Positive bound states of fractional Choquard equations with upper Hardy-Littlewood-Sobolev critical exponent

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Abstract

We are interested in the existence of positive bound solutions for the following fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + V(x)u = \Big(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\Big)|u|^{2^*_{\mu,s}-2}u, x\in\Omega, \\ u=0, x\in\mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is an unbounded exterior domain, $\partial\Omega\neq\emptyset$, $\mathbb{R}^N\backslash\Omega$ is bounded, $s\in(0,1)$, $N>2s$, $0<\mu<\min\{N,4s\}$, $2^*_{\mu,s}=\frac{2N-\mu}{N-2s}$ is the fractional upper Hardy-Littlewood-Sobolev critical exponent, and $V\in L^{\frac{N}{2s}}(\Omega)$ is a non-negative function. By combining variational methods and the Brouwer degree theory, we investigate the existence of positive bound solutions to this equation when $V(x)$ and the hole $\mathbb{R}^N\backslash\Omega$ are suitable small in some senses. The result obtained in this paper extend and improve some recent works. Our result also holds true in the case $\Omega=\mathbb{R}^N$, hence this paper can be viewed as an extension of recent contributions on the Benci-Cerami problem for the fractional Choquard equation.

Keywords: bound state, fractional Choquard equation, Berestycki-Lions condition, Brouwer degree, upper Hardy-Littlewood Sobolev critical exponent.

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1 Introduction and main results

In this article, we are interested in the following fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + V(x)u = \Big(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} dy\Big)|u|^{2^*_{\mu,s}-2}u, x \in \Omega, \\ u = 0, x \in \mathbb{R}^N \backslash \Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded exterior domain, $\partial\Omega \neq \emptyset$, $\mathbb{R}^N \backslash \Omega$ is bounded, $s \in (0,1)$, $N > 2s$, $0 < \mu < \min\{N, 4s\}$, $2^*_{\mu,s} = \frac{2N-\mu}{N-2s}$ is the fractional upper Hardy-Littlewood-Sobolev critical exponent and $(-\Delta)^s$ is the fractional Laplace operator. The fractional Laplace operator was first introduced in the pioneering work by Laskin [26, 27], for more details about the fractional Laplacian and fractional Sobolev spaces we refer the interested reader to the monograph [39].

On the one hand, equation (1.1) stems from the following Choquard equation or nonlinear Schrödinger-Newton equation

$$-\Delta u + u = \Big(\frac{1}{|x|} * |u|^2\Big)u, x \in \mathbb{R}^3. \tag{1.2}$$

In the framework of quantum mechanics, equation (1.2) was elaborated by Pekar [46] in 1954. In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (1.2) to describe an electron trapped in its own hole. As an approximation of the Hartree-Fock theory, Bongers also investigated equation (1.2) in [6]. It is remarked that, as a model of self gravitating matter and is known in that context as the Schrödinger-Newton equation, this equation was studied by Penrose [47, 48]. To the best of our knowledge, Lieb [33] and Lions [36] first studied the existence and symmetry of positive solutions to equation (1.2). Since then, many authors pay their much attentions to the studying of existence, multiplicity and properties of the solutions of the nonlinear Choquard equations, and indeed, many interesting results were obtained in the last decades. By using rearrangements technique, the existence and uniqueness, up to translations, were investigated by Lieb and Lions in [35, 37]; Furthermore, they proved the existence of a sequence of radially symmetric solutions by variational methods. In [52], Wei and Winter first proved the non-degeneracy and uniqueness of the ground state, and then they succeeded to obtain the multi-bump solutions for (1.2). Classification of solutions of (1.2) was first studied by Ma and Zhao [38]. More recently, Moroz and Van Schaftingen [42] completely studied the qualitative properties of solutions for the following Choquard equation

$$-\Delta u + u = (K_\alpha * |u|^p)|u|^{p-2}u, x \in \mathbb{R}^N, \tag{1.3}$$

where $p > 1$, $N \in \{1, 2, \ldots\}$ and K_α is a Riesz potential defined by

$$K_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}} := \frac{\mathcal{C}_\alpha}{|x|^{N-\alpha}}.$$

Subsequently, Moroz and Van Schaftingen [43] gave a broad survey about Choquard equations. Especially, Gao et al. [15] and Guo et al. [17] independently studied positive high-energy solutions for the

Benci-Cerami problem of Choquard equation

$$-\Delta u+V(x)u=(K_\alpha*|u|^{2^*_\mu})|u|^{2^*_\mu-2}u, u\in D^{1,2}(\mathbb{R}^N) \tag{1.4}$$

when $|V|_{\frac{N}{2}}$ is suitable small, where $0<\mu<N$ if $N=3$ or $N=4$, and $N-4\le\mu<N$ if $N\ge5$, $2^*_\mu=\frac{2N-\mu}{N-2}$ is the upper Hardy-Littlewood-Sobolev critical exponent. Recently, Alves, Figueiredo and Molle [2] considered the Choquard equation (1.4) with $V(x)=\lambda+V_0(x)$ and $\lambda\ge0, V_0\in L^{\frac{N}{2}}(\mathbb{R}^N)$, $0<\mu<\min\{N,4\}$ and $N\ge3$, under V_0 and λ are suitable small, they obtained the existence of two positive solutions to equation (1.4). In fact, the results obtained in [2, 15, 17] extended the classical results due to Benci and Cerami [5] for the Schrödinger equation to the Choquard equation.

Compared with classical Choquard equations, the studying of the existence and multiplicity of solutions for fractional Choquard equations is not much in the literature. Especially, in the following, some articles related to our topic must cite here. In [44], Mukherjee and Sreenadh studied the existence of weak solutions of the following doubly nonlocal fractional elliptic problem:

$$\begin{cases} (-\Delta)^s u=\Big(\int_\Omega \frac{|u|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\Big)|u|^{2^*_{\mu,s}-2}u+\lambda u, \text{ in } \Omega,\\ u=0, \text{ in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.5}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, λ is a real parameter, $0<\mu<N$ and $N>2s$. They obtained some existence, nonexistence and regularity results for weak solution of the above problem using variational methods. In [19], He and Rădulescu were concerned with the qualitative analysis of positive solutions to the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u+V(x)u=(I_\alpha*|u|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, x\in\mathbb{R}^N,\\ u\in D^{s,2}(\mathbb{R}^N),\ u(x)>0,\ x\in\mathbb{R}^N, \end{cases} \tag{1.6}$$

where $s\in(0,1)$, $2s<N$, $0<\alpha<\min\{N,4s\}$, $2^*_{\alpha,s}=\frac{2N-\alpha}{N-2s}$, I_α is a Riesz potential defined by

$$I_\alpha(x)=\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N-\alpha}{2})\pi^{N/2}2^{N-\alpha}|x|^\alpha}:=\frac{A_\alpha}{|x|^\alpha}, \tag{1.7}$$

and $V(x)$ satisfies the following conditions:

(i) The function V is positive on a set of positive measure;

(ii) $V\in L^q(\mathbb{R}^N)$ for all $q\in[p_1,p_2]$, where $1<p_1<\frac{2N-\alpha}{4s-\alpha}<p_2$ with $p_2<\frac{N}{4s-N}$ if $2s<N<4s$;

(iii) $|V|_{\frac{N}{2s}}<\Big(2^{\frac{4s-\alpha}{2N-\alpha}}-1\Big)S_s^{\frac{(2s-N)[(N-\alpha)(1-s)+2s]+(2N-\alpha)2s}{2s(N-\alpha+2s)}}$,

where S_s is the best Sobolev constant for the embedding $D^{s,2}(\mathbb{R}^N)\hookrightarrow L^{2^*_s}(\mathbb{R}^N)$. By proving a version of the global compactness result of Struwe [50] for the case of fractional operators in $\mathbb{R}^N$, they showed that equation (1.6) has at least one bound state solution. Subsequently, He, Zhao and Zou [20] also studied positive solutions to the fractional Choquard equation (1.6) under following conditions:

(iv) $V \geq 0, \not\equiv 0$, and $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$;

(v) $|V|_{\frac{N}{2s}} < \left(2^{\frac{(4s-\alpha)^2}{(2N-\alpha)(N+2s-\alpha)}} - 1\right) S_s$.

It is noticed that the results obtained in [19, 20] are strongly dependent on the condition $V \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, which means that $V(x)$ may vanish at the infinity. In [16], Guan, first and second author of this paper obtained multiple bound state solutions for the fractional Choquard equation (1.6) when $V(x)$ is a positive potential bounded from below. In fact, the results obtained in [16] extended and improved some works in [19, 20] in the case where the coefficient $V(x)$ vanishes at infinity.

On the other hand, equation (1.1) is closely related to the following classic local problems in exterior domain

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, x \in D, \\ u = 0, x \in \partial D, \end{cases} \tag{1.8}$$

where $D \subset \mathbb{R}^N (N \geq 3)$ is an unbounded domain, $\partial D \neq \emptyset$ is bounded, $2 < p < \frac{2N}{N-2}$. In classical paper due to Benci and Cerami [4], authors showed that (1.8) does not have any ground state solution. So, they only find a bound state solution. For their purpose, the authors first analyzed the behavior of Palais-Smale sequences and proved a precise estimate of the energy levels where the Palais-Smale condition fails, then using variational method and Brouwer degree theory succeeded to obtain that the problem (1.8) has at least one positive solution for λ sufficiently small or for $\mathbb{R}^N \backslash D$ small enough. After this pioneer work, many local problems involving exterior domains were considered, we refer to [3, 7, 8, 9, 11, 21, 25, 31, 32, 40, 41] and the references therein.

In recent years, some scholars have begun to extend the classic results obtained in [4] to some nonlocal problems. Specifically, Alves et al. [1] and Correia et al. [12] independently extended results obtained in [4] to the following fractional elliptic problems in exterior domain

$$\begin{cases} (-\Delta)^s u + u = |u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N \backslash D, \end{cases} \tag{1.9}$$

where $D \subset \mathbb{R}^N$ is an exterior domain with smooth boundary such that $\mathbb{R}^N \backslash D$ is bounded, $s \in (0, 1)$, $N > 2s$, $p \in (2, 2_s^*)$ and $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. In [1, 12], authors first proved a version to the fractional operator in unbounded domain of the global compactness result due to Struwe (see [50, 53]), then combining with Brouwer degree, barycentric functions and minimax argument, they obtained the existence of positive solutions for equation (1.9) provided that $\mathbb{R}^N \backslash D$ is contained in a small ball. Subsequently, Correia and Oliveira [13] investigated positive solution for a class of fractional elliptic problems in exterior domains with small critical perturbation. In [10], Chen and

Liu extended the classic results of [4] to Kirchhoff type nonlocal problem with $N = 3$. Soon afterwards, Wang et al. [51] generalized the Kirchhoff type nonlocal problem with subcritical nonlinearity discussed in [10] to Kirchhoff type nonlocal problem with small critical perturbation. In [23], Jia et al. studied the existence of positive solutions for a class of Kirchhoff type problem in exterior domains with general nonlinear term. Very recently, by establishing global compact lemma, combining variational method with Brouwer degree, Jia et al. [24] obtained a positive solution for a class of Kirchhoff type nonlocal problem with critical exponent and nonconstant potential function in exterior domains when the hole suitable small. In [29], Ledesma and Miyagaki concerned with the existence of positive solutions for the following fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + u = \Big(\int_D \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\Big)|u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N \backslash D \end{cases} \tag{1.10}$$

where $N > 2s$, $s \in (0, 1)$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$ and $p \in (2, 2_s^*)$. Firstly, to overcome the loss of uniqueness, as in [49], authors investigated limit profiles of ground states of limit problem of (1.10) as α sufficiently close to 0; Then, combined with Splitting Lemma due to Struwe (see [50, 53]) and arguments used by Benci and Cerami [4], they succeed to obtain a positive solution for (1.10) when $\mathbb{R}^N \backslash D$ small enough. In [30], Ledesma devoted to studying the existence of a positive solution for the fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + u = |u|^{q-2}u + \varepsilon\Big(\int_D \frac{|u(y)|^p}{|x-y|^{\alpha}} dy\Big)|u|^{p-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N \backslash D \end{cases} \tag{1.11}$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$, $N > 2s$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$, $q \in (2, 2_s^*)$ and $p \in (\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2s})$. In fact, equation (1.11) could be viewed as a extension of fractional elliptic problems (1.9) with small Choquard type nonlocal perturbation. Recently, Ye et al. [55] investigated the existence of positive solutions to the following fractional Choquard equation

$$\begin{cases} (-\Delta)^s u + u = \Big(\int_D \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy\Big)|u|^{p-2}u + \varepsilon\Big(\int_D \frac{|u(y)|^{2^*_{\alpha,s}}}{|x-y|^{N-\alpha}} dy\Big)|u|^{2^*_{\alpha,s}-2}u, x \in D, \\ u = 0, x \in \mathbb{R}^N \backslash D \end{cases} \tag{1.12}$$

where $\varepsilon > 0$ is a parameter, $s \in (0, 1)$, $N > 2s$, $0 < \alpha < N$, $D \subset \mathbb{R}^N$ is an unbounded exterior domain with smooth boundary $\partial D \neq \emptyset$, $p \in (2, 2^*_{\alpha,s})$ and $2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$. Authors first obtained the limit profiles and uniqueness of positive radial ground states for the limit equation of problems (1.12) without small critical perturbations as $\alpha \to N$. Then, using variational method and Brouwer degree theory, they obtained the existence of positive bound state solutions for equation (1.12) in case $\varepsilon > 0$ is sufficiently small.

Motivated by the above works, especially by [1, 4, 5, 12, 16, 19, 20, 24], in this paper, we are interested in the existence of positive bound state solutions to Choquard equation (1.1) with nonconstant potential function in exterior domain.

The main result, in the case of small perturbations from infinity of the indefinite potential, establishes the following existence property of bound states.

Theorem 1.1. *Suppose that V satisfies the following conditions:*
(V_1) $V \in L^{\frac{N}{2s}}(\Omega), V(x) \geq 0$ *and* $V(x) \not\equiv 0, x \in \Omega.$
(V_2)

$$0 < |V|_{\frac{N}{2s}} < \Big(2^{\frac{4s-\mu}{2N-\mu}} - 1\Big) S_s.$$

Then there is a small $\lambda > 0$ such that if $\mathbb{R}^N \backslash \Omega \subset B_\lambda(0)$, the equation (1.1) has at least one positive bound state solution.

Remark 1.1. *If V is a constant, it is obvious that $V \notin L^{\frac{N}{2s}}(\mathbb{R}^N)$. However, the results about fractional Choquard equation in exterior domains obtained in [29, 30] are strongly dependent on V is constant potential function. So, the methods used in [29, 30] seem to be not valid for our case.*

Remark 1.2. *To the best of our knowledge, when discussing critical problems (local problems or nonlocal problems) in exterior domain, the critical term is basically used as small critical perturbation except [24]. Inspired by [4, 5, 24], in the case of small perturbations from infinity of the indefinite potential and non small critical perturbation, we obtain the existence of positive bound states of fractional Choquard equation* (1.1) *in exterior domains. Since there are double nonlocal characteristics in our equation which come from the nonlocal operator $(-\Delta)^s$ and the fractional Choquard nonlinear term, some refined estimates for our problem are very necessary and delicate.*

Remark 1.3. *It is particularly worth noting that our result also holds true in the case $\Omega = \mathbb{R}^N$, hence can be viewed as a extension of a recent results for Benci-Cerami problem for the fractional Choquard equation by X. He, V. D. Rădulescu, Small linear perturbations of fractional Choquard equations with critical exponent, J. Differential Equations, 2021 (Specific details, please seen [19]) and X. He, X. Zhao, W. Zou, The Benci-Cerami problem for the fractional Choquard equation with critical exponent, Manuscript Math.,2023 (Specific details, please seen [20]). Furthermore, since it is known, but not completely trivial, that $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$ as $s \to 1^-$, our result also extend the results obtained in [15, 17].*

2 Preliminary results

Without any loss of generality, we may assume that $0 \in \mathbb{R}^N \backslash \Omega$. As usual, for any $s \in (0, 1)$, let

$$H^{s,2}(\mathbb{R}^N) = \Big\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx < \infty \Big\}$$

where the inner product and norm defined by

$$(u,v)_{H^s} = \int_{\mathbb{R}^N} u(x)v(x)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx, \|u\|^2_{H^s} = (u,u)_{H^s}.$$

The norm of u in $L^r(\Omega)$ and $L^r(\mathbb{R}^N)$ are denoted by $|u|_r$ and $|u|_{r,\mathbb{R}^N}$, $1 \le r < \infty$. For any $s \in (0,1)$, defined

$$D^{s,2}(\mathbb{R}^N) = \Big\{u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx < \infty\Big\}$$

with the Gagliardo seminorm

$$\|u\|^2_{\mathbb{R}^N} = (u,u)_{\mathbb{R}^N} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx.$$

According to Propositions 3.4 and 3.6 of [45], by omitting the normalization constant we have

$$\|u\|^2_{\mathbb{R}^N} = |(-\Delta)^{\frac{s}{2}}u|^2_{2,\mathbb{R}^N} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx.$$

Then set

$$D_0^{s,2}(\Omega) = \Big\{u \in D^{s,2}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\Big\}$$

with the norm

$$\|u\|^2 = (u,u) = \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx,$$

where $\mathcal{Q} := (\mathbb{R}^N \times \mathbb{R}^N)\backslash(\Omega^c \times \Omega^c)$, $\Omega^c = \mathbb{R}^N\backslash\Omega$. According to definition of $D_0^{s,2}(\Omega)$, it is obvious that $D_0^{s,2}(\Omega) \subset D^{s,2}(\mathbb{R}^N)$.

Proposition 2.1. *([34, 35]) Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,N,\mu,r)$ independent of f,h such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\mu}dxdy\Big| \le C(t,N,\mu,r)|f|_t \cdot |g|_r. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t,N,\mu,r) = C(N,\mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{\pi-\mu}{2})}{\Gamma(\frac{2N-\mu}{2})}\Big(\frac{\Gamma(\frac{\pi}{2})}{\Gamma(N)}\Big)^{-1+\frac{\mu}{N}}.$$

In this case, the equality in (2.1) is achieved if and only if $f \equiv (const.)g$ and

$$g(x) = A(\gamma^2 + |x-a|^2)^{-\frac{2N-\mu}{2}}$$

for some $A \in \mathbb{C}$, $0 \ne \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Lemma 2.1. *([53]) If $N > 2s$ and $a \in L^{\frac{N}{2s}}(\mathbb{R}^N)$, $\psi : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}, u \mapsto \int_{\mathbb{R}^N} a(x)u^2dx$ is weakly continuous.*

Let $f = g = |u|^q$, then according to Proposition 2.1 we conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)u(y)|^q}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

is well defined if $|u|^q \in L^t(\mathbb{R})$ for some $t > 1$ with $\frac{2}{t}+\frac{\mu}{N} = 2$. Therefore, for $u \in D^{s,2}(\mathbb{R}^N)$, thanks to Sobolev embedding theorems we have

$$\frac{2N-\mu}{N} \le q \le \frac{2N-\mu}{N-2s}. \tag{2.2}$$

Hence, for any $u \in D^{s,2}(\mathbb{R}^N)$, we get

$$\Big(\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x\Big)^{\frac{1}{2^*_{\mu,s}}} \le (\mathcal{C}(N,\mu))^{\frac{1}{2^*_{\mu,s}}}|u|^2_{2^*_{\mu,s}},$$

where $\mathcal{C}(N,\mu) := A_\mu C(N,\mu)$.

From above arguments, the energy functional associated with equation (1.1) is defined by

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x + \frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x\\ &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x,\ u \in D^{s,2}_0(\Omega).\end{aligned}$$

Furthermore, $\mathcal{J}(u) \in C^1(D^{s,2}_0(\Omega),\mathbb{R})$ and

$$\langle\mathcal{J}'(u),v\rangle = (u,v) + \int_\Omega V(x)uv\mathrm{d}x - \int_\Omega\int_\Omega\frac{|u(y)|^{2^*_{\mu,s}}|u(x)|^{2^*_{\mu,s}-2}u(x)v(x)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x$$

for $u, v \in D^{s,2}_0(\Omega)$.

Define the Nehari manifold as

$$\mathcal{N} := \{u \in D^{s,2}_0(\Omega))\backslash\{0\} : \mathcal{G}(u) = 0\},\ \text{where } \mathcal{G}(u) := \langle\mathcal{J}'(u),u\rangle.$$

Moreover, we have the following results about $\mathcal{N}$.

Lemma 2.2. *Suppose that (V_1) holds, then we have that*
(a) $\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the unit sphere of $D^{s,2}_0(\Omega)$;
(b) $\mathcal{J}$ has a positive bound from below on $\mathcal{N}$;
(c) u is a critical point of $\mathcal{J}$ if and only if u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$.

Proof. Choose $u \in D^{s,2}_0(\Omega)$ with $\|u\| = 1$, defined $f_u(t)$ by

$$f_u(t) = \frac{t^2}{2}\|u\|^2 + \frac{t^2}{2}\int_\Omega V(x)u^2\mathrm{d}x - \frac{t^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x, t > 0.$$

Obviously, $f_u(0)$ and $f'_u(0) = 0$. Thanks to $V(x) \ge 0$, we have that

$$f_u(t) > 0 \text{ for } t > 0 \text{ small enough and } f_u(t) < 0 \text{ for } t > 0 \text{ large enough.}$$

Hence, there exists $t_u > 0$ such that $f_u(t_u) = \max_{t\geq 0} f_u(t)$ with $f_u'(t_u) = 0$ and $t_u u \in \mathcal{N}$. It is easy to see that t_u is unique. Furthermore, we have $f_u'(t) > 0$ for $0 < t < t_u$, $f_u'(t) < 0$ for $t > t_u$.

Since $\mathcal{J} \in C^2(D_0^{s,2}(\Omega), \mathbb{R})$, $\mathcal{G}$ is a C^1 functional. For any $u \in \mathcal{N}$, we have

$$\begin{aligned}\langle \mathcal{G}'(u), v\rangle &= 2\|u\|^2 + 2\int_\Omega V(x)u^2 \mathrm{d}x - 2\cdot 2^*_{\mu,s}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ &= (2 - 2\cdot 2^*_{\mu,s})\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x < 0,\end{aligned} \tag{2.3}$$

which implies that (a) holds.

For any $u \in \mathcal{N}$, since $V(x) \geq 0$, using Sobolev inequality, there is $C_0 > 0$ such that

$$0 = \|u\|^2 + \int_\Omega V(x)u^2\mathrm{d}x - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \geq \|u\|^2 - C_0\|u\|^{2\cdot 2^*_{\mu,s}}.$$

Then, there is $C_1 > 0$ such that

$$\|u\| \geq C_1 \text{ for any } u \in \mathcal{N}. \tag{2.4}$$

Consequently, by (2.4) we conclude that

$$\begin{aligned}\mathcal{J}(u) =& \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega V(x)u^2\mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\ =& (\frac{1}{2} - \frac{1}{2\cdot 2^*_{\mu,s}})\|u\|^2 > (\frac{1}{2} - \frac{1}{2\cdot 2^*_{\mu,s}})C_1, \text{ for any } u \in \mathcal{N},\end{aligned}$$

which shows that (b) is satisfied.

If u is a critical point of $\mathcal{J}$ with $u \not\equiv 0$, then $\mathcal{J}'(u) = 0$ and thus $\mathcal{G}(u) = 0$. So u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$. If u is a critical point of $\mathcal{J}$ constrained on $\mathcal{N}$, then there is $\varsigma \in \mathbb{R}$ satisfying $\mathcal{J}'(u) = \varsigma\mathcal{G}'(u)$. By $u \in \mathcal{N}$, we have

$$\langle \varsigma\mathcal{G}'(u), u\rangle = \langle \mathcal{J}'(u), u\rangle = 0.$$

Hence, due to (2.3), we get $\varsigma = 0$. That is, $\mathcal{J}'(u) = 0$. □

Let $S_{\mu,s}$ be the best constant

$$S_{\mu,s} := \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathbb{R}^N}}{\left(\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x\right)^{\frac{1}{2^*_{\mu,s}}}},$$

where I_μ defined as in (1.7), and S_s be the best Sobolev constant for the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$, that is,

$$S_s = \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathbb{R}^N}}{|u|^2_{2^*_s,\mathbb{R}^N}}.$$

It is well-known that $S_{\mu,s}$ and S_s are both achieved at u if and only if

$$u(x) = C(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|})^{\frac{N-2s}{2}}, x \in \mathbb{R}^N$$

for some $x_0 \in \mathbb{R}^N$, $C > 0$ and $\varepsilon > 0$ (see Theorem 2.15 of [14]). Furthermore,

$$S_{\mu,s} = \frac{S_s}{(\mathcal{C}(N,\mu))^{\frac{1}{2^*_{\mu,s}}}}.$$

Let

$$\Psi = \frac{\nu_1}{(1+|x|^2)^{\frac{N-2s}{2}}}, \text{ where } \nu_1 = (\frac{S_s^{\frac{N}{2s}}\Gamma(N)}{\pi^{\frac{N}{2}}\Gamma(N)}).$$

Then, from [45], we have that

$$S_s\|\Psi\|^2_{2^*_s,\mathbb{R}^N} = \|\Psi\|^2_{\mathbb{R}^N}, \tag{2.5}$$

and

$$\|\Psi\|^2_{\mathbb{R}^N} = \|\Psi\|^{2^*_s}_{2^*_s,\mathbb{R}^N} = S_s^{\frac{N}{2s}}.$$

Set

$$\widetilde{\Psi}(x) = S_s^{\frac{(N-\mu)(2s-N)}{4s(N-\mu+2s)}}(\mathcal{C}(N,\mu))^{\frac{2s-N}{2(N-\mu+2s)}}\Psi(x),$$

then $\widetilde{\Psi}(x)$ is the unique minimizer for $S_{\mu,s}$ and satisfies

$$\|\widetilde{\Psi}\|^2_{\mathbb{R}^N} = \int_{\mathbb{R}^N}(I_\mu * |\widetilde{\Psi}|^{2^*_{\mu,s}})|\widetilde{\Psi}|^{2^*_{\mu,s}}\mathrm{d}x = S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}. \tag{2.6}$$

Let

$$\begin{aligned}\varphi_{\delta,z}(x) &= \delta^{\frac{2s-N}{N}}\widetilde{\Psi}(\frac{x-z}{\delta}) \\ &= \frac{a_{\mu,s}\delta^{\frac{N-2s}{2}}}{(\delta^2+|x-z|^2)^{\frac{N-2s}{2}}}, \forall \delta > 0,\ z \in \mathbb{R}^N.\end{aligned} \tag{2.7}$$

where $a_{\mu,s} = S_s^{\frac{(N-\mu)(2s-N)}{4s(N-\mu+2s)}}(\mathcal{C}(N,\mu))^{\frac{2s-N}{2(N-\mu+2s)}}\nu_1$.

Now, we introduce the following equation

$$(-\Delta)^s u = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u, \text{ in } \mathbb{R}^N, \tag{2.8}$$

and its energy functional $\mathcal{J}_\infty : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathcal{J}_\infty(u) = \frac{1}{2}\|u\|^2_{\mathbb{R}^N} - \frac{1}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\mathrm{d}x.$$

It follows from [28] that the positive solutions of equation (2.8) are unique. Furthermore, by the invariance of the scaling, the function $\varphi_{\delta,z}$ defined as (2.7) solves equation (2.8) and satisfies (2.6).

Let

$$\mathcal{N}_\infty := \{u \in D^{s,2}(\mathbb{R}^N)\backslash\{0\} : \langle \mathcal{J}'_\infty(u), u\rangle = 0\},$$

and then we have that

$$\mathcal{J}_\infty(\varphi_{\delta,z}) = m_\infty := \min_{u\in\mathcal{N}_\infty} \mathcal{J}_\infty(u) = \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}.$$

Then a ground state solution of (2.8) is a nontrivial solution $u \in D^{s,2}(\mathbb{R}^N)$ satisfying

$$\mathcal{J}_\infty(u) = m_\infty \text{ and } \mathcal{J}'_\infty(u) = 0.$$

Furthermore, from [19, 20] we have the following result about nodal solution of equation (2.8).

Lemma 2.3. *If $u \in D^{s,2}(\mathbb{R}^N)$ is a nodal solution of equation* (2.8)*, then*

$$\mathcal{J}_\infty(u) \geq 2^{\frac{4s-\mu}{N-\mu+2s}} \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}} = 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

The following proposition indicates that there is no ground state solution to the equation (1.1).

Proposition 2.2. *Suppose that (V_1) holds, then $m := \min_{u\in\mathcal{N}} \mathcal{J}(u) = m_\infty$ and m is not achieved.*

Proof. Step 1: To prove $m = m_\infty$. For any $u \in \mathcal{N}$, there is $t_u > 0$ satisfying $t_u u \in \mathcal{N}_\infty$. So we conclude that

$$\begin{aligned}
m_\infty \leq \mathcal{J}_\infty(t_u u) &= \frac{1}{2}\|t_u u\|^2_{\mathbb{R}^N} - \frac{1}{2\cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |t_u u|^{2^*_{\mu,s}})|t_u u|^{2^*_{\mu,s}} \mathrm{d}x \\
&= \frac{1}{2}\|t_u u\|^2 - \frac{1}{2\cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|t_u u(x)|^{2^*_{\mu,s}}|t_u u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\leq \frac{1}{2}\|t_u u\|^2 + \frac{1}{2}\int_\Omega V(x)(t_u u)^2\mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}} \int_\Omega\int_\Omega \frac{|t_u u(x)|^{2^*_{\mu,s}}|t_u u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&= \mathcal{J}(t_u u) \leq \mathcal{J}(u),
\end{aligned}$$

which shows that $m \geq m_\infty$.

In the following, we prove that $m \leq m_\infty$. Let $\widetilde{u}_n \subset D^{s,2}_0(\Omega)$ be defined by $\widetilde{u}_n := \zeta(x)\widetilde{\varphi}_n$, here $\widetilde{\varphi}_n(\cdot) = \varphi(\cdot - z_n)$ and $\varphi = \varphi_{1,0} \in D^{s,2}(\mathbb{R}^N)$ defined in (2.7) is a positive solution of (2.8), $\{z_n\} \subset \Omega$ with $|z_n| \to +\infty$ as $n \to +\infty$, $\zeta : \mathbb{R}^N \to [0,1]$ is defined by

$$\zeta(x) = \xi\Big(\frac{|x|}{\lambda}\Big), \lambda := \inf\{\tau : \mathbb{R}^N\backslash\Omega \subset \overline{B_\tau(0)}\},$$

where $B_\tau(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \tau\}$ and $\xi(t) : \mathbb{R}^+ \cup \{0\} \to [0,1]$ is a non-decreasing function satisfying

$$\xi(t) = 0, t \leq 1 \text{ and } \xi(t) = 1, t \geq 2.$$

Firstly, we claim that

$$\mathcal{J}(\widetilde{u}_n) \to m_\infty \text{ and } \langle \mathcal{J}'(\widetilde{u}_n), \widetilde{u}_n\rangle \to 0 \text{ as } n \to +\infty. \tag{2.9}$$

Obviously, $\|\widetilde{\varphi}_n\|_{\mathbb{R}^N} = \|\varphi\|_{\mathbb{R}^N}$ and $\widetilde{\varphi}_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to +\infty$. Hence, it follows from Lemma 2.1 that

$$\int_\Omega V(x)(\widetilde{u}_n)^2 \mathrm{d}x \to 0 \text{ as } n \to +\infty.$$

Set

$$\int_{\mathbb{R}^N} A_n(x,y)\mathrm{d}y\mathrm{d}x := \int_{\mathbb{R}^N} \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N} \frac{|\varphi(x-z_n)|^{2^*_{\mu,s}}|\varphi(y-z_n)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

where

$$A_n(x,y) = \frac{[\zeta^{2^*_{\mu,s}}(x+z_n)\zeta^{2^*_{\mu,s}}(y+z_n) - 1]|\varphi|^{2^*_{\mu,s}}|\varphi|^{2^*_{\mu,s}}}{|x-y|^\mu}.$$

Since $|z_n| \to +\infty$ as $n \to +\infty$ and

$$|x+z_n| \geq |z_n| - |x| \text{ and } |y+z_n| \geq |z_n| - |y|,$$

for each $x, y \in \mathbb{R}^N$ there is $N_0 \in \mathbb{N}$ satisfying

$$|x+z_n| \geq 2\lambda \text{ and } |y+z_n| \geq 2\lambda, n \geq N_0.$$

Hence $\zeta(x+z_n) = 1 = \zeta(y+z_n)$ for all $n \geq N_0$, and we have that

$$A_n(x,z) \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \text{ as } n \to +\infty.$$

Furthermore,

$$|A_n(x,y)| \leq C\frac{|\varphi(x)|^{2^*_{\mu,s}}|\varphi(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \in L^1(\mathbb{R}^N \times \mathbb{R}^N).$$

Therefore, by the Lebesgue's theorem, we conclude that

$$\int_{\mathbb{R}^N} \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{|\varphi(x)|^{2^*_{\mu,s}}|\varphi(z)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \text{ as } n \to +\infty. \tag{2.10}$$

Similar to the proof as in [1] we can show that

$$\|\widetilde{u}_n - \varphi(\cdot - z_n)\|^2_{\mathbb{R}^N} \to 0 \text{ as } n \to +\infty. \tag{2.11}$$

Thus, combining with (2.10) and (2.11), the claim (2.9) holds.

For $\widetilde{u}_n \subset D_0^{s,2}(\Omega)$, arguing as in Lemma 2.2, there exists unique $t_n > 0$ such that $t_n\widetilde{u}_n \in \mathcal{N}$ and then $\langle \mathcal{J}'(t_n\widetilde{u}_n), t_n\widetilde{u}_n\rangle = 0$.

We claim that $t_n \to 1$ as $n \to +\infty$. Indeed, according to definition of $\widetilde{u}_n$, it is easy to see that

$$a \leq \|\widetilde{u}_n\| \leq b, a \leq \int_\Omega\int_\Omega \frac{|\widetilde{u}_n|^{2^*_{\mu,s}}|\widetilde{u}_n|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \leq b,$$

where $a, b > 0$ are constants. Hence, we can get that there is $C > 0$ such that $|t_n| \geq C$. Suppose that $t_n \to +\infty$, thanks to $t_n\widetilde{u}_n \in \mathcal{N}$, we have that

$$\|\widetilde{u}_n\|^2 = t_n^{(2\cdot 2^*_{\mu,s}-2)}\int_\Omega\int_\Omega \frac{|\widetilde{u}_n(x)|^{2^*_{\mu,s}}|\widetilde{u}_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x + o_n(1) \text{ as } n\to+\infty,$$

which is a contradiction. Thus, $\{t_n\}$ is bounded from above. So, according to fact $\langle \mathcal{J}'(\widetilde{u}_n), \widetilde{u}_n\rangle \to 0$ as $n\to+\infty$, we can easy to get $t_n \to 1$ as $n\to+\infty$. Then, it follows from (2.9) that $\mathcal{J}(t_n\widetilde{u}_n)\to m_\infty$. Due to $t_n\widetilde{u}_n \in \mathcal{N}$, we deduce that $m \le m_\infty$.

Consequently, from above arguments, we conclude that $m = m_\infty$.

Step 2: To prove that m is not achieved. Suppose that, by contradiction, there exists $u^\star \in \mathcal{N}$ such that $\mathcal{J}(u^\star) = m = m_\infty$ and $\mathcal{J}'(u^\star) = 0$. Let $t_{u^\star} > 0$ be such that $t_{u^\star}u^\star \in \mathcal{N}_\infty$, then we have

$$\begin{aligned}
m_\infty \le \mathcal{J}_\infty(t_{u^\star}u^\star) &= \frac{1}{2}\|t_{u^\star}u^\star\|^2_{\mathbb{R}^N} - \frac{1}{2\cdot 2^*_{\mu,s}}\int_{\mathbb{R}^N}(I_\mu * |t_{u^\star}u^\star|^{2^*_{\mu,s}})|t_u u^\star|^{2^*_{\mu,s}}\mathrm{d}x \\
&= \frac{1}{2}\|t_{u^\star}u^\star\|^2 - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|t_u u^\star(x)|^{2^*_{\mu,s}}|t_u u^\star(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&\le \frac{1}{2}\|t_{u^\star}u^\star\|^2 + \frac{1}{2}\int_\Omega V(x)(t_{u^\star}u^\star)^2\mathrm{d}x - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|t_u u^\star(x)|^{2^*_{\mu,s}}|t_u u^\star(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \\
&= \mathcal{J}(t_{u^\star}u^\star) \le \mathcal{J}(u) = m_\infty.
\end{aligned}$$

Hence, we deduce that

$$t_{u^\star} = 1, \int_\Omega V(x)(u^\star)^2\mathrm{d}x = 0$$

which implies that $u^\star$ is a minimizer of m_∞. Without loss of generality, we can assume that $u^\star \ge 0$. Therefore, by the maximum principle we get that $u^\star > 0$ in $\mathbb{R}^N$, which is impossible since $u^\star = 0$ in $\mathbb{R}^N\backslash\Omega$. □

In the following, we cite some useful Lemmas for prove our main results.

Lemma 2.4. *([19, 42]) Let $N > 2s$ and $\mu \in (0, N)$. If $\{u_n\}$ is a bounded sequence in $L^{2^*_s}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n\to\infty$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^N}(I_\mu * |u_n|^{2^*_{\mu,s}})|u_n|^{2^*_{\mu,s}}\,dx - \int_{\mathbb{R}^N}(I_\mu * |u_n-u|^{2^*_{\mu,s}})|u_n-u|^{2^*_{\mu,s}}\,dx \\
&\to \int_{\mathbb{R}^N}(I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}}\,dx.
\end{aligned}$$

Lemma 2.5. *([19]) Let $\{u_n\}$ is a bounded sequence in $D^{s,2}(\mathbb{R}^N)$ such that $u_n \to 0$ almost everywhere in $\mathbb{R}^N$ as $n\to\infty$. Denote $g(u) = (I_\mu * |u|^{2^*_{\mu,s}})|u|^{2^*_{\mu,s}-2}u$. Then for each $v \in D^{s,2}(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N}|g(u_n+v) - g(u_n) - g(v)|^{\frac{2^*_s}{2^*_s-1}}\,dx = o_n(1).$$

By arguing as Lemma 3.4 in [20], we can obtain the following result.

Lemma 2.6. *Suppose that $\{u_n\} \subset D^{s,2}(\mathbb{R}^N)$ is a sequence of (P.S.)c sequence for $\mathcal{J}_\infty$ such that $u_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ and $u_n \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. Then there exists a sequence of points $\{z_n\} \subset \mathbb{R}^N$ and a sequence of positive numbers $\{\sigma_n\}$ such that*

$$v_n(x) = \sigma_n^{\frac{N-2s}{2}} u_n(\sigma_n x + z_n) \tag{2.12}$$

converges weakly in $D^{s,2}(\mathbb{R}^N)$ to a nontrivial solution v of (2.8).

It follows from Proposition 2.2 that the equation (1.1) does not have any ground state solution. So, we only to find a bound state solution. For this purpose, we need to obtain the global compactness result.

Theorem 2.1. *Suppose that (V_1) holds, let $\{u_n\} \subset D_0^{s,2}(\Omega)$ is a sequence of (P.S.)c sequence for $\mathcal{J}$, that is*

$$\mathcal{J}(u_n) \to c \text{ and } \mathcal{J}'(u_n) \to 0 \text{ as } n \to +\infty. \tag{2.13}$$

Then there exist a number $l \in \mathbb{N} := \{0, 1, 2, \ldots\}$, l sequences of numbers $\{\sigma_n^i\} \subset \mathbb{R}^+$, points $\{z_n^i\} \subset \mathbb{R}^N$, $1 \le i \le l$, $l+1$ sequences of functions $\{u_n^{(j)}\} \subset D^{s,2}(\mathbb{R}^N), 0 \le j \le l$, such that for some subsequence, still denoted by $\{u_n\}$,

$$\begin{aligned} &(a)\ u_n(x) = u_n^{(0)}(x) + \sum_{i=1}^{l} (\sigma_n^i)^{-\frac{N-2s}{2}} u_n^{(i)}\Big(\frac{\cdot - z_n^i}{\sigma_n^i}\Big); \\ &(b)\ u_n^{(0)} \rightharpoonup u^{(0)} \text{ in } D_0^{s,2}(\Omega) \text{ as } n \to +\infty; \\ &(c)\ u_n^{(i)} \rightharpoonup u^{(i)} \ne 0 \text{ in } D^{s,2}(\mathbb{R}^N) \text{ as } n \to +\infty, 1 \le i \le l; \end{aligned} \tag{2.14}$$

where $u^{(0)}, u^{(i)} (1 \le i \le l)$ satisfy

$$(-\Delta)^s u^{(0)} + V(x) u^{(0)} = \Big(\int_\Omega \frac{|u^{(0)}(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} dy\Big) |u^{(0)}|^{2^*_{\mu,s}-2} u^{(0)}, x \in \Omega; u = 0, x \in \mathbb{R}^N \backslash \Omega. \tag{2.15}$$

$$(-\Delta)^s u^{(i)} = (I_\mu * |u^{(i)}|^{2^*_{\mu,s}}) |u^{(i)}|^{2^*_{\mu,s}-2} u^{(i)}, x \in \mathbb{R}^N, 1 \le i \le l. \tag{2.16}$$

Moreover, we have

$$\|u_n - u^{(0)} - \sum_{i=1}^{l} (\sigma_n^i)^{-\frac{N-2s}{2}} u^{(i)}\Big(\frac{\cdot - z_n^i}{\sigma_n^i}\Big)\|^2 \to 0 \text{ as } n \to +\infty, \tag{2.17}$$

and

$$\mathcal{J}(u_n) \to \mathcal{J}(u^{(0)}) + \sum_{i=1}^{l} \mathcal{J}_\infty(u^{(i)}) \text{ as } n \to +\infty. \tag{2.18}$$

Proof. First, we prove that $\{u_n\}$ is bounded in $D_0^{s,2}(\Omega)$. It follows from (2.13) that

$$
\begin{aligned}
c + o_n(1) + o_n(1)\|u_n\| &= \mathcal{J}(u_n) - \frac{1}{2 \cdot 2^*_{\mu,s}} \langle \mathcal{J}'(u_n), u_n \rangle \\
&\le (\frac{1}{2} - \frac{1}{2 \cdot 2^*_{\mu,s}})\|u_n\|^2 + (\frac{1}{2} - \frac{1}{2 \cdot 2^*_{\mu,s}}) \int_\Omega V(x) u_n^2 \mathrm{d}x,
\end{aligned}
$$

which combine with $V(x) \geq 0$ show that $\{u_n\}$ is bounded in $D_0^{s,2}(\Omega)$. So there is $u^{(0)} \in D_0^{s,2}(\Omega)$ such that, up to a subsequence, still denoted by $\{u_n\}$,

$$
u_n \rightharpoonup u^{(0)} \text{ in } D_0^{s,2}(\Omega);\ u_n \to u^{(0)} \text{ a.e. in } \Omega. \tag{2.19}
$$

For any $\psi \in C_0^\infty(\mathbb{R}^N)$, by Lemma 2.1 and Lemma 2.5 we have that

$$
\begin{aligned}
\langle \mathcal{J}'(u_n), \psi \rangle =& (u_n, \psi) + \int_\Omega V(x) u_n \psi \mathrm{d}x - \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}} |u_n(y)|^{2^*_{\mu,s}-2} u_n(y)\psi(y)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
=& (u^{(0)}, \psi) + \int_\Omega V(x) u^{(0)} \psi \mathrm{d}x - \int_\Omega \int_\Omega \frac{|u^{(0)}(x)|^{2^*_{\mu,s}} |u^{(0)}(y)|^{2^*_{\mu,s}-2} u^{(0)}(y)\psi(y)}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x + o_n(1) \\
=& \langle \mathcal{J}'(u^{(0)}), \psi \rangle + o_n(1),
\end{aligned}
$$

which shows $\langle \mathcal{J}'(u^{(0)}), \psi \rangle = 0$. That is, $u^{(0)}$ satisfies (2.15).

Let

$$
v_n^{(1)}(x) = \begin{cases} (u_n - u^{(0)})(x), x \in \Omega, \\ 0, x \in \mathbb{R}^N \backslash \Omega. \end{cases} \tag{2.20}
$$

Then it follows from (2.19) that $v_n^{(1)} \rightharpoonup 0$ in $D_0^{s,2}(\Omega)$ as $n \to +\infty$. Thanks to Lemma 2.1, we have

$$
\int_{\mathbb{R}^N} V(x) (v_n^{(1)})^2 \mathrm{d}x \to 0 \text{ as } n \to +\infty. \tag{2.21}
$$

Furthermore, by Lemma 2.4 and Lemma 2.4, we have that

$$
\begin{aligned}
&\|v_n^{(1)}\|^2_{\mathbb{R}^N} = \|v_n^{(1)}\|^2 = \|u_n\|^2 - \|u^{(0)}\|^2 + o_n(1), \\
&\mathcal{J}_\infty(v_n^{(1)}) = \frac{1}{2}\|v_n^{(1)}\|^2_{\mathbb{R}^N} + \frac{1}{2}\int_\Omega V(x)(v_n^{(1)})^2 \mathrm{d}x - \frac{1}{2 \cdot 2^*_{\mu,s}} \int_{\mathbb{R}^N} (I_\mu * |v_n^{(1)}|^{2^*_{\mu,s}}) |v_n^{(1)}|^{2^*_{\mu,s}} \mathrm{d}x + o_n(1) \\
&\qquad = \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_\Omega V(x) u_n^2 \mathrm{d}x - \frac{1}{2 \cdot 2^*_{\mu,s}} \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}} |u_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x \\
&\qquad - \frac{1}{2}\|u^{(0)}\|^2 - \frac{1}{2}\int_\Omega V(x)(u^{(0)})^2 \mathrm{d}x + \frac{1}{2 \cdot 2^*_{\mu,s}} \int_\Omega \int_\Omega \frac{|u^{(0)}(x)|^{2^*_{\mu,s}} |u^{(0)}(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} \mathrm{d}y\mathrm{d}x + o_n(1) \\
&\qquad = \mathcal{J}(u_n) - \mathcal{J}(u^{(0)}) + o_n(1).
\end{aligned} \tag{2.22}
$$

If $v_n^{(1)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, the Theorem is proved with $l = 0$.

If $v_n^{(1)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. For any $\psi \in C_0^\infty(\mathbb{R}^N)$, thanks to Lemma 2.1 and (2.13), we have that

$$\begin{aligned}\langle \mathcal{J}_\infty'(v_n^{(1)}), \psi\rangle =&(v_n^{(1)}, \psi)_{\mathbb{R}^N} + \int_\Omega V(x)v_n^{(1)}\psi \mathrm{d}x + o_n(1)\|\psi\|_{\mathbb{R}^N}\\ &- \int_{\mathbb{R}^N}(I_\mu * |v_n^{(1)}|^{2_{\mu,s}^*})|v_n^{(1)}|^{2_{\mu,s}^*-2}v_n^{(1)}\psi \mathrm{d}x\\ =&\langle \mathcal{J}'(v_n^{(1)}), \psi\rangle + o_n(1)\|\psi\|_{\mathbb{R}^N}\\ =&\langle \mathcal{J}'(u_n), \psi\rangle - \langle \mathcal{J}'(u^{(0)}), \psi\rangle + o_n(1)\\ =&o_n(1),\end{aligned} \tag{2.23}$$

which shows that $\mathcal{J}_\infty'(v_n^{(1)}) \to 0$ as $n \to +\infty$. Hence, $\{v_n^{(1)}\}$ is a Palais-Smale sequence for $\mathcal{J}_\infty$, and satisfies

$$v_n^{(1)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N);\ v_n^{(1)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then, by Lemma 2.6, there exist $\{z_n^1\} \subset \mathbb{R}^N$, $\{\sigma_n^1\} \subset \mathbb{R}^+$ and $u^{(1)} \in D^{s,2}(\mathbb{R}^N)$ such that

$$\begin{aligned}&u_n^{(1)} := (\sigma_n^1)^{\frac{N-2s}{2}} v_n^{(1)}(\sigma_n^1 \cdot + z_n^1),\\ &u_n^{(1)} \rightharpoonup u^{(1)} \text{ in } D^{s,2}(\mathbb{R}^N),\\ &\mathcal{J}_\infty'(u^{(1)}) = 0,\ u^{(1)} \neq 0.\end{aligned}$$

Thus $u^{(1)}$ is a nontrivial solution of the equation (2.16). Moreover, we have

$$\begin{aligned}\|v_n^{(1)}\|_{\mathbb{R}^N}^2 &= \|u_n^{(1)}\|_{\mathbb{R}^N}^2 = \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1),\\ \mathcal{J}_\infty(v_n^{(1)}) &= \mathcal{J}_\infty(u_n^{(1)}) = \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1).\end{aligned}$$

Combine with above equality and (2.22), we conclude that

$$\begin{aligned}\|u_n\|^2 =&\|v_n^{(1)}\|_{\mathbb{R}^N}^2 + \|u^{(0)}\|^2 + o_n(1)\\ =&\|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ \mathcal{J}(u_n) =&\mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(v_n^{(1)}) + o_n(1)\\ =&\mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1).\end{aligned} \tag{2.24}$$

Let $v_n^{(2)} := u_n^{(1)} - u^{(1)}$, if $v_n^{(2)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, the Theorem is proved with $l = 1$.

If $v_n^{(2)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. Similarly, we can conclude that $\{v_n^{(2)}\}$ is a Palais-Smale sequence for $\mathcal{J}_\infty$ and satisfies

$$v_n^{(2)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N);\ v_n^{(2)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then, it follows from Lemma 2.6 that there are $\{z_n^2\} \subset \mathbb{R}^N$, $\{\sigma_n^2\} \subset \mathbb{R}^+$ and $u^{(2)} \in D^{s,2}(\mathbb{R}^N)$ so that

$$\begin{aligned}&u_n^{(2)} := (\sigma_n^2)^{\frac{N-2s}{2}} v_n^{(2)}(\sigma_n^2 \cdot + z_n^2),\\ &u_n^{(2)} \rightharpoonup u^{(2)} \text{ in } D^{s,2}(\mathbb{R}^N),\\ &\mathcal{J}_\infty'(u^{(2)}) = 0,\ u^{(2)} \neq 0,\end{aligned}$$

which implies that $u^{(2)}$ is a nontrivial solution of the equation (2.16). Moreover, one has

$$\|v_n^{(2)}\|_{\mathbb{R}^N}^2 = \|u_n^{(2)}\|_{\mathbb{R}^N}^2 = \|u^{(2)}\|_{\mathbb{R}^N}^2 + \|u_n^{(2)} - u^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1),$$
$$\mathcal{J}_\infty(v_n^{(2)}) = \mathcal{J}_\infty(u_n^{(2)}) = \mathcal{J}_\infty(u^{(2)}) + \mathcal{J}_\infty(u_n^{(2)} - u^{(2)}) + o_n(1),$$

together with (2.24), we conclude that

$$\begin{aligned}\|u_n\|^2 =&\|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u_n^{(1)} - u^{(1)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ =&\|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|v_n^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ =&\|u^{(0)}\|^2 + \|u^{(1)}\|_{\mathbb{R}^N}^2 + \|u^{(2)}\|_{\mathbb{R}^N}^2 + \|v_n^{(2)} - u^{(2)}\|_{\mathbb{R}^N}^2 + o_n(1)\\ \mathcal{J}(u_n) =&\mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u_n^{(1)} - u^{(1)}) + o_n(1)\\ =&\mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(v_n^{(2)}) + o_n(1)\\ =&\mathcal{J}(u^{(0)}) + \mathcal{J}_\infty(u^{(1)}) + \mathcal{J}_\infty(u^{(2)}) + \mathcal{J}_\infty(v_n^{(2)} - u^{(2)}) + o_n(1)\end{aligned} \tag{2.25}$$

Iterating the above procedures, we can obtain sequences $\{u_n^{(k-1)}\}$ in this way. Let $v_k^{(k)} := u_n^{(k-1)} - u^{(k-1)}$, if $v_n^{(k)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$, then Theorem is proved with $l = k$.

If $v_n^{(k)} \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$. Arguing as before, $\{v_n^{(k)}\}$ is a Palais-Smale sequence for $\mathcal{J}_\infty$ such that

$$v_n^{(k)} \rightharpoonup 0 \text{ in } D^{s,2}(\mathbb{R}^N);\ v_n^{(k)} \nrightarrow 0 \text{ in } D^{s,2}(\mathbb{R}^N).$$

Then, according to Lemma 2.6, there exist $\{z_n^k\} \subset \mathbb{R}^N$, $\{\sigma_n^k\} \subset \mathbb{R}^+$ and $u^{(k)} \in D^{s,2}(\mathbb{R}^N)$ satisfying

$$u_n^{(k)} := (\sigma_n^k)^{\frac{N-2s}{2}} v_n^{(k)}(\sigma_n^k \cdot + z_n^k),$$
$$u_n^{(k)} \rightharpoonup u^{(k)} \text{ in } D^{s,2}(\mathbb{R}^N),$$
$$\mathcal{J}'_\infty(u^{(k)}) = 0,\ u^{(k)} \neq 0.$$

Thus $u^{(k)}$ is a nontrivial solution of the equation (2.16). Furthermore,

$$\begin{aligned}\|u_n\| =&\|u^{(0)}\|^2 + \Sigma_{i=1}^k \|u^{(i)}\|_{\mathbb{R}^N}^2 + \|u_n^{(k)} - u^{(k)}\|_{\mathbb{R}^N}^2 + o_n(1),\\ \mathcal{J}(u_n) =&\mathcal{J}(u^{(0)}) + \Sigma_{i=1}^k \mathcal{J}_\infty(u^{(i)}) + \mathcal{J}_\infty(u_n^{(k)} - u^{(k)}) + o_n(1).\end{aligned} \tag{2.26}$$

Thanks to

$$0 = \langle \mathcal{J}'_\infty(u^{(i)}), u^{(i)} \rangle = \|u^{(i)}\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N} (I_\mu * |u^{(i)}|^{2^*_{\mu,s}})|u^{(i)}|^{2^*_{\mu,s}} \mathrm{d}x, i \in \{1, 2 \dots, k\}$$

and the definition of $S_{\mu,s}$, we obtain that $\|u^{(i)}\|_{\mathbb{R}^N} \geq S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}, i = 1, 2, \dots, k$. Then, we conclude that the iteration must terminate at a finite index $l \geq 1$, that is, $v_n^{(l+1)} := u_n^{(l)} - u^{(l)} \to 0$ in $D^{s,2}(\mathbb{R}^N)$. Then, we have

$$\begin{aligned}\|u_n\| =&\|u^{(0)}\|^2 + \Sigma_{i=1}^l \|u^{(i)}\|_{\mathbb{R}^N}^2 + o_n(1),\\ \mathcal{J}(u_n) =&\mathcal{J}(u^{(0)}) + \Sigma_{i=1}^l \mathcal{J}_\infty(u^{(i)}) + o_n(1).\end{aligned}$$

Moreover, it is easy to obtain from above discission that

$$\begin{aligned}u_n =&(u^{(0)}+o_n(1))+(\sigma_n^1)^{-\frac{N-2s}{2}}(u^{(1)}+o_n(1))\big(\frac{x-z_n^1}{\sigma_n^1}\big)\\&+(\sigma_n^1\sigma_n^2)^{-\frac{N-2s}{2}}(u^{(2)}+o_n(1))\big(\frac{x-z_n^1-\sigma_n^1z_n^2}{\sigma_n^1\sigma_n^2}\big)\\&+(\sigma_n^1\sigma_n^2\sigma_n^3)^{-\frac{N-2s}{2}}(u^{(3)}+o_n(1))\big(\frac{x-z_n^1-\sigma_n^1z_n^2-\sigma_n^1\sigma_n^2z_n^3}{\sigma_n^1\sigma_n^2\sigma_n^3}\big)\\&+\cdots\\&+(\sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^l)^{-\frac{N-2s}{2}}(u^{(l)}+o_n(1))\big(\frac{x-z_n^1-\sigma_n^1z_n^2-\sigma_n^1\sigma_n^2z_n^3-\cdots-\sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^{l-1}z_n^l}{\sigma_n^1\sigma_n^2\sigma_n^3\cdots\sigma_n^l}\big).\end{aligned}$$

So, it follows from rewrite the notations that (2.14)-(2.16) are satisfied. □

Corollary 2.1. *Suppose that* (V_1) *holds, let* $\{u_n\}\subset D_0^{s,2}(\Omega)$ *be a non-negative sequence such that*

$$\mathcal{J}(u_n)\to m,\ \langle\mathcal{J}'(u_n),u_n\rangle=0 \ \textit{as}\ n\to+\infty, \tag{2.27}$$

then we have

$$u_n=w_n+\varphi_{\delta_n,z_n}, \tag{2.28}$$

where $\{w_n\}\subset D^{s,2}(\mathbb{R}^N)$ *such that* $w_n\to 0$ *in* $D^{s,2}(\mathbb{R}^N)$*, and* φ_{δ_n,z_n} *defined in* (2.7) *is the positive function realizing* m_∞*.*

Proof. It follows from (2.27) that $\{u_n\}$ is a minimizing sequence for $\mathcal{J}|_{\mathcal{N}}$. Then, it follows from variational principle ([53]) that there is a sequence $\{v_n\}\subset\mathcal{N}$ satisfying

$$\mathcal{J}(v_n)\to m,\ \mathcal{J}'(v_n)-\varsigma_n\mathcal{G}'(v_n)\to 0\ ,\|u_n-v_n\|\to 0 \text{ as } n\to+\infty, \tag{2.29}$$

where $\varsigma_n\in\mathbb{R}$. We may assume $v_n\geq 0$.

Next, we prove that $\mathcal{J}'(v_n)\to 0$ as $n\to+\infty$. From (2.29), ones has that

$$\langle\mathcal{J}'(v_n),v_n\rangle-\varsigma_n\langle\mathcal{G}'(v_n),v_n\rangle=o_n(1)\|v_n\|.$$

It follows from (2.4), (2.5) and the fact $\{v_n\}\subset\mathcal{N}$ that $\langle\mathcal{G}'(v_n),v_n\rangle<\widetilde{C}<0$. Therefore, thanks to $\{v_n\}\subset\mathcal{N}$ and $\mathcal{J}(v_n)\to m$, it is easy to see that $\{v_n\}$ is bounded. So, $\varsigma_n\to 0$ as $n\to+\infty$. For any $\phi\in D_0^{s,2}(\Omega)$,

$$\langle\mathcal{G}'(v_n),\phi\rangle=2(v_n,\phi)+2\int_\Omega V(x)v_n\phi\mathrm{d}x-2\cdot 2^*_{\mu,s}\int_\Omega\int_\Omega\frac{|v_n(x)|^{2^*_{\mu,s}}|v_n(y)|^{2^*_{\mu,s}-2}v_n(y)\phi(y)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x.$$

Thus, according to the boundedness of $|V|_{\frac{N}{2s}}$, using the Hölder inequality we deduce that

$$|\langle\mathcal{G}'(v_n),\phi\rangle|\leq\big(C_1\|v_n\|+C_2\|v_n\|^{\frac{3N+2s-2\mu}{N-2s}}\big)\|\phi\|,$$

which combine with boundedness of $\{v_n\}$ show that $\mathcal{G}'(v_n)$ is bounded. Then we can easy to obtain that $\mathcal{J}'(v_n) \to 0$ as $n \to +\infty$.

Now, for any $\phi \in D_0^{s,2}(\Omega)$, we could easy to get that

$$\langle \mathcal{J}'(u_n) - \mathcal{J}'(v_n), \phi \rangle \to 0 \text{ as } n \to +\infty,$$

from which we conclude that $\mathcal{J}'(u_n) \to 0$ as $n \to +\infty$.

By Theorem 2.1, there exist a number $l \in \mathbb{N}$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (2.15)-(2.18) hold.

If $u^{(0)} \not\equiv 0$ and $l \geq 1$, thanks to (2.15)-(2.16), we have

$$\langle \mathcal{J}'(u^{(0)}), u^{(0)} \rangle = 0, \langle \mathcal{J}'_\infty(u^{(i)}), u^{(i)} \rangle = 0, 1 \leq i \leq l,$$

which shows that $u^{(0)} \in \mathcal{N}, u^{(i)} \in \mathcal{N}_\infty, 1 \leq i \leq l$. Hence, we have that $\mathcal{J}(u^{(0)}) \geq m = m_\infty, \mathcal{J}_\infty(u^{(i)}) \geq m_\infty, 1 \leq i \leq l$. We deduce that

$$m = \mathcal{J}(u^{(0)}) + \Sigma_{i=1}^{l} \mathcal{J}_\infty(u^{(i)}) \geq (l+1)m_\infty \geq 2m_\infty,$$

which contradicts with the fact $m = m_\infty$.

If $u^{(0)} \equiv 0$ and $l \geq 2$, similar to the above discussion, we obtain a contradiction

$$m = \Sigma_{i=1}^{l} \mathcal{J}_\infty(u^{(i)}) \geq l m_\infty \geq 2m_\infty.$$

If $u^{(0)} \not\equiv 0$ and $l = 0$, then $u^{(0)}$ is a ground state solution of (1.1), which contradicts with Proposition 2.2.

If $u^{(0)} \equiv 0$ and $l = 0$, we also get a contradiction due to the fact $\mathcal{J}(u^{(0)}) = m$.

According to above arguments, we must have that $u^{(0)} \equiv 0$ and $l = 1$. That is, $u^{(1)}$ satisfies (2.8) and then $\mathcal{J}_\infty(u^{(1)}) = m = m_\infty$. This fact shows that $u^{(1)}$ is a ground state solution of (2.8). Then it follows from Lemma 2.3 and the nonnegativity of $\{u_n\}$ that $u^{(1)} \geq 0$. Arguing as in Corollary 4.2 of [18], we can get $u^{(1)} > 0$. Consequently, it follows from above discussion and the unique of positive solution of (2.8), the (2.28) holds. □

Corollary 2.2. *Suppose that* (V_1) *holds, let* $\{u_n\} \subset D_0^{s,2}(\Omega)$ *be a non-negative sequence of (P.S.)c sequence for* $\mathcal{J}$*, that is*

$$\mathcal{J}(u_n) \to c \text{ and } \mathcal{J}'(u_n) \to 0 \text{ as } n \to +\infty, \tag{2.30}$$

if $c \in (m_\infty, 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty)$*, then the functional* $\mathcal{J}$ *satisfying the* $(P.S.)_c$ *condition.*

Proof. According to Theorem 2.1, there are $l \in \mathbb{N}$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that (2.15)-(2.18) hold. Thanks to $0 < \frac{4s-\mu}{N-\mu+2s} < 1$ and $c \in (m_\infty, 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty)$, similar to the arguments of Corollary 2.1, we can conclude that $u^{(0)} \not\equiv 0$ and $l = 0$. So, it follows from (2.17) that $u_n \to u^{(0)}$ in $D_0^{s,2}(\Omega)$. □

3 Main technique and some basic estimates

Let $\varphi_{\delta,z}$ defined by (2.7) be the ground state solution of the equation (2.8). Without any loss of generality, we assume that $0 \in \mathbb{R}^N \backslash \Omega$, then $\lambda = \inf\{\tau : \mathbb{R}^N \backslash \Omega \subset \overline{B_\tau(0)}\} > 0$. Let

$$v_\lambda := \zeta(x)\varphi_{\delta,z} = \xi\Big(\frac{|x|}{\lambda}\Big)\varphi_{\delta,z},$$

where ζ, ξ are defined as in Proposition 2.2. Defined $\mathcal{K}_\lambda : \mathbb{R}^N \times \mathbb{R}^+ \to D^{s,2}(\mathbb{R}^N)$ by

$$\mathcal{K}_\lambda(z,\delta) = t_{v_\lambda} v_\lambda,$$

where $t_{v_\lambda} > 0$ satisfies $\langle \mathcal{J}'(t_{v_\lambda} v_\lambda), t_{v_\lambda} v_\lambda \rangle = 0$. According to definitions, v_λ and $\mathcal{K}_\lambda(z,\delta)$ can be seen as elements in $D_0^{s,2}(\Omega)$ and $L^{2_s^*}(\Omega)$. Moreover, we have that

$$\|\mathcal{K}_\lambda(z,\delta)\| = \|\mathcal{K}_\lambda(z,\delta)\|_{\mathbb{R}^N},\ \|v_\lambda\| = \|v_\lambda\|_{\mathbb{R}^N},$$

$$|\mathcal{K}_\lambda(z,\delta)|_{2_s^*} = |\mathcal{K}_\lambda(z,\delta)|_{2^*,\mathbb{R}^N},\ |v_\lambda|_{2_s^*} = |v_\lambda|_{2_s^*,\mathbb{R}^N}.$$

Lemma 3.1. *Suppose that (V_1) holds with $|V|_{\frac{N}{2s}} \neq 0$, then $\mathcal{K}_\lambda(z,\delta)$ satisfies*
(a) $\mathcal{K}_\lambda(z,\delta)$ is continuous in (z,δ) for every λ;
(b) $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \to m_\infty$ and $\langle \mathcal{J}'(\mathcal{K}_\lambda(z,\delta)), \mathcal{K}_\lambda(z,\delta)\rangle \to 0$ as $|z| \to +\infty$, uniformly for every bounded λ, and bounded δ away from 0;
(c) as $\lambda \to 0$, $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \to m_\infty$ and $\langle \mathcal{J}'(\mathcal{K}_\lambda(z,\delta)), \mathcal{K}_\lambda(z,\delta)\rangle \to 0$ as $\delta \to 0$ or $\delta \to +\infty$,uniformly in $z \in \mathbb{R}^N$.

Proof. (a) is obviously hold. By the similar arguments as in the proof of Proposition 2.2, we easily obtain (b). In the what following, we prove (c).

$$\begin{aligned}
\|v_\lambda - \varphi_{\delta,z}\|^2_{\mathbb{R}^N} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big|\Big(\xi(\frac{|x|}{\lambda}) - \xi(\frac{|y|}{\lambda})\Big)\varphi_{\delta,0}(x-z) + \Big(\xi(\frac{|y|}{\lambda}) - 1\Big)\Big(\varphi_{\delta,0}(x-z) - \varphi_{\delta,0}(y-z)\Big)\Big|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&\leq 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big|\xi(\frac{|x|}{\lambda}) - \xi(\frac{|y|}{\lambda})\Big|^2 \Big|\varphi_{\delta,0}(x-z)\Big|^2}{|x-y|^{N+2s}} dxdy + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big|\xi(\frac{|y|}{\lambda}) - 1\Big|^2 \Big|\varphi_{\delta,0}(x-z) - \varphi_{\delta,0}(y-z)\Big|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\
&:= 2(I_1 + I_2),
\end{aligned}$$

where

$$I_1 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big|\xi(\frac{|x|}{\lambda}) - \xi(\frac{|y|}{\lambda})\Big|^2 \Big|\varphi_{\delta,0}(x-z)\Big|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x,$$

$$I_2 := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\Big|\xi(\frac{|y|}{\lambda}) - 1\Big|^2 \Big|\varphi_{\delta,0}(x-z) - \varphi_{\delta,0}(y-z)\Big|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x.$$

First, we claim $I_2 \to 0$ as $\lambda \to 0$. In fact,

$$I_2 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\xi(\frac{|y+z|}{\lambda}) - 1\right|^2 \left|\varphi_{\delta,0}(x) - \varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x.$$

By definition of ξ, we have that

$$\frac{\left|\xi(\frac{|y+z|}{\lambda}) - 1\right|^2 \left|\varphi_{\delta,0}(x) - \varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}} \le 4 \frac{\left|\varphi_{\delta,0}(x) - \varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N),$$

$$\frac{\left|\xi(\frac{|y+z|}{\lambda}) - 1\right|^2 \left|\varphi_{\delta,0}(x) - \varphi_{\delta,0}(y)\right|^2}{|x-y|^{N+2s}} \to 0 \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N \text{ as } \lambda \to 0.$$

Hence, the Lebesgue's theorem ensures that

$$I_2 \to 0 \text{ as } \lambda \to 0, \text{ for every } z \in \mathbb{R}^N. \tag{3.1}$$

Next, arguing as in Lemma 4.1 of [1] (see also Lemma 2.3 of [54]), we prove $I_1 \to 0,$ as $\lambda \to 0,$ for $z \in \mathbb{R}^N$. Let $\mathbb{R}^N \times \mathbb{R}^N = \Pi_1 \cup \Pi_2 \cup \Pi_3$, it is easy to see that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\ &= \sum_{i=1}^{3} \int_{\Pi_i} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x, \end{aligned}$$

where

$$\Pi_1 := (\mathbb{R}^N \backslash B_{2\lambda}(-z)) \times (\mathbb{R}^N \backslash B_{2\lambda}(-z)); \ \Pi_2 := B_{2\lambda}(-z) \times \mathbb{R}^N;$$
$$\Pi_3 := (\mathbb{R}^N \backslash B_{2\lambda}(-z)) \times B_{2\lambda}(-z).$$

For any $(x, y) \in \Pi_1$, $|x+z| \ge 2\lambda, |y+z| \ge 2\lambda$, hence $\xi(\frac{|x+z|}{\lambda}) = \xi(\frac{|y+z|}{\lambda}) = 1$. Then we have that

$$\int_{\Pi_1} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x = 0. \tag{3.2}$$

Denoted

$$\begin{aligned} &\int_{\Pi_2} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\ &= \int_{B_{2\lambda}(-z)}\int_{B_\lambda(x)} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x \\ &+ \int_{B_{2\lambda}(-z)}\int_{\mathbb{R}^N \backslash B_\lambda(x)} \frac{\left|\xi(\frac{|x+z|}{\lambda}) - \xi(\frac{|y+z|}{\lambda})\right|^2 \left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}} \mathrm{d}y\mathrm{d}x. \end{aligned}$$

On the one hand, using mean value theorem, we conclude that

$$\int_{B_{2\lambda}(-z)}\int_{B_{\lambda}(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\le\frac{C_1}{(2-2s)\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx. \tag{3.3}$$

On the other hand, we obtain that

$$\begin{aligned}&\int_{B_{2\lambda}(-z)}\int_{\mathbb{R}^N\setminus B_{\lambda}(x)}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&=4\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2\int_{\mathbb{R}^N\setminus B_{\lambda}(x)}\frac{1}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\le\frac{C_2}{s\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.\end{aligned} \tag{3.4}$$

Combine with (3.3) and (3.4), we have that

$$\int_{\Pi_2}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\le\frac{C_3}{\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx. \tag{3.5}$$

Define

$$\mathcal{A}:=\{y\in B_{2\lambda}(-z):|x-y|\le\lambda\},\mathcal{A}^c:=\{y\in B_{2\lambda}(-z):|x-y|>\lambda\},$$

then

$$\begin{aligned}&\int_{\Pi_3}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&=\int_{\mathbb{R}^N\setminus B_{2\lambda}(-z)}\int_{\mathcal{A}}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&+\int_{\mathbb{R}^N\setminus B_{2\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

Arguing as before, we can obtain that

$$\begin{aligned}&=\int_{\mathbb{R}^N\setminus B_{2\lambda}(-z)}\int_{\mathcal{A}}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\&\le\frac{C_4}{\lambda^{2s}}\int_{B_{3\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.\end{aligned} \tag{3.6}$$

Choose $\gamma>4$, it is easy to see that

$$(\mathbb{R}^N\setminus B_{2\lambda}(-z))\times B_{2\lambda}(-z)\subset[(B_{\gamma\lambda}(-z)\times B_{2\lambda}(-z))\cup(\mathbb{R}^N\setminus B_{\gamma\lambda}(-z)\times B_{2\lambda}(-z))].$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N\backslash B_{2\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\le\int_{B_{\gamma\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&+\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x.
\end{aligned}\tag{3.7}$$

By direct computation, one gets

$$\begin{aligned}
&\int_{B_{\gamma\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\le\frac{C_5}{s\lambda^{2s}}\int_{B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.
\end{aligned}\tag{3.8}$$

If $(x,y)\in\mathbb{R}^N\backslash B_{\gamma\lambda}(-z))\times B_{2\lambda}(-z)$, we have

$$|x-y|\ge|x+z|-|z+y|\ge\frac{|x+z|}{2}+\frac{\gamma\lambda}{2}-2\lambda\ge\frac{|x+z|}{2}.$$

So, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\le\frac{C_6}{\gamma^N}\Big(\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}},
\end{aligned}\tag{3.9}$$

Hence, it follows from (3.7)-(3.9) that

$$\begin{aligned}
&\int_{\mathbb{R}^N\backslash B_{2\lambda}(-z)}\int_{\mathcal{A}^c}\frac{\left|\xi(\frac{|x+z|}{\lambda})-\xi(\frac{|y+z|}{\lambda})\right|^2\left|\varphi_{\delta,0}(x)\right|^2}{|x-y|^{N+2s}}\mathrm{d}y\mathrm{d}x\\
&\le\frac{C_6}{\gamma^N}\Big(\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}+\frac{C_7}{\lambda^{2s}}\int_{B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx.
\end{aligned}\tag{3.10}$$

Therefore, combining with (3.2), (3.5), (3.6) and (3.10), we conclude that

$$\begin{aligned}
I_1&\le\frac{C_3}{\lambda^{2s}}\int_{B_{2\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx+\frac{C_4}{\lambda^{2s}}\int_{B_{3\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx\\
&+\frac{C_7}{\lambda^{2s}}\int_{B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx+\frac{C_6}{\gamma^N}\Big(\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}\\
&\le\frac{C_8}{\lambda^{2s}}\int_{B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^2dx+\frac{C_6}{\gamma^N}\Big(\int_{\mathbb{R}^N\backslash B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}\\
&\le C_9\gamma^{2s}\Big(\int_{B_{\gamma\lambda}(-z)}|\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}}+\frac{C_{10}}{\gamma^N}.
\end{aligned}$$

Given $\varepsilon > 0$, we can fix γ large enough such that $\frac{C_{10}}{\gamma^N} < \frac{\varepsilon}{2}$. So,

$$I_1 \le C_9\gamma^{2s}\Big(\int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*}dx\Big)^{\frac{2}{2_s^*}} + \frac{\varepsilon}{2}.$$

Now let us fix λ small enough such that

$$\int_{B_{\gamma\lambda}(-z)} |\varphi_{\delta,0}(x)|^{2_s^*}dx < \Big(\frac{\varepsilon}{2}\Big)^{\frac{2}{2_s^*}}\frac{1}{C_9\gamma^{2s}}.$$

From above arguments, we get that $I_1 \le \varepsilon$ uniformly in z for λ small enough, which shows that

$$I_1 \to 0, \text{ as } \lambda \to 0, \text{ for } z \in \mathbb{R}^N.$$

Then, together with (3.2), we deduce that

$$\|v_\lambda\|^2 = \|v_\lambda\|^2_{\mathbb{R}^N} \to \|\varphi_{\delta,z}\|^2_{\mathbb{R}^N} \text{ as } \lambda \to 0, \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{3.11}$$

It follows from simple calculation that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\lambda(x)|^{2^*_{\mu,s}}|v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{\delta,z}(x)|^{2^*_{\mu,s}}|\varphi_{\delta,z}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\lambda(x)|^{2^*_{\mu,s}}|v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{\delta,0}(x-z)|^{2^*_{\mu,s}}|\varphi_{\delta,0}(y-z)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\Big[\Big|\xi(\frac{|x+z|}{\lambda})\Big|^{2^*_{\mu,s}}\Big|\xi(\frac{|y+z|}{\lambda})\Big|^{2^*_{\mu,s}} - 1\Big]|\varphi_{\delta,0}(x)|^{2^*_{\mu,s}}|\varphi_{\delta,0}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x\\
&:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Pi_\lambda(x,y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Since

$$\Pi_\lambda(x,y) \le C\frac{|\varphi_{\delta,0}(x)|^{2^*_{\mu,s}}|\varphi_{\delta,0}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}} \in L^1(\mathbb{R}^N\times\mathbb{R}^N)$$

and

$$\Pi_\lambda(x,y) \to 0 \text{ a.e. in } \mathbb{R}^N\times\mathbb{R}^N \text{ as } \lambda \to 0,$$

using the Lebesgue's dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_\lambda(x)|^{2^*_{\mu,s}}|v_\lambda(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{\delta,z}(x)|^{2^*_{\mu,s}}|\varphi_{\delta,z}(y)|^{2^*_{\mu,s}}}{|x-y|^{N-\mu}}\mathrm{d}y\mathrm{d}x \tag{3.12}$$

uniformly in $z \in \mathbb{R}^N$ as $\lambda \to 0$.

Thanks to

$$\begin{aligned}
|v_\lambda - \varphi_{\delta,z}|^{2_s^*}_{2_s^*,\mathbb{R}^N} &= \int_{B_{2\lambda}(0)}\Big|\Big(\xi(\frac{|y|}{\lambda}) - 1\Big)\varphi_{\delta,0}(x)\Big|^{2_s^*}dx\\
&\le C\int_{B_{2\lambda}(0)}|\varphi_{\delta,z}(x)|^{2_s^*}dx\\
&\to 0, \text{ for } (\delta,z) \in \mathbb{R}^N\times\mathbb{R}^+,
\end{aligned}$$

we obtain that

$$|v_\lambda|_{2_s^*}^{2_s^*} = |v_\lambda|_{2_s^*,\mathbb{R}^N}^{2_s^*} \to |\varphi_{\delta,z}|_{2_s^*,\mathbb{R}^N}^{2_s^*} \text{ as } \lambda \to 0, \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{3.13}$$

which together with $V \in L^{\frac{N}{2s}}(\Omega)$ concludes that

$$\int_{\mathbb{R}^N} V(x)|v_\lambda|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} V(x)|\varphi_{\delta,z}|^2 \mathrm{d}x, \text{ as } \lambda \to 0, \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+. \tag{3.14}$$

Thanks to $V \in L^{\frac{N}{2s}}(\Omega)$, arguing as in the proof of Lemma 4.3 in [20], for any $\varepsilon > 0$, there exist $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ such that

$$\int_{\mathbb{R}^N} V(x)|\varphi_{\delta,z}|^2 \mathrm{d}x < \varepsilon,$$

for $z \in \mathbb{R}^N$ and $\delta \in (0,\delta_1] \cup [\delta_2,+\infty)$. An then, combined with (3.14), we obtain that as $\lambda \to 0$,

$$\int_{\mathbb{R}^N} V(x)|v_\lambda|^2 \mathrm{d}x \to 0 \text{ as } \delta \to 0 \text{ or } \delta \to +\infty, \ \forall z \in \mathbb{R}^N. \tag{3.15}$$

So, it follows from (3.11), (3.12), (3.15), $\mathcal{J}_\infty(\varphi_{\delta,z}) = m_\infty$ and $\langle \mathcal{J}_\infty'(\varphi_{\delta,z}), \varphi_{\delta,z}\rangle = 0$ that, as $\lambda \to 0$,

$$\mathcal{J}(v_\lambda) \to m_\infty \text{ and } \langle \mathcal{J}'(v_\lambda, v_\lambda\rangle \to 0 \text{ as } \delta \to 0 \text{ or } \delta \to +\infty \text{ for } z \in \mathbb{R}^N.$$

Then similar to the proof as in Proposition 2.2, we can conclude that as $\lambda \to 0$, $t_{v_\lambda} \to 1$, as $\delta \to 0$ or $\delta \to +\infty$ for $z \in \mathbb{R}^N$. Consequently, (c) follows directly from the definition of $\mathcal{K}_\lambda(z,\delta)$. □

Lemma 3.2. *Let* $(V_1) - (V_2)$ *hold, then there is* $\lambda^* \in (0, \frac{1}{8})$ *such that for any* $\lambda < \lambda^*$

$$\sup_{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+} \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

Proof. It follows from (3.11), (3.12), (3.14) and $\langle \mathcal{J}'(\mathcal{K}_\lambda(z,\delta)), \mathcal{K}_\lambda(z,\delta)\rangle = \langle \mathcal{J}'(t_{v_\lambda} v_\lambda), t_{v_\lambda} v_\lambda\rangle = 0$ that

$$\begin{aligned} t_{v_\lambda}^{(2\cdot 2_{\mu,s}^*-2)} &= \frac{\|v_\lambda\|^2 + \int_\Omega V(x) v_\lambda^2 \mathrm{d}x}{\int_\Omega (I_\mu * |v_\lambda|^{2_{\mu,s}^*})|v_\lambda|^{2_{\mu,s}^*}\mathrm{d}x} \le \frac{\|v_\lambda\|^2 + |V|_{\frac{N}{2s}} |v_\lambda|_{2_s^*}^2}{\int_\Omega (I_\mu * |v_\lambda|^{2_{\mu,s}^*})|v_\lambda|^{2_{\mu,s}^*}\mathrm{d}x} \\ &\to \frac{\|\varphi_{\delta,z}\|_{\mathbb{R}^N}^2 + |V|_{\frac{N}{2s}} |\varphi_{\delta,z}|_{2_s^*,\mathbb{R}^N}^{2_s^*}}{\int_{\mathbb{R}^N} (I_\mu * |\varphi_{\delta,z}|^{2_{\mu,s}^*})|\varphi_{\delta,z}|^{2_{\mu,s}^*}\mathrm{d}x} \text{ as } \lambda \to 0, \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+. \end{aligned}$$

Hence, by (V_1) and (V_2) we derive that

$$\begin{aligned} \lim_{\lambda\to 0} \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) &= \lim_{\lambda\to 0} \Big(\frac{t_{v_\lambda}^2}{2}\|v_\lambda\|^2 + \frac{t_{v_\lambda}^2}{2}\int_\Omega V(x) v_\lambda^2 \mathrm{d}x - \frac{t_{v_\lambda}^{2\cdot 2_{\mu,s}^*}}{2\cdot 2_{\mu,s}^*}\int_\Omega (I_\mu * |v_\lambda|^{2_{\mu,s}^*})|v_\lambda|^{2_{\mu,s}^*}\mathrm{d}x \Big) \\ &= \lim_{\lambda\to 0} \Big(\frac{1}{2} - \frac{1}{2\cdot 2_{\mu,s}^*}\Big) t_{v_\lambda}^{2\cdot 2_{\mu,s}^*} \int_{\mathbb{R}^N} (I_\mu * |v_\lambda|^{2_{\mu,s}^*})|v_\lambda|^{2_{\mu,s}^*}\mathrm{d}x \\ &\le \frac{N-\mu+2s}{2(2N-\mu)} \Big(\frac{\|\varphi_{\delta,z}\|_{\mathbb{R}^N}^2 (\frac{1}{S_s}|V|_{\frac{N}{2s}} + 1)}{\int_{\mathbb{R}^N} (I_\mu * |\varphi_{\delta,z}|^{2_{\mu,s}^*})|\varphi_{\delta,z}|^{2_{\mu,s}^*}\mathrm{d}x} \Big)^{\frac{2\cdot 2_{\mu,s}^*}{2\cdot 2_{\mu,s}^*-2}} \int_{\mathbb{R}^N} (I_\mu * |\varphi_{\delta,z}|^{2_{\mu,s}^*})|\varphi_{\delta,z}|^{2_{\mu,s}^*}\mathrm{d}x \\ &\le \frac{N-\mu+2s}{2(2N-\mu)} \Big(\frac{1}{S_s}|V|_{\frac{N}{2s}} + 1\Big)^{\frac{2N-\mu}{N-\mu+2s}} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}} \\ &< 2^{\frac{4s-\mu}{N-\mu+2s}} \frac{N-\mu+2s}{2(2N-\mu)} S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}} \\ &= 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty \ \forall (z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+. \end{aligned}$$

Consequently, we conclude that there exists constant $\lambda^* \in (0, \frac{1}{8})$ such that for any $\lambda < \lambda^*$

$$\sup_{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+} \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

$\square$

In subsequent discussions, we always assume Ω fixed with $diam(\mathbb{R}^N \setminus \Omega) := \sup\{|x-y| : x, y \in \mathbb{R}^N \setminus \Omega\} < \lambda^*$, where $\lambda^* \in (0, 1/8)$ is the constant obtained in Lemma 3.2. Hence, for any $x_0 \in \mathbb{R}^N \backslash \Omega$, $\mathbb{R}^N \backslash \Omega \subset B_{\lambda^*}(x_0)$. Thus, we have that

$$\lambda = \inf\{\tau : \mathbb{R}^N \backslash \Omega \subset \overline{B_\tau(0)}\} < \lambda^* < \frac{1}{8},\ \mathbb{R}^N \backslash \Omega \subset B_{\frac{1}{8}}(0).$$

Defined $\chi_i : \mathbb{R}^+ \to \mathbb{R}, i = 1, 2$ by

$$\chi_1(t) = \begin{cases} 4, t \le \frac{1}{4}, \\ \frac{1}{t}, t > \frac{1}{4}, \end{cases} \text{ and } \chi_2(t) = \begin{cases} 0, t < 1, \\ 1, t \ge 1. \end{cases}$$

Define a barycenter type map $\alpha : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}^N$ and a functional $\beta : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ as

$$\alpha(u) = \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} \chi_1(|x|)x|(-\Delta)^{-\frac{s}{2}}u|^2 \mathrm{d}x, \beta(u) = \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} \chi_2(|x|)|(-\Delta)^{-\frac{s}{2}}u|^2 \mathrm{d}x.$$

Let

$$\mathcal{M} := \left\{u \in \mathcal{N} : (\alpha(u), \beta(u)) = (0, \frac{1}{2})\right\} \subset D_0^{s,2}(\Omega).$$

Lemma 3.3. *If $|z| \ge \frac{1}{2}$, then we have*

$$\alpha(\varphi_{\delta,z}) = \frac{z}{|z|} + o(1) \ \textit{as}\ \delta \to 0.$$

Proof. Fix $|z| \ge \frac{1}{2}$, we have $B_{\frac{1}{4}}(0) \cap B_\varepsilon(z) = \emptyset$ for any $\varepsilon > 0$ small enough. Then by the definition of $\varphi_{\delta,z}$ and thanks to Proposition 2.2 in [45], we have that

$$\begin{aligned} \Big|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_{\frac{1}{4}}(0)} 4x|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x\Big| &\le \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_{\frac{1}{4}}(0)} 4|x||(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \\ &\le C \int_{B_{\frac{1}{4}}(0)} |\nabla\varphi_{\delta,z}|^2 \mathrm{d}x \\ &\le C\delta^{\frac{N-2s}{2}}|z|^2 \to 0 \text{ as } \delta \to 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \Big|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathcal{A}} \frac{x}{|x|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x\Big| &\le \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N\backslash B_{\frac{1}{4}}(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \\ &\le C_1 \int_{\mathbb{R}^N\backslash B_{\frac{1}{4}}(0)} |\nabla\varphi_{\delta,z}|^2 \mathrm{d}x \\ &\le C_2\delta^{N-2s} \to 0 \text{ as } \delta \to 0, \end{aligned} \tag{3.17}$$

where $\mathcal{A} = \mathbb{R}^N \backslash (B_\varepsilon(z) \cup B_{\frac{1}{4}}(0))$.

For any $x \in B_\varepsilon(z)$, considering $|z| \geq \frac{1}{2}$ and $\varepsilon > 0$ small enough, we have

$$\Big|\frac{x}{|x|} - \frac{z}{|z|}\Big| < C_3\varepsilon.$$

Therefore, we deduce that

$$\begin{aligned}
&\Big|\frac{z}{|z|} - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_\varepsilon(z)} \frac{x}{|x|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x\Big| \\
&= \Big|\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_\varepsilon(z)} \Big(\frac{z}{|z|} - \frac{x}{|x|}\Big)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x + \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N \backslash B_\varepsilon(z)} \frac{z}{|z|}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x\Big| \\
&\leq \frac{C_3\varepsilon}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x + \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N \backslash B_\varepsilon(z)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \to 0 \text{ as } \delta \to 0.
\end{aligned} \tag{3.18}$$

Hence, combining with (3.16), (3.17) and (3.18), we obtain that

$$\alpha(\varphi_{\delta,z}) = \frac{z}{|z|} + o(1) \text{ as } \delta \to 0.$$

□

Lemma 3.4. *Suppose that* (V_1) *holds with* $|V|_{\frac{N}{2s}} \neq 0$*, then we have*

$$c_0 := \inf_{u \in \mathcal{M} \cap \mathcal{P}} \mathcal{J}(u) > m_\infty, \tag{3.19}$$

where $\mathcal{P}$ *is the cone of non-negative functions of* $D_0^{s,2}(\Omega)$*. Furthermore, as* $\lambda \to 0$*, there exist* $T > \frac{1}{2}, 0 < \delta_1 < \frac{1}{2}, \delta_2 > \frac{1}{2}$ *such that*

$$\begin{aligned}
&(a)\ \beta(\mathcal{K}_\lambda(z,\delta)) < \frac{1}{2}, |z| < \frac{1}{2} \text{ and } \delta \leq \delta_1; \\
&(b)\ \Big|\alpha(\mathcal{K}_\lambda(z,\delta)) - \frac{z}{|z|}\Big| < \frac{1}{4}, |z| \geq \frac{1}{2} \text{ and } \delta \leq \delta_1; \\
&(c)\ \beta(\mathcal{K}_\lambda(z,\delta)) > \frac{1}{2}, z \in \mathbb{R}^N \text{ and } \delta \geq \delta_2; \\
&(d)\ \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < \frac{c_0 + m_\infty}{2}, z \in \mathbb{R}^N \text{ and } \delta = \delta_1 \text{ or } \delta = \delta_2; \\
&(e)\ \mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \in (m_\infty, \frac{c_0 + m_\infty}{2}), |z| \geq T \text{ and and } \delta \in [\delta_1, \delta_2]; \\
&(f)\ \Big(\alpha(\mathcal{K}_\lambda(z,\delta)), z\Big)_{\mathbb{R}^N} > 0, |z| = T \text{ and } \delta \in [\delta_1, \delta_2].
\end{aligned} \tag{3.20}$$

Proof. First, we prove (3.19). Obviously, $c_0 \geq m_\infty$, so to obtain (3.19), we suppose $c_0 = m_\infty$ by contradiction. Hence, there is $\{u_n\} \subset \mathcal{M} \cap \mathcal{P}$ such that

$$\lim_{n\to\infty} \mathcal{J}(u_n) = m_\infty,\ \langle \mathcal{J}'(u_n), u_n \rangle = 0,\ \alpha(u_n) = 0,\ \beta(u_n) = \frac{1}{2}. \tag{3.21}$$

Thanks to Proposition 2.2, $\{u_n\}$ is not relatively compact. Then it follows from Corollary 2.1 that

$$u_n(x) = \varphi_{\delta_n,z_n}(x) + w_n(x), x \in \mathbb{R}^N$$

where $\{z_n\} \in \mathbb{R}^N, \{\delta_n\} \in \mathbb{R}^+$, $\{w_n\} \subset D^{s,2}(\mathbb{R}^N)$ with $w_n \to 0$ in $D^{s,2}(\mathbb{R}^N)$, and φ_{δ_n,z_n} is a positive ground state solution of (2.8) realizing m_∞. In subsequence sense, for (δ_n, z_n), one of the following conditions holds:

$$\begin{aligned}
&(1)\ \delta_n \to +\infty \text{ as } n \to +\infty;\ (2)\ \delta_n \to \delta \neq 0 \text{ as } n \to +\infty;\\
&(3)\ \delta_n \to 0 \text{ and } z_n \to z \text{ with } |z| < \frac{1}{2} \text{ as } n \to +\infty;\\
&(4)\ \delta_n \to 0\ \text{ as } n \to +\infty \text{ and } |z_n| \geq \frac{1}{2} \text{ for } n \text{ large.}
\end{aligned} \tag{3.22}$$

By definitions of $\alpha(u)$ and $\beta(u)$, thanks to (3.21), we have that

$$\alpha(\varphi_{\delta_n,z_n}) \to 0,\ \beta(\varphi_{\delta_n,z_n}) \to \frac{1}{2} \text{ as } n \to +\infty. \tag{3.23}$$

If $\delta_n \to +\infty$ as $n \to +\infty$, we obtain that

$$\begin{aligned}
\beta(\varphi_{\delta_n,z_n}) &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} \chi_2(|x|)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N\setminus B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,z_n}|^2 \mathrm{d}x\\
&= 1 + o_n(1) \text{ as } n \to +\infty,
\end{aligned}$$

which is a contradiction.

If $\delta_n \to \delta \neq 0$ as $n \to +\infty$, by Proposition 2.2 we can obtain that $z_n \to +\infty$ as $n \to +\infty$. Then we have

$$\begin{aligned}
\beta(\varphi_{\delta_n,z_n}) &= \beta(\varphi_{\delta,z_n}) + o_n(1)\\
&= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N} \chi_2(|x+z_n|)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2 \mathrm{d}x + o_n(1)\\
&= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{\mathbb{R}^N\setminus B_1(-z_n)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2 \mathrm{d}x + o_n(1)\\
&= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_1(-z_n)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2 \mathrm{d}x + o_n(1)\\
&= 1 + o_n(1) \text{ as } n \to +\infty.
\end{aligned}$$

Thanks to (3.23), we get a contradiction.

If $\delta_n \to 0$ and $z_n \to z$ with $|z| < \frac{1}{2}$ as $n \to +\infty$, then

$$\begin{aligned}\beta(\varphi_{\delta_n,z_n}) &= \beta(\varphi_{\delta_n,z}) + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N}\chi_2(|x+z|)|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,0}|^2\mathrm{d}x + o_n(1)\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(-z)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta_n,0}|^2\mathrm{d}x + o_n(1)\\ &= o_n(1) \text{ as } n \to +\infty,\end{aligned}$$

which contradicts with (3.23).

If $\delta_n \to 0$ as $n \to +\infty$ and $|z_n| \geq \frac{1}{2}$ for n large, according to Lemma 3.3, we get

$$\alpha(\varphi_{\delta_n,z_n}) = \frac{z_n}{|z_n|} + o_n(1) \text{ as } n \to +\infty.$$

Obviously, it is impossible duo to (3.23).

Consequently, it follows from above discussions that $c_0 > m_\infty$.

In the what following, we give the prove of (3.20).

By (c) of Lemma 3.1, as $\lambda \to 0$, one has

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda}v_\lambda)) \to m_\infty \text{ as } \delta \to 0, \text{ uniformly in } z \in \mathbb{R}^N.$$

Then it follows from $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$ and Lemma 3.3 that, as $\lambda \to 0$

$$\beta(\mathcal{K}_\lambda(z,\delta)) \to \beta(\varphi_{\delta,z}) \text{ as } \delta \to 0, \text{ uniformly in } z \in \mathbb{R}^N. \tag{3.24}$$

Hence we conclude that

$$\begin{aligned}\beta(\varphi_{\delta,z}) &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(0)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2\mathrm{d}x\\ &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}}\int_{\mathbb{R}^N\setminus B_1(-z)}|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,0}|^2\mathrm{d}x\\ &= o(1) \text{ as } \delta \to 0, \text{ for } |z| < \frac{1}{2},\end{aligned}$$

which shows that, as $\lambda \to 0$, there is $0 < \delta_1 < \frac{1}{2}$ such that (a) holds.

Similarly, it follows from (c) of Lemma 3.1 and Corollary 2.1 that, as $\lambda \to 0$

$$\alpha(\mathcal{K}_\lambda(z,\delta)) \to \alpha(\varphi_{\delta,z}) \text{ as } \delta \to 0, \text{ uniformly in } z \in \mathbb{R}^N.$$

Then according to Lemma 3.3, as $\lambda \to 0$, there is $0 < \delta_1 < \frac{1}{2}$ such that (b) holds.

By (c) of Lemma 3.1 and Corollary 2.1, as $\lambda \to 0$, we also obtain that

$$\beta(\mathcal{K}_\lambda(z,\delta)) \to \beta(\varphi_{\delta,z}) \text{ as } \delta \to 0, \text{ uniformly in } z \in \mathbb{R}^N. \tag{3.25}$$

Thanks to

$$\int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \le C\delta^{-\frac{N}{2}-1+s} \to 0 \text{ as } \delta \to +\infty,$$

we have that

$$\begin{aligned}\beta(\varphi_{\delta,z}) &= 1 - \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_1(0)} |(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \mathrm{d}x \\ &= 1 + o(1) \text{ as } \delta \to +\infty.\end{aligned}$$

Hence, as $\lambda \to 0$, it follows from (3.25) that there is $\delta_2 > \frac{1}{2}$ such that (c) holds.

According to (c) of Lemma 3.1, as $\lambda \to 0$, we have

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda} v_\lambda)) \to m_\infty \text{ as } \delta \to 0 \text{ or } \delta \to +\infty \text{ uniformly in } z \in \mathbb{R}^N.$$

Thanks to (3.19), as $\lambda \to 0$, there exist $0 < \delta_1 < \frac{1}{2} < \delta_2$ such that (d) holds.

By $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$, we have $\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) \ge m$. Hence, thanks to Proposition 2.2, we obtain that

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) > m = m_\infty, \forall z \in \mathbb{R}^N, \delta > 0.$$

By (b) of Lemma 3.1, one has

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) = \mathcal{J}(t_{v_\lambda} v_\lambda)) \to m_\infty \text{ as } |z| \to +\infty$$

uniformly for every bounded λ, and bounded δ away from 0. Thanks to (3.19), there is $T_1 > \frac{1}{2}$ satisfying

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta)) < \frac{c_0 + m_\infty}{2}, |z| \ge T_1, \delta_1 \le \delta \le \delta_2, \lambda < 1.$$

That is, (e) holds.

According to (b) of Lemma 3.1, Corollary 2.1 and the fact $\mathcal{K}_\lambda(z,\delta) \in \mathcal{N}$, we have that

$$\Big(\alpha(\mathcal{K}_\lambda(z,\delta)), z\Big)_{\mathbb{R}^N} \to \Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} \text{ as } |z| \to +\infty \tag{3.26}$$

uniformly in $\delta_1 \le \delta \le \delta_2, \lambda < 1$.

Let

$$(\mathbb{R}^N)_z^+ := \{x \in \mathbb{R}^N : (x,z)_{\mathbb{R}^N} > 0\} \text{ and } (\mathbb{R}^N)_z^- := \{x \in \mathbb{R}^N : (x,z)_{\mathbb{R}^N} \le 0\}.$$

Since $\delta \in [\delta_1, \delta_2]$, there exist a large $T_2 > 0$ and $r \in (0, 1/4)$ such that if $|z| \ge T_2$, the ball $B_r(\widetilde{z}) \subset (\mathbb{R}^N)_z^+$ with $\widetilde{z}$ satisfying $|z - \widetilde{z}| = \frac{1}{2}$, and by the definition of $\varphi_{\delta,z}$ one has

$$|(-\Delta)^{-\frac{s}{2}}\varphi_{\delta,z}|^2 \ge C_0 > 0, x \in B_r(\widetilde{z}).$$

Hence, for any $|z| \geq T_2$, $\delta \in [\delta_1, \delta_2]$, one has

$$\begin{aligned}\Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^+} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N} \mathrm{d}x \\ &+ \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^-} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N} \mathrm{d}x \\ &:= I_1 + I_2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}I_1 &= \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^+} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N} \mathrm{d}x \\ &\geq \frac{|z|}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{B_r(\widetilde{z})} \frac{C_0(x,z)_{\mathbb{R}^N}}{|x||z|} \mathrm{d}x \\ &\geq C_1|z| \int_{B_r(\widetilde{z})} \frac{1}{|x|} \mathrm{d}x \\ &= C_2|z|.\end{aligned}$$

Arguing as Lemma 4.7 in [20], choose $T_3 > 0$ large enough such that $z \in \mathbb{R}^N$ with $|z| > T_3$, we have

$$\frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^-} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \mathrm{d}x < \frac{C_2}{2}.$$

Then we have

$$\begin{aligned}I_2 &\geq \frac{1}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^-} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \chi_1(|x|)(x,z)_{\mathbb{R}^N} \mathrm{d}x \\ &\geq -\frac{|z|}{S_{\mu,s}^{\frac{2N-\mu}{N-\mu+2s}}} \int_{(\mathbb{R}^N)_z^-} |(-\Delta)^{-\frac{s}{2}} \varphi_{\delta,z}|^2 \mathrm{d}x \\ &> -\frac{C_2|z|}{2}.\end{aligned}$$

Therefor

$$\Big(\alpha(\varphi_{\delta,z}), z\Big)_{\mathbb{R}^N} = I_1 + I_2 \geq \frac{C_2}{2}|z| > 0,$$

for all $z \in \mathbb{R}^N$ with $|z| > T = \max\{T_1, T_2, T_3\} > 0$ and for all $\delta_1 \leq \delta \leq \delta_2$.

□

Let

$$\Pi_{\delta,z} := \Big\{(z,\delta) \in \mathbb{R}^N \times \mathbb{R}^+ : |z| < T, \delta \in [\delta_1, \delta_1]\Big\},$$

where δ_1, δ_2 and T are constants defined as in Lemma 3.4. Denoted

$$\partial\Pi_{\delta,z} := \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4,$$

where

$$\Pi_1=\Big\{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+:|z|<\frac{1}{2},\delta=\delta_1\Big\};\ \Pi_2=\Big\{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+:\frac{1}{2}\le|z|<T,\delta=\delta_1\Big\};$$

$$\Pi_3=\Big\{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+:|z|\le T,\delta=\delta_2\Big\};\ \Pi_4=\Big\{(z,\delta)\in\mathbb{R}^N\times\mathbb{R}^+:|z|=T,\delta\in[\delta_1,\delta_1]\Big\}.$$

Defined $\Theta\subset D_0^{s,2}(\Omega)$ by

$$\Theta:=\Big\{\mathcal{K}_\lambda(z,\delta):(z,\delta)\in\Pi_{\delta,z}\Big\}.$$

It is easy to see that $\Theta\subset\mathcal{P}\cap\mathcal{N}$. Set

$$\mathcal{E}:=\Big\{\gamma:\gamma\in C(\mathcal{P}\cap\mathcal{N},\mathcal{P}\cap\mathcal{N}),\gamma(u)=u\text{ for any }u\text{ with }\mathcal{J}(u)<\frac{c_0+m_\infty}{2}\Big\},$$

$$\mathcal{F}:=\Big\{D\subset\mathcal{P}\cap\mathcal{N}:D=\gamma(\Theta),\gamma\in\mathcal{E}\Big\}.$$

Lemma 3.5. *Let* (V_1) *hold and* $D\in\mathcal{F}$, *then as* $\lambda\to0$ *we have*

$$D\cap\mathcal{M}\neq\emptyset.$$

Proof. To prove this Lemma, we just prove that for any $\gamma\in\mathcal{E}$, as $\lambda\to0$, there is $(z^*,\delta^*)\in\Pi_{\delta,z}$ so that

$$((\alpha,\beta)\circ\gamma\circ\mathcal{K}_\lambda)(z^*,\delta^*)=(0,\frac{1}{2}).\tag{3.27}$$

Defined $\mathcal{L}_\gamma:\mathbb{R}^N\times\mathbb{R}^+\to\mathbb{R}^N\times\mathbb{R}^+$ and $\mathcal{L}:\Pi_{\delta,z}\to\mathbb{R}^N\times\mathbb{R}^+$ by

$$\mathcal{L}_\gamma=(\alpha,\beta)\circ\gamma\circ\mathcal{K}_\lambda,\quad\mathcal{L}=(\alpha,\beta)\circ\mathcal{K}_\lambda.$$

Firstly, we prove that, as $\lambda\to0$

$$deg(\mathcal{L}_\gamma,\Pi_{\delta,z},(0,\frac{1}{2}))=deg(\mathcal{L},\Pi_{\delta,z},(0,\frac{1}{2})).\tag{3.28}$$

It follows from (d) and (e) of (3.20) that, as $\lambda\to0$

$$\mathcal{J}(\mathcal{K}_\lambda(z,\delta))<\frac{c_0+m_\infty}{2},(z,\delta)\in\partial\Pi_{\delta,z}.$$

Thus, as $\lambda\to0$, we have

$$\gamma(\mathcal{K}_\lambda(z,\delta))=\mathcal{K}_\lambda(z,\delta),(z,\delta)\in\partial\Pi_{\delta,z}.$$

So, we obtain that

$$\mathcal{L}_\gamma(z,\delta)=((\alpha,\beta)\circ\mathcal{K}_\lambda)(z,\delta)=\mathcal{L}(z,\delta),(z,\delta)\in\partial\Pi_{\delta,z},$$

which concludes that (3.28) holds.

Secondly, we need to prove that as $\lambda\to0$

$$deg(\mathcal{L},\Pi_{\delta,z},(0,\frac{1}{2}))=1.\tag{3.29}$$

Let

$$\Upsilon(z,\delta,t)=t\mathcal{L}(z,\delta)+(1-t)(z,\delta), t\in[0,1].$$

It follows from the homotopy invariance property of the topological degree and the fact

$$deg(Id_{\Pi_{\delta,z}},\Pi_{\delta,z},(0,\frac{1}{2}))=1,$$

that, for get (3.29), we just to prove that as $\lambda\to 0$

$$\Upsilon(z,\delta,t)\neq(0,\frac{1}{2}),\ \text{for}\ \forall(z,\delta)\in\partial\Pi_{\delta,z}\ \text{and}\ t\in[0,1]. \tag{3.30}$$

If $(z,\delta)\in\Pi_1$, thanks to $\delta_1<\frac{1}{2}$ and (a) of (3.20), we have

$$t(\beta\circ\mathcal{K}_\lambda)(z,\delta_1)+(1-t)\delta_1<\frac{1}{2},\ t\in[0,1]. \tag{3.31}$$

If $(z,\delta)\in\Pi_2$, since $\delta_1<\frac{1}{2}$, it follows from (b) of (3.20) that

$$\left|\alpha(\mathcal{K}_\lambda(z,\delta))-\frac{z}{|z|}\right|<\frac{1}{4},$$

which concludes that

$$\begin{aligned}|(1-t)z+t(\alpha\circ\mathcal{K}_\lambda)(z,\delta_1)|&\geq\left|(1-t)z+t\frac{z}{|z|}\right|-\left|t(\alpha\circ\mathcal{K}_\lambda)(z,\delta_1)-t\frac{z}{|z|}\right|\\&\geq(1-t)|z|+t-\frac{t}{4}\\&\geq\frac{1}{2}+\frac{t}{4}>0,\ t\in[0,1].\end{aligned} \tag{3.32}$$

If $(z,\delta)\in\Pi_3$, by $\delta_2>\frac{1}{2}$ and (c) of (3.20), we deduce that

$$t(\beta\circ\mathcal{K}_\lambda)(z,\delta_2)+(1-t)\delta_2>(1-t)\frac{1}{2}+\frac{t}{2}=\frac{1}{2},\ t\in[0,1]. \tag{3.33}$$

If $(z,\delta)\in\Pi_4$, according to (f) of (3.20), we have

$$\Big(t(\alpha\circ\mathcal{K}_\lambda)(z,\delta)+(1-t)z,z\Big)_{\mathbb{R}^N}=(1-t)|z|^2+t\Big((\alpha\circ\mathcal{K}_\lambda)(z,\delta),z\Big)_{\mathbb{R}^N}>0,t\in[0,1]. \tag{3.34}$$

Consequently, by (3.31)-(3.34), we get (3.30), and then (3.29) holds. Therefore, combine (3.28) with (3.29), we obtain that $D\cap\mathcal{M}\neq\emptyset$. □

4 The proof of main results

Proof. Let

$$c^*:=\inf_{D\in\mathcal{F}}\sup_{u\in D}\mathcal{J}(u).$$

$$K_{c^*}:=\{u\in\mathcal{P}\cap\mathcal{N}:\mathcal{J}(u)=c^*,\ \mathcal{J}'(u)=0\}.$$

$$\mathcal{J}^r := \{u \in \mathcal{N} : \mathcal{J}(u) \leq r\},\ r \in \mathbb{R}.$$

Let λ^* small enough such that Lemmas 3.2 and 3.5 hold for any $\lambda < \lambda^*$. Fix λ with $\lambda < \lambda^*$. To prove the Theorem 1.1, we only prove that $K_{c^*} \neq \emptyset$. Suppose by contradiction $K_{c^*} = \emptyset$. It follows from Lemma 3.4 and Lemma 3.5 that

$$c^* \geq \inf_{u \in \mathcal{M} \cap \mathcal{P}} \mathcal{J}(u) = c_0 > m_\infty.$$

By Lemma 3.2, we obtain that $c^* < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty$ due to $\Theta \subset \mathcal{F}$. Therefore, we have

$$m_\infty < c^* < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty.$$

According to Corollary 2.2, $\mathcal{J}$ satisfies Palais-Smale condition in

$$\mathcal{P} \cap \mathcal{N} \cap \{u \in D_0^{s,2}(\Omega) : m_\infty < \mathcal{J}(u) < 2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty\}.$$

So, it follows from a variant due to Hofer [22] of the classical deformation lemma (see [50, 53]) that there is a continuous map

$$\psi : [0,1] \times \mathcal{P} \cap \mathcal{N} \to \mathcal{P} \cap \mathcal{N}$$

and $\varepsilon_0 > 0$ satisfying:
(a) $\mathcal{J}^{c^*+\varepsilon_0} \backslash \mathcal{J}^{c^*-\varepsilon_0} \subset\subset \mathcal{J}^{2^{\frac{4s-\mu}{N-\mu+2s}} m_\infty} \backslash \mathcal{J}^{\frac{c_0+m_\infty}{2}}$;
(b) $\psi(0,u) = u$;
(c) $\psi(t,u) = u, u \in \mathcal{J}^{c^*-\varepsilon_0} \cup \{\mathcal{P} \cap \mathcal{N} \backslash \mathcal{J}^{c^*+\varepsilon_0}\},\ t \in [0,1]$;
(d) $\psi(1, \mathcal{J}^{c^*+\frac{\varepsilon_0}{2}}) \subset \mathcal{J}^{c^*-\frac{\varepsilon_0}{2}}$.

By definition of c^*, there exists $D^* \in \mathcal{F}$ such that

$$c^* \leq \sup_{u \in D^*} \mathcal{J}(u) < c^* + \frac{\varepsilon_0}{2},$$

hence $\psi(1, D^*) \in \mathcal{F}$, and it follows from (d) that

$$c^* \leq \sup_{u \in \psi(1,D^*)} \mathcal{J}(u) < c^* - \frac{\varepsilon_0}{2},$$

which is a contradiction. Thus, $K_{c^*} \neq \emptyset$ and Theorem 1.1 is proved. □

Declarations

Conflict of interest

The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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