

Normalized solutions of the critical Schrödinger-Bopp-Podolsky system with logarithmic nonlinearity

Sihua Liang^a, Xueqi Sun^b, Shuaishuai Liang^c, Vicențiu D. Rădulescu^{d,e,f,g,h,*}

^aCollege of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China

^bCollege of Mathematics, Harbin Institute of Technology, Harbin, Harbin, 150001, P.R. China

^cSchool of Mathematics, Jilin University, Changchun 130022, Jilin, PR China

^dFaculty of Applied Mathematics, AGH University of Kraków, al. Mickiewicza 30, Kraków 30-059, Poland

^eBrno University of Technology, Faculty of Electrical Engineering and Communication,
Technická 3058/10, Brno 61600, Czech Republic

^fScientific Research Center, Baku Engineering University, Baku AZ0102, Azerbaijan

^gSimion Stoilow Institute of Mathematics of the Romanian Academy, 010702 Bucharest, Romania

^hDepartment of Mathematics, University of Craiova, Street A.I. Cuza 13, 200585 Craiova, Romania

Abstract

In this paper, we study the following critical Schrödinger-Bopp-Podolsky system driven by the p -Laplace operator and with a logarithmic nonlinearity:

$$\begin{cases} -\Delta_p u + \mathcal{V}(\varepsilon x)|u|^{p-2}u + \kappa\phi u = \lambda|u|^{p-2}u + \vartheta|u|^{p-2}u \log|u|^p + |u|^{p^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi^2u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

The analysis is developed under the prescribed mass assumption $\int_{\mathbb{R}^3} |u|^p dx = \rho^p$, where $\vartheta, a, \rho > 0, \varepsilon > 0, \lambda \in \mathbb{R}, \kappa > 0$ and $p \in (1, 3)$. The potential $\mathcal{V} \geq 0$ is a bounded and continuous function that satisfies some suitable global conditions. The main results establish the existence, multiplicity and concentration of normalized solutions to the above system and the proofs combine suitable variational and topological methods. This seems to be the first paper dealing with the existence and concentration of solutions with prescribed mass for critical Schrödinger-Bopp-Podolsky systems involving the p -Laplacian and logarithmic nonlinearity. In the final part of this paper, we are interested in the asymptotic behaviour of normalized solutions as $\lambda \rightarrow 0$ and $a \rightarrow 0$, respectively. The main feature of this paper is given by the combined effects generated by the simultaneous appearance of a quasilinear operator, critical exponent, and the logarithmic nonlinearity.

Keywords: Schrödinger-Bopp-Podolsky system; p -Laplacian; Critical growth; Logarithmic nonlinearity; Lusternik-Schnirelmann category; Variational methods; Normalized solutions.

2020 Mathematics Subject Classification: 35A15; 35B33; 35R11; 46N50; 47G20; 47J30; 58E05.

* Corresponding author: Vicențiu D. Rădulescu. Email addresses: liangsihua@163.com (Sihua Liang), sunxueqi1@126.com (Xueqi Sun), liangss22@mails.jlu.edu.cn (Shuaishuai Liang), radulescu@agh.edu.pl (Vicențiu D. Rădulescu)

1 Introduction and main result

This paper is devoted to the study of the following critical Schrödinger-Bopp-Podolsky system involving the p -Laplacian and logarithmic nonlinearity:

$$\begin{cases} -\Delta_p u + \mathcal{V}(\varepsilon x)|u|^{p-2}u + \kappa\phi u = \lambda|u|^{p-2}u + \vartheta|u|^{p-2}u \log|u|^p + |u|^{p^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi^2u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

under the mass constraint

$$\int_{\mathbb{R}^3} |u|^p dx = \rho^p, \quad (1.2)$$

where $\Delta_p \cdot = \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ is the usual p -Laplacian, $\vartheta, a, \rho > 0$, $\varepsilon > 0$, $\lambda \in \mathbb{R}$, $\kappa > 0$, $p \in (1, 3)$ and the potential $\mathcal{V} \geq 0$ is a bounded continuous function.

In recent years, D'Avenia and Siciliano [7] first considered the following Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ \Delta^2\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.3)$$

where $a, \omega > 0$. This system is formed when we couple a Schrödinger field $\psi = \psi(t, x)$ with the electromagnetic field in the Bopp-Podolsky electromagnetic theory, especially in the electrostatic case for standing waves $\psi(t, x) = e^{i\omega t}u(x)$. The Bopp-Podolsky theory is a second order gauge theory of the electromagnetic field, and is proposed to solve the so-called infinity problem that occurs in the classical Maxwell theory, see [13, 14, 15]. Note that this theory was first developed by Bopp [12], and independently by Podolsky [41]. Furthermore, the Bopp-Podolsky theory can be interpreted as a valid theory for short distances (see [23]), whereas for long distances it is experimentally indistinguishable from Maxwell one. Therefore, the Bopp-Podolsky parameter $a > 0$ with the inverse dimension of mass can be regarded as the cut-off distance or can also be related to an effective radius of the electron. For more background of this topic, we refer to [9, 16]. On the other hand, system (1.1) involving p -Laplacian has a wide range of physical applications, such as the study of flows through porous media ($p = \frac{3}{2}$), nonlinear elasticity ($p \geq 2$) and glaciology ($p \in (1, \frac{4}{3}]$); see [20, 24, 31] and references. This is the reason why it is more interesting and meaningful to consider the Schrödinger-Bopp-Podolsky system driven by the p -Laplace operator.

Generally, there are two ways to consider system (1.1):

- (i) the frequency λ is a given constant.
- (ii) the frequency λ is unknown to system (1.1).

In case (i), we are usually interested in investigating ground state solutions of system (1.1) since such solutions possess more properties, such as stability, positivity, and symmetry. They can be regarded as minimizers of the energy functional $\mathcal{I}_\lambda$ among its nontrivial critical points, that is, the minimizers of

$$c_\lambda := \{\mathcal{I}_\lambda : u \in \mathbb{X}(\mathbb{R}^3) \setminus \{0\}, \mathcal{I}'_\lambda(u) = 0\},$$

where $\mathcal{I}_\lambda$ is the action functional of system (1.1). For example, Bahrouni and Missaoui [8] obtained the existence of a ground state solution with a fixed sign and a least energy nodal solution for system (1.3) by minimization techniques. Also, they showed the relationship between the energy of the nodal solution and that of the ground state solution. In [58], Yao et al. obtained the existence of ground state solution for Schrödinger-Bopp-Podolsky system involving Choquard nonlinearity via Nehari-Pohozaev manifold method. Chen et al. [18] considered the existence of ground state solutions for the non-autonomous Schrödinger-Bopp-Podolsky system by variational methods. Some other interesting results on this topic can be found in [28, 29, 30, 37, 50] and the references therein.

In case (ii), the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, which is determined by the solution rather than being given in advance. Such solutions always correspond to a couple (u, λ) that satisfies some fixed problems. We call these solutions normalized solutions, that is, solutions of system (1.1) satisfying the normalized condition (1.2). From a physical point of view, $\int_{\mathbb{R}^3} |u|^p dx$ is regarded as the mass, which represents the number of particles of each component in Bose-Einstein condensates or the power supply in the nonlinear optics framework. On the other hand, normalized solutions provide a better perspective for understanding the dynamical properties, such as orbital stability or

instability, and describe attractive Bose-Einstein condensates. That is the reason why there are many scholars exploring the normalized solutions in recent years.

Nevertheless, there are not many results about normalized solutions of Schrödinger-Bopp-Podolsky systems. In [1], Afonso and Siciliano first considered the Schrödinger-Bopp-Podolsky system under Neumann boundary conditions. Under appropriate conditions, they obtained the existence of multiple normalized solutions by the Lusternik-Schnirelmann theory and Krasnoselskii genus. Siciliano and Silva [48] explored the non-existence and qualitative properties of solutions for Schrödinger-Bopp-Podolsky systems by the fibering approach and variational methods. de Paula Ramos and Siciliano [44] studied the following Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + \omega u + \phi u = |u|^{p-2}u, \\ -\Delta^2 \phi + a^2 \Delta \phi = 4\pi u^2, \\ \|u\|_{L^2} = \rho, \end{cases} \quad (1.4)$$

where $a, \rho > 0$ are fixed, with unknowns being $u, \phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\omega \in \mathbb{R}$. The authors proved that system (1.4) has a least energy solution if $2 < p < 3$ (resp. $3 < p < \frac{10}{3}$) and ρ is small enough. Furthermore, they also obtained that if $2 < p < \frac{14}{5}$ and ρ is sufficiently small, least energy solutions are radially symmetric up to translation, and as $a \rightarrow 0$, these solutions converge to least energy solutions of Schrödinger-Poisson-Slater systems under the same L^2 -norm constraint. For the non-existence results, we refer to [49]. Huang and Wang [27] considered system (1.4) in the L^2 -mass supercritical case. More precisely, by Mountain-pass arguments developed on the L^2 -spheres, they obtained the existence of normalized ground states when the spherical radius of L^2 -spheres is small enough, the asymptotic behavior of normalized ground states as the mass vanishes or tends to infinity as $a \rightarrow 0$, the radial symmetry and uniqueness of normalized ground states, and the instability of normalized ground states. Zhang et al. [59] considered Schrödinger-Bopp-Podolsky system involving a negative potential. Indeed, they obtained the existence of normalized solutions with positive energy by Mountain-pass argument and also proved the non-existence of normalized solutions with negative energy. Li et al. [28] studied the following critical Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + \phi u = \lambda u + \mu |u|^{p-2}u + |u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.5)$$

on the mass constraint

$$\int_{\mathbb{R}^3} |u|^2 dx = c^2,$$

where $c > 0$, $2 < p < \frac{8}{3}$, $\mu > 0$ is a parameter, and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. By a constraint minimizing approach, they obtained the existence of normalized ground state solutions for the above system. Peng [42] studied the existence and multiplicity of normalized solutions for Schrödinger-Bopp-Podolsky system of the following type:

$$\begin{cases} -\Delta u + \lambda u - \kappa \left(\frac{1-e^{-\frac{|x|}{a}}}{|x|} * |u|^2 \right) u = |u|^4u & \text{in } \mathbb{R}^3, \\ u > 0, \quad \int_{\mathbb{R}^3} |u|^2 dx = c^2, \end{cases} \quad (1.6)$$

where $c, a > 0$ and $\kappa \in \mathbb{R} \setminus \{0\}$ are fixed constants and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. More precisely, for $\kappa > 0$, combined with the truncation argument and a measure representation lemma proposed by Lions [40], they obtained that system (1.6) has at least n pairs of radial normalized solutions with negative energy. Furthermore, they also obtained a non-existence result of the above system via a Liouville-type Theorem and the Pohozaev identity.

On the other hand, system (1.1) contains logarithmic nonlinearity. It is essential to introduce the background of logarithmic Schrödinger equations. Such equations have a wide range of applications in physics, such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, effective quantum gravity and so on. For more information on this topic, see [10, 25].

In the following, we shall present several relevant references for this paper. Alves and Ji [6] obtained the existence and multiplicity of normalized solutions for logarithmic Schrödinger equations by minimization techniques and the Lusternik-Schnirelmann category. Shen and Squassina [47] supplemented and expanded the results of [6]. They considered logarithmic Schrödinger systems involving p -Laplacian:

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x) |u|^{p-2}u = \lambda |u|^{p-2}u + |u|^{p-2}u \log |u|^p & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = a^p \varepsilon^N, \end{cases} \quad (1.7)$$

where $a, \varepsilon > 0, \lambda \in \mathbb{R}$ is known as the Lagrange multiplier. More precisely, they obtained the existence and concentration of normalized solutions for system (1.7). At the same time, they also proved the existence of normalized solutions for system (1.7) in both L^p -mass subcritical and L^p -mass supercritical cases. In [43], Peng and Jia considered the following logarithmic Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\varepsilon \Delta u + V(x)u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ -\varepsilon \phi + \varepsilon^4 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.8)$$

where $\varepsilon > 0$ and the potential function $V(x) \in C(\mathbb{R}^3, \mathbb{R})$. By the variational methods introduced by Szulkin [51], they obtained the existence and concentration of solutions for system (1.8). For more results on this topic, we refer to [19, 35, 36, 38, 52, 53, 54].

To the best of our knowledge, there are no relevant results in the existing literature regarding the normalized solutions of logarithmic Schrödinger-Bopp-Podolsky systems involving p -Laplacian. Therefore, it is natural to consider that whether there exist normalized solutions of system (1.1) involving p -Laplacian and Sobolev critical exponent? Moreover, whether these solutions are concentrated, or possess some good properties? Inspired by the above works, we first show that it is possible to obtain that the number of normalized solutions is not less than the number of global minimum points of $\mathcal{V}$ as the parameter ε is small enough by minimization techniques and the truncated argument. Next, by borrowing the arguments in Li and Zhang [34] and fixing the parameter a , we shall obtain the asymptotic behavior of normalized solutions as $\lambda \rightarrow 0$. Finally, inspired by references [7, 46], we will show the found solutions strongly converge to solutions of Schrödinger-Poisson systems when λ is fixed and $a \rightarrow 0$. As far as we know, this is the first study on the concentration and asymptotic behavior of normalized solutions for system (1.1).

Now, we are ready to state the main results in this paper. We suppose that the potential $\mathcal{V} \geq 0$ is a bounded and continuous function satisfying the following conditions:

$$(V_1) \quad \mathcal{V} \in L^\infty(\mathbb{R}^3), \mathcal{V}(x) \geq 0 \text{ for all } x \in \mathbb{R}^3.$$

$$(V_2) \quad \mathcal{V}_\infty = \lim_{|x| \rightarrow +\infty} \mathcal{V}(x) > \mathcal{V}_0 := \min_{x \in \mathbb{R}^3} \mathcal{V}(x) = 0.$$

$$(V_3) \quad \mathcal{V}^{-1}(\{0\}) = \{c_1, c_2, \dots, c_l\} \text{ with } c_1 = 0 \text{ and } c_j \neq c_s \text{ if } j \neq s.$$

Theorem 1.1. *Let conditions $(V_1) - (V_3)$ be satisfied and $a > 0$. For any $l \in \mathbb{N}$, then there exist $\tilde{\varepsilon}$ and $\mathcal{V}_*$ such that system (1.1) admits at least l couples $(u_\varepsilon^i, \lambda_\varepsilon^i)$ of weak solutions for $|\mathcal{V}|_\infty < \mathcal{V}_*$ and $\varepsilon \in (0, \tilde{\varepsilon})$ with $\int_{\mathbb{R}^3} |u_\varepsilon^i|^p dx = \rho^p, \lambda_\varepsilon^i < 0$ for $i = 1, 2, \dots, l$. Furthermore, each u_ε^i has a maximum point $z_\varepsilon^i \in \mathbb{R}^3$ such that $\mathcal{V}(z_\varepsilon^i) \rightarrow \mathcal{V}(x^i) = \mathcal{V}_0$ as $\varepsilon \rightarrow 0^+$.*

In what follows, we consider the asymptotic behavior of solutions for system (1.1) as $\lambda \rightarrow 0$ and $a \rightarrow 0$, respectively.

Theorem 1.2. *Suppose that conditions $(V_1) - (V_3)$ hold, and for a suitable fixed parameter $a^* > 0$, $(u_{\varepsilon, \lambda}^{a^*}, \lambda)$ is a family of weak solutions of system (1.1) obtained in Theorem 1.1. Then, up to subsequences, $\lim_{\lambda \rightarrow 0} u_{\varepsilon, \lambda}^{a^*} = u_{\varepsilon, 0}^{a^*}$, where $u_{\varepsilon, 0}^{a^*}$ is a normalized solution of the following system:*

$$\begin{cases} -\Delta_p u + \mathcal{V}(\varepsilon x)|u|^{p-2}u + \kappa \phi u = \vartheta |u|^{p-2}u \log |u|^p + |u|^{p^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi^2 u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = \rho^p. \end{cases} \quad (1.9)$$

Moreover, each $u_{\varepsilon, 0}^{a^*}$ has a maximum point $\bar{z}_\varepsilon \in \mathbb{R}^3$ such that $\mathcal{V}(\bar{z}_\varepsilon) \rightarrow \mathcal{V}(\bar{z}) = \mathcal{V}_0$ as $\varepsilon \rightarrow 0^+$.

Theorem 1.3. *Suppose conditions $(V_1) - (V_3)$ are satisfied, and $(u_{\varepsilon, \lambda^*}^a, \lambda^*)$ is a family of weak solutions of system (1.1) obtained in Theorem 1.1 for a suitable fixed parameter $\lambda^* < 0$. Then, up to subsequences, $\lim_{a \rightarrow 0} u_{\varepsilon, \lambda^*}^a = u_{\varepsilon, \lambda^*}^0$, where $u_{\varepsilon, \lambda^*}^0$ is a normalized solution of the following Schrödinger-Poisson system:*

$$\begin{cases} -\Delta_p u + \mathcal{V}(\varepsilon x)|u|^{p-2}u + \kappa \phi u = \lambda |u|^{p-2}u + \vartheta |u|^{p-2} \log |u|^p + |u|^{p^*-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi^2 u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^p dx = \rho^p. \end{cases} \quad (1.10)$$

The proofs of Theorems 1.1-1.3 are based on suitable variational and topological arguments. It is more interesting and meaningful to consider Theorem 1.1-Theorem 1.3 due to the appearance of non-local terms (ϕ, u) , logarithmic nonlinearity and critical nonlinearity, so we have to establish some new estimates. In addition, since λ is not prescribed, the sequences of approximated Lagrange multipliers have to be controlled. To clearly compare the contributions of this paper with previous results, we emphasize the following points.

- (i) It seems that this work is the first contribution to obtaining the concentration and asymptotic behavior of normalized solutions for critical Schrödinger-Bopp-Podolsky systems involving p -Laplacian and logarithmic nonlinearity. In the previous results, there are no such results for Schrödinger-Bopp-Podolsky systems involving p -Laplacian and logarithmic nonlinearity, even in the critical case. Therefore, our results are completely new.
- (ii) Inspired by the ideas of [34, 42, 47], we consider the concentration and asymptotic behavior of normalized solutions. However, we do not simply repeat the methods in [34, 47] to get the desired results. More precisely, Li and Zhang [34] considered the existence, multiplicity and asymptotic behavior of system (1.1) without logarithmic nonlinearity in the case $p = 2$. In contrast the equations in Shen and Squadssina [47] merely contain logarithmic nonlinearity, while system (1.1) includes the non-local (u, ϕ) and critical nonlinearity. For non-local term (u, ϕ) , we shall use reduction arguments proposed in D'Avenia and Siciliano [7] to reduce energy functional to a single variable functional $\mathcal{J}_\varepsilon$. In addition, to obtain our main results in this paper, inspired by the ideas about the method of decomposition of logarithmic nonlinearity in Shen and Squadssina [55], we complete the proof of Theorems 1.1-1.3. On the other hand, Peng and Jia [43] only obtained the concentration of solutions for logarithmic Schrödinger-Bopp-Podolsky system involving the classical Laplacian, even not in the critical case, and the solutions they studied are not normalized solutions. To some extent, we supplement and extend the results of [34, 42, 47].
- (iii) The appearance of the non-zero potential function $\mathcal{V}(x)$ makes the proof of the existence of the normalized solution extremely complicated. In the L^p -subcritical case, we cannot directly obtain the energy functional corresponding to system (1.1) on the sphere (see reference [32]). At the same time, the compactness of the minimizing sequence also requires detailed estimates to handle. To overcome these difficulties, borrowing of the arguments as references [3, 5], we use the penalization method proposed by Del Pino and Felmer [21]. We achieved this goal by modifying the nonlinearity outside the set. However, the most important way to overcome this difficulty is that we must find a new appropriate penalization for system (1.1), because the the original penalization in Del Pino and Felmer [21] has a linear growth at infinity, and this growth is controlled by the lower bound $\mathcal{V}_0$ of the potential function $\mathcal{V} > 0$. Here, the lower bound of $\mathcal{V}$ can be equal to zero, so the penalization found in Del Pino and Felmer [21] cannot be directly used in our case.
- (iv) The main obstacle in this paper is to verify that (PS) condition holds. In fact, system (1.1) contains critical nonlinearity in the whole space $\mathbb{R}^N$, which prevents the bounded (PS) sequence from converging. Hence, we have to consider how the interaction between the nonlocal term and the nonlinear term will affect the existence of normalized solutions for system (1.1). In addition, we shall encounter the other difficulty that sequences of approximated Lagrange multipliers have to be controlled, since λ is not prescribed. Furthermore, weak limits of the Palais-Smale sequences could leave a constraint. In conclusion, we use the concentration-compactness principles due to Lions [40] to overcome the lack of compactness in $\mathbb{R}^N$. At the same time, we shall establish some new techniques to overcome the difficulties brought by the emergence of the nonlocal term.

The paper is organized as follows. In Sect. 2, we introduce the variational setting and present some preliminary lemmas. In Sect. 3, we adopt a truncation function and consider the autonomous problem with truncation. In Sect. 4, we prove Theorems 1.1-1.3 by the Lusternik-Schnirelmann category and variational methods.

2 Preliminaries

In this section, we shall introduce some notations, fundamental concepts and properties concerning the Orlicz spaces which will be used later. For more details, we refer to Rao and Ren [45].

Definition 2.1. *An N -function is a continuous function $\Phi : \mathbb{R} \rightarrow [0, +\infty)$ that satisfies the following conditions:*

- (i) Φ is a convex and even function;
- (ii) $\Phi(t) = 0 \iff t = 0$;
- (iii) $\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t \rightarrow \infty} \frac{\Phi(t)}{t} = +\infty$.

We say that the N -function Φ satisfies the Δ_2 -condition, denoted by $\Phi \in (\Delta_2)$, if

$$\Phi(2t) \leq k\Phi(t), \quad \forall t \geq t_0$$

for some constants $k > 0$ and $t_0 \geq 0$.

The conjugate function $\tilde{\Phi}$ associated with Φ is obtained through the Legendre's transformation, defined as

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

It can be shown that $\tilde{\Phi}$ is also an N -function. The functions Φ and $\tilde{\Phi}$ are mutually complementary, that is, $\tilde{\tilde{\Phi}} = \Phi$. For an open set $\Omega \subset \mathbb{R}^3$, we define the Orlicz space associated with the N -function Φ as follows

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} \Phi\left(\frac{|u|}{\tilde{\lambda}}\right) dx < +\infty, \text{ for some } \tilde{\lambda} > 0 \right\}.$$

Note that $L^\Phi(\Omega)$ is a Banach space equipped with the Luxemburg norm define by

$$\|u\|_\Phi = \inf \left\{ \tilde{\lambda} > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\tilde{\lambda}}\right) dx \leq 1 \right\}.$$

Then there also holds the Hölder and Young type inequalities, namely

$$st \leq \Phi(t) + \tilde{\Phi}(s), \quad \forall s, t \geq 0$$

and

$$\left| \int_{\Omega} uv dx \right| \leq 2\|u\|_\Phi \|v\|_{\tilde{\Phi}}, \quad \text{for } \forall u \in L^\Phi(\Omega) \text{ and } \forall v \in L^{\tilde{\Phi}}(\Omega).$$

The space is the reflexive and separable space $L^\Phi(\Omega)$ provided that $\Phi, \tilde{\Phi} \in (\Delta_2)$. Furthermore, the Δ_2 -condition shows that

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} \Phi(|u|) dx < +\infty \right\}$$

and

$$u_n \rightarrow u \text{ in } L^\Phi(\Omega) \iff \int_{\Omega} \Phi(|u_n - u|) dx \rightarrow 0.$$

In addition, we recall some properties of N -functions. Let Φ be an N -function of $\mathcal{C}^1$ class and $\tilde{\Phi}$ be its conjugate function. If

$$1 < l \leq \frac{\Phi'(t)t}{\Phi(t)} \leq m, \quad t \neq 0, \quad (2.1)$$

then $\Phi, \tilde{\Phi} \in (\Delta_2)$. Finally, we consider the functions

$$\xi_0(t) = \min\{t^l, t^m\} \text{ and } \xi_1(t) = \max\{t^l, t^m\}, \quad \forall t \geq 0.$$

By (2.1), we can see that the function Φ satisfies

$$\xi_0(\|u\|_\Phi) \leq \Phi(u) \leq \xi_1(\|u\|_\Phi), \quad \forall u \in L^\Phi(\Omega). \quad (2.2)$$

In order to deal with the logarithmic nonlinearity in system (1.1), borrowing from the arguments of [6], we define the functions F_1 and F_2 as follows

$$F_1(s) = \begin{cases} F_1(-s), & s \leq 0, \\ -\frac{1}{p}s^p \log s^p, & 0 < s < (p-1)\delta, \\ -\frac{1}{p}s^p [\log((p-1)\delta)^p + p+1] + p\delta s^{p-1} - \frac{1}{p(p-1)}((p-1)\delta)^p, & s \geq (p-1)\delta, \end{cases}$$

and

$$F_2(s) = \begin{cases} 0, & |s| < (p-1)\delta, \\ \frac{1}{p}|s|^p \log(|s|^p/((p-1)\delta)^p) + p\delta|s|^{p-1} - \frac{p+1}{p}|s|^p - \frac{1}{p(p-1)}((p-1)\delta)^p, & |s| \geq (p-1)\delta, \end{cases}$$

where $\delta > 0$ is sufficiently small but fixed, and then we obtain the decomposition

$$|s|^p \log |s|^p = F_2(s) - F_1(s). \quad (2.3)$$

Furthermore, F_1 and F_2 satisfy the following properties:

($\mathcal{F}_1$) F_1 is even with $F_1'(s)s \geq 0$ and $F_1(s) \geq 0$ for all $s \in \mathbb{R}$. Moreover, $F_1 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is convex if $\delta \approx 0^+$;

($\mathcal{F}_2$) $F_2 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{C}^2((\delta, +\infty), \mathbb{R})$ and for each $\tilde{q} \in (p, p + \frac{p^2}{3})$, there exists a $C_{\tilde{q}} > 0$ such that

$$|F_2'(s)| \leq C_{\tilde{q}}|s|^{\tilde{q}-1}, \quad \forall s \in \mathbb{R};$$

($\mathcal{F}_3$) $s \mapsto \frac{F_2'(s)}{s^{p-1}}$ is a nondecreasing function for $s > 0$ and a strictly increasing function for $s > \delta$;

($\mathcal{F}_4$) $\lim_{s \rightarrow \infty} \frac{F_2'(s)}{s^{p-1}} = \infty$.

Lemma 2.1. [47, Lemma 2.2] *The function F_1 is an N -function. Moreover, if $2 \leq p < N$, it holds that $F_1, \tilde{F}_1 \in (\Delta_2)$.*

Replacing Φ and Ω in the above discussions with F_1 and $\mathbb{R}^3$, respectively, we define the Orlicz Space $L^{F_1}(\mathbb{R}^3)$ and it is standard to prove the following result.

Corollary 2.1. *The functional $\Theta : L^{F_1}(\mathbb{R}^3) \rightarrow \mathbb{R}$ given by $u \mapsto \int_{\mathbb{R}^3} F_1(u)dx$ is of class $\mathcal{C}^1(L^{F_1}(\mathbb{R}^3))$ with*

$$\Theta'(u)v = \int_{\mathbb{R}^3} F_1'(u)vdx, \quad \forall u, v \in L^{F_1}(\mathbb{R}^3),$$

where $L^{F_1}(\mathbb{R}^3)$ is the Orlicz space associated with F_1 endowed with the Laremburg norm $\|\cdot\|_{F_1}$.

In the sequel, in order to avoid the points $u \in W^{1,p}(\mathbb{R}^3)$ that satisfy $F_1(u) \notin L^1(\mathbb{R}^3)$, we should consider the workspace $\mathbb{X}(\mathbb{R}^3) = W^{1,p}(\mathbb{R}^3) \cap L^{F_1}(\mathbb{R}^3)$ equipped with the following norm

$$\|\cdot\| \triangleq \|\cdot\|_{W^{1,p}(\mathbb{R}^3)} + \|\cdot\|_{F_1},$$

where $\|\cdot\|_{W^{1,p}(\mathbb{R}^3)}$ denotes the usual norm in $W^{1,p}(\mathbb{R}^3)$. Moreover, the space of radially symmetric functions $\mathbb{X}_{rad}(\mathbb{R}^3)$ is defined by $\mathbb{X}_{rad}(\mathbb{R}^3) = \{u \in \mathbb{X}(\mathbb{R}^3) : u(x) = u(|x|)\}$ with the norm $\|\cdot\|$.

In the following, we also define the functional space $H^1(\mathbb{R}^3)$ by

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3)\}$$

endowed with the norm

$$\|u\|_{H^1(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx.$$

In addition, we also define $\mathcal{D}$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ introduced by the scalar product

$$\langle u, v \rangle_{\mathcal{D}} = \int_{\mathbb{R}^3} (\nabla u \nabla v + a^2 \Delta u \Delta v)dx.$$

Therefore, $\mathcal{D}$ is a Hilbert space, and there exists a continuous embeddings $\mathcal{D} \hookrightarrow D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. Finally, we denote $L^r(\mathbb{R}^3)$ as the usual Lebesgue space endowed with the standard norm.

$$|u|_r^r := \int_{\mathbb{R}^3} |u|^r dx$$

for $r \in [1, +\infty)$.

As shown in d'Avenia and Siciliano [7], the continuous embedding $\mathcal{D} \hookrightarrow L^\infty(\mathbb{R}^3)$ holds, where

$$\mathcal{D} := \{\phi \in D^{1,2}(\mathbb{R}^3) : \Delta \phi \in L^2(\mathbb{R}^3)\}.$$

The energy functional $\mathcal{I}_{\varepsilon,a}(u, \phi)$ corresponding to system (1.1) is defined by

$$\begin{aligned} \mathcal{I}_{\varepsilon,a}(u, \phi) = & \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon x) |u|^p dx + \frac{\kappa}{2} \int_{\mathbb{R}^3} \phi |u|^2 dx - \frac{\kappa}{16\pi} |\nabla \phi|_2^2 - \frac{\kappa a^2}{16\pi} |\Delta \phi|_2^2 \\ & - \frac{\vartheta}{p} \int_{\mathbb{R}^3} |u|^p \log |u|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \quad \text{in } \mathbb{X}(\mathbb{R}^3) \times \mathcal{D}. \end{aligned}$$

Note that the critical points of $\mathcal{I}_{\varepsilon,a}(u, \phi)$ are weak solutions of system (1.1). By (2.3), the energy functional $\mathcal{I}_{\varepsilon,a}(u, \phi)$ can be rewritten as

$$\begin{aligned}\mathcal{I}_{\varepsilon,a}(u, \phi) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u|^p dx + \frac{\kappa}{2} \int_{\mathbb{R}^3} \phi u^2 dx - \frac{\kappa}{16\pi} |\nabla \phi|_2^2 - \frac{\kappa a^2}{16\pi} |\Delta \phi|_2^2 \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(u) dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx.\end{aligned}$$

In fact, if $(u, \phi) \in \mathbb{X}(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $\mathcal{I}_{\varepsilon,a}$, then for every $(v, \zeta) \in \mathbb{X}(\mathbb{R}^3) \times \mathcal{D}$, one has

$$\begin{aligned}0 = \partial_u \mathcal{I}_{\varepsilon,a}(u, \phi)[v] &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^{p-2} \nabla u \nabla v dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u|^{p-2} u v dx \\ &\quad + \kappa \int_{\mathbb{R}^3} \phi u v dx + \vartheta \int_{\mathbb{R}^3} (F'_1(u) - F'_2(u)) v dx - \int_{\mathbb{R}^3} |u|^{p^*-2} u v dx\end{aligned}$$

and

$$0 = \partial_\phi \mathcal{I}_{\varepsilon,a}(u, \phi)[\zeta] = \frac{\mu}{2} \int_{\mathbb{R}^3} u^2 \zeta dx - \frac{\mu}{8\pi} \int_{\mathbb{R}^3} \nabla \phi \nabla \zeta dx - \frac{\mu a^2}{8\pi} \int_{\mathbb{R}^3} \Delta \phi \Delta \zeta dx$$

which implies that (u, ϕ) is a weak solution of system (1.1).

From this fact, together with the standard argument, we research for the critical points of a functional of a single variable. Now, for every fixed $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique solution $\phi_u \in \mathcal{D}$ such that

$$-\Delta \phi + a^2 \Delta^2 \phi = u^2.$$

In order to write explicitly this solution (see Podolsky [41]), we consider

$$\mathcal{K}(x) = \frac{1 - e^{-|x|/a}}{4\pi|x|}.$$

From d'Avenia and Siciliano [7], we have the fundamental properties for $\mathcal{K}$.

Lemma 2.2. *For every $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta \phi + a^2 \Delta^2 \phi = \delta_y.$$

Moreover, we also have

(i) if $f \in L^1_{loc}(\mathbb{R}^3)$ and for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \rightarrow \frac{f(y)}{|x-y|}$ is summated, then $\mathcal{K} * f \in L^1_{loc}(\mathbb{R}^3)$,

(ii) if $f \in L^p(\mathbb{R}^3)$ with $1 \leq p < \frac{3}{2}$, then $\mathcal{K} * f \in L^q(\mathbb{R}^3)$ for $q \in (\frac{3p}{3-2p}, +\infty]$.

In both cases $\mathcal{K} * f$ solves

$$-\Delta \phi + a^2 \Delta^2 \phi = \delta_y$$

in the sense of distributions and we have the following distributional derivatives

$$\nabla(\mathcal{K} * f) = (\nabla \mathcal{K}) * f \quad \text{and} \quad \Delta(\mathcal{K} * f) = (\Delta \mathcal{K}) * f \quad \text{a.e. in } \mathbb{R}^3.$$

Then, if we fix $u \in E$, the unique solution of the second equation for system (1.1) in $\mathcal{D}$ is

$$\phi_u^a(x) := \mathcal{K} * u^2 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|/a}}{|x-y|} u^2(y) dy.$$

Therefore, the function ϕ_u possesses the following good properties.

Lemma 2.3. [7, Lemma 3.4] *For every $u \in H^1(\mathbb{R}^3)$, then $\phi_u^a \in \mathcal{D}$ and the following conclusions are satisfied:*

(i) for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)}^a = \phi_u^a(\cdot + y)$;

(ii) $\phi_u^a \geq 0$;

(iii) for any $t > 0$, $\phi_{tu}^a = t^2 \phi_u^a$;

(iv) $\|\phi_u^a\|_{\mathcal{D}} \leq C|u|_{\frac{12}{5}}^2 \leq C\|u\|_{\mathbb{X}}^p$ and $\int_{\mathbb{R}^3} \phi_u^a |u|^2 dx \leq C|u|_{\frac{12}{5}}^4 \leq C\|u\|_{\mathbb{X}}^4$;

(v) if $u_n \rightharpoonup u$ in $\mathbb{X}(\mathbb{R}^3)$, then $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$.

In what follows, We define a smooth functional $\mathcal{P}(u) : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$\mathcal{P}(u) = \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

In fact, the functional $\mathcal{P}(u)$ has the following useful properties, similar to the Brézis-Lieb Lemma [56, Lemma 1.32].

Lemma 2.4. [57, Lemma 2.3] Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \rightarrow u$ a.e. in $\mathbb{R}^3$. Then

$$\mathcal{P}(u_n - u) = \mathcal{P}(u_n) - \mathcal{P}(u) + o(1) \text{ as } n \rightarrow \infty.$$

Now, taking the same arguments as d'Avenia and Siciliano [7], we can reduce system (1.1) into the following single equation:

$$-\Delta_p u + \mathcal{V}(\varepsilon x)|u|^{p-2}u + \kappa \phi_u^a u = \lambda|u|^{p-2}u + |u|^{p-2}u \log |u|^p + |u|^{p^*-2}u \text{ in } \mathbb{R}^3. \quad (2.4)$$

It is standard that for any $\rho > 0$, a solution of problem (2.4) restricted to (1.2) can be regarded as a critical point of the corresponding energy functional

$$\begin{aligned} \mathcal{J}_\varepsilon(u) := & \mathcal{I}_{\varepsilon,a}(u, \phi_u^a) = \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1)|u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx \\ & + \vartheta \int_{\mathbb{R}^3} F_1(u) dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx. \end{aligned} \quad (2.5)$$

Then the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. It is easy to verify that the energy functional $\mathcal{J}_\varepsilon$ is well-defined and is of class C^1 on $\mathbb{X}(\mathbb{R}^3)$. Furthermore, the pair $(u, \phi_u^a) \in \mathbb{X}(\mathbb{R}^3) \times \mathcal{D}$ is a critical point of $\mathcal{I}_{\varepsilon,a}$ is equivalent to the fact that u is a critical point of $\mathcal{J}_\varepsilon$ and $\phi = \phi_u^a$.

In the following, we introduce the well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.5. ([39]) Let $t, r > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, n, \alpha, r)$ independent of f, h such that

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\alpha} dx dy \leq C(t, N, \alpha, r) |f|_t |h|_r.$$

In fact, if $r = t = \frac{6}{5}$, then by the Hardy-Littlewood-Sobolev inequality, we deduce

$$\int_{\mathbb{R}^3} \phi_u^a |u|^2 dx \leq \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * u^2 \right) |u|^2 dx \leq C|u|_{\frac{12}{5}}^4. \quad (2.6)$$

In order to prove that the $(PS)_c$ condition are satisfied, we use the second concentration compactness principle and the concentration compactness principle at infinity. Now, we recall the concentration compactness principle due to Lions [40].

Lemma 2.6. ([40]) Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^3)$ such that $|u_n|^{p^*} \rightharpoonup \nu$ and $|\nabla u_n|^p \rightharpoonup \kappa$ in the sense of measures. Then, there exists a (at most countable) set of distinct points $\{x_i\} \subset \mathbb{R}^3$, for some at most countable index set I ,

(i) $\nu = |u|^{p^*} + \sum_{i \in I} \delta_{x_i} \nu_i$, $\nu_i > 0$;

(ii) $\mu \geq |\nabla u|^p + \sum_{i \in I} \delta_{x_i} \mu_i$, $\mu_i > 0$;

(iii) $\mu_i \geq S \nu_i^{p/p^*}$,

where δ_{x_i} are Dirac measures at x_i , S is the best Sobolev constant defined by

$$S = \inf \left\{ \int_{\mathbb{R}^3} |u|^p dx : \int_{\mathbb{R}^3} |u|^{p^*} dx = 1 \right\} \quad (2.7)$$

and κ_i, ν_i are some positive constants.

However, Lemma 2.6 is only concerned with the possible concentration of a weakly convergent sequences at finite points and it does not provide any information about the loss of mass of a sequence at infinity. The following concentration compactness principle at infinity was established by Chabrowski [17] and provides some quantitative information about the mass loss of the sequences at infinity.

Lemma 2.7. ([17]) *Let $\{u_n\}$ be a weakly convergent sequence to u in $W^{1,p}(\mathbb{R}^3)$, and define*

$$(i) \quad \nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{p^*} dx;$$

$$(ii) \quad \mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^p dx.$$

The quantities ν_∞ and μ_∞ exist and satisfy

$$(iii) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^{p^*} dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty;$$

$$(iv) \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^p dx = \int_{\mathbb{R}^3} d\mu + \mu_\infty;$$

$$(v) \quad \mu_\infty \geq S\nu_\infty^{p/p^*}.$$

We will need the following Gagliardo-Nirenberg inequality, which plays a crucial role in proving that the truncated functional $\mathcal{J}_{\varepsilon,T}$ is bounded from below on S_ρ .

Lemma 2.8. (Gagliardo-Nirenberg inequality) [2, 55] *For every $p < s < p^*$, there exists an optimal constant $\mathbb{C}_{p,s} > 0$ depending only on p and s such that*

$$|u|_s \leq \mathbb{C}_{p,s} |\nabla u|_p^{\beta_s} |u|_p^{1-\beta_s}, \quad \forall u \in W^{1,p}(\mathbb{R}^3), \quad (2.8)$$

where

$$\beta_s := 3 \left(\frac{1}{p} - \frac{1}{s} \right) = \frac{3(s-p)}{ps}. \quad (2.9)$$

By (2.8), we see that

$$\bar{p} = p + \frac{p^2}{3}$$

is the L^p -mass critical exponent with respect to p -Laplace equations.

Lemma 2.9. *Let $(\mathcal{F}_1) - (\mathcal{F}_3)$ and (V_1) be satisfied, then there exists a unique maximum point of $\mathcal{H}_\rho(t)$, where*

$$\mathcal{H}_\rho(t) = \frac{1}{p} t^p - \vartheta \mathbb{C}_{p,\bar{q}} \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} t^{\bar{q}\beta_{\bar{q}}} - \frac{1}{p^*} S^{-\frac{p^*}{p}} t^{p^*}$$

for all $t > 0$.

Proof. By Lemma 2.8, Lemma 2.3-(ii), $(\mathcal{F}_1) - (\mathcal{F}_2)$ and (V_1) , for any $u \in S_\rho$, we deduce that

$$\begin{aligned} \mathcal{J}_\varepsilon(u) &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \\ &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \vartheta \int_{\mathbb{R}^3} |u|^{\bar{q}} dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \\ &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \vartheta \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} \mathbb{C}_{\bar{q},p} |\nabla u|_p^{\bar{q}\beta_{\bar{q}}} - \frac{1}{p^*} S^{-\frac{p^*}{p}} |\nabla u|_p^{p^*} \\ &:= \mathcal{H}(\rho, |\nabla u|_p). \end{aligned} \quad (2.10)$$

Let

$$\mathcal{H}_\rho(t) = \mathcal{H}(\rho, t) := \frac{1}{p}t^p - \vartheta \mathbb{C}_{p, \tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} t^{\tilde{q}\beta_{\tilde{q}}} - \frac{1}{p^*} S^{-\frac{p^*}{p}} t^{p^*}$$

for all $t > 0$. Therefore, we deduce that

$$\begin{aligned} \mathcal{H}'(\varrho, t) &= t^{\tilde{q}\beta_{\tilde{q}}-1} \left(t^{p-\tilde{q}\beta_{\tilde{q}}} - S^{-\frac{p^*}{p}} t^{p^*-\tilde{q}\beta_{\tilde{q}}} - \vartheta \mathbb{C}_{p, \tilde{q}} \beta_{\tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} \right) \\ &= t^{\tilde{q}\beta_{\tilde{q}}-1} \mathcal{G}_1(t), \end{aligned}$$

where

$$\mathcal{G}_1(t) = t^{p-\tilde{q}\beta_{\tilde{q}}} - S^{-\frac{p^*}{p}} t^{p^*-\tilde{q}\beta_{\tilde{q}}} - \vartheta \mathbb{C}_{p, \tilde{q}} \beta_{\tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}}$$

for all $t \in [0, \infty)$. Moreover, we obtain that

$$\lim_{t \rightarrow 0^+} \mathcal{G}_1(t) = -\vartheta \mathbb{C}_{p, \tilde{q}} \beta_{\tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathcal{G}_1(t) = -\infty.$$

Therefore, there exists a local maximum point $t_0 \in (0, \infty)$ of $\mathcal{G}_1(t)$. It is clear to see that t_0 is also a local maximum point of $\mathcal{G}_2(t)$ defined by

$$\mathcal{G}_2(t) = t^{p-\tilde{q}\beta_{\tilde{q}}} - S^{-\frac{p^*}{p}} t^{p^*-\tilde{q}\beta_{\tilde{q}}} = t^{p-\tilde{q}\beta_{\tilde{q}}} (1 - \vartheta S^{-\frac{p^*}{p}} t^{p^*-p})$$

and $\mathcal{G}_2(t) > 0$ for $t > 0$ sufficiently small. Therefore, $\mathcal{G}_2(t_0) > 0$ and t_0 is the unique maximum point of $\mathcal{G}_2(t)$ since $\mathcal{G}_3(t)$ is a decreasing function on $(0, \infty)$, where

$$\mathcal{G}_3(t) := 1 - S^{-\frac{p^*}{p}} t^{p^*-p}.$$

Therefore, t_0 is the unique maximum point of $\mathcal{G}_1(t)$ on the interval $(0, \infty)$.

In the following, we may suppose that $\vartheta \mathbb{C}_{p, \tilde{q}} \beta_{\tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} < \mathcal{G}_2(t_0)$, then $\mathcal{G}_1(t_0) > 0$. Clearly, t_0 is the unique maximum point of $\mathcal{H}'(\rho, t)$ with positive maximum value. Therefore, there exist $0 < t_1 < t_0 < t_2 < +\infty$ such that $\mathcal{H}'(\rho, t) > 0$ if $t \in (t_1, t_2)$ and $\mathcal{H}'(\rho, t) < 0$ if $t \in (0, t_1) \cup (t_2, \infty)$. Then $\mathcal{H}_\rho(t)$ attains the maximum point at t_2 and local minimum point at t_1 . For $p < q < p + \frac{p^2}{3}$, we define the function

$$\mathcal{G}(t) = \frac{1}{p} t^{p-\tilde{q}\beta_{\tilde{q}}} - \frac{1}{p^*} S^{-\frac{p^*}{p}} t^{p^*-\tilde{q}\beta_{\tilde{q}}},$$

where $\beta_{\tilde{q}}$ is given in Lemma 2.8. Together with the definition of $\mathcal{H}(\rho, t)$, we obtain

$$\mathcal{H}(\rho, t) = t^{\tilde{q}\beta_{\tilde{q}}} (\mathcal{G}(t) - \vartheta \mathbb{C}_{p, \tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}}).$$

Therefore, we obtain that t_2 is the unique maximum point of $\mathcal{G}(t)$. Moreover, we deduce that $\mathcal{G}(t) > 0$ for $t > 0$ small enough. Then if we assume that $\vartheta \mathbb{C}_{p, \tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} < \mathcal{G}(t_2)$, we can deduce that the maximum value $\mathcal{H}_\rho(t) > 0$ and there exist $0 < R_0 < t_2 < R_1 < +\infty$ such that

$$\mathcal{H}_\rho(R_0) = \mathcal{H}_\rho(R_1) = 0. \tag{2.11}$$

In the following, we denote

$$\tilde{\rho}_1 = \min \left\{ \left(\frac{\mathcal{G}_2(t_0)}{\vartheta \mathbb{C}_{p, \tilde{q}} \beta_{\tilde{q}}} \right)^{\frac{p}{\tilde{q}(1-\beta_{\tilde{q}})}}, \left(\frac{\tilde{q} \mathcal{G}(t_2)}{\vartheta \mathbb{C}_{p, \tilde{q}}} \right)^{\frac{p}{\tilde{q}(1-\beta_{\tilde{q}})}} \right\},$$

we have $\mathcal{H}_\rho(t) < 0$ in the intervals $(0, R_0)$, $(R_1, +\infty)$ and $\mathcal{H}_\rho(t) > 0$ on (R_0, R_1) and all $\rho_1 \in (0, \tilde{\rho}_1)$. Therefore, we finish the proof of Lemma 2.9. $\square$

Let $\tau(x) \in C^\infty(\mathbb{R}^+, [0, 1])$ be an non-increasing function such that

$$\tau(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq R_0, \\ 0 & \text{if } x \geq R_1. \end{cases}$$

Now, we study the following truncated functional

$$\begin{aligned}\mathcal{J}_{\varepsilon,T}(u) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1 dx - \vartheta \int_{\mathbb{R}^3} F_2 dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx.\end{aligned}\tag{2.12}$$

By $(\mathcal{F}_1) - (\mathcal{F}_2)$, Lemma 2.8 and (V_1) , we obtain

$$\begin{aligned}\mathcal{J}_{\varepsilon,T}(u) &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \vartheta \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} \mathbb{C}_{p,\bar{q}} |\nabla u|_{\bar{q}}^{\beta_{\bar{q}}} - \frac{\tau(|\nabla u|_p)}{p^*} S^{-p^*/p} |\nabla u|_p^{p^*} \\ &:= \mathcal{H}_T(\rho, |\nabla u|_p),\end{aligned}$$

that is,

$$\mathcal{H}_T(\rho, t) := \frac{t^p}{p} - \vartheta \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} \mathbb{C}_{p,\bar{q}} t^{\bar{q}\beta_{\bar{q}}} - \frac{\tau(t)}{p^*} S^{-p^*/p} t^{p^*}.\tag{2.13}$$

In what follows, we give some properties of $\mathcal{H}_T(\rho, t)$.

Lemma 2.10. *Let $(\mathcal{F}_1) - (\mathcal{F}_2)$ and (V_1) be satisfied. Then the function $\mathcal{H}_{\rho,T}$ satisfies the following properties:*

- (i) $\mathcal{H}_T(\rho, t) \equiv \mathcal{H}_{\rho}(t)$ for all $t \in (0, R_0]$,
- (ii) $\mathcal{H}_T(\rho, t)$ is positive and strictly increasing in $(R_0, +\infty)$.

Proof. In the following, we consider the sign of $\mathcal{H}_T(\rho, t)$ as $t \geq R_1$. For all $t \in [R_1, +\infty)$, we have

$$\mathcal{H}_T(\rho, t) = \mathcal{H}_{T,\rho}(t) := \frac{1}{p} t^p - \vartheta \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} \mathbb{C}_{p,\bar{q}} t^{\bar{q}\beta_{\bar{q}}}.$$

Now, we may suppose that

$$0 < \rho < \min \left\{ \tilde{\rho}_1, \left(\frac{1}{\vartheta p \mathbb{C}_{p,\bar{q}}} \right)^{\frac{p}{\bar{q}(1-\beta_{\bar{q}})}} R_1^{\frac{p(p-\bar{q}\beta_{\bar{q}})}{\bar{q}(1-\beta_{\bar{q}})}} \right\}.$$

Consequently, we have $\mathcal{H}_{T,\rho}(t) > 0$ for all $t \geq R_1$. In addition, we have that

$$\mathcal{H}'_T(\rho, t) = t \left(1 - \vartheta \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} \mathbb{C}_{p,\bar{q}} \beta_{\bar{q}} t^{\bar{q}\beta_{\bar{q}}-p} \right) > 0$$

for all $t \geq R_1$ and $0 < \rho < \left(\frac{1}{\vartheta \mathbb{C}_{p,\bar{q}} \beta_{\bar{q}}} \right)^{\frac{p}{\bar{q}(1-\beta_{\bar{q}})}} R_1^{\frac{p(p-\bar{q}\beta_{\bar{q}})}{\bar{q}(1-\beta_{\bar{q}})}}$. Then we may suppose that

$$0 < \rho < \tilde{\rho} := \min \left\{ \tilde{\rho}_1, \left(\frac{1}{\vartheta p \mathbb{C}_{p,\bar{q}}} \right)^{\frac{p}{(1-\beta_{\bar{q}})}} R_1^{\frac{p(p-\bar{q}\beta_{\bar{q}})}{\bar{q}(1-\beta_{\bar{q}})}}, \left(\frac{1}{\vartheta \mathbb{C}_{p,\bar{q}} \beta_{\bar{q}}} \right)^{\frac{p}{\bar{q}(1-\beta_{\bar{q}})}} R_1^{\frac{p(p-\bar{q}\beta_{\bar{q}})}{\bar{q}(1-\beta_{\bar{q}})}} \right\}.$$

Hence, we deduce that $\mathcal{H}_T(\rho, t)$ has the following properties

$$\begin{cases} \mathcal{H}_T(\rho, t) \equiv \mathcal{H}_{\rho}(t) & \text{for all } t \in (0, R_0], \\ \mathcal{H}_T(\rho, t) & \text{is positive and strictly increasing in } (R_0, +\infty). \end{cases}$$

Hence, we complete the proof of Lemma 2.10. $\square$

Without loss of generality, we may assume that we can take $R_0 > 0$ sufficiently small such that

$$\frac{1}{p} t_1^2 - \frac{1}{p^*} S^{-\frac{p^*}{p}} t_1^{p^*} \geq 0 \text{ for all } t_1 \in [0, R_0] \quad \text{and} \quad R_0^p < S^{\frac{3}{p}}.\tag{2.14}$$

In what follows, for any $\mathcal{V}_1 \in [0, |\mathcal{V}|_{\infty}]$, we denote by $\mathcal{J}_{\mathcal{V}_1}, \mathcal{J}_{\mathcal{V}_1,T} : \mathbb{X}(\mathbb{R}^3) \rightarrow \mathbb{R}$ the following functionals

$$\begin{aligned}\mathcal{J}_{\mathcal{V}_1}(u) &:= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{(\mathcal{V}_1 + 1)}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_{\mathcal{V}_1, T}(u) &:= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{(\mathcal{V}_1 + 1)}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx.\end{aligned}$$

3 The autonomous problem with truncated.

In this section, we study the properties of the truncated functional $\mathcal{J}_{\mathcal{V}_1, T}$ on S_ρ .

Lemma 3.1. *The functional $\mathcal{J}_{\mathcal{V}_1, T}$ is bounded from below in S_ρ and coercive.*

Proof. For each $u \in S_\rho$, we deduce that

$$\mathcal{J}_{\mathcal{V}_1, T}(u) \geq \mathcal{H}_{T, \rho}(|\nabla u|_p) \geq \inf_{t>0} \mathcal{H}_{T, \rho}(t) > -\infty.$$

Moreover, $\mathcal{J}_{\mathcal{V}_1, T}(u) \rightarrow \infty$ as $|\nabla u|_p \rightarrow \infty$. Therefore, we obtain the desired result. $\square$

In the following, we can define that

$$\tilde{d}_{\mathcal{V}_1, T, \rho} := \inf_{u \in S_\rho} \mathcal{J}_{\mathcal{V}_1, T}(u). \quad (3.1)$$

The following result presents an important property of $c_{\mathcal{V}_1, T, \rho}$.

Lemma 3.2. *For $\rho < \tilde{\rho}$, there exists $\mathcal{V}_* > 0$ such that $\tilde{d}_{\mathcal{V}_1, T, \rho} < 0$ if $\mathcal{V}_1 < \mathcal{V}_*$.*

Proof. Fix $\psi \in \mathbb{X}(\mathbb{R}^3) \setminus \{0\}$ and $s > 0$. By some calculations, we obtain that

$$\begin{aligned}\mathcal{J}_{\mathcal{V}_1, T}(e^{\frac{3s}{p}} \psi) &= \frac{e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p dx + \frac{\mathcal{V}_1 + 1}{p} e^{3s} \int_{\mathbb{R}^3} |\psi|^p dx + \frac{e^{\frac{4}{p} \cdot 3s}}{4} \int_{\mathbb{R}^3} \phi_\psi^a \psi^2 dx \\ &\quad - \vartheta \frac{3s}{p} e^{3s} \log |e| \int_{\mathbb{R}^3} |\psi|^p dx - \frac{\vartheta e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p \log |\psi|^p dx - e^{\frac{p^*}{p} \cdot 3s} \cdot \frac{\tau(|\nabla e^{\frac{3s}{p}} \psi|_p)}{p^*} \int_{\mathbb{R}^3} |\psi|^{p^*} dx \\ &= \frac{e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p dx + \frac{\mathcal{V}_1 + 1}{p} e^{3s} \int_{\mathbb{R}^3} |\psi|^p dx + \frac{e^{\frac{4}{p} \cdot 3s}}{4} \int_{\mathbb{R}^3} \phi_\psi^a |\psi|^2 dx \\ &\quad - \vartheta \frac{3s}{p} e^{3s} \log |e| \int_{\mathbb{R}^3} |\psi|^p dx - \frac{\vartheta e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p \log |\psi|^p dx - \frac{e^{\frac{p^*}{p} \cdot 3s}}{p^*} \int_{\mathbb{R}^3} |\psi|^{p^*} dx < 0\end{aligned}$$

for $s > 0$ small enough. Therefore, we deduce that

$$\frac{e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p dx + \frac{e^{\frac{4}{p} \cdot 3s}}{4} \int_{\mathbb{R}^3} \phi_\psi^a \psi^2 dx - \vartheta \frac{3s}{p} e^{3s} \log |e| \int_{\mathbb{R}^3} |\psi|^p dx - \frac{\vartheta e^{3s}}{p} \int_{\mathbb{R}^3} |\psi|^p \log |\psi|^p dx - \frac{e^{\frac{p^*}{p} \cdot 3s}}{p^*} \int_{\mathbb{R}^3} |\psi|^{p^*} dx := \mathcal{A} < 0$$

for $s > 0$ small enough. Consequently, setting $\mathcal{V}_1 < \mathcal{V}_* := -\frac{p(\mathcal{A}+2)}{\rho^p}$, we obtain that

$$\mathcal{J}_{\mathcal{V}_1, T}(e^{\frac{3s}{p}} \psi) \leq \mathcal{A} - (\mathcal{A} + 2) + 1 < 0,$$

which shows that $\tilde{d}_{\mathcal{V}_1, T, \rho} < 0$, and completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Assume that $\mathcal{V}_1 < \mathcal{V}_*$ holds. The energy functional $\mathcal{J}_{\mathcal{V}_1, T}$ has the properties as follows:*

(i) $\mathcal{J}_{\mathcal{V}_1, T} \in C^1(\mathbb{X}(\mathbb{R}^3), \mathbb{R})$.

(ii) If $\mathcal{J}_{\mathcal{V}_1, T} \leq 0$ then $|\nabla u|_p < R_0$, and $\mathcal{J}_{\mathcal{V}_1, T}(v) = \mathcal{J}_{\mathcal{V}_1}(v)$ for all v in a small neighborhood of u in $\mathbb{X}(\mathbb{R}^3)$.

Proof. The results of Lemma 3.3 can be obtained by the similar arguments to [33, Lemma 3.3], so we omit here. $\square$

We recall the definition of $\mathcal{H}_T(\rho, t)$ given in (2.13), and define

$$\mathcal{F}_T(\rho, t) := \frac{1}{p} - \vartheta \mathbb{C}_{p, \tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} t^{\gamma_1} - \frac{\tau(t)}{p^*} S^{-\frac{p^*}{p}} t^{p^*-p},$$

where $\gamma_1 := \tilde{q}\beta_{\tilde{q}} - p < 0$. At the same time, we also define the following function

$$\mathcal{F}(\rho, t) = \frac{1}{p} - \vartheta \mathbb{C}_{p, \tilde{q}} \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} t^{\gamma_1} - \frac{1}{p^*} S^{-\frac{p^*}{p}} t^{p^*-p}. \quad (3.2)$$

In what follows, we study the properties of $\mathcal{F}_T(\rho, t)$ for all $t \in (0, \infty)$.

Lemma 3.4. *Let $(\rho_2, t_2) \in (0, \infty) \times (0, \infty)$ satisfy $\mathcal{F}(\rho_2, t_2) \geq 0$. Then for all $\rho_1 \in (0, \rho_2]$, it holds that*

$$\mathcal{F}_T(\rho_1, t_1) \geq 0, \quad \forall t_1 \in \left[\left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{p}} t_2, t_2\right].$$

Proof. Since $\rho \rightarrow \mathcal{F}_T(\rho, t)$ is a non-increasing function on the interval $(0, \infty)$, we can deduce that

$$\mathcal{F}_T(\rho_1, t_2) \geq \mathcal{F}_T(\rho_2, t_2) \geq \mathcal{F}(\rho_2, t_2) \geq 0.$$

By some calculations, we infer that

$$\begin{aligned} \mathcal{F}_T\left(\rho_1, \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{p}} t_2\right) &\geq \mathcal{F}\left(\rho_1, \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{p}} t_2\right) \\ &= \frac{1}{p} - \vartheta \mathbb{C}_{p, \tilde{q}} \left(\frac{\rho_1}{\rho_2}\right)^{\gamma_2} \rho_2^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} t_2^{\gamma_1} - \frac{1}{p^*} \left(\frac{\rho_1}{\rho_2}\right)^{\frac{p^*-p}{p}} S^{-\frac{p^*}{p}} t_2^{p^*-p} \\ &\geq \frac{1}{p} - \vartheta \mathbb{C}_{p, \tilde{q}} \rho_2^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} t_2^{\gamma_1} - \frac{1}{p^*} S^{-\frac{p^*}{p}} t_2^{p^*-p} \\ &= \mathcal{F}(\rho_2, t_2) \geq 0, \end{aligned}$$

where $\gamma_2 := \frac{\tilde{q}(1-\beta_{\tilde{q}})+\gamma_1}{p} > 0$ and γ_1 is given in Lemma 3.3. Therefore, we obtain that

$$\mathcal{F}_T\left(\rho_1, \left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{p}} t_2\right) \geq 0 \quad \text{and} \quad \mathcal{F}_T(\rho_1, t_2) \geq 0.$$

By the definition of τ , we conclude that $\mathcal{F}_T(\rho_1, t_1) \geq 0$ for $t_1 \in \left[\left(\frac{\rho_1}{\rho_2}\right)^{\frac{1}{p}} t_2, t_2\right]$. Consequently, we complete the proof of Lemma 3.4. $\square$

Lemma 3.5. *For each $u \in S_\rho$, we deduce that*

$$\mathcal{J}_{\mathcal{V}_1, T}(u) \geq |\nabla u|_p^p \mathcal{F}_T(\rho, |\nabla u|_p).$$

Proof. By the Gagliardo-Nirenberg inequality (see Lemma 2.8), (V_1) , $(\mathcal{F}_1) - (\mathcal{F}_2)$ and Lemma 2.3-(ii), we obtain that

$$\begin{aligned} \mathcal{J}_{\mathcal{V}_1, T}(u) &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx - \vartheta \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} \mathbb{C}_{p, \tilde{q}} |\nabla u|_p^{\tilde{q}\beta_{\tilde{q}}} - \frac{\tau(|\nabla u|_p)}{p^*} S^{-\frac{p^*}{p}} |\nabla u|_p^{p^*} \\ &= |\nabla u|_p^p \left[\frac{1}{p} - \vartheta \rho^{\frac{\tilde{q}(1-\beta_{\tilde{q}})}{p}} \mathbb{C}_{p, \tilde{q}} |\nabla u|_p^{\gamma_1} - \frac{\tau(|\nabla u|_p)}{p^*} S^{-\frac{p^*}{p}} |\nabla u|_p^{p^*-p} \right] \\ &= |\nabla u|_p^p \mathcal{F}_T(\rho, |\nabla u|_p) \end{aligned}$$

for each $u \in S_\rho$. Therefore, the proof of Lemma 3.5 is finished. $\square$

We recall the definition of $\tilde{d}_{\mathcal{V}_1, T, \rho}$ in (3.1), and we will show that it possesses some nice properties.

Lemma 3.6. *$\tilde{d}_{\mathcal{V}_1, T, \rho}$ is continuous with regard to $\rho \in (0, \tilde{\rho})$.*

Proof. By the similar arguments to [26, Lemma 2.3], we can obtain the desired result. Therefore, we omit it here. $\square$

For any $\rho < \tilde{\rho}$, by setting of (2.11) and $\mathcal{H}(\tilde{\rho}, t) = t^p \mathcal{F}(\tilde{\rho}, t)$, it follows that $\mathcal{F}(\tilde{\rho}, R_0) = 0$. Moreover, $\rho \rightarrow \mathcal{F}(\rho, t)$ is an non-increasing function. It follows that $\mathcal{F}(\rho, R_0) \geq 0$ for all $\rho \in (0, \tilde{\rho})$.

Lemma 3.7. $\frac{\rho_1^p}{\rho_2^p} \tilde{d}_{\mathcal{V}_1, T, \rho_2} < \tilde{d}_{\mathcal{V}_1, T, \rho_1} < 0$, where $0 < \rho_1 < \rho_2 < \tilde{\rho}$.

Proof. Set $\zeta = \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{p}} > 1$. Let $\{u_n\} \subset S_{\rho_1}$ be a minimizing sequence with respect to $\tilde{d}_{\mathcal{V}_1, T, \rho_1}$, i.e., $\mathcal{J}_{\mathcal{V}_1, T}(u_n) \rightarrow \tilde{d}_{\mathcal{V}_1, T, \rho_1} < 0$ as $n \rightarrow +\infty$ by Lemma 3.2. Therefore, there exists n_0 such that

$$\mathcal{J}_{\mathcal{V}_1, T}(u_n) < 0 \quad \text{for } n \geq n_0. \quad (3.3)$$

By Lemma 3.4, $\mathcal{F}(\rho_2, R_0) \geq 0$, and we obtain that $\mathcal{F}_T(\rho_1, t) \geq 0$ for any $t \in \left[\left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{p}} R_0, R_0\right]$. Moreover, it follows from (3.3) and Lemma 3.5 that

$$|\nabla u|_p < \left(\frac{\rho_2}{\rho_1}\right)^{\frac{1}{p}} R_0 \quad \text{for } n \geq n_0. \quad (3.4)$$

Let $v_n = \zeta u_n$, so $v_n \in S_{\rho_2}$. Then we can obtain that $|\nabla v_n|_p = \zeta |\nabla u_n|_p < R_0$ by Lemma 3.3. Consequently, we deduce that

$$\tau(|\nabla u_n|_p) = \tau(|\nabla v_n|_p) = 1.$$

Moreover, by $\zeta > 1$, we obtain that

$$\tilde{d}_{\mathcal{V}_1, T, \rho_2} \leq \mathcal{J}_{\mathcal{V}_1, T}(u_n) \leq \zeta^{p^2} \mathcal{J}_{\mathcal{V}_1, T}(u_n) - \frac{\zeta^p}{p} \log \zeta^p \rho_1^p < \zeta^{p^2} \mathcal{J}_{\mathcal{V}_1, T}(u_n).$$

Passing the limit as $n \rightarrow \infty$, we infer that $\tilde{d}_{\mathcal{V}_1, T, \rho_2} \leq \zeta^{p^2} \tilde{d}_{\mathcal{V}_1, T, \rho_1}$, i.e.,

$$\frac{\rho_1^p}{\rho_2^p} \tilde{d}_{\mathcal{V}_1, T, \rho_2} \leq \tilde{d}_{\mathcal{V}_1, T, \rho_1}.$$

Therefore, we finish the proof of Lemma 3.7. $\square$

Lemma 3.8. Let $\{u_n\} \subset S_\rho$ be a minimizing sequence with respect to $\tilde{d}_{\mathcal{V}_1, T, \rho} < 0$, then $u_n \rightarrow u$ in $L^{p^*}(\mathbb{R}^3)$.

Proof. Since $|\nabla u_n|_p^p \leq R_0$ for n large enough, by Lemma 2.6 and Lemma 2.7, there exist two positive measures ν and μ such that

$$|\nabla u_n|^2 \rightharpoonup \mu, \quad \text{and} \quad |u_n|^{p^*} \rightharpoonup \nu \quad (3.5)$$

in the space $\mathcal{M}(\mathbb{R}^3)$ as $n \rightarrow \infty$.

We define that $\varphi_\varrho(x) := \varphi\left(\frac{x-x_i}{\varrho}\right)$ for all $\varrho > 0$, where x_i is given in Lemma 2.6 and $\varphi \in C_c^\infty(\mathbb{R}^3)$, $\varphi = 1$ in B_1 , $\varphi = 0$ in B_2^c and $|\nabla \varphi|_{L^\infty(\mathbb{R}^3)} \leq 2$. Note that $\{u_n \varphi_\varrho\}$ is bounded in $\mathbb{X}(\mathbb{R}^3)$ and φ_ϱ take values in $\mathbb{R}$, we have $\langle \mathcal{J}'_{\mathcal{V}_1}(u_n), \varphi_\varrho u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, for $n \in \mathbb{N}$ sufficiently large. Together with these facts, $(\mathcal{F}_1) - (\mathcal{F}_2)$, Lemma 2.3-(ii) and (V_1) , we deduce that

$$\int_{\mathbb{R}^3} |\nabla u_n|^p \varphi_\varrho(x) dx + \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \varphi_\varrho(x) dx \leq \int_{\mathbb{R}^3} |u_n|^{\tilde{q}} \varphi_\varrho dx + \int_{\mathbb{R}^3} |u_n|^{p^*} \varphi_\varrho(x) dx + o_n(1). \quad (3.6)$$

By the Hölder inequality, we can obtain that

$$\left| \lim_{\varrho \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \varphi_\varrho(x) dx \right| = 0. \quad (3.7)$$

By $\tilde{q} \in (p, p + \frac{p^2}{3})$ and the definition of φ_ϱ , we have

$$\lim_{\varrho \rightarrow 0} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^{\tilde{q}} \varphi_\varrho dx = 0.$$

Therefore, by (3.5) and Lemma 2.6, we have

$$\begin{aligned}\lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} \varphi_\varrho \int_{\mathbb{R}^3} |\nabla u_n|^p dx &= \lim_{\varrho \rightarrow 0} \int_{\mathbb{R}^3} \varphi_\varrho d\mu = \mu(\{x_i\}) = \mu_i, \\ \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \varphi_\varrho |u_n|^{p^*} dx &= \lim_{\varrho \rightarrow 0} \int_{\mathbb{R}^3} \varphi_\varrho d\nu = \nu(\{x_i\}) = \nu_i.\end{aligned}$$

Letting $\varrho \rightarrow 0$, we obtain that $\mu_i = \nu_i$. Together with $\mu_i \geq S\nu_i^{\frac{p}{p^*}}$, it implies that

$$(i) \mu_i = 0 \quad \text{or} \quad (ii) \mu_i \geq S^{\frac{3}{p}}. \quad (3.8)$$

Arguing by contradiction that there exists $i_0 \in \mathcal{I}$ such that $\mu_{i_0} \geq S^{\frac{3}{p}}$, we obtain that

$$R_0^p \geq \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} |\nabla u_n|_p^p \geq \lim_{\varrho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \varphi_\varrho |\nabla u_n|^p dx = \lim_{\varrho \rightarrow 0} \int_{\mathbb{R}^3} \varphi_\varrho d\mu = \mu_{i_0} \geq S^{\frac{3}{p}},$$

which contradicts (2.14). Therefore, we can deduce that

$$u_n \rightarrow u \quad \text{in} \quad L_{loc}^{p^*}(\mathbb{R}^3).$$

In what follows, we define a cut off function $\eta \in C^\infty(\mathbb{R}^3)$ such that $\eta = 0$ in B_1 and $\eta = 1$ in $\mathbb{R}^3 \setminus B_2$, and we set $\eta_R(x) = \eta(x/R)$ for all $R > 0$. Note that $\{u_n \eta_R\}$ is also bounded in $\mathbb{X}(\mathbb{R}^3)$ and η_R takes values in $\mathbb{R}$. By a direct calculation, we deduce that $\langle \mathcal{J}'_{V_1}(u_n), \eta_R u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, for $n \in \mathbb{N}$ sufficiently large. Combining with these facts, $(\mathcal{F}_1) - (\mathcal{F}_2)$, Lemma 2.3-(ii) and (V_1) , we deduce that

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_n|^p \eta_R(x) dx + \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \eta_R(x) dx \\ \leq \int_{\mathbb{R}^3} |u_n|^{\tilde{q}} \eta_R(x) dx + \int_{\mathbb{R}^3} |u_n|^{p^*} \eta_R(x) dx + o_n(1).\end{aligned} \quad (3.9)$$

From the Hölder inequality, it holds that

$$\left| \lim_{R \rightarrow +\infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \eta_R(x) dx \right| = 0.$$

By the definition of η_R , we have

$$\int_{\{x \in \mathbb{R}^3 : |x| > R\}} |\nabla u_n|^p dx \leq \int_{\mathbb{R}^3} \eta_R |\nabla u_n|^p dx \leq \int_{\{x \in \mathbb{R}^3 : |x| > R/2\}} |\nabla u_n|^p dx.$$

Thus, by Lemma 2.7, we obtain

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \eta_R |\nabla u_n|^p dx = \mu_\infty. \quad (3.10)$$

Similarly, we obtain that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \eta_R |u_n|^{p^*} dx = \nu_\infty$$

and

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \eta_R |u_n|^{\tilde{q}} dx = \lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} \eta_R |u|^{\tilde{q}} dx = \lim_{R \rightarrow \infty} \int_{|x| > R/2} \eta_R |u|^{\tilde{q}} dx = 0.$$

Letting $R \rightarrow \infty$, we deduce that $\mu_\infty = \nu_\infty$. We obtain $\mu_\infty \geq S\nu_\infty^{\frac{p}{p^*}}$. Furthermore, we deduce that

$$(iii) \mu_\infty = 0 \quad \text{or} \quad (iv) \mu_\infty \geq S^{\frac{3}{p}}. \quad (3.11)$$

Similarly, we deduce that case (iv) cannot occur. Therefore, we have

$$u_n \rightarrow u \quad \text{in} \quad L^{p^*}(\mathbb{R}^3 \setminus B_R(0)).$$

Hence, we know that

$$u_n \rightarrow u \quad \text{in} \quad L^{p^*}(\mathbb{R}^3).$$

This completes the proof of Lemma 3.12. □

Lemma 3.9. (Compactness theorem on $S(\rho)$). Let $\{u_n\} \subset S_\rho$ be a minimizing sequence with respect to $\tilde{d}_{\mathcal{V}_1, T, \rho}$. Then, for some subsequence, either

(i) $\{u_n\}$ is strongly convergent, or

(ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \rightarrow \infty$ such that the sequence $\tilde{u}_n(x) = u_n(x + y_n)$ is strongly convergent to a function $\tilde{u} \in S_\rho$ with $\mathcal{J}_{\mathcal{V}_1, T}(\tilde{u}) = \tilde{d}_{\mathcal{V}_1, T, \rho}$.

Proof. By Lemma 3.1 and Lemma 3.2, we obtain that $\{u_n\}$ is bounded in $\mathbb{X}(\mathbb{R}^3)$. Therefore, there exists $u \in \mathbb{X}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $\mathbb{X}(\mathbb{R}^3)$ for some subsequence. In the following, we consider the following case:

Case 1. Assume $u \not\equiv 0$.

If $|u|_p := b \neq \rho$, then we must have $b \in (0, \rho)$. Let $v_n := u_n - u$. From Lemma 2.3, the Fatou's Lemma, the Brézis-Lieb Lemma [56, Lemma 1.32] and Lemma 2.4, we obtain that

$$\begin{aligned} |u_n|_p^p &= |v_n|_p^p + |u|_p^p + o_n(1), \\ \int_{\mathbb{R}^3} \phi_{u_n}^a u_n^2 dx &= \int_{\mathbb{R}^3} \phi_{v_n}^a v_n^2 dx + \int_{\mathbb{R}^3} \phi_u^a u^2 dx + o_n(1), \\ |\nabla u_n|_p^p &= |\nabla u|_p^p + |\nabla v_n|_p^p + o_n(1) \end{aligned}$$

and

$$|u_n|_{p^*}^{p^*} = |v_n|_{p^*}^{p^*} + |u|_{p^*}^{p^*} + o_n(1).$$

We suppose that $e_n = |v_n|_p$, then $|v_n|_p \rightarrow e$, where $\rho^p = e^p + b^p$. Noting that $e_n \in (0, \rho)$ for n large enough, and by the fact that τ is continuous, non-increasing function and Lemma 3.7, we deduce that

$$\begin{aligned} \tilde{d}_{\mathcal{V}_1, T, \rho} + o_n(1) &= \mathcal{J}_{\mathcal{V}_1, T}(u_n) \\ &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla v_n|^p dx + \frac{1}{p} (\mathcal{V}_1 + 1) \int_{\mathbb{R}^3} |v_n|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_{v_n}^a v_n^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(v_n) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(v_n) dx - \frac{\tau(|\nabla v_n|_p)}{p^*} \int_{\mathbb{R}^3} |v_n|^{p^*} dx \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} (\mathcal{V}_1 + 1) \int_{\mathbb{R}^3} |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \\ &\geq \mathcal{J}_{\mathcal{V}_1, T}(v_n) + \mathcal{J}_{\mathcal{V}_1, T}(u) + o_n(1) \\ &\geq \tilde{d}_{\mathcal{V}_1, T, e_n} + \tilde{d}_{\mathcal{V}_1, T, b} + o_n(1) \\ &\geq \frac{e_n^p}{\rho^p} \tilde{d}_{\mathcal{V}_1, T, \rho} + \tilde{d}_{\mathcal{V}_1, T, b} + o_n(1). \end{aligned}$$

Passing the limit as $n \rightarrow +\infty$, by Lemma 3.7, we obtain that

$$\tilde{d}_{\mathcal{V}_1, T, \rho} \geq \frac{e^p}{\rho^p} \tilde{d}_{\mathcal{V}_1, T, \rho} + \tilde{d}_{\mathcal{V}_1, T, b} > \frac{e^p}{\rho^p} \tilde{d}_{\mathcal{V}_1, T, \rho} + \frac{b^p}{\rho^p} \tilde{d}_{\mathcal{V}_1, T, \rho} = \tilde{d}_{\mathcal{V}_1, T, \rho}$$

which is impossible. This asserts that $|u|_p = \rho$, or equivalently, $u \in S(\rho)$.

As $|u_n|_p = |u|_p = \rho$, $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ is reflexive, we have that

$$u_n \rightarrow u \text{ in } L^p(\mathbb{R}^3). \quad (3.12)$$

This combined with interpolation theorem in the Lebesgue space and $(\mathcal{F}_2)$ gives

$$\int_{\mathbb{R}^3} F_2(u_n) dx \rightarrow \int_{\mathbb{R}^3} F_2(u) dx. \quad (3.13)$$

Moreover, by [7, Lemma 5.1], we obtain that

$$\int_{\mathbb{R}^3} \phi_{u_n}^a u_n^2 dx \rightarrow \int_{\mathbb{R}^3} \phi_u^a u^2 dx. \quad (3.14)$$

These above facts together with $\tilde{d}_{\mathcal{V}_1, T, \rho} = \lim_{n \rightarrow +\infty} \mathcal{J}_{\mathcal{V}_1, T}(u_n)$ and $F_1 \geq 0$ in $(\mathcal{F}_1)$, we obtain that

$$\begin{aligned} \tilde{d}_{\mathcal{V}_1, T, \rho} &= \lim_{n \rightarrow +\infty} \mathcal{J}_{\mathcal{V}_1, T}(u_n) \\ &= \lim_{n \rightarrow +\infty} \left(\frac{1}{p} \int_{\mathbb{R}^3} |\nabla u_n|^p dx + \frac{\mathcal{V}_1 + 1}{p} \int_{\mathbb{R}^3} |u_n|^p dx + \kappa \int_{\mathbb{R}^3} \phi_{u_n}^a u_n^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u_n) dx \right. \\ &\quad \left. - \vartheta \int_{\mathbb{R}^3} F_2(u_n) dx - \frac{\tau(|\nabla u_n|_p)}{p^*} |u_n|^{p^*} dx \right) \\ &\geq \mathcal{J}_{\mathcal{V}_1, T}(u). \end{aligned}$$

As $u \in S_\rho$, we conclude that $\mathcal{J}_{\mathcal{V}_1, T}(u) = \tilde{d}_{\mathcal{V}_1, T, \rho}$, then

$$\lim_{n \rightarrow \infty} \mathcal{J}_{\mathcal{V}_1, T}(u_n) = \mathcal{J}_{\mathcal{V}_1, T}(u),$$

together with (3.12) and (3.13), we obtain that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^p dx = \int_{\mathbb{R}^3} |\nabla u|^p dx$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} F_1(u_n) dx = \int_{\mathbb{R}^3} F_1(u) dx.$$

Combining with the fact that $F_1 \in (\Delta_2)$ by Lemma 2.1 and (3.12), we obtain that $u_n \rightarrow u$ in $\mathbb{X}(\mathbb{R}^3)$, which implies that (i) occurs.

Case 2. Assume $u \equiv 0$, i.e., $u_n \rightharpoonup 0$ in $\mathbb{X}(\mathbb{R}^3)$.

Now we claim that there exists $R_1, \varsigma > 0$, and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_{R_1}(y_n)} |u_n|^p dx \geq \varsigma \quad \text{for all } n. \quad (3.15)$$

If not, we must obtain $u_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ for all $s \in (p, p^*)$ by the Vanishing Lemma. Noting that $|\nabla u_n|_p \leq R_0$ for n large enough, by $(\mathcal{F}_1) - (\mathcal{F}_2)$, Lemma 2.3-(ii) and (3.2), we have that

$$\begin{aligned} 0 &> \mathcal{J}_{\mathcal{V}_1, T}(u_n) \geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u_n|^p dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u_n|^{p^*} dx + o_n(1) \\ &\geq \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u_n|^p dx - \vartheta \mathbb{C}_{p, \bar{q}} \int_{\mathbb{R}^3} |u|^{\bar{q}} dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |u_n|^{p^*} dx + o_n(1) \\ &\geq \frac{1}{p} |\nabla u_n|_p^p - \frac{1}{p^*} S^{-\frac{p^*}{p}} |\nabla u_n|_p^{p^*} + o_n(1) \\ &\geq |\nabla u_n|_p^p \left(\frac{1}{p} - \vartheta \mathbb{C}_{p, \bar{q}} C \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} |\nabla u|_p^{\bar{q}\beta_{\bar{q}}-p} - \frac{1}{p^*} S^{-\frac{p^*}{p}} |\nabla u_n|_p^{p^*-p} \right) + o_n(1) \\ &\geq |\nabla u_n|_p^p \left(\frac{1}{p} - \vartheta \mathbb{C}_{p, \bar{q}} C \rho^{\frac{\bar{q}(1-\beta_{\bar{q}})}{p}} R_0^{\bar{q}\beta_{\bar{q}}-p} - \frac{1}{p^*} S^{-\frac{p^*}{p}} R_0^{p^*-p} \right) + o_n(1) \\ &= |\nabla u_n|_p^p \mathcal{F}(\rho, R_0) \geq 0 \end{aligned}$$

which is impossible. Therefore, (3.15) holds and $|y_n| \rightarrow +\infty$. Consequently, we may define $\tilde{u}_n(x) = u_n(x + y_n)$. It is obvious to see that $\{\tilde{u}_n\} \subset S_\rho$ and it is also a minimizing sequence of $\tilde{d}_{\mathcal{V}_1, T, \rho}$. Moreover, there exists $\tilde{u} \in \mathbb{X}(\mathbb{R}^3) \setminus \{0\}$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $\mathbb{X}(\mathbb{R}^3)$. Following the similar arguments to the first part of this proof, we deduce that $\tilde{u}_n \rightarrow \tilde{u}$ in $\mathbb{X}(\mathbb{R}^3)$, which implies that (ii) is true. Therefore, we complete the proof of Lemma 3.9. $\square$

Lemma 3.10. $\tilde{d}_{\mathcal{V}_1, T, \rho}$ is attained.

Proof. From Lemma 3.1 and Lemma 3.9, there exists a bounded minimizing sequence $\{u_n\} \subset S_\rho$ and $u_n \rightarrow u$ in $\mathbb{X}(\mathbb{R}^3)$ with respect to $\tilde{d}_{\mathcal{V}_1, T, \rho} = \mathcal{J}_{\mathcal{V}_1, T}(u) < 0$. Therefore, it follows from Lemma 3.3 that $\{u_n\}$ is also a minimizing sequence for $\mathcal{J}_{\mathcal{V}_1}(u)$ and $\tilde{d}_{\mathcal{V}_1, T, \rho}(u) = \mathcal{J}_{\mathcal{V}_1}(u)$. Consequently, we finish the proof of Lemma 3.10. $\square$

The following result is a consequence of Lemma 3.10.

Corollary 3.1. If $\mathcal{V}_1 < \mathcal{V}_2 < \mathcal{V}_*$, then $\tilde{d}_{\mathcal{V}_1, T, \rho} < \tilde{d}_{\mathcal{V}_2, T, \rho}$.

Proof. Let $u \in S_\rho$ satisfy $\mathcal{J}_{\mathcal{V}_2, T}(u) = \tilde{d}_{\mathcal{V}_2, T, \rho}$. Therefore, $\tilde{d}_{\mathcal{V}_1, T, \rho} \leq \mathcal{J}_{\mathcal{V}_1, T}(u) < \mathcal{J}_{\mathcal{V}_2, T}(u) = \tilde{d}_{\mathcal{V}_2, T, \rho}$. $\square$

4 The nonautonomous problem

Now we denote by $\mathcal{J}_{0,T}, \mathcal{J}_{\infty,T} : \mathbb{X}(\mathbb{R}^3) \rightarrow \mathbb{R}$ the following functionals:

$$\begin{aligned} \mathcal{J}_{0,T}(u) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_{\infty,T}(u) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}_{\infty} + 1) |u|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx + \vartheta \int_{\mathbb{R}^3} F_1(u) dx \\ &\quad - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx. \end{aligned}$$

By $(V_1) - (V_3)$, $\mathcal{V}_{\infty} < \mathcal{V}_*$ and Lemma 3.10, we define the minimum value $\tilde{d}_{0,T,\rho}$ and $\tilde{d}_{\infty,T,\rho}$

$$\tilde{d}_{0,T,\rho} := \inf_{u \in S_{\rho}} \mathcal{J}_{0,T}(u) \quad \text{and} \quad \tilde{d}_{\infty,T,\rho} := \inf_{u \in S_{\rho}} \mathcal{J}_{\infty,T}(u),$$

respectively, which are attained. Therefore, there exist $u_0, u_{\infty} \in S_{\rho}$ such that $\mathcal{J}_{0,T}(u_0) = \tilde{d}_{0,T,\rho}$ and $\mathcal{J}_{\infty,T}(u_{\infty}) = \tilde{d}_{\infty,T,\rho}$, respectively. Furthermore, by Corollary 3.1 and $\mathcal{V}_0 < \mathcal{V}_{\infty}$, we know that

$$\tilde{d}_{0,T,\rho} < \tilde{d}_{\infty,T,\rho} < 0. \quad (4.1)$$

In what follows, we consider the truncated functional $\mathcal{J}_{\varepsilon,T} : \mathbb{X}(\mathbb{R}^3) \rightarrow \mathbb{R}$ given in (2.12). Note that we study the following the minimum value:

$$\tilde{d}_{\varepsilon,T,\rho} := \inf_{u \in S_{\rho}} \mathcal{J}_{\varepsilon,T}(u),$$

where $\tilde{d}_{\varepsilon,T,\rho}$ is well defined since $\tilde{d}_{0,T,\rho} < \tilde{d}_{\varepsilon,T,\rho} < \tilde{d}_{\infty,T,\rho} < 0$ and $\mathcal{J}_{\varepsilon,T}$ is coercive on S_{ρ} being greater than $\mathcal{J}_{0,T}$.

Lemma 4.1. *Let $\rho > 0$, then $\limsup_{\varepsilon \rightarrow 0^+} \tilde{d}_{\varepsilon,T,\rho} \leq \tilde{d}_{0,T,\rho}$ and there exists $\varepsilon_* > 0$ such that $\mathcal{J}_{\varepsilon,\rho} < \mathcal{J}_{\infty,\rho}$ for all $\varepsilon \in (0, \varepsilon_*)$.*

Proof. Let $u_0 \in S_{\rho}$ with $\mathcal{J}_{0,T}(u_0) = \tilde{d}_{0,T,\rho}$. Therefore, we deduce that

$$\begin{aligned} \tilde{d}_{\varepsilon,T,\rho} \leq \mathcal{J}_{\varepsilon,T}(u_0) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u_0|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u_0|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_{u_0}^a u_0^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(u_0) dx - \vartheta \int_{\mathbb{R}^3} F_2(u_0) dx - \frac{\tau(|\nabla u_0|_p)}{p^*} \int_{\mathbb{R}^3} |u_0|^{p^*} dx. \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0^+$, we have that

$$\limsup_{\varepsilon \rightarrow 0^+} \tilde{d}_{\varepsilon,T,\rho} \leq \limsup_{\varepsilon \rightarrow 0^+} \mathcal{J}_{\varepsilon,T}(u_0) = \mathcal{J}_{0,T}(u_0) = \tilde{d}_{0,T,\rho}.$$

Together with (4.1), then there exists $\varepsilon_* > 0$ small enough such that

$$\tilde{d}_{\varepsilon,T,\rho} < \tilde{d}_{\infty,T,\rho} \quad \text{for all } \varepsilon \in (0, \varepsilon_*)$$

which completes the proof of Lemma 4.1. □

Taking the same arguments as Lemma 3.3, we obtain the following result.

Lemma 4.2. *The energy functional $\mathcal{J}_{\varepsilon,T}$ satisfies the following properties:*

- (i) $\mathcal{J}_{\varepsilon,T} \in C^1(\mathbb{X}(\mathbb{R}^3), \mathbb{R})$.
- (ii) If $\mathcal{J}_{\varepsilon,T}(u) \leq 0$ then $|\nabla u|_p < R_0$, and $\mathcal{J}_{\varepsilon,T}(v) = \mathcal{J}_{\varepsilon}(v)$ for all v in a small neighborhood of u in $\mathbb{X}(\mathbb{R})^3$.

Let $\{u_n\} \subset S_\rho$ be a minimizing sequence of $\mathbb{X}_{\varepsilon,T}(u_n)$ with respect to any $\tilde{c} < \tilde{d}_{\infty,T,\rho} < 0$. Taking the same arguments as Lemma 3.1 and Lemma 3.2, we can deduce that $\{|\nabla u_n|_p\}$ is bounded. Therefore, there exists $u \in \mathbb{X}(\mathbb{R}^3)$ and a subsequence of $\{u_n\}$, still denoted by itself, such that

$$u_n \rightharpoonup u_\varepsilon \text{ in } \mathbb{X}(\mathbb{R}^3) \quad \text{and} \quad u_n(x) \rightarrow u_\varepsilon(x) \quad \text{a.e. in } \mathbb{R}^3.$$

Lemma 4.3. *For $\varepsilon > 0$, the weak limit u_ε of $\{u_n\}$ is nontrivial.*

Proof. By contradiction, we suppose that $u_\varepsilon = 0$. Therefore,

$$\tilde{c} + o_n(1) = \mathcal{J}_{\varepsilon,T}(u_n) = \mathcal{J}_{\infty,T}(u_n) + \frac{1}{p} \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) - \mathcal{V}_\infty) |u_n|^p dx.$$

From $(V_1) - (V_3)$, for any given $\xi > 0$, there exists $R > 0$ such that

$$\mathcal{V}(x) \geq \mathcal{V}_\infty - \xi \quad \text{for all } |x| \geq R.$$

Therefore, we deduce that

$$\tilde{c} + o_n(1) = \mathcal{J}_{\varepsilon,T}(u_n) \geq \mathcal{J}_{\infty,T}(u_n) + \frac{1}{p} \int_{B_{R/\varepsilon}(0)} (\mathcal{V}(\varepsilon x) - \mathcal{V}_\infty) |u_n|^p dx - \frac{\xi}{p} \int_{B_{R/\varepsilon}^c(0)} |u_n|^p dx.$$

Recalling that $\{u_n\}$ is bounded in $\mathbb{X}(\mathbb{R}^3)$ and $u_n \rightarrow 0$ in $L^p(B_{R/\varepsilon}(0))$, it holds that

$$\tilde{c} + o_n(1) \geq \mathcal{J}_{\infty,T}(u_n) - \xi C \geq \tilde{d}_{\infty,T,\rho} - \xi C. \quad (4.2)$$

By the arbitrariness of $\xi > 0$, we obtain that $\tilde{c} \geq \tilde{d}_{\infty,T,\rho}$, which is impossible. So we deduce that the weak limit u_ε of $\{u_n\}$ is nontrivial. $\square$

Lemma 4.4. *Let $\{u_n\}$ be a $(PS)_{\tilde{c}}$ sequence of $\mathcal{J}_{\varepsilon,T}$ restricted to S_ρ with $\tilde{c} < \tilde{d}_{\infty,T,\rho}$ and u_ε is the weak limit of $\{u_n\}$ in $\mathbb{X}(\mathbb{R}^3)$. If $u_n \rightharpoonup u_\varepsilon$ in $\mathbb{X}(\mathbb{R}^3)$, then there exists $\gamma > 0$ independent of ε such that*

$$\liminf_{n \rightarrow +\infty} \left(\int_{\mathbb{R}^3} |u_n - u_\varepsilon|^p dx \right)^{\frac{1}{p}} \geq \gamma.$$

Proof. From Lemma 4.2 and $\tilde{c} < \tilde{d}_{\infty,T,\rho} < 0$, we obtain that $|\nabla u_n|_p < R_0$ for n sufficiently large. Therefore, the sequence $\{u_n\}$ is also a $(PS)_{\tilde{c}}$ sequence of $\mathcal{J}_\varepsilon$ constrained to S_ρ , i.e.,

$$\mathcal{J}_\varepsilon(u_n) \rightarrow \tilde{c} \quad \text{and} \quad \left\| \mathcal{J}_\varepsilon|_{S_\rho}(u_n) \right\|'_{(\mathbb{X}(\mathbb{R}^3))^*} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Define $\Psi : \mathbb{X}(\mathbb{R}^3) \rightarrow \mathbb{R}$ by $\Psi(u) = \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$, we observe that $S_\rho = \Psi^{-1}(\varrho/p)$. By [56, Proposition 5.12], then there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|\mathcal{J}'_\varepsilon(u_n) - \lambda_n \Psi'(u_n)\|_{(\mathbb{X}(\mathbb{R}^3))^*} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (4.3)$$

Due to the boundedness of $\{u_n\}$ in $\mathbb{X}(\mathbb{R}^3)$, we obtain that $u_n \rightharpoonup u_\varepsilon$. Let $v_n := u_n - u_\varepsilon$, then $\{\lambda_n\}$ is also bounded. For some subsequence, there exists λ_ε such that $\lambda_n \rightarrow \lambda_\varepsilon$ as $n \rightarrow +\infty$. Together with (4.2), we deduce that

$$\mathcal{J}'_\varepsilon(u_\varepsilon) - \lambda_\varepsilon \Psi'(u_\varepsilon) = 0 \text{ in } (\mathbb{X}(\mathbb{R}^3))^*, \quad \|\mathcal{J}'_\varepsilon(v_n) - \lambda_\varepsilon \Psi'(v_n)\|_{(\mathbb{X}(\mathbb{R}^3))^*} \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (4.4)$$

By (2.6) and Lemma 2.8, choosing $\kappa > 0$ small enough, then there exists $C_\kappa > 0$ such that

$$\begin{aligned} 0 &> \tilde{d}_{\infty,T,\rho} > \lim_{n \rightarrow \infty} \mathcal{J}_\varepsilon(u_n) \\ &= \lim_{n \rightarrow \infty} \left(\mathcal{J}_\varepsilon(u_n) - \frac{1}{p} \mathcal{J}'_\varepsilon(u_n) u_n + \frac{\lambda_n}{p} |u_n|^p + o_n(1) \right) \\ &\geq \frac{\lambda_\varepsilon}{p} \int_{\mathbb{R}^3} |u|^p dx + \kappa \left(\frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} \phi_u^\alpha u^2 dx \\ &\geq \frac{\lambda_\varepsilon}{p} \int_{\mathbb{R}^3} |u|^p dx + \kappa \left(\frac{1}{4} - \frac{1}{p} \right) \mathbb{C}_p R_0^{4\beta_{12/5}} \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{4(1-\beta_{12/5})} \\ &= \left(\frac{\lambda_\varepsilon}{p} + \kappa \tilde{C} \right) \min\{\rho^p, \rho^{4p(1-\beta_{12/5})}\} \\ &\geq C_\kappa \lambda_\varepsilon \min\{\rho^p, \rho^{4p(1-\beta_{12/5})}\}, \end{aligned} \quad (4.5)$$

where $\tilde{C} := \frac{1}{4} - \frac{1}{p} \mathbb{C}_p R_0^{4\beta_{12/5}} < 0$. By (4.4), the above facts imply that

$$|\nabla v_n|_p^p + \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon x) |v_n|^p dx + \int_{\mathbb{R}^3} \phi_{v_n}^a v_n^2 dx - \lambda_\varepsilon |v_n|_p^p = \vartheta \int_{\mathbb{R}^3} F_2'(v_n) v_n dx - \vartheta \int_{\mathbb{R}^3} F_1'(v_n) v_n dx + |v_n|_{p^*}^{p^*} + o_n(1).$$

Together with (4.5) and $(\mathcal{F}_1) - (\mathcal{F}_2)$, we obtain that

$$|\nabla v_n|_p^p + \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon x) |v_n|^p dx - \frac{p\tilde{d}_{\infty,T,\rho}}{\rho} |v_n|_p^p \leq \vartheta |v_n|_{\tilde{q}}^{\tilde{q}} + |v_n|_{p^*}^{p^*} + o_n(1). \quad (4.6)$$

Invoking (4.6) and the Sobolev inequality, we can get that

$$C_0 \|v_n\|_\varepsilon^p \leq \vartheta |v_n|_{\tilde{q}}^{\tilde{q}} + |v_n|_{p^*}^{p^*} + o_n(1) \leq \vartheta C_1 \|v_n\|_\varepsilon^{\tilde{q}} + C_2 \|v_n\|_\varepsilon^{p^*} + o_n(1).$$

Since $v_n \rightharpoonup 0$ in $\mathbb{X}(\mathbb{R}^3)$, there exists C_4 independent of ε such that $\|v_n\|_\varepsilon \geq C_4$. Moreover, we have that

$$\liminf_{n \rightarrow +\infty} \left(\vartheta \|v_n\|_\varepsilon^{\tilde{q}} + \|v_n\|_\varepsilon^{p^*} \right) \geq C_5 \quad (4.7)$$

for some $C_5 > 0$. By (4.7) and the Gagliardo-Nirenberg inequality (see Lemma 2.8) and the Sobolev inequality, there exists $\gamma > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\liminf_{n \rightarrow +\infty} |v_n|_p \geq \gamma.$$

Therefore, we complete the proof of Lemma 4.4. □

From now on, we fix

$$0 < \tilde{\varrho}_0 < \min \left\{ \tilde{d}_{\infty,T,\rho} - \tilde{d}_{0,T,\rho}, \frac{\gamma^p}{\rho^p} \left(\tilde{d}_{\infty,T,\rho} - \tilde{d}_{0,T,\rho} \right) \right\}.$$

Lemma 4.5. $\mathcal{J}_{\varepsilon,T}$ satisfies the $(PS)_{\tilde{c}}$ condition restricted to S_ρ if $\tilde{c} < \tilde{d}_{0,T,\rho} + \tilde{\varrho}_0$.

Proof. Let $\{u_n\} \subset S_\rho$ be a $(PS)_{\tilde{c}}$ sequence of $\mathcal{J}_{\varepsilon,T}$ on S_ρ . Noting that $\tilde{c} < \tilde{d}_{\infty,T,\rho} < 0$, we obtain the boundedness of $\{u_n\}$ in $\mathbb{X}(\mathbb{R}^3)$ by Lemma 4.2. Let $u_n \rightharpoonup u_\varepsilon$ in $\mathbb{X}(\mathbb{R}^3)$ and $u_\varepsilon \not\equiv 0$ from Lemma 4.3. A straightforward computation gives that $v_n := u_n - u_\varepsilon$ also is a $(PS)_{\tilde{c}'}$ sequence of $\mathcal{J}_{\varepsilon,T}$ on S_ρ and $\tilde{c}' < \tilde{c}$. If $v_n \rightharpoonup 0$ in $\mathbb{X}(\mathbb{R}^3)$, we have $\liminf_{n \rightarrow +\infty} |v_n|_p \geq \gamma$ due to Lemma 4.4.

In what follows, let $\mathfrak{b} = |u_\varepsilon|_p$, $\mathfrak{e}_n = |v_n|_p$ and assume that $|v_n|_p \rightarrow \mathfrak{e}$, so we get $\mathfrak{e} \geq \gamma > 0$ and $\rho^p = \mathfrak{b}^p + \mathfrak{e}^p$. By the fact that $v_n \rightharpoonup 0$, and the similar arguments to (4.2), we have that $\mathcal{J}_{\varepsilon,T}(v_n) \geq \tilde{d}_{\infty,T,\rho} + o_n(1)$. From $\mathfrak{e}_n \in (0, \rho)$ for n large enough, we obtain that

$$\tilde{c} + o_n(1) = \mathcal{J}_{\varepsilon,T}(u_n) \geq \mathcal{J}_{\varepsilon,T}(v_n) + \mathcal{J}_{\varepsilon,T}(u_\varepsilon) \geq \tilde{d}_{\infty,T,\rho} + \tilde{d}_{0,T,\mathfrak{b}} + o_n(1). \quad (4.8)$$

By Lemma 3.7 and (4.7), we obtain that

$$\tilde{d}_{0,T,\rho} + \tilde{\varrho}_0 \geq \tilde{c} + o_n(1) \geq \frac{\mathfrak{e}_n^p}{\rho^p} \tilde{d}_{\infty,T,\rho} + \frac{\mathfrak{b}^p}{\rho^p} \tilde{d}_{0,T,\rho}.$$

Passing the limit as $n \rightarrow +\infty$, we have that

$$\tilde{\varrho}_0 \geq \frac{\mathfrak{e}^p}{\rho^p} \left(\tilde{d}_{\infty,T,\rho} - \tilde{d}_{0,T,\rho} \right) \geq \frac{\gamma^p}{\rho^p} \left(\tilde{d}_{\infty,T,\rho} - \tilde{d}_{0,T,\rho} \right)$$

which contradicts $\tilde{\varrho}_0 < \frac{\gamma^p}{\rho^p} \left(\tilde{d}_{\infty,T,\rho} - \tilde{d}_{0,T,\rho} \right)$. Therefore, we obtain $u_n \rightarrow u_\varepsilon$ in $\mathbb{X}(\mathbb{R}^3)$. Consequently, we finish the proof of Lemma 4.5. □

4.1 Multiplicity result

In this section, we shall apply some arguments found in Alves [4] to obtaining the multiplicity result of system (1.1). In what follows, we fix $\hat{\rho}, \tilde{r} > 0$ satisfying

- $\overline{B_{\hat{\rho}}(c_i)} \cap \overline{B_{\hat{\rho}}(c_j)} = \emptyset$, for i, j and c_i, c_j are defined in (V_3) .
- $\bigcup_{i=1}^l B_{\hat{\rho}}(c_i) \subset B_{\tilde{r}}(0)$.
- $K_{\hat{\rho}/p} = \bigcup_{i=1}^l \overline{B_{\hat{\rho}/p}(c_i)}$.

Set the function $\mathcal{Q}_\varepsilon : \mathbb{X}(\mathbb{R}^3) \setminus \{0\} \rightarrow \mathbb{R}^3$ by

$$\mathcal{Q}_\varepsilon(u) := \frac{\int_{\mathbb{R}^3} \mathcal{X}(\varepsilon x) |u|^2 dx}{\int_{\mathbb{R}^3} |u|^2 dx},$$

where $\mathcal{X} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by

$$\mathcal{X}(x) := \begin{cases} x & \text{if } |x| \leq \tilde{r}, \\ \tilde{r} \frac{x}{|x|} & \text{if } |x| > \tilde{r}. \end{cases}$$

The following lemmas play essential roles in obtaining (PS) sequences of $\mathcal{J}_{\varepsilon,T}$ on S_ρ .

Lemma 4.6. *There exist $\varepsilon_1 \in (0, \varepsilon_0]$ and $\varrho_1 \in (0, \tilde{\varrho}_0]$ such that if $\varepsilon \in (0, \varepsilon_1)$, $u \in S_\rho$ and $\mathcal{J}_{\varepsilon,T}(u) \leq \tilde{d}_{0,T,\rho} + \varrho_1$, then $\mathcal{Q}_\varepsilon(u) \in K_{\hat{\rho}/p}$.*

Proof. Arguing by contradiction that there exist sequences $\varrho_n \rightarrow 0, \varepsilon_n \rightarrow 0$ and $\{u_n\} \subset S_\rho$ such that

$$\mathcal{J}_{\varepsilon_n,T}(u_n) \leq \tilde{d}_{0,T,\rho} + \varrho_n, \quad \mathcal{Q}_{\varepsilon_n}(u_n) \notin K_{\hat{\rho}/p}. \quad (4.9)$$

Therefore, we obtain that

$$\tilde{d}_{0,T,\rho} \leq \mathcal{J}_{0,T}(u_n) \leq \mathcal{J}_{\varepsilon_n,T}(u_n) \leq \tilde{d}_{0,T,\rho} + \varrho_n.$$

This implies that $\{u_n\} \subset S_\rho$ is a minimizing sequence of $\mathcal{J}_{0,T}(u_n)$. From Lemma 3.9, passing to a subsequence if necessary, we consider two cases:

- (i) there exists a function $u \in S_\rho$ such that $u_n \rightarrow u$ in $\mathbb{X}(\mathbb{R}^3)$ as $n \rightarrow \infty$, or
- (ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \rightarrow +\infty$ such that $v_n(x) = u_n(x + y_n)$ converges to some $v \in S_\rho$.

Analysis of (i). By the Lebesgue dominated convergence theorem, we obtain that

$$\mathcal{Q}_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^3} \mathcal{X}(\varepsilon_n x) |u_n|^p dx}{\int_{\mathbb{R}^3} |u_n|^p dx} \rightarrow \frac{\int_{\mathbb{R}^3} \mathcal{X}(0) |u|^p dx}{\int_{\mathbb{R}^3} |u|^p dx} = c_i \in K_{\hat{\rho}/p}$$

which contradicts $\mathcal{Q}_{\varepsilon_n}(u_n) \notin K_{\hat{\rho}/p}$.

Analysis of (ii). Now, we shall consider two cases:

$$(I) \quad |\varepsilon_n y_n| \rightarrow +\infty \quad \text{and} \quad (II) \quad \varepsilon_n y_n \rightarrow y \text{ for some } y \in \mathbb{R}^3.$$

For case (I), from the limit $v_n \rightarrow v$ in $\mathbb{X}(\mathbb{R}^3)$, we obtain

$$\begin{aligned} \mathcal{J}_{\varepsilon_n,T}(u_n) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla v_n|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon_n x + \varepsilon_n y_n) |v_n|^p dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{v_n}^a v_n^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(u) dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla v_n|_p)}{p^*} \int_{\mathbb{R}^3} |v_n|^{p^*} dx \\ &\rightarrow \mathcal{J}_{\infty,T}(v). \end{aligned} \quad (4.10)$$

Since $\mathcal{J}_{\varepsilon_n, T}(u_n) \leq \tilde{d}_{0, T, \rho} + \varrho_n$, we obtain that

$$\tilde{d}_{\infty, T, \rho} \leq \mathcal{J}_{\infty, T}(v) \leq \tilde{d}_{0, T, \rho}$$

which is impossible.

For case (II), taking the similar arguments of (4.10), we can deduce $\mathcal{J}_{\varepsilon_n, T}(u_n) \rightarrow \mathcal{J}_{y, T}(v)$, which combined with $\mathcal{J}_{\varepsilon_n, T}(u_n) \leq \tilde{d}_{0, T, \rho} + \varrho_n$ implies that $\tilde{d}_{y, T, \rho} \leq \mathcal{J}_{y, T}(v) \leq \tilde{d}_{0, T, \rho}$. It follows from Corollary 3.1 that $\mathcal{V}(y) = \mathcal{V}_0$ and $y = c_i$ for some $i = 1, 2, \dots, l$. Therefore, we deduce that

$$\mathcal{Q}_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^3} \mathcal{H}(\varepsilon_n x + \varepsilon_n y_n) |v_n|^p dx}{\int_{\mathbb{R}^3} |v_n|^p dx} \rightarrow \frac{\int_{\mathbb{R}^3} \mathcal{H}(y) |v|^p dx}{\int_{\mathbb{R}^3} |v|^p dx} = c_i \in K_{\hat{\varrho}/p}$$

which yields that $\mathcal{Q}_{\varepsilon_n}(u_n) \in K_{\hat{\varrho}/p}$ for n sufficiently large. This contradicts (4.9), so we finish the proof of Lemma 4.6. $\square$

In what follows, we give some notations which will be used later.

- $\Upsilon_{\varepsilon}^i := \{u \in S_{\rho} : |\mathcal{Q}_{\varepsilon}(u) - c_i| < \hat{\varrho}\}$, $\partial\Upsilon_{\varepsilon}^i := \{u \in S_{\rho} : |\mathcal{Q}_{\varepsilon}(u) - c_i| = \hat{\varrho}\}$.
- $\gamma_{\varepsilon}^i := \inf_{u \in \Upsilon_{\varepsilon}^i} \mathcal{J}_{\varepsilon, T}(u)$, $\tilde{\gamma}_{\varepsilon}^i := \inf_{u \in \partial\Upsilon_{\varepsilon}^i} \mathcal{J}_{\varepsilon, T}(u)$.

Lemma 4.7. *There exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that*

$$\gamma_{\varepsilon}^i < \tilde{d}_{0, T, \rho} + \frac{\varrho_1}{p} \quad \text{and} \quad \gamma_{\varepsilon}^i < \tilde{\gamma}_{\varepsilon}^i \quad \text{for any } \varepsilon \in (0, \varepsilon_2). \quad (4.11)$$

Proof. Let $u \in S_{\rho}$ satisfy $\mathcal{J}_{0, T}(u) = \tilde{d}_{0, T, \rho}$. Now, for $i \in \{0, 1, \dots, l\}$, we define the function $\hat{u}_{\varepsilon}^i : \mathbb{R}^3 \rightarrow \mathbb{R}$ as

$$\hat{u}_{\varepsilon}^i(x) := u\left(x - \frac{c_i}{\varepsilon}\right).$$

Therefore, $\hat{u}_{\varepsilon}^i \in S_{\rho}$ for all $\varepsilon > 0$ and $i \in \{0, 1, \dots, l\}$. By the change of variable, it holds that

$$\begin{aligned} \mathcal{J}_{\varepsilon, T}(\hat{u}_{\varepsilon}^i) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla u|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon x + c_i) |u|^p dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^a u^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(u) dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^3} |u|^{p^*} dx \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{J}_{\varepsilon, T}(\hat{u}_{\varepsilon}^i) = \mathcal{J}_{0, T}(u) = \tilde{d}_{0, T, \rho}. \quad (4.12)$$

Noting that as $\varepsilon \rightarrow 0^+$, $\mathcal{Q}_{\varepsilon}(\hat{u}_{\varepsilon}^i) \rightarrow c_i$, we obtain that $\hat{u}_{\varepsilon}^i \in \Upsilon_{\varepsilon}^i$ as ε is small enough. By (4.12), there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$\gamma_{\varepsilon}^i < \tilde{d}_{0, T, \rho} + \frac{\varrho_1}{p} \quad \text{for any } \varepsilon \in (0, \varepsilon_2)$$

showing the first inequality in (4.11).

For any $v \in \partial\Upsilon_{\varepsilon}^i$, we deduce that $\mathcal{Q}_{\varepsilon}(v) \notin K_{\hat{\varrho}/p}$. Therefore, it follows from Lemma 4.6 that

$$\mathcal{J}_{\varepsilon, T}(v) > \tilde{d}_{0, T, \rho} + \varrho_1 \quad \text{for all } v \in \partial\Upsilon_{\varepsilon}^i \text{ and } \varepsilon \in (0, \varepsilon_2).$$

This implies that

$$\tilde{\gamma}_{\varepsilon}^i = \inf_{v \in \partial\Upsilon_{\varepsilon}^i} \mathcal{J}_{\varepsilon, T}(v) \geq \tilde{d}_{0, T, \rho} + \varrho_1$$

for all $\varepsilon \in (0, \varepsilon_2)$. Moreover, we deduce that

$$\gamma_{\varepsilon}^i < \tilde{\gamma}_{\varepsilon}^i \quad \text{for all } \varepsilon \in (0, \varepsilon_2).$$

Therefore, this completes the proof of Lemma 4.7. $\square$

Proposition 4.1. *Suppose $\varepsilon \in (0, \tilde{\varepsilon})$ to be fixed, where $\tilde{\varepsilon} := \varepsilon_2$ is given in Lemma 4.7, decreasing $\tilde{\varepsilon} > 0$ if necessary, then $\mathcal{J}_\varepsilon|_{S(\rho)}$ has at least l different nontrivial critical points.*

Proof. For each $i \in \{1, 2, \dots, l\}$, we can apply the Ekeland's variational principle to finding a sequence $\{u_n^i\} \subset \Upsilon_\varepsilon^i$ satisfying

$$\mathcal{J}_{\varepsilon,T}(u_n^i) \rightarrow \gamma_\varepsilon^i \quad \text{and} \quad \left\| \mathcal{J}_{\varepsilon,T}'|_{S_\rho}(u_n^i) \right\|_{(\mathbb{X}(\mathbb{R}^3))^*} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Since $\gamma_\varepsilon^i < \tilde{d}_{0,T,\rho} + \varrho_0$ and by Lemma 4.5, we can obtain that there exists u^i such that $u_n^i \rightarrow u^i$ in $\mathbb{X}(\mathbb{R}^3)$. Therefore,

$$u^i \in \Upsilon_\varepsilon^i, \quad \mathcal{J}_{\varepsilon,T}(u^i) = \gamma_\varepsilon^i \quad \text{and} \quad \mathcal{J}_{\varepsilon,T}'|_{S_\rho}(u^i) = 0.$$

For

$$\mathcal{Q}_\varepsilon(u^i) \in \overline{B_{\hat{\rho}}(c_i)}, \quad \mathcal{Q}_\varepsilon(u^j) \in \overline{B_{\hat{\rho}}(c_j)} \quad \text{and} \quad \overline{B_{\hat{\rho}}(c_i)} \cap \overline{B_{\hat{\rho}}(c_j)} = \emptyset \quad \text{for } i \neq j,$$

then $u^i \neq u^j$ for $i \neq j$, where $1 \leq i, j \leq l$. This shows that $\mathcal{J}_{\varepsilon,T}$ possesses at least l nontrivial critical points for any $\varepsilon \in (0, \tilde{\varepsilon})$. $\square$

Using Lemma 4.2 and the fact that $\mathcal{J}_{\varepsilon,T}(u^i) < 0$ for any $i = 1, 2, \dots, l$, we obtain that u^i are in fact the critical points of $\mathcal{J}_\varepsilon$ on S_ρ with $\mathcal{J}_\varepsilon(u^i) = \gamma_\varepsilon^i < 0$ and $\mathcal{J}_\varepsilon'(u^i)u^i = \lambda_i \rho$. Then, taking the similar arguments to (4.5), we can obtain that $\lambda_i < 0$.

Now, we will study the concentrating behavior of positive solutions for system (1.1). To this end, by Proposition 4.1, we know that there are k couples of $(v_\varepsilon^i, \lambda_\varepsilon^i) \in \mathbb{X} \times \mathbb{R}$ such that

$$v_\varepsilon^i \in \Theta_\varepsilon^i, \quad \mathcal{J}_\varepsilon(v_\varepsilon^i) = \gamma_\varepsilon^i \quad \text{and} \quad \mathcal{J}_\varepsilon'(v_\varepsilon^i) - \lambda_\varepsilon^i \Psi'(v_\varepsilon^i) = 0 \quad \text{in } \mathbb{X},$$

where $i \in \{1, 2, \dots, l\}$, $v_\varepsilon^i(x) > 0$ for all $x \in \mathbb{R}^3$ and $\lambda^i < 0$.

Lemma 4.8. *Suppose $\varepsilon \in (0, \tilde{\varepsilon})$ to be fixed, decreasing $\tilde{\varepsilon} > 0$ if necessary, there are $y_\varepsilon^i \in \mathbb{R}^3$, $R_0^i > 0$ and $\gamma_0^i > 0$ such that*

$$\int_{B_{R_0^i}(y_\varepsilon^i)} |v_\varepsilon^i|^p dx \geq \gamma_0^i$$

for $i \in \{1, 2, \dots, l\}$. Moreover, the family $\{\varepsilon y_\varepsilon^i\}_i$ is bounded and, passing to a subsequence if necessary, $\varepsilon y_\varepsilon^i \rightarrow x^i$ as $\varepsilon \rightarrow 0^+$.

Proof. By contradiction, we suppose that there exists a sequence $\{\varepsilon_n\}_n$ with $\varepsilon_n \rightarrow 0^+$ such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |v_{\varepsilon_n}^i|^2 dx = 0$$

for all $R > 0$. It follows from Lion's Vanishing Lemma [11] that

$$v_{\varepsilon_n}^i \rightarrow 0 \quad \text{in } L^s(\mathbb{R}^3) \quad \text{for all } s \in (p, p^*). \quad (4.13)$$

Taking the same arguments as Lemma 3.9, together with the fact that $F_1(t) \geq 0$ for all $t \in \mathbb{R}$ by $(\mathcal{F}_1)$, we have $\lim_{n \rightarrow \infty} \mathcal{J}_{\varepsilon_n}(v_{\varepsilon_n}^i) \geq 0$ which contradicts the fact that

$$\lim_{n \rightarrow \infty} \mathcal{J}_{\varepsilon_n}(v_{\varepsilon_n}^i) = \lim_{n \rightarrow \infty} \gamma_{\varepsilon_n}^i \leq \tilde{d}_{0,T,\rho} + \varrho_1 < 0. \quad (4.14)$$

Consequently, we may define $\tilde{v}_\varepsilon^i(\cdot) = v_\varepsilon^i(\cdot + y_\varepsilon^i)$ and $\{\tilde{v}_\varepsilon^i\}_i$ is bounded with respect to $\varepsilon \in (0, \tilde{\varepsilon})$. Therefore, there exists $\tilde{v} \in \mathbb{X}(\mathbb{R}^3) \setminus \{0\}$ such that $\tilde{v}_\varepsilon^i \rightarrow \tilde{v}^i$ in $\mathbb{X}(\mathbb{R}^3)$ as $\varepsilon \rightarrow 0^+$ along a subsequence. Since $\{\tilde{v}_\varepsilon^i\}_i \subset S_\rho$ and

$$\mathcal{J}_\varepsilon(v_\varepsilon^i) \geq \mathcal{J}_0(v_\varepsilon^i) = \mathcal{J}_0(\tilde{v}_\varepsilon^i) \geq \tilde{d}_{0,T,\rho}$$

jointly with (4.12), it yields that $\lim_{\varepsilon \rightarrow 0^+} \mathcal{J}_0(\tilde{v}_\varepsilon^i) = \tilde{d}_{0,T,\rho}$. Applying Lemma 3.9, we know that $\tilde{v}_\varepsilon^i \rightarrow \tilde{v}$ in $\mathbb{X}(\mathbb{R}^3)$ as $\varepsilon \rightarrow 0^+$. Suppose that $\{\varepsilon y_\varepsilon^i\}_i$ is unbounded with respect to $\varepsilon \in (0, \tilde{\varepsilon})$, then there exists a subsequence $\{\varepsilon_n y_{\varepsilon_n}^i\}_i$ such

that $|\varepsilon_n y_{\varepsilon_n}^i| \rightarrow +\infty$ as $n \rightarrow \infty$. Exploiting $\tilde{v}_{\varepsilon_n}^i \rightarrow \tilde{v}$ in $\mathbb{X}(\mathbb{R}^3)$,

$$\begin{aligned} \mathcal{J}_{\varepsilon_n}(v_{\varepsilon_n}^i) &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla v_{\varepsilon_n}^i|^p dx + \frac{1}{p} \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon_n x) |v_{\varepsilon_n}^i|^p dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_{v_{\varepsilon_n}^i} (v_{\varepsilon_n}^i)^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(u) dx - \vartheta \int_{\mathbb{R}^3} F_2(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |v_{\varepsilon_n}^i|^{p^*} dx \\ &= \frac{1}{p} \int_{\mathbb{R}^3} |\nabla v_{\varepsilon_n}^i|^p dx + \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{V}(\varepsilon_n x + \varepsilon_n y_n) |\tilde{v}_{\varepsilon_n}^i|^2 dx + \frac{\kappa}{4} \int_{\mathbb{R}^3} \phi_{\tilde{v}_{\varepsilon_n}^i} (\tilde{v}_{\varepsilon_n}^i)^2 dx \\ &\quad + \vartheta \int_{\mathbb{R}^3} F_1(\tilde{v}_{\varepsilon_n}^i) dx - \vartheta \int_{\mathbb{R}^3} F_2(\tilde{v}_{\varepsilon_n}^i) dx - \frac{1}{p^*} \int_{\mathbb{R}^3} |\tilde{v}_{\varepsilon_n}^i|^{p^*} dx \\ &\rightarrow \mathcal{J}_{\infty}(\tilde{v}). \end{aligned}$$

Combined with (4.14), it holds that

$$\tilde{d}_{0,T,\rho} + \varrho_1 \geq \mathcal{J}_{\infty}(\tilde{v}) \geq \tilde{d}_{\infty,T,\rho}.$$

From Lemma 3.7, it contradicts the definition of ϱ_1 given in Lemma 4.6. Therefore, up to a subsequence, $\varepsilon y_{\varepsilon}^i \rightarrow x_0^i$ in $\mathbb{R}^3$ as $\varepsilon \rightarrow 0^+$.

In what follows, we verify that $x_0^i = x^i$. In fact, we may take a similar argument to the case (ii) in Lemma 4.6, and we have that $\mathcal{V}(x_0^i) = \mathcal{V}_0$. Recalling $v_{\varepsilon}^i \in \Upsilon_{\varepsilon}^i$, we know that $\lim_{n \rightarrow \infty} \mathcal{Q}_{\varepsilon_n}(v_{\varepsilon_n}^i) = x_0^i$. Furthermore, we deduce that $|x^i - x_0^i| \leq \hat{\rho}$. Therefore, we obtain that $x_0^i = x^i$. The proof of Lemma 4.8 is finished. $\square$

Lemma 4.9. *Let $\varepsilon \in (0, \tilde{\varepsilon})$ be fixed, decreasing $\varepsilon^* > 0$. There exists v_{ε}^i possessing a maximum η_{ε}^i satisfying $\mathcal{V}(\varepsilon \eta_{\varepsilon}^i) \rightarrow \mathcal{V}(x^i)$ as $\varepsilon \rightarrow 0^+$ for $i \in \{1, 2, \dots, l\}$.*

Proof. Since $\tilde{v}_{\varepsilon}^i(\cdot) = v_{\varepsilon}^i(\cdot + y_{\varepsilon}^i)$, the definition of v_{ε}^i implies that a couple of weak solution $(\tilde{v}_{\varepsilon}^i, \lambda_{\varepsilon}^i)$ solves the following problem

$$\begin{cases} -\Delta_p \tilde{v}_{\varepsilon}^i + \mathcal{V}(\varepsilon x + \varepsilon x_{\varepsilon}^i) |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i + \kappa \phi_{\tilde{v}_{\varepsilon}^i}^a \tilde{v}_{\varepsilon}^i = \lambda_{\varepsilon}^i |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i + \vartheta |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i \log |\tilde{v}_{\varepsilon}^i|^p + |\tilde{v}_{\varepsilon}^i|^{p^*-2} \tilde{v}_{\varepsilon}^i & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |\tilde{v}_{\varepsilon}^i|^p dx = \rho^p. \end{cases} \quad (4.15)$$

Arguing as Proposition 4.1 and Lemma 4.8, it holds that $\tilde{v}_{\varepsilon}^i \rightarrow \tilde{v}^i$ in $\mathbb{X}(\mathbb{R}^3)$, $\lambda_{\varepsilon}^i \rightarrow \lambda^i$ in $\mathbb{R}^3$ and $\varepsilon x_{\varepsilon}^i \rightarrow x^i$ in $\mathbb{R}^3$ as $\varepsilon \rightarrow 0^+$. Therefore, by (4.15), we obtain that $(\tilde{v}^i, \lambda^i)$ is a nontrivial solution to the problem of type:

$$-\Delta_p \tilde{v}^i + \mathcal{V}_0 |\tilde{v}^i|^{p-2} \tilde{v}^i + \phi_{\tilde{v}^i}^a \tilde{v}^i = \lambda^i |\tilde{v}^i|^{p-2} \tilde{v}^i + \vartheta |\tilde{v}^i|^{p-2} \tilde{v}^i \log |\tilde{v}^i|^p + |\tilde{v}^i|^{p^*-2} \tilde{v}^i \quad \text{in } \mathbb{R}^3.$$

In what follows, we shall divide the proof into two steps.

Step 1. We verify that $|\tilde{v}_{\varepsilon}^i|_{\infty} \geq \varrho^i$ and

$$\lim_{|x| \rightarrow \infty} \tilde{v}_{\varepsilon}^i(x) = 0, \quad (4.16)$$

where $\varrho^i > 0$ is independent of $\varepsilon \in (0, \tilde{\varepsilon})$.

We prove the first conclusion. Arguing by contradiction that $|\tilde{v}_{\varepsilon}^i|_{\infty} \rightarrow 0$ as $\varepsilon \rightarrow 0^+$ in the sense of a subsequence. Then, we obtain that $\tilde{v}_{\varepsilon}^i \rightarrow 0$ in $\mathbb{X}(\mathbb{R}^3)$ which is impossible. Consequently, we have

$$|\tilde{v}_{\varepsilon}^i|_{\infty} \geq \varrho^i,$$

where $\varrho^i > 0$ is independent of $\varepsilon \in (0, \tilde{\varepsilon})$.

In what follows, we verify that (4.16) holds. Indeed, for every $R > 0$ and $0 < r \leq \frac{R}{2}$, we choose a cut-off function $\psi \in C_0^{\infty}(\mathbb{R}^3, [0, 1])$ such that $\psi(x) = 1$ if $|x| \geq R$, and $\psi(x) = 0$ if $|x| \leq R - r$ as well as $|\nabla \psi| \leq \frac{2}{r}$. Given $\varepsilon \in (0, \tilde{\varepsilon})$ and $\mathfrak{h} > 1$, define

$$\tilde{v}_{\varepsilon, \mathfrak{h}}^i(x) = \begin{cases} \tilde{v}_{\varepsilon}^i(x), & \tilde{v}_{\varepsilon}^i(x) < \mathfrak{h}, \\ \mathfrak{h}, & \tilde{v}_{\varepsilon}^i(x) \geq \mathfrak{h}, \end{cases}$$

and

$$\tilde{z}_{\varepsilon, \mathfrak{h}}^i = \psi^p(\tilde{v}_{\varepsilon, \mathfrak{h}}^i)^{p(\vartheta-1)} \tilde{v}_{\varepsilon}^i \quad \text{and} \quad \tilde{\omega}_{\varepsilon, \mathfrak{h}}^i = \psi \tilde{v}_{\varepsilon}^i (\tilde{v}_{\varepsilon, \mathfrak{h}}^i)^{\vartheta-1}$$

with $\theta > 1$ to be determined later. Let $\tilde{z}_{\varepsilon, \mathbf{h}}^i$ be a test function in (4.15), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \psi^p(\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\nabla \tilde{v}_{\varepsilon}^i|^p dx + \int_{\mathbb{R}^3} \psi^p \phi_{\tilde{v}_{\varepsilon}^i}^a (\tilde{v}_{\varepsilon}^i)^2 (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} \\ &= -p(\theta-1) \int_{\mathbb{R}^3} (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p\theta-p-1} \psi^p \tilde{v}_{\varepsilon}^i |\nabla \tilde{v}_{\varepsilon}^i|^{p-2} \nabla \tilde{v}_{\varepsilon}^i \nabla \tilde{v}_{\varepsilon, \mathbf{h}}^i dx \\ & \quad - p \int_{\mathbb{R}^3} \psi^{p-1} (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} \tilde{v}_{\varepsilon}^i |\nabla \tilde{v}_{\varepsilon}^i|^{p-2} \nabla \tilde{v}_{\varepsilon}^i \nabla \psi dx \\ & \quad + \int_{\mathbb{R}^3} g(\tilde{v}_{\varepsilon}^i) \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} \tilde{v}_{\varepsilon}^i dx - \int_{\mathbb{R}^3} \mathcal{V}_{\varepsilon}(x) |\tilde{v}_{\varepsilon}^i|^p \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} dx, \end{aligned}$$

where $\mathcal{V}_{\varepsilon}(x) = \mathcal{V}(\varepsilon x + \varepsilon x_{\varepsilon}^i)$ and

$$\begin{aligned} g(\tilde{v}_{\varepsilon}^i) &= \lambda_{\varepsilon}^i |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i + |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i \log |\tilde{v}_{\varepsilon}^i|^p - \kappa \phi_{\tilde{v}_{\varepsilon}^i}^a \tilde{v}_{\varepsilon}^i + |\tilde{v}_{\varepsilon}^i|^{p^*-2} \tilde{v}_{\varepsilon}^i \\ &= (\lambda_{\varepsilon}^i - 1) |\tilde{v}_{\varepsilon}^i|^{p-2} \tilde{v}_{\varepsilon}^i + \vartheta F_2'(\tilde{v}_{\varepsilon}^i) - \vartheta F_1'(\tilde{v}_{\varepsilon}^i) - \kappa \phi_{\tilde{v}_{\varepsilon}^i}^a \tilde{v}_{\varepsilon}^i + |\tilde{v}_{\varepsilon}^i|^{p^*-2} \tilde{v}_{\varepsilon}^i. \end{aligned}$$

By $(\mathcal{F}_1)$, $(\mathcal{F}_2)$ with $\tilde{q} \in (p, p^*)$ and Lemma 2.3-(ii), we obtain that

$$\int_{\mathbb{R}^3} g(\tilde{v}_{\varepsilon}^i) \tilde{v}_{\varepsilon}^i dx \leq \int_{\mathbb{R}^3} (\lambda_{\varepsilon}^i - 1) |\tilde{v}_{\varepsilon}^i|^p dx + p C_{\tilde{q}} \int_{\mathbb{R}^3} |\tilde{v}_{\varepsilon}^i|^{\tilde{q}} dx + \int_{\mathbb{R}^3} |\tilde{v}_{\varepsilon}^i|^{p^*} dx.$$

Therefore, we can deduce that

$$\begin{aligned} & \int_{\mathbb{R}^3} \psi^p(\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\nabla \tilde{v}_{\varepsilon}^i|^p dx \leq p \int_{\mathbb{R}^3} \psi^{p-1} (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} \tilde{v}_{\varepsilon}^i |\nabla \tilde{v}_{\varepsilon}^i|^{p-1} |\nabla \psi| dx \\ & \quad + \vartheta p C_{\tilde{q}} \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{\tilde{q}} dx + (\lambda_{\varepsilon}^i - 1 - \mathcal{V}_0) \int_{\mathbb{R}^3} |\tilde{v}_{\varepsilon}^i|^p \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} dx + \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{p^*} dx. \end{aligned}$$

It follows from the Young's inequality that

$$\begin{aligned} & \int_{\mathbb{R}^3} \left[\psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\nabla \tilde{v}_{\varepsilon}^i|^p - (\lambda_{\varepsilon}^i - 1 - \mathcal{V}_0) |\tilde{v}_{\varepsilon}^i|^p \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} \right] dx \\ & \leq C_p \int_{\mathbb{R}^3} (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^p |\nabla \psi|^p dx + \vartheta p C_p C_{\tilde{q}} \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{\tilde{q}} dx + \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{p^*} dx. \end{aligned}$$

Arguing as the proof of (4.5), we have that $\lambda_{\varepsilon}^i < 0$ for all $\varepsilon \in (0, \tilde{\varepsilon})$. By (V_2) , it holds that $\mathcal{V}_0 + 1 > 0$. Moreover, it holds that

$$|\nabla \tilde{\omega}_{\varepsilon, \mathbf{h}}^i|^p \leq C_p \vartheta^p \left(\psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\nabla \tilde{v}_{\varepsilon}^i|^p + |\nabla \psi|^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^p \right).$$

Together with the Sobolev inequality, we deduce that

$$\begin{aligned} \left(\int_{\mathbb{R}^3} |\tilde{\omega}_{\varepsilon, \mathbf{h}}^i|^{p^*} dx \right)^{\frac{p}{p^*}} & \leq \tilde{C}_p \vartheta^p \left(\int_{\mathbb{R}^3} (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^p |\nabla \psi|^p dx + \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{\tilde{q}} dx \right. \\ & \quad \left. + \int_{\mathbb{R}^3} \psi^p (\tilde{v}_{\varepsilon, \mathbf{h}}^i)^{p(\theta-1)} |\tilde{v}_{\varepsilon}^i|^{p^*} dx \right) \\ & \leq \tilde{C}_{p,r} \vartheta^p \left(\int_{R-r \leq |x| \leq R} |\tilde{v}_{\varepsilon}^i|^{p\vartheta} dx + \int_{|x| \geq R-r} (\tilde{v}_{\varepsilon}^i)^{p(\vartheta-1)} |\tilde{v}_{\varepsilon}^i|^{\tilde{q}} dx \right. \\ & \quad \left. + \int_{|x| \geq R-r} (\tilde{v}_{\varepsilon}^i)^{p(\vartheta-1)} |\tilde{v}_{\varepsilon}^i|^{p^*} dx \right). \end{aligned}$$

In the following, we shall fix $t = \sqrt{r}$, $p^* > \frac{pt}{t-1}$ and $\chi = \frac{p^*(t-1)}{pt} > 1$. Therefore,

$$\begin{aligned} \left(\int_{\mathbb{R}^3} |\tilde{\omega}_{\varepsilon, l}^i|^{p^*} dx \right)^{\frac{p}{p^*}} &\leq \tilde{C}_p \vartheta^p \left\{ \left(\int_{R-r \leq |x| \leq R} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}} \left(\int_{R-r \leq |x| \leq R} dx \right)^{\frac{1}{t}} \right. \\ &\quad + \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}} \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{(\tilde{q}-p)t} dx \right)^{\frac{1}{t}} \\ &\quad \left. + \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}} \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{(p^*-p)t} dx \right)^{\frac{1}{t}} \right\}. \end{aligned}$$

Taking $\tilde{q} = \frac{p(1+t)}{t}$ and using the Sobolev inequality, we obtain that

$$\left(\int_{\mathbb{R}^3} |\tilde{\omega}_{\varepsilon, h}^i|^{p^*} dx \right)^{\frac{p}{p^*}} \leq \tilde{C}_{p, \rho} \theta^p \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}}.$$

Since $\tilde{\omega}_{\varepsilon, h}^i = \psi \tilde{v}_\varepsilon^i (\tilde{v}_{\varepsilon, h}^i)^{\theta-1}$, we obtain that

$$\begin{aligned} \left(\int_{|x| \geq R} |\tilde{v}_{\varepsilon, h}^i|^{p^*} dx \right)^{\frac{p}{p^*}} &\leq \left(\int_{|x| \geq R} \psi^{p^*} |\tilde{v}_\varepsilon^i|^{p^*} |\tilde{v}_{\varepsilon, h}^i|^{p^*(\theta-1)} dx \right)^{\frac{p}{p^*}} \\ &\leq \left(\int_{\mathbb{R}^3} |\tilde{\omega}_{\varepsilon, h}^i|^{p^*} dx \right)^{\frac{p}{p^*}} \leq \tilde{C}_{p, \rho} \theta^p \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}}. \end{aligned}$$

Passing the limit as $h \rightarrow +\infty$ in the above inequality, we deduce that

$$\left(\int_{|x| \geq R} |\tilde{v}_\varepsilon^i|^{p^* \theta} dx \right)^{\frac{p}{p^*}} \leq \tilde{C}_{p, \rho} \theta^p \left(\int_{|x| \geq R-r} |\tilde{v}_\varepsilon^i|^{\frac{p\theta t}{t-1}} dx \right)^{\frac{t-1}{t}}. \quad (4.17)$$

Setting $\chi = \frac{p^*(t-1)}{pt}$ and $s = \frac{pt}{t-1}$, by (4.17), we obtain that

$$|\tilde{v}_\varepsilon^i|_{\chi^{m+1} s(|x| \geq R)} \leq \tilde{C}_{p, \rho}^{\sum_{i=j}^m \chi^{-j}} \chi^{\sum_{j=1}^m i \chi^{-j}} |\tilde{v}_\varepsilon^i|_{p^*(|x| \geq R-r)}.$$

Therefore, we can deduce that

$$|\tilde{v}_\varepsilon^i|_{L^\infty(|x| \geq R)} \leq \tilde{C}_{p, \rho}^{\sum_{j=1}^m \chi^{-j}} \chi^{\sum_{j=1}^m i \chi^{-j}} |\tilde{v}_\varepsilon^i|_{p^*(|x| \geq R-r)}. \quad (4.18)$$

Since $\tilde{v}_\varepsilon^i \rightarrow \tilde{v}^i$ in $\mathbb{X}(\mathbb{R}^3)$, (4.18) implies that we obtain the desired result.

Step 2. We verify that v_ε^i possesses a maximum η_ε^i satisfying $\mathcal{V}(\varepsilon \eta_\varepsilon^i) \rightarrow \mathcal{V}(x^i)$.

In the following, letting α_ε^i be a maximum of $\tilde{v}_\varepsilon^i$, we have that $|\tilde{v}_\varepsilon^i(\alpha_\varepsilon^i)|_\infty \geq \varrho^i$. By the fact that $\lim_{|x| \rightarrow \infty} \tilde{v}_\varepsilon^i(x) = 0$ uniformly in ε , then there exists $\mathcal{R}_0^i > 0$ independent of ε such that $|\alpha_\varepsilon^i| \leq \mathcal{R}_0^i$. Recalling $\tilde{v}_\varepsilon^i(\cdot) = v_\varepsilon^i(\cdot + y_\varepsilon^i)$, then $y_\varepsilon^i + \alpha_\varepsilon^i$ is a maximum of v_ε^i . Define $\eta_\varepsilon^i = y_\varepsilon^i + \alpha_\varepsilon^i$, according to Lemma 4.8 and $|\alpha_\varepsilon^i| \leq \mathcal{R}_0^i$, we obtain that $\varepsilon \eta_\varepsilon^i \rightarrow x^i$ as $\varepsilon \rightarrow 0^+$ and hence $\mathcal{V}(\varepsilon \eta_\varepsilon^i) \rightarrow \mathcal{V}(x^i)$ by the continuity of $\mathcal{V}$. Therefore, we complete the proof of Lemma 4.9. $\square$

Proof of Theorem 1.1. From Proposition 4.1 and Lemma 4.9, we obtain that system (1.1) possesses at least l different couples of solutions $(v_\varepsilon^i, \lambda_\varepsilon^i) \in \mathbb{X}(\mathbb{R}^3) \times \mathbb{R}$ with $v_\varepsilon^i(x) > 0$ for every $x \in \mathbb{R}^3$ and $\lambda_\varepsilon^i < 0$, where $i \in \{1, 2, \dots, l\}$. Let $u_\varepsilon^i(\cdot) = v_\varepsilon^i(\cdot/\varepsilon)$ and $z_\varepsilon^i = \varepsilon \eta_\varepsilon^i$ for $i \in \{1, 2, \dots, l\}$, then $(u_\varepsilon^i, \lambda_\varepsilon^i)$ is the desired solution for $i \in \{1, 2, \dots, l\}$. The proof of Theorem 1.1 is completed. $\square$

4.2 Proof of Theorem 1.2

Fixing a suitable parameter $a^* > 0$, we take $(u_{\varepsilon,\lambda}^{a^*}, \lambda)$ to be a family of weak solutions of system (1.1) by Theorem 1.1. Obviously, we can obtain the boundedness of $\{u_{\varepsilon,\lambda}^{a^*}\}$ in $\mathbb{X}(\mathbb{R}^3)$. Then, going to a subsequence if necessary, there exists $u_{\varepsilon,0}^{a^*} \in \mathbb{X}(\mathbb{R}^3)$ satisfying $u_{\varepsilon,\lambda}^{a^*} \rightarrow u_{\varepsilon,0}^{a^*}$ as $\lambda \rightarrow 0$.

In the following, for any $\omega \in \mathbb{X}(\mathbb{R}^3)$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\lambda}^{a^*}|^{p-2} \nabla u_{\varepsilon,\lambda}^{a^*} \nabla \omega dx + \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u_{\varepsilon,\lambda}^{a^*}|^{p-2} u_{\varepsilon,\lambda}^{a^*} \omega dx - \lambda \int_{\mathbb{R}^3} |u_{\varepsilon,\lambda}^{a^*}|^{p-2} u_{\varepsilon,\lambda}^{a^*} \omega dx + \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,\lambda}^{a^*}}^a u_{\varepsilon,\lambda}^{a^*} \omega dx \\ &= \vartheta \int_{\mathbb{R}^3} F'_2(u_{\varepsilon,\lambda}^{a^*}) \omega dx - \vartheta \int_{\mathbb{R}^3} F'_1(u_{\varepsilon,\lambda}^{a^*}) \omega dx + \int_{\mathbb{R}^3} |u_{\varepsilon,\lambda}^{a^*}|^{p^*-2} u_{\varepsilon,\lambda}^{a^*} \omega dx. \end{aligned} \quad (4.19)$$

From Lemma 4.3, Lemma 4.5, and Lemma 3.12, letting $\lambda \rightarrow 0$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\lambda}^{a^*}|^{p-2} \nabla u_{\varepsilon,\lambda}^{a^*} \nabla \omega dx \rightarrow \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,0}^{a^*}|^{p-2} \nabla u_{\varepsilon,0}^{a^*} \nabla \omega dx, \quad \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,\lambda}^{a^*}}^a u_{\varepsilon,\lambda}^{a^*} \omega dx \rightarrow \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,0}^{a^*}}^a u_{\varepsilon,0}^{a^*} \omega dx, \\ & \int_{\mathbb{R}^3} F'_2(u_{\varepsilon,\lambda}^{a^*}) u_{\varepsilon,\lambda}^{a^*} \omega dx \rightarrow \int_{\mathbb{R}^3} F'_1(u_{\varepsilon,0}^{a^*}) u_{\varepsilon,0}^{a^*} \omega dx, \quad \int_{\mathbb{R}^3} F'_2(u_{\varepsilon,\lambda}^{a^*}) u_{\varepsilon,\lambda}^{a^*} \omega dx \rightarrow \int_{\mathbb{R}^3} F'_1(u_{\varepsilon,0}^{a^*}) u_{\varepsilon,0}^{a^*} \omega dx \end{aligned}$$

and

$$\int_{\mathbb{R}^3} |u_{\varepsilon,\lambda}^{a^*}|^{p^*-2} u_{\varepsilon,\lambda}^{a^*} \omega dx \rightarrow \int_{\mathbb{R}^3} |u_{\varepsilon,0}^{a^*}|^{p^*-2} u_{\varepsilon,0}^{a^*} \omega dx.$$

From this fact, we get

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla u_{\varepsilon,0}^{a^*}|^{p-2} \nabla u_{\varepsilon,0}^{a^*} \nabla \omega dx - \lambda \int_{\mathbb{R}^3} |u_{\varepsilon,0}^{a^*}|^{p-2} u_{\varepsilon,0}^{a^*} \omega dx + \int_{\mathbb{R}^3} (\mathcal{V}(\varepsilon x) + 1) |u_{\varepsilon,0}^{a^*}|^{p-2} u_{\varepsilon,0}^{a^*} \omega dx + \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,0}^{a^*}}^a u_{\varepsilon,0}^{a^*} \omega dx \\ &= \vartheta \int_{\mathbb{R}^3} F'_2(u_{\varepsilon,0}^{a^*}) u_{\varepsilon,0}^{a^*} \omega dx - \vartheta \int_{\mathbb{R}^3} F'_1(u_{\varepsilon,0}^{a^*}) u_{\varepsilon,0}^{a^*} \omega dx + \int_{\mathbb{R}^3} |u_{\varepsilon,0}^{a^*}|^{p^*-2} u_{\varepsilon,0}^{a^*} \omega dx \end{aligned}$$

which implies that $u_{\varepsilon,0}^{a^*}$ is solutions of system (1.9).

Finally, with the same arguments as in the proof of Theorem 1.1, the solutions $u_{\varepsilon,0}^{a^*}$ of system (1.9) have a maximum point $\bar{z}_\varepsilon \in \mathbb{R}^3$ such that $\mathcal{V}(\bar{z}_\varepsilon) \rightarrow \mathcal{V}(\bar{z}) = \mathcal{V}_0$ as $\varepsilon \rightarrow 0^+$. $\square$

4.3 Proof of Theorem 1.3

Let $(u_{\varepsilon,\lambda^*}^a, \lambda^*)$ be a family of weak solutions of system (1.1) obtained in Theorem 1.1 for a suitable fixed parameter $\lambda^* < 0$. On the other hand, by [22, Lemma 6.1], as $a \rightarrow 0$, we have $\phi_{u_{\varepsilon,\lambda^*}^a}^a \rightarrow \phi_{u_{\varepsilon,\lambda^*}^0}^0$ in $D^{1,2}(\mathbb{R}^3)$, which solves $-\Delta \phi_{u_{\varepsilon,\lambda^*}^0}^0 = (u_{\varepsilon,\lambda^*}^0)^2$. Finally, as in the proof of Theorem 1.2, we conclude that $u_{\varepsilon,\lambda^*}^0$ is a solution of system (1.10). $\square$

Declarations

Acknowledgements

Sihua Liang was supported by the Young outstanding talents project of Scientific Innovation and entrepreneurship in Jilin (Grant No. 20240601048RC), the National Natural Science Foundation of China (Grant No. 12571114). V.D. Rădulescu was supported by grant “Nonlinear Differential Systems in Applied Sciences” of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8/22. He also acknowledges the support of the AGH University of Kraków under grant no. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education.

Data Availability Statement

Date sharing is not applicable to this article as no new data were created or analyzed in this study.

Competing interests

The authors declare that they have no competing interests.

Author contributions

All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

Conflict of interest

The authors state no conflict of interest.

References

- [1] D.G. Afonso, G. Siciliano, Normalized solutions to a Schrödinger-Bopp-Podolsky system under Neumann boundary conditions, *Commun. Contemp. Math.* (2021) 2150100.
- [2] M. Agueh, Sharp Gagliardo-Nirenberg inequalities via p -Laplacian type equations, *NoDEA Nonlinear Differential Equations Appl.* 15 (2008) 457–472.
- [3] C.O. Alves, C. Ji, O.H. Miyagaki, Normalized solutions for a Schrödinger equation with critical growth in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations* 61 (2022) 18.
- [4] C.O. Alves, On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$, *Z. Angew. Math. Phys.* 73 (2022) 17.
- [5] C.O. Alves, N. V. Thin, On existence of multiple normalized solution to a class of elliptic problems in whole $\mathbb{R}^N$ via Lusternik-Schnirelmann category, *SIAM J. Math. Anal.* 55 (2023) 1264–1283.
- [6] C.O. Alves, C. Ji, Multiple normalized solutions to a logarithmic Schrödinger equation via Lusternik-Schnirelmann category, *J. Geom. Anal.* 34 (2024), Paper No. 198, 29 pp.
- [7] P. D’Avenia, G. Siciliano, Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: Solutions in the electrostatic case, *J. Differential Equations* 267 (2019) 1025–1065.
- [8] A. Bahrouni, H. Missaoui, On the Schrödinger-Bopp-Podolsky system: Ground state and least energy nodal solutions with nonsmooth nonlinearity, *J. Math. Phys.* 65 (2024) Paper No. 121504.
- [9] M.C. Bertin, B.M. Pimentel, C.E. Valcárcel, G.E.R. Zambrano, Hamilton-Jacobi formalism for Podolsky’s electromagnetic theory on the null-plane, *J. Math. Phys.* 58 (2017) 082902.
- [10] I. Bialynicki-Birula, J. Mycielski, Nonlinear wave mechanics, *Ann. Physics* 100 (1976), 62–93.
- [11] G.M. Bisci, N.V. Thin, L. Vilasi, On a class of nonlocal Schrödinger equations with exponential growth, *Adv. Differential Equations* 27 (2022) 571–610.
- [12] F. Bopp, Eine lineare Theorie des Elektrons, *Ann. Phys.* 38 (1940) 345–384.
- [13] M. Born, Modified field equations with a finite radius of the electron, *Nature* 132 (1933) 282.
- [14] M. Born, On the quantum theory of the electromagnetic field, *Proc. R. Soc. Lond. Ser.* 143 (1934) 410–437.
- [15] M. Born, L. Infeld, Foundations of the new field theory, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 144 (1934) 425–451.
- [16] R. Bufalo, B. M. Pimentel, D. E. Soto, Causal approach for the electron-positron scattering in generalized quantum electrodynamics, *Phys. Rev. D* 90 (2014) 085012.
- [17] J. Chabrowski, Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, *Calc. Var. Partial Differential Equations* 3 (1995) 493–512.
- [18] S. Chen, L. Lin, V.D. Rădulescu, X. Tang, Ground state solutions of the non-autonomous Schrödinger-Bopp-Podolsky system, *Anal. Math. Phys.* 12 (2022), Paper No. 17, 32 pp.
- [19] L. Cui, A. Mao, Existence and asymptotic behavior of positive solutions to some logarithmic Schrödinger-Poisson system, *Z. Angew. Math. Phys.* 75 (2024), Paper No. 30, 25 pp.
- [20] J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries: I. Elliptic Equations* Research Notes in Mathematics vol 106 (1985), Boston, MA: Pitman.

- [21] M. del Pino, P.L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* 4 (1996) 121–137.
- [22] G. de Paula Ramos, G. Siciliano, Existence and limit behavior of least energy solutions to constrained Schrödinger-Bopp-Podolsky systems in $\mathbb{R}^3$. *Z. Angew. Math. Phys.* 74 (2023) Paper No. 56, 17 pp.
- [23] J. Frenkel, $\frac{3}{4}$ problem in classical electrodynamics, *Phys. Rev. E* 54 (1996) 5859–5862.
- [24] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, (2001), Berlin: Springer.
- [25] E.F. Hefter, Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics, *Phys. Rev. A* 32 (1985), 1201–1204.
- [26] J. Hu, A. Mao, Normalized solutions to the Kirchhoff equation with a perturbation term, *Differential Integral Equations* 36 (2023) 289–312.
- [27] J. Huang, S. Wang, Normalized ground states for the mass supercritical Schrödinger-Bopp-Podolsky system: existence, uniqueness, limit behavior, strong instability, *J. Differential Equations* 437 (2025) Paper No. 113282, 66 pp.
- [28] Y. Li, X. Chang, Z. Feng, Normalized solutions for Sobolev critical Schrödinger-Bopp-Podolsky systems, *Electron. J. Differential Equations* 2023, Paper No. 56, 19 pp.
- [29] L. Lin, P. Pucci, X. Tang, Ground state solutions for the nonlinear Schrödinger-Bopp-Podolsky system with critical Sobolev exponent, *Adv. Nonlinear Stud.* 20 (2020) 511–538.
- [30] L. Lin, X. Tang, Nehari type ground state solution for Schrödinger-Bopp-Podolsky system, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* 82 (2020) 139–152.
- [31] N.E. Mastorakis, H. Fathabadi, On the solution of p -Laplacian for non-Newtonian fluid flow, *WSEAS Trans. Math.* 8 (2009) 238–245.
- [32] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.* 28 (1997) 1633–1659.
- [33] X. Li, L. Xu, M. Zhu, Multiplicity and stability of normalized solutions to non-autonomous Schrödinger equation with mixed nonlinearities, *Proc. Edinb. Math. Soc.* (2) 67 (2024) 1–27.
- [34] Y. Li, B. Zhang, Critical Schrödinger-Bopp-Podolsky System with prescribed mass, *J. Geom. Anal.* 33 (2023) 220.
- [35] Q. Li, W. Zou, The existence and multiplicity of the normalized solutions for fractional Schrödinger equations involving Sobolev critical exponent in the L^2 -subcritical and L^2 -supercritical cases, *Adv. Nonlinear Anal.* 11 (2022), 1531–1551.
- [36] Q. Li, J. Nie, W. Wang, J. Zhou, Normalized solutions for Sobolev critical fractional Schrödinger equation, *Adv. Nonlinear Anal.* 13 (2024) Paper No. 20240027, 24 pp.
- [37] Sh. Liang, Y. Song, S. Shi, Concentrating solutions for double critical fractional Schrödinger-Poisson system with p -Laplacian in $\mathbb{R}^3$, *Adv. Nonlinear Anal.* 14 (2025) Paper No. 20240063, 37 pp.
- [38] Sh. Liang, J. Ma, S. Shi, Y. Song, Multiple normalized solutions for Choquard equation involving the biharmonic operator and competing potentials in $\mathbb{R}^N$, *Bull. Math. Sci.* 15 (2025) Paper No. 2450017, 22 pp.
- [39] E. Lieb, M. Loss, *Analysis*, *Grad. Stud. Math.* (2001).
- [40] P.L. Lions, The concentration compactness principle in the calculus of variations, the locally compact case, I, II, *Ann. Inst. H. Poincaré Anal. Non. Linéaire*. 1 (1984) 109–145, 223–283.
- [41] B. Podolsky, A generalized electrodynamics, I. Nonquantum, *Phys. Rev.* 62 (1942) 68–71.
- [42] X. Peng, Normalized solutions for the critical Schrödinger-Bopp-Podolsky system, *Bull. Malays. Math. Sci. Soc.* 47 (2024) 9.
- [43] X. Peng, G. Jia, Existence and concentration behavior of solutions for the logarithmic Schrödinger-Bopp-Podolsky system, *Z. Angew. Math. Phys.* 72 (2021) Paper No. 198, 18 pp.
- [44] G. de Paula Ramos, G. Siciliano, Existence and limit behavior of least energy solutions to constrained Schrödinger-Bopp-Podolsky system in $\mathbb{R}^3$, *Z. Angew. Math. Phys.* 56 (2023) Paper No. 56, 17 pp.
- [45] M.N. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1985.

- [46] D. Santos, M. Heydy, G. Siciliano, Critical Schrödinger-Bopp-Podolsky systems: solutions in the semiclassical limit, *Calc. Var. Partial Differential Equations* 63 (2024) Paper No. 155, 23 pp.
- [47] L. Shen, M. Squassina, Existence and concentration of normalized solutions for p -Laplacian equations with logarithmic nonlinearity, *J. Differential Equations* 421 (2025) 1–49.
- [48] G. Siciliano, K. Silva, The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field, *Publ. Mat.* 64 (2020) 373–390.
- [49] K. Silva, On an abstract bifurcation result concerning homogeneous potential operators with applications to PDEs, *J. Differential Equations* 269 (2020) 7643–7675.
- [50] X. Sun, Y. Fu, S. Liang, Multiplicity and concentration of solutions for Kirchhoff equations with exponential growth, *Bull. Math. Sci.* 14 (2024) Paper No. 2450004, 42 pp.
- [51] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 3 (1986) 77–109.
- [52] C. Wang, J. Sun, Normalized solutions for the p -Laplacian equation with a trapping potential, *Adv. Nonlinear Anal.* 12 (2023) Paper No. 20220291, 14 pp.
- [53] L. Wang, S. Feng, K. Cheng, Existence and concentration of positive solutions for a fractional Schrödinger logarithmic equation, *Complex Var. Elliptic Equ.* 69 (2024) 317–348.
- [54] M. Xiang, Y. Ma, M. Yang, Normalized homoclinic solutions of discrete nonlocal double phase problems, *Bull. Math. Sci.* 14 (2024) Paper No. 2450003, 18 pp.
- [55] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* 87 (1983) 567–576.
- [56] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, (1996).
- [57] J. Yang, H. Chen, S. Liu, The existence of nontrivial solution of a class of Schrödinger-Bopp-Podolsky system with critical growth, *Bound. Value Probl.* (2020) Paper No. 144, 16 pp.
- [58] X. Yao, S. Chen, M. Shu, The existence of ground state solutions for a Schrödinger-Bopp-Podolsky system with convolution nonlinearity, *J. Geom. Anal.* 33 (2023) Paper No. 374, 28 pp.
- [59] R. Zhang, S. Yao, J. Sun, Normalized solutions for Schrödinger-Bopp-Podolsky system with a negative potential, *Appl. Math. Lett.* 161 (2025) Paper No. 109368, 6 pp.