde{u}_t(x) := \tilde{u}(x/t)$ for $t > 0$. Hence, it follows from (2.41) and (3.6) that

$$\begin{aligned}\frac{d}{d\theta}\Phi(\theta\tilde{u}_t) &= \langle \Phi'(\theta\tilde{u}_t), \tilde{u}_t\rangle \\ &= \theta^{N-1}\left[\|\nabla\tilde{u}\|_N^N + t^N\left(\int_{\mathbb{R}^N} V(tx)|\tilde{u}|^N dx - \int_{\mathbb{R}^N} \frac{f(\theta\tilde{u})}{|\theta\tilde{u}|^{N-2}\theta\tilde{u}}|\tilde{u}|^N dx\right)\right] \\ &\geqslant \theta^{N-1}[\|\nabla\tilde{u}\|_N^N + t^N(V_*\|\tilde{u}\|_N^N - C_2)], \quad \forall\, t > 0, \quad \theta \in (0,1].\end{aligned} \tag{3.7}$$

Since $\tilde{u} \neq 0$, we can choose $t_0 \in (0,1)$ such that

$$\|\nabla \tilde{u}\|_N^N + t_0^N (V_* \|\tilde{u}\|_N^N - C_2) > 0. \tag{3.8}$$

Now from (V0), (2.40), (3.4), and $0 < s_0 < 1$, we obtain

$$\begin{aligned}
\Phi(\tilde{u}_t) &= \frac{1}{N}\|\nabla \tilde{u}\|_N^N + \frac{t^N}{N}\int_{\mathbb{R}^N} V(tx)|\tilde{u}|^N dx - t^N \int_{\mathbb{R}^N} F(\tilde{u}) dx \\
&= \frac{1}{N}\|\nabla \tilde{u}\|_N^N + \frac{t^N}{N}\int_{\mathbb{R}^N} [V(tx) - V^* - \varepsilon_0]|\tilde{u}|^N dx \\
&\leqslant \frac{1}{N}\|\nabla \tilde{u}\|_N^N - \frac{\varepsilon_0 t^N}{N}\|\tilde{u}\|_N^N \qquad (3.9\text{a}) \\
&\leqslant \frac{s_0^N}{N}\|\nabla \hat{u}\|_N^N < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \quad \forall\, t > 0. \qquad (3.9\text{b})
\end{aligned}$$

(3.9a) shows that there exists a $T > 1$ such that $\Phi(\tilde{u}_T) < 0$. Let

$$\gamma^*(t) = \begin{cases} t_0^{-1} t \tilde{u}_{t_0}, & 0 \leqslant t \leqslant t_0, \\ \tilde{u}_{t_0 + (T - t_0)(t - t_0)/(1 - t_0)}, & t_0 \leqslant t \leqslant 1. \end{cases} \tag{3.10}$$

Then it is easy to see that $\gamma^* \in \Gamma$ is defined by (2.44). From (3.7) and (3.8), we deduce that $\Phi(t_0^{-1} t \tilde{u}_{t_0})$ is increasing on $t \in [0, t_0]$. Hence it follows from (2.40) and (3.4) that

$$\begin{aligned}
\Phi(t_0^{-1} t \tilde{u}_{t_0}) &\leqslant \Phi(\tilde{u}_{t_0}) \\
&= \frac{1}{N}\|\nabla \tilde{u}\|_N^N + \frac{t_0^N}{N}\int_{\mathbb{R}^N} V(t_0 x)|\tilde{u}|^N dx - t_0^N \int_{\mathbb{R}^N} F(\tilde{u}) dx \\
&= \frac{1}{N}\|\nabla \tilde{u}\|_N^N + \frac{t_0^N}{N}\int_{\mathbb{R}^N} [V(t_0 x) - V^* - \varepsilon_0]|\tilde{u}|^N dx \\
&\leqslant \frac{1}{N}\|\nabla \tilde{u}\|_N^N - \frac{\varepsilon_0 t_0^N}{N}\|\tilde{u}\|_N^N \\
&\leqslant \frac{s_0^N}{N}\|\nabla \hat{u}\|_N^N < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \quad \forall\, 0 \leqslant t \leqslant t_0.
\end{aligned} \tag{3.11}$$

Combining (3.9b) with (3.11), we derive that

$$c^* \leqslant \max_{t \in [0,1]} \Phi(\gamma^*(t)) \leqslant \frac{s_0^N}{N}\|\nabla \hat{u}\|_N^N < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

This completes the proof. □

Let $d > 0$. We define Moser's functions $w_n(x)$ supported in B_d as follows:

$$w_n(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log n)^{(N-1)/N}, & 0 \leqslant |x| \leqslant d/n, \\ \log(d/|x|)/(\log n)^{1/N}, & d/n \leqslant |x| \leqslant d, \\ 0, & |x| \geqslant d. \end{cases} \tag{3.12}$$

It is easy to verify that $w_n \in W^{1,N}(\mathbb{R}^N)$. By an elemental computation, we have

$$\|\nabla w_n\|_N^N = \int_{B_d} |\nabla w_n|^N dx = \omega_{N-1}\int_0^d r^{N-1}|\nabla w_n|^N dr = 1 \tag{3.13}$$

and (see [48, Lemma 3.2])

$$\|w_n\|_N^N = \int_{B_d} w_n^N dx = \omega_{N-1}\int_0^d r^{N-1}|w_n|^N dr$$

$$
\begin{aligned}
&= \frac{d^N}{Nn^N}(\log n)^{N-1} + \frac{1}{\log n}\int_{d/n}^{d} r^{N-1}\log^N\left(\frac{d}{r}\right)dr \\
&= \frac{d^N}{Nn^N}(\log n)^{N-1} + \frac{d^N}{\log n}\int_0^{\log n} s^N \mathrm{e}^{-Ns}ds \\
&= \frac{(N-1)!d^N}{N^N\log n} + O\biggl(\frac{1}{n\log n}\biggr). \qquad (3.14)
\end{aligned}
$$

Combining (3.13) with (3.14), we have

$$
\|w_n\|^N = \int_{\mathbb{R}^N}[|\nabla w_n|^N + V(x)|w_n|^N]dx \leqslant 1 + \frac{(N-1)!V^* d^N}{N^N\log n} + O\biggl(\frac{1}{n\log n}\biggr). \tag{3.15}
$$

In the following two lemmas, we show that both the conditions (F6) and (F7) yield (VF) by means of delicate analyses involving the Trudinger-Moser ratio $\mathcal{C}^*_{\mathrm{TM}}$ and Moser's functions.

Lemma 3.2. *Assume that* (F1), (F2), *and* (F3′) *hold. If*

$$
\liminf_{t\to+\infty}\frac{t^{N/(N-1)}F(t)}{\mathrm{e}^{\alpha_0 t^{N/(N-1)}}} \geqslant \kappa, \tag{3.16}
$$

then it holds that

$$
\mathcal{C}^*_{\mathrm{TM}}\begin{cases} \geqslant \dfrac{2\kappa\alpha_0^N}{(N-1)!}, & \text{if } 0<\kappa<+\infty, \\ = +\infty, & \text{if } \kappa = +\infty. \end{cases} \tag{3.17}
$$

Proof. We only consider the case $0<\kappa<+\infty$ since the other case is similar. Let $t_0 = (\frac{\alpha_N}{\alpha_0})^{(N-1)/N}$. For any $\varepsilon>0$, it follows from (3.16) that there exists a $t_\varepsilon>0$ such that

$$
t^{N/(N-1)}F(t) \geqslant (\kappa-\varepsilon)\mathrm{e}^{\alpha_0 t^{N/(N-1)}}, \quad \forall\, t\geqslant t_\varepsilon. \tag{3.18}
$$

Then from (3.12)–(3.14) and (3.18), we have

$$
\|t_0\nabla w_n\|_N^N = \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}, \tag{3.19}
$$

$$
\|t_0 w_n\|_N^N = \frac{(N-1)!d^N}{N^N\log n}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} + O\biggl(\frac{1}{n\log n}\biggr), \tag{3.20}
$$

and for n large,

$$
\begin{aligned}
\int_{\mathbb{R}^N}F(t_0w_n)dx &\geqslant \int_{B_{d/n^{(N-1)/N}}}F(t_0w_n)dx \\
&\geqslant (\kappa-\varepsilon)\int_{B_{d/n^{(N-1)/N}}}\frac{\mathrm{e}^{\alpha_0 t_0^{N/(N-1)}w_n^{N/(N-1)}}}{t_0^{N/(N-1)}w_n^{N/(N-1)}}dx \\
&\geqslant \frac{(\kappa-\varepsilon)\alpha_0}{N\log n}\int_{B_{d/n^{(N-1)/N}}}\mathrm{e}^{\alpha_0 t_0^{N/(N-1)}w_n^{N/(N-1)}}dx \\
&= \frac{(\kappa-\varepsilon)\alpha_0\omega_{N-1}d^N}{N\log n}\Biggl[\frac{\exp(\frac{N\alpha_0 t_0^{N/(N-1)}\log n}{\alpha_N})}{Nn^N} \\
&\quad + d^{-N}\int_{d/n}^{d/n^{(N-1)/N}} r^{N-1}\exp\left(\frac{N\alpha_0 t_0^{N/(N-1)}(\log(d/r))^{N/(N-1)}}{\alpha_N(\log n)^{1/(N-1)}}\right)dr\Biggr] \\
&= \frac{(\kappa-\varepsilon)\alpha_0\omega_{N-1}d^N}{N\log n}\left[\frac{1}{N} + \log n\int_{(N-1)/N}^{1}\exp(N(s^{N/(N-1)}-s)\log n)ds\right] \\
&\geqslant \frac{(\kappa-\varepsilon)\alpha_0\omega_{N-1}d^N}{N\log n}\left[\frac{1}{N} + \log n\int_{(N-1)/N}^{1}\mathrm{e}^{N(s-1)\log n}ds\right]
\end{aligned}
$$

$$= \frac{(\kappa-\varepsilon)\alpha_0\omega_{N-1}d^N}{N\log n}\bigg(\frac{2}{N}-\frac{1}{Nn}\bigg)$$
$$= \frac{2(\kappa-\varepsilon)\omega_{N-1}\alpha_0 d^N}{N^2\log n} - O\bigg(\frac{1}{n\log n}\bigg). \tag{3.21}$$

Combining (3.20) and (3.21), we obtain

$$\frac{N\int_{\mathbb{R}^N}F(t_0w_n)dx}{\|t_0w_n\|_N^N} \geqslant \frac{2(\kappa-\varepsilon)\alpha_0^N}{(N-1)!} - O\bigg(\frac{1}{n}\bigg). \tag{3.22}$$

This, together with the definition of $\mathcal{C}^*_{\rm TM}$, implies that

$$\mathcal{C}^*_{\rm TM} \geqslant \frac{2(\kappa-\varepsilon)\alpha_0^N}{(N-1)!},$$

which implies (3.17) due to the arbitrariness of $\varepsilon>0$. □

Lemma 3.3. *Assume that* (F1), (F2), *and* (F3′) *hold. If there exist* $\beta>0$ *and* $p>N$ *such that*

$$F(t)\geqslant \beta|t|^p, \quad \forall\, t\in\mathbb{R}, \tag{3.23}$$

then it holds that

$$\mathcal{C}^*_{\rm TM} \geqslant \frac{\beta N}{\mathcal{C}^p_{p,N}}\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{\frac{(p-N)(N-1)}{N}}. \tag{3.24}$$

Proof. By the definition of $\mathcal{C}_{p,N}$, we can choose $u_n\in W^{1,N}(\mathbb{R}^N)$ such that

$$\mathcal{C}_{p,N} \leqslant \frac{\|\nabla u_n\|_N^{1-N/p}\|u_n\|_N^{N/p}}{\|u_n\|_p} < \mathcal{C}_{p,N}+\frac{1}{n}, \quad \forall\, n\in\mathbb{N}. \tag{3.25}$$

Let $t_n=\frac{1}{\|\nabla u_n\|_N}(\frac{\alpha_N}{\alpha_0})^{(N-1)/N}$. Then we have

$$\|t_n\nabla u_n\|_N^N = \bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{N-1}. \tag{3.26}$$

From (3.25), we obtain

$$\frac{N}{\|t_nu_n\|_N^N}\int_{\mathbb{R}^N}F(t_nu_n)dx \geqslant \frac{\beta N\|u_n\|_p^p}{\|u_n\|_N^N}t_n^{p-N} = \frac{\beta N\|u_n\|_p^p}{\|u_n\|_N^N\|\nabla u_n\|_N^{p-N}}\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{\frac{(p-N)(N-1)}{N}}$$
$$> \frac{\beta N}{(\mathcal{C}_{p,N}+\frac{1}{n})^p}\bigg(\frac{\alpha_N}{\alpha_0}\bigg)^{\frac{(p-N)(N-1)}{N}}, \quad \forall\, n\in\mathbb{N}. \tag{3.27}$$

This, together with (3.26) and the definition of $\mathcal{C}^*_{\rm TM}$, implies that (3.24) holds. □

The relationship between the constants $\mathcal{C}_{p,N}$ in (F7) and $\mathcal{S}_{p,N;c}$ defined by (1.9) is established in the following lemma.

Lemma 3.4. *It holds that*

$$\mathcal{C}_{p,N} \leqslant c^{-1/p}\bigg[\bigg(\frac{N}{p}\bigg)^{\frac{N}{p}}\bigg(\frac{p-N}{p}\bigg)^{\frac{p-N}{p}}\bigg]^{\frac{1}{N}}\mathcal{S}_{p,N;c}. \tag{3.28}$$

Proof. By Young's inequality and the definition of $\mathcal{C}_{p,N}$, we have

$$\mathcal{C}_{p,N}c^{1/p} \leqslant \frac{\|\nabla u\|_N^{1-N/p}\|u\|_N^{N/p}}{\|u\|_p}c^{1/p} = \frac{\|\nabla u\|_N^{1-N/p}(c\|u\|_N^N)^{1/p}}{\|u\|_p}$$
$$= \frac{\Big[(\|\nabla u\|_N^N)^{1-N/p}(c\|u\|_N^N)^{N/p}\Big]^{1/N}}{\|u\|_p}$$

$$\leqslant \left[\left(\frac{N}{p}\right)^{\frac{N}{p}} \left(\frac{p-N}{p}\right)^{\frac{p-N}{p}} \right]^{\frac{1}{N}} \frac{(\|\nabla u\|_N^N + c\|u\|_N^N)^{1/N}}{\|u\|_p}, \quad \forall\, u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}.$$

It follows from the definition of $\mathcal{S}_{p,N;c}$ that (3.28) holds. □

From Lemmas 3.1 and 3.2, we get immediately the following corollary.

Corollary 3.5. *Assume that* (V0), (F1), (F2), (F3′), *and* (F6) *hold. Then*

$$c^* < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{3.29}$$

4 Ground states for the periodic potential case

In this section, we consider the periodic potential case and show Theorems 1.4 and 1.5. Using the estimates for the minimax level c^* (see Lemma 3.1) and applying the concentration compactness argument, the compactness lemma (i.e., Lemma 2.4), and the Trudinger-Moser inequality (see Lemma 1.2), we first show that the Cerami sequence given by Lemma 2.8 is nonvanishing and obtain a nontrivial solution for (1.1) by the weak to weak* continuity of Φ' for such a sequence (see Lemmas 2.5 and 2.17). Then, by virtue of the technical condition (F3) and the monotonicity condition (F5), we show the existence of ground state solutions by constraining the functional on the critical points set $\mathcal{K}$ and the Nehari manifold $\mathcal{N}$, respectively. Such existence results, to the best of authors' knowledge, seem to be new.

Proof of Theorem 1.4. (i) We complete the proof by three steps.

Step 1. Show that $\mathcal{K} \neq \emptyset$ and there exists a $\tilde{u} \in \mathcal{K}$ such that $\Phi(\tilde{u}) \leqslant c^*$. Applying Lemmas 2.8, 2.14, and 3.1, we deduce that there exists a bounded sequence $\{u_n\} \subset W^{1,N}(\mathbb{R}^N)$ satisfying (2.42) with

$$\kappa_0 \leqslant c^* < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} \tag{4.1}$$

and $\|u_n\|_N^N \leqslant C_1$ for some constant $C_1 > 0$. It follows from (2.41) and (2.42) that

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \leqslant C_2. \tag{4.2}$$

We may thus assume, passing to a subsequence if necessary, that $u_n \rightharpoonup \bar{u}$ in $W^{1,N}(\mathbb{R}^N)$, $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [2,\infty)$, and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^N$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^N dx = 0,$$

then by Lions' concentration compactness principle (see [46, Lemma 1.21]), we know that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $N < s < \infty$. By virtue of Lemma 2.4, we can obtain

$$\int_{\mathbb{R}^N} F(u_n)dx = o(1). \tag{4.3}$$

Hence, it follows from (2.40), (2.42), and (4.3) that

$$\begin{aligned} \frac{1}{N}(\|\nabla u_n\|_N^N + V_*\|u_n\|_N^N) &\leqslant \frac{1}{N}\int_{\mathbb{R}^N}[|\nabla u_n|^N + V(x)|u_n|^N]dx \\ &= c^* + \int_{\mathbb{R}^N} F(u_n)dx + o(1) = c^* + o(1), \end{aligned} \tag{4.4}$$

which, together with (4.1), implies that $\limsup_{n\to\infty}(\|\nabla u_n\|_N^N + V_*\|u_n\|_N^N) < (\frac{\alpha_N}{\alpha_0})^{N-1}$. Hence, there exist $\bar{\varepsilon} > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|\nabla u_n\|_N^N + V_*\|u_n\|_N^N \leqslant \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}(1-3\bar{\varepsilon})^{N-1}, \quad \forall\, n \geqslant n_0. \tag{4.5}$$

Let us choose $q \in (1, N/(N-1))$ such that

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})q}{1-\bar{\varepsilon}} < 1. \tag{4.6}$$

By (F1) and (F2), there exists a $C_3 > 0$ such that

$$|f(t)t| \leqslant \frac{c^*}{2C_1}|t|^N + C_3|t|\phi_N(\alpha_0(1+\bar{\varepsilon})|t|^{N/(N-1)}), \quad \forall\, t \in \mathbb{R}. \tag{4.7}$$

Let $q' = q/(q-1)$. Then $q' > N$, and it follows from (4.5)–(4.7) and Lemmas 1.2(ii) and 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^N} f(u_n)u_n dx &\leqslant \frac{c^*}{2C_1}\|u_n\|_N^N + C_3\int_{\mathbb{R}^N} |u_n|\phi_N(\alpha_0(1+\bar{\varepsilon})|u_n|^{N/(N-1)})dx \\ &\leqslant \frac{c^*}{2} + C_3\left[\int_{\mathbb{R}^N} \phi_N(\alpha_0(1+\bar{\varepsilon})q|u_n|^{N/(N-1)})dx\right]^{\frac{1}{q}}\|u_n\|_{q'} \\ &\leqslant \frac{c^*}{2} + o(1). \end{aligned}\tag{4.8}$$

Now from (F3), (2.40), (2.41), (2.42), and (4.8), we derive

$$c^* + o(1) = \Phi(u_n) - \frac{1}{N}\langle\Phi'(u_n), u_n\rangle = \int_{\mathbb{R}^N}\left[\frac{1}{N}f(u_n)u_n - F(u_n)\right]dx \leqslant \frac{c^*}{2N} + o(1). \tag{4.9}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{y_n\} \subset \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(y_n)} |u_n|^N dx > \frac{\delta}{2}$. Let us define $\tilde{u}_n(x) = u_n(x+y_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)} |\tilde{u}_n|^N dx > \frac{\delta}{2}. \tag{4.10}$$

Since $V(x)$ is 1-periodic on x, we have $\|\tilde{u}_n\| = \|u_n\|$ and

$$\Phi(\tilde{u}_n) \to c^*, \quad \|\Phi'(\tilde{u}_n)\|(1+\|\tilde{u}_n\|) \to 0. \tag{4.11}$$

Passing to a subsequence, we have $\tilde{u}_n \rightharpoonup \tilde{u}$ in $W^{1,N}(\mathbb{R}^N)$, $\tilde{u}_n \to \tilde{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leqslant s < \infty$, and $\tilde{u}_n \to \tilde{u}$ a.e. on $\mathbb{R}^N$. Thus, (4.10) implies that $\tilde{u} \neq 0$. In view of Lemma 2.5, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(\tilde{u}_n)\phi dx = \int_{\mathbb{R}^2} f(\tilde{u})\phi dx, \quad \forall\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{4.12}$$

By (2.42), (4.12) and Lemma 2.17, it is easy to deduce that $\Phi'(\tilde{u}) = 0$. This shows that $\tilde{u} \in \mathcal{K} \neq \emptyset$. Moreover, by (F3), (2.40), (2.41), (4.11), and Fatou's lemma, we have

$$\begin{aligned} c^* &= \lim_{n\to\infty}\left[\Phi(\tilde{u}_n) - \frac{1}{N}\langle\Phi'(\tilde{u}_n), \tilde{u}_n\rangle\right] = \lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{N}f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)\right]dx \\ &\geqslant \int_{\mathbb{R}^N}\left[\frac{1}{N}f(\tilde{u})\tilde{u} - F(\tilde{u})\right]dx = \Phi(\tilde{u}) - \frac{1}{N}\langle\Phi'(\tilde{u}), \tilde{u}\rangle = \Phi(\tilde{u}). \end{aligned}$$

Step 2. Verify that $\bar{m} := \inf_{\mathcal{K}} \Phi > 0$. It follows from (2.40), (2.41), and (F3) that

$$\Phi(u) = \Phi(u) - \frac{1}{N}\langle\Phi'(u), u\rangle = \int_{\mathbb{R}^N}\left[\frac{1}{N}f(u)u - F(u)\right]dx \geqslant 0, \quad \forall\, u \in \mathcal{K}. \tag{4.13}$$

Thus $\bar{m} \geqslant 0$. Let $\{u_n\} \subset \mathcal{K}$ such that

$$\Phi(u_n) \to \bar{m}, \quad \Phi'(u_n) = 0. \tag{4.14}$$

By Lemma 2.14, we know that $\{u_n\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$. By using reduction to absurdity, we assume that $\bar{m}=0$. Then (4.14) implies that

$$\Phi(u_n)\to 0, \quad \Phi'(u_n)=0. \tag{4.15}$$

By (F3) and (F4), there exists an $R>\bar{t}_0$ such that (2.55) holds. Let A, B, and $\mathcal{F}(t)$ be defined by (2.56) and (2.57), respectively. Then it follows from (F1), (F3), and (2.57) that there exists a $c_0>0$ such that (2.58) holds. From (2.40), (2.41), (2.55), and (4.15), we have

$$\begin{aligned} o(1) &= \Phi(u_n)-\frac{1}{N}\langle\Phi'(u_n),u_n\rangle=\int_{\mathbb{R}^N}\left(\frac{1}{N}f(u_n)u_n-F(u_n)\right)dx\\ &\geqslant \int_{|u_n|\leqslant R}\mathcal{F}(u_n)dx+\frac{1}{2N}\int_{|u_n|>R}f(u_n)u_ndx. \end{aligned} \tag{4.16}$$

Set $\delta_1:=\liminf_{n\to\infty}\|u_n\|$. There are two possible cases to distinguish:

Case (1) $\delta_1=0$. In this case, passing to a subsequence if necessary, we may assume that $\|u_n\|\to 0$. From (F1), (F2), (2.41), (4.15), and Lemmas 1.2(ii) and 2.7, we get

$$\begin{aligned} \|u_n\|^N &= \int_{\mathbb{R}^N} f(u_n)u_ndx\\ &\leqslant \int_{\mathbb{R}^N}\left[\frac{V_*}{2}|u_n|^N+C_4\phi_N(2\alpha_0|u_n|^{N/(N-1)})|u_n|^{N+1}\right]dx\\ &\leqslant \frac{1}{2}\|u_n\|^N+C_5\|u_n\|^{N+1}, \end{aligned} \tag{4.17}$$

which, together with the fact that $u_n\neq 0$, yields that $\|u_n\|\geqslant \frac{1}{2C_5}$. A contradiction is derived.

Case (2) $\delta_1>0$. In this case, then it follows from (2.41), (2.56)–(2.58), and (4.16) that

$$\begin{aligned} 1 &= \frac{1}{\|u_n\|^N}\int_{\mathbb{R}^N} f(u_n)u_ndx\\ &\leqslant \frac{1}{\|u_n\|^N}\left(\frac{V_*}{2}\int_A|u_n|^Ndx+\int_B f(u_n)u_ndx\right)+o(1)\\ &\leqslant \frac{1}{2}+\frac{c_0^{(N-1)/N}}{\|u_n\|^N}\left(\int_{|u_n|\leqslant R}\mathcal{F}(u_n)dx\right)^{(N-1)/N}\|u_n\|_{N+1}^{(N+1)/N}+o(1)\\ &= \frac{1}{2}+o(1). \end{aligned}$$

This is also a contradiction. Both Cases (1) and Case (2) show that $\bar{m}>0$.

Step 3. Prove that there exists a $\tilde{u}\in\mathcal{K}$ such that $\Phi(\tilde{u})=\bar{m}$. Repeating the same arguments as in Step 1, we show that there exists a $\tilde{u}\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ such that $\Phi'(\tilde{u})=0$ and $\Phi(\tilde{u})\leqslant m$. Since $\tilde{u}\in\mathcal{K}$ and $\Phi(\tilde{u})\geqslant m$, we have $\Phi(\tilde{u})=\bar{m}$.

(ii) By virtue of Lemmas 3.2 and 3.3, both (F6) and (F7) imply (VF), so (ii) follows directly from (i). □

Proof of Theorem 1.5. (i) Applying Lemmas 2.8, 2.13, 2.15, and 3.1, we deduce that there exists a bounded sequence $\{u_n\}\subset W^{1,N}(\mathbb{R}^N)$ satisfying (2.42) with

$$\kappa_0\leqslant c^*=b<\frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{4.18}$$

By the same arguments as in the proof of Theorem 1.4, we have

$$\Phi(\tilde{u}_n)\to b, \quad \|\Phi'(\tilde{u}_n)\|(1+\|\tilde{u}_n\|)\to 0, \tag{4.19}$$

where $\tilde{u}_n$ is a $\mathbb{Z}^N$-translation of u_n, $\tilde{u}_n \rightharpoonup \tilde{u} \neq 0$ in $W^{1,N}(\mathbb{R}^N)$, $\tilde{u}_n \to \tilde{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leqslant s < \infty$; $\tilde{u}_n \to \tilde{u}$ a.e. on $\mathbb{R}^N$ and $\Phi'(\tilde{u}) = 0$. Since $\tilde{u} \in \mathcal{N}$, one has $\Phi(\tilde{u}) \geqslant b$. From (2.40), (2.41), (2.50) with $t = 0$, (4.19), and Fatou's lemma, we deduce that

$$\begin{aligned} b &= \lim_{n\to\infty} \left[\Phi(\tilde{u}_n) - \frac{1}{N}\langle \Phi'(\tilde{u}_n), \tilde{u}_n\rangle\right] = \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{N} f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)\right] dx \\ &\geqslant \int_{\mathbb{R}^N} \left[\frac{1}{N} f(\tilde{u})\tilde{u} - F(\tilde{u})\right] dx = \Phi(\tilde{u}) - \frac{1}{N}\langle \Phi'(\tilde{u}), \tilde{u}\rangle = \Phi(\tilde{u}). \end{aligned}$$

Therefore, $\Phi(\tilde{u}) = b$.

(ii) From Lemmas 3.2 and 3.3, we deduce that (VF) holds if either (F6) or (F7) is satisfied, then we get (ii) from (i). □

5 Mountain-pass type solutions for the radially symmetric potential case

The radially symmetric potential case is studied in this section where the proof of Theorem 1.6 will be given. In particular, we employ the radial lemma, the Brézis-Lieb lemma and the compactness lemma (i.e., Lemma 2.4) to show that the Cerami sequence given by Lemma 2.8 is nonvanishing and that it has a convergent subsequence.

Proof of Theorem 1.6. (i) To achieve of goal, we divide the process into two steps.

Step 1. Show that $\mathcal{K} \neq \emptyset$. Applying Lemmas 2.8, 2.13, 2.14, and 3.1, and using the radially symmetric space $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ instead of $W^{1,N}(\mathbb{R}^N)$ in these lemmas, we deduce that there exists a bounded sequence $\{u_n\} \subset W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ satisfying (2.42) with

$$\kappa_0 \leqslant c^* < \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{5.1}$$

It follows from (2.41) and (2.42) that

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \leqslant C_1. \tag{5.2}$$

Passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup \bar{u}$ in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^N)$ for $s \in (N, \infty)$, and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^N$. By virtue of Lemma 2.4, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = \int_{\mathbb{R}^N} F(\bar{u})dx. \tag{5.3}$$

If $\bar{u} = 0$, then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $s \in (N, \infty)$. Hence, it follows from (2.40), (2.42), and (5.3) that

$$\begin{aligned} \frac{1}{N}(\|\nabla u_n\|_N^N + V_*\|u_n\|_N^N) &\leqslant \frac{1}{N}\int_{\mathbb{R}^N} [|\nabla u_n|^N + V(x)|u_n|^N] dx \\ &= c^* + \int_{\mathbb{R}^N} F(u_n)dx + o(1) = c^* + o(1), \end{aligned} \tag{5.4}$$

which, together with (5.1), implies that

$$\limsup_{n\to\infty}(\|\nabla u_n\|_N^N + V_*\|u_n\|_N^N) < \left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.$$

Hence we can repeat the same arguments as in the proof of Theorem 1.4(i) to get a contradiction. Therefore, $\bar{u} \neq 0$. In view of Lemma 2.5, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(u_n)\phi dx = \int_{\mathbb{R}^N} f(\bar{u})\phi dx, \quad \forall\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{5.5}$$

By (2.42), (5.5), and Lemma 2.17, it is easy to deduce that $\Phi'(\bar{u})=0$. Thus $\mathcal{K}\neq\emptyset$.

Step 2. Prove that $u_n\to\bar{u}$ in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$. By (2.41), (2.42), and the fact that $\Phi'(\bar{u})=0$, we get

$$0=\lim_{n\to\infty}\langle\Phi'(u_n),u_n\rangle=\lim_{n\to\infty}\left[\|u_n\|^N-\int_{\mathbb{R}^N}f(u_n)u_ndx\right]\tag{5.6}$$

and

$$\int_{\mathbb{R}^N}(|\nabla\bar{u}|^{N-2}\nabla\bar{u}\cdot\nabla\varphi+V(x)|\bar{u}|^{N-2}\bar{u}\varphi)dx=\int_{\mathbb{R}^N}f(\bar{u})\varphi dx,\quad\forall\,\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^N).\tag{5.7}$$

Note that $\bar{u}\in W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, and there exists $\{\varphi_n\}\subset\mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\|\varphi_n-\bar{u}\|=o(1)$. This, together with (F1), (F2), and Lemma 1.2(i), yields that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}f(\bar{u})(\varphi_n-\bar{u})dx\right|&\leqslant\int_{\mathbb{R}^N}|f(\bar{u})||\varphi_n-\bar{u}|dx\\&\leqslant C_2\int_{\mathbb{R}^N}[|\bar{u}|^{N-1}+\phi_N(2\alpha_0|\bar{u}|^{N/(N-1)})]|\varphi_n-\bar{u}|dx\\&\leqslant C_2\left\{\|\bar{u}\|_N^{N-1}+\left[\int_{\mathbb{R}^N}\phi_N\left(\frac{2N\alpha_0}{N-1}|\bar{u}|^{N/(N-1)}\right)dx\right]^{\frac{N-1}{N}}\right\}\|\varphi_n-\bar{u}\|_N\\&=o(1).\end{aligned}\tag{5.8}$$

It follows from (5.7) and (5.8) that

$$\begin{aligned}\|\bar{u}\|^N&=\int_{\mathbb{R}^N}(|\nabla\bar{u}|^N+V(x)\bar{u}^N)dx\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla\bar{u}|^{N-2}\nabla\bar{u}\cdot\nabla\varphi_n+V(x)|\bar{u}|^{N-2}\bar{u}\varphi_n)dx\\&=\lim_{n\to\infty}\int_{\mathbb{R}^N}f(\bar{u})\varphi_ndx=\int_{\mathbb{R}^N}f(\bar{u})\bar{u}dx,\end{aligned}\tag{5.9}$$

which, together (5.6), implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[f(u_n)u_n-f(\bar{u})\bar{u}]dx=\lim_{n\to\infty}\|u_n\|^N-\|\bar{u}\|^N.\tag{5.10}$$

From (F3), (2.40)–(2.42), (5.3), and (5.10), we have

$$\begin{aligned}c^*&=\lim_{n\to\infty}\left[\Phi(u_n)-\frac{1}{N}\langle\Phi'(u_n),u_n\rangle\right]=\frac{1}{N}\lim_{n\to\infty}\int_{\mathbb{R}^N}[f(u_n)u_n-NF(u_n)]dx\\&=\frac{1}{N}\lim_{n\to\infty}\int_{\mathbb{R}^N}f(u_n)u_ndx-\int_{\mathbb{R}^N}F(\bar{u})dx\geqslant\frac{1}{N}\lim_{n\to\infty}\int_{\mathbb{R}^N}[f(u_n)u_n-f(\bar{u})\bar{u}]dx\\&=\frac{1}{N}\Big(\lim_{n\to\infty}\|u_n\|^N-\|\bar{u}\|^N\Big).\end{aligned}\tag{5.11}$$

Note that $u_n\rightharpoonup\bar{u}$ in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$, $u_n\to\bar{u}$ in $L^s(\mathbb{R}^N)$ for $s\in(N,\infty)$, and $u_n\to\bar{u}$ a.e. on $\mathbb{R}^N$, and it follows from Lemma 2.17 and the Brézis-Lieb lemma [46] that

$$\lim_{n\to\infty}(\|u_n\|^N-\|u_n-\bar{u}\|^N-\|\bar{u}\|^N)=0,\tag{5.12}$$

which, together with with (5.1) and (5.11), implies

$$\lim_{n\to\infty}\|u_n-\bar{u}\|^N\leqslant Nc^*<\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}.\tag{5.13}$$

Hence, there exist $\bar{\varepsilon}>0$ and $n_1\in\mathbb{N}$ such that

$$\|u_n-\bar{u}\|^N\leqslant\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}(1-3\bar{\varepsilon})^{N-1},\quad\forall\,n\geqslant n_1.\tag{5.14}$$

Now we choose $q \in (1, N/(N-1))$ such that

$$\frac{(1+\bar{\varepsilon})^2(1-3\bar{\varepsilon})q^2}{1-\bar{\varepsilon}} < 1. \tag{5.15}$$

By an element inequality, one has

$$|u_n|^{N/(N-1)} \leqslant (1+\bar{\varepsilon})|u_n - \bar{u}|^{N/(N-1)} + C(\bar{\varepsilon})|\bar{u}|^{N/(N-1)}. \tag{5.16}$$

It follows from (5.14)–(5.16) and Lemmas 1.2(i) and 2.7(ii) that

$$\begin{aligned}\int_{\mathbb{R}^N} \phi_N(\alpha_0(1+\bar{\varepsilon})q|u_n|^{N/(N-1)})dx &\leqslant \frac{q-1}{q}\int_{\mathbb{R}^N} \phi_N\left(\frac{q^2\alpha_0(1+\bar{\varepsilon})C(\bar{\varepsilon})}{q-1}|\bar{u}|^{N/(N-1)}\right)dx \\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N} \phi_N(q^2\alpha_0(1+\bar{\varepsilon})^2|u_n-\bar{u}|^{N/(N-1)})dx \\ &\leqslant C_3. \end{aligned} \tag{5.17}$$

Let $q' = q/(q-1)$. Then $q' > N$. Thus for any $\varepsilon > 0$, by (F1), (F2), (5.17), and Hölder's inequality, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} f(u_n)(u_n-\bar{u})dx\right| \\ &\quad \leqslant \int_{\mathbb{R}^N}[\varepsilon|u_n|^{N-1} + C_\varepsilon\phi_N(\alpha_0(1+\bar{\varepsilon})|u_n|^{N/(N-1)})]|u_n-\bar{u}|dx \\ &\quad \leqslant \varepsilon\|u_n\|_N^{N-1}\|u_n-\bar{u}\|_N + C_\varepsilon\left[\int_{\mathbb{R}^N}\phi_N(\alpha_0(1+\bar{\varepsilon})q|u_n|^{N/(N-1)})dx\right]^{1/q}\|u_n-\bar{u}\|_{q'} \\ &\quad \leqslant C_4\varepsilon + o(1).\end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$, we deduce that

$$\left|\int_{\mathbb{R}^N} f(u_n)(u_n-\bar{u})dx\right| = o(1). \tag{5.18}$$

Therefore, it follows from (2.41), (2.42), (2.74), and (5.18) that

$$\begin{aligned} o(1) &= \langle \Phi'(u_n) - \Phi'(\bar{u}), u_n - \bar{u}\rangle \\ &= \int_{\mathbb{R}^N}[(|\nabla u_n|^{N-2}\nabla u_n - |\nabla\bar{u}|^{N-2}\nabla\bar{u})\cdot(\nabla u_n - \nabla\bar{u}) \\ &\quad + V(x)(|u_n|^{N-2}u_n - |\bar{u}|^{N-2}\bar{u})(u_n-\bar{u})]dx - \int_{\mathbb{R}^N}[f(u_n)-f(\bar{u})](u_n-\bar{u})dx \\ &\geqslant 2^{2-N}\int_{\mathbb{R}^N}[|\nabla u_n - \nabla\bar{u}|^N + V(x)|u_n-\bar{u}|]dx + o(1) \\ &= 2^{2-N}\|u_n-\bar{u}\|^N + o(1), \end{aligned}$$

which implies that $u_n \to \bar{u}$ in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, $\Phi(\bar{u}) = c^*$.

(iii) If (F5) holds, using Lemma 2.15 instead of Lemma 2.14 in the above argument, we show that $\Phi(\bar{u}) = c^*$; moreover, $\Phi(\bar{u}) = b = c^*$ by Lemma 2.13.

(ii) and (iv): In view of Lemmas 3.2 and 3.3, both (F6) and (F7) imply (VF), so (ii) and (iv) follow directly from (i) and (iii), respectively. □

6 Ground states for the asymptotical constant potential case

In this section, we study the asymptotical constant potential case and show Theorem 1.7 by studying the limit problem of (1.1) and comparing Φ with the energy functional associated with the limit problem.

Proof of Theorem 1.7. (i) Define

$$\Phi_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+V^*|u|^{N/(N-1)})dx-\int_{\mathbb{R}^N}F(u)dx,\quad \forall\, u\in W^{1,N}(\mathbb{R}^N) \tag{6.1}$$

and

$$\mathcal{N}_\infty:=\{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}:\langle\Phi'_\infty(u),u\rangle=0\},\quad b_\infty:=\inf_{\mathcal{N}_\infty}\Phi_\infty. \tag{6.2}$$

Applying Theorem 1.5 to Φ_∞, one gets that Φ_∞ has a critical point $u_\infty\in\mathcal{N}_\infty$, i.e.,

$$u_\infty\in\mathcal{N}_\infty,\quad \Phi'_\infty(u_\infty)=0\quad\text{and}\quad b_\infty=\Phi_\infty(u_\infty). \tag{6.3}$$

Since Φ_∞ is autonomous, $V\in\mathcal{C}(\mathbb{R}^N,\mathbb{R})$, $V(x)\leqslant V^*$, and $V_*<V^*$, there exist $\bar{x}\in\mathbb{R}^N$ and $\bar{r}>0$ such that

$$V^*-V(x)>0\quad\text{and}\quad |u_\infty(x)|>0\quad\text{for a.e. } |x-\bar{x}|\leqslant\bar{r}. \tag{6.4}$$

In view of Lemma 2.11, there exists a $t_\infty>0$ such that $t_\infty u_\infty\in\mathcal{N}$. Hence, by Corollary 2.10 and (6.4), we have

$$\begin{aligned}
b_\infty&=\Phi_\infty(u_\infty)\geqslant\Phi_\infty(t_\infty u_\infty)\\
&=\Phi(t_\infty u_\infty)+\frac{t_\infty^N}{N}\int_{\mathbb{R}^N}[V^*-V(x)]|u_\infty|^N dx\\
&\geqslant b+\frac{t_\infty^N}{N}\int_{\mathbb{R}^N}[V^*-V(x)]|u_\infty|^N dx>b.
\end{aligned}\tag{6.5}$$

Applying Lemmas 2.8, 2.13, 2.15, and 3.1, we deduce that there exists a bounded sequence $\{u_n\}\subset W^{1,N}(\mathbb{R}^N)$ satisfying (2.42) with

$$\kappa_0\leqslant c^*=b<\frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1}. \tag{6.6}$$

Hence it follows from (2.41) and (2.42) that

$$\int_{\mathbb{R}^N}f(u_n)u_n dx\leqslant C_1 \tag{6.7}$$

for some $C_1>0$. We may assume, up to a subsequence, that $u_n\rightharpoonup\bar{u}$ in $W^{1,N}(\mathbb{R}^N)$, $u_n\to\bar{u}$ in $L^s_{\rm loc}(\mathbb{R}^N)$ for $s\in[2,\infty)$, and $u_n\to\bar{u}$ a.e. on $\mathbb{R}^N$. For any $n\in\mathbb{N}$, it follows from Lemma 2.11 that there exists a $t_n>0$ such that $t_nu_n\in\mathcal{N}_\infty$. Consequently,

$$\Phi_\infty(t_nu_n)\geqslant b_\infty\quad\text{and}\quad\langle\Phi'_\infty(t_nu_n),t_nu_n\rangle=0. \tag{6.8}$$

If

$$\delta:=\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^N dx=0,$$

then by the same arguments as in the Proof of Theorem 1.4(i), we get a contradiction. Thus $\delta>0$. We may assume, up a subsequence, that there exists $\{y_n\}\subset\mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(y_n)}|u_n|^N dx>\frac{\delta}{2}$. Let us define $\tilde{u}_n(x)=u_n(x+y_n)$ so that

$$\int_{B_{1+\sqrt{N}}(0)}|\tilde{u}_n|^N dx>\frac{\delta}{2}. \tag{6.9}$$

Note that $\|\tilde{u}_n\|_{W^{1,N}(\mathbb{R}^N)}=\|u_n\|_{W^{1,N}(\mathbb{R}^N)}$. Passing to a subsequence, we have $\tilde{u}_n\rightharpoonup\tilde{u}$ in $W^{1,N}(\mathbb{R}^N)$, $\tilde{u}_n\to\tilde{u}$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2\leqslant s<\infty$, and $\tilde{u}_n\to\tilde{u}$ a.e. on $\mathbb{R}^N$. By (6.9), one has $\tilde{u}\neq 0$. Hence, it follows from (2.50) with $t=0$, (6.1), and (6.8) that

$$\begin{aligned}
0&=t_n^{-N}\langle\Phi'_\infty(t_nu_n),t_nu_n\rangle\\
&=t_n^{-N}\langle\Phi'_\infty(t_n\tilde{u}_n),t_n\tilde{u}_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} (|\nabla \tilde{u}_n|^N + V^*|\tilde{u}_n|^N)dx - t_n^{-N} \int_{\mathbb{R}^N} f(t_n\tilde{u}_n)t_n\tilde{u}_n dx \\
&\leqslant \int_{\mathbb{R}^N} (|\nabla \tilde{u}_n|^N + V^*|\tilde{u}_n|^N)dx - N \int_{\mathbb{R}^N} \frac{F(t_n\tilde{u}_n)}{|t_n|^N} dx,
\end{aligned}
$$

which, together with (F1) and the boundedness of $\|\tilde{u}_n\|_{W^{1,N}(\mathbb{R}^N)}$, implies that $\{t_n\}$ is bounded. Therefore, there exists a constant $K > 0$ such that

$$0 \leqslant t_n \leqslant K. \tag{6.10}$$

We claim that $\bar{u} \neq 0$. Otherwise, we have $\bar{u} = 0$, i.e., $u_n \rightharpoonup 0$ in $W^{1,N}(\mathbb{R}^N)$. Passing to a subsequence if necessary, one has $u_n \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for $s \in [1, \infty)$ and $u_n \to 0$ a.e. on $\mathbb{R}^N$. By (2.41), (2.42), (2.49), (6.1), (6.5), (6.6), (6.8), and (6.10), we have

$$
\begin{aligned}
b + o_n(1) &= \Phi(u_n) \\
&\geqslant \Phi(t_n u_n) + \frac{1 - t_n^N}{N} \langle \Phi'(u_n), u_n \rangle \\
&= \Phi(t_n u_n) + o_n(1) \\
&= \Phi_\infty(t_n u_n) + \frac{t_n^N}{N} \int_{\mathbb{R}^N} [V(x) - V^*]|u_n|^N dx + o_n(1) \\
&\geqslant b_\infty + \frac{t_n^N}{N} \int_{|x| \leqslant R} [V(x) - V^*]|u_n|^N dx + \frac{t_n^N}{N} \int_{|x|>R} [V(x) - V^*]|u_n|^N dx + o_n(1) \\
&\geqslant b_\infty - \frac{(V^* - V_*)K^N}{N} \int_{|x| \leqslant R} |u_n|^N dx - \frac{K^N}{N} \sup_{|x|>R} [V^* - V(x)]\|u_n\|_N^N + o_n(1) \\
&\geqslant b + \frac{t_\infty^N}{N} \int_{\mathbb{R}^N} [V^* - V(x)]|u_\infty|^N dx - \frac{K^N}{N} \sup_{|x|>R} [V^* - V(x)]\|u_n\|_N^N + o_n(1) \\
&\geqslant b + \frac{t_\infty^N}{2N} \int_{\mathbb{R}^N} [V^* - V(x)]|u_\infty|^N dx + o_R(1) + o_n(1),
\end{aligned} \tag{6.11}
$$

which leads to a contradiction by (6.4), and thus $\bar{u} \neq 0$. In view of Lemma 2.5, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} f(u_n)\phi dx = \int_{\mathbb{R}^N} f(\bar{u})\phi dx, \quad \forall\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{6.12}$$

From (2.42), (6.12), and Lemma 2.17, it is easy to deduce that $\Phi'(\bar{u}) = 0$. Furthermore, we show that $\Phi(\bar{u}) = b = c^*$ by using the same argument as in the proof of Theorem 1.5(i).

(ii) By Lemmas 3.2 and 3.3, (F6) or (F7) implies (VF), so (ii) is a direct corollary of (i). □

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