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GROUND STATES OF BIHARMONIC EQUATIONS ON LATTICE GRAPHS

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Abstract. In this paper, we are concerned with the existence of ground states to the following biharmonic equation on the lattice graph

$$\Delta^2 u - \Delta u + V(x)u = f(x,u), \quad x \in \mathbb{Z}^N.$$

The analysis is performed if the potential V and the reaction f are T-periodic in x, and the mapping $u \mapsto f(x,u)/|u|$ is non-decreasing on $\mathbb{R}\setminus\{0\}$. By using the variational methods, we establish the existence of ground states for the above problem. Moreover, if the potential V has a bounded potential well and $f(x,u) = f(u)$ with $u \mapsto f(u)/|u|$ non-decreasing on $\mathbb{R}\setminus\{0\}$, the ground states are also obtained for the above equation. Finally, we extend the main results on the lattice graph $\mathbb{Z}^N$ to quasi-transitive graphs. In our analysis, the mappings $u \mapsto f(x,u)/|u|$ or $u \mapsto f(u)/|u|$ are only non-decreasing on $\mathbb{R} \setminus \{0\}$, which allows to consider larger classes of nonlinearities in the reaction.

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1. Introduction

There is a vast literature concerning the existence of ground state solutions to nonlinear Schrödinger equations. First of all, let us recall the following semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N). \tag{1.1}$$

The functional corresponding to (1.1) is

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Assume that the following assumption holds.

(S_1) V is continuous, 1-periodic in $x_1,\dots,x_N$ and $0 \notin \sigma(-\Delta + V)$, the spectrum of $-\Delta + V$.

Let $E := H^1(\mathbb{R}^N)$. By (S_1), there is an equivalent inner product $\langle\cdot\,,\cdot\rangle$ in E such that

$$\Phi(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx$$

where $E = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum, and $u = u^+ + u^- \in E^+ \oplus E^-$. If $\sigma(-\Delta + V) \subset (0,+\infty)$, then $\dim E^- = 0$, otherwise $E^-$is infinite-dimensional. Pankov in [8, Section 2.2] introduced the following set:

$$\mathcal{M} := \big\{u \in E \setminus E^- : \Phi'(u)u = 0 \text{ and } \Phi'(u)v = 0 \text{ for all } v \in E^-\big\}.$$

It is clear to see that $\mathcal{M}$ contains all nontrivial critical points of Φ. If $f \in C^1$, $\big|f'_u(x,u)\big| \le \widetilde{a}\big(1 + |u|^{p-2}\big)$ where $\widetilde{a} > 0$, $2 < p < 2^*, 2^* = 2N/(N-2)$ if $N \ge 3$ and $2^* = +\infty$ if $N = 1,2$, and

$$0 < \frac{f(x,u)}{u} < \theta f'_u(x,u) \quad \text{for some } \theta \in (0,1) \text{ and all } u \ne 0. \tag{1.2}$$

Pankov [8] showed that $\mathcal{M}$ is a C^1-manifold, and it is a natural constraint in the sense that u is a nontrivial critical point of Φ if and only if $u \in \mathcal{M}$ and it is a critical point of $\Phi|_{\mathcal{M}}$. Since $c := \inf \Phi|_{\mathcal{M}} > -\infty$, Ekeland's variational principle yields a Palais–Smale sequence (a (PS) sequence for short) for $\Phi|_{\mathcal{M}}$ at the level c. Then, by (1.2), Pankov showed that this (PS) sequence is bounded, and he obtained a minimizer by the concentration-compactness arguments. After that, in the seminar paper [10], Szulkin and Weth did not assume that f is differentiable and satisfies (1.2), $\mathcal{M}$ need not be of class C^1, and therefore Pankov's method was not applicable. Nevertheless, $\mathcal{M}$ is still a topological manifold, naturally homeomorphic to the unit sphere in E^+. In this case, the following assumption:

(f'_4) $u \mapsto \dfrac{f(x,u)}{|u|}$ is increasing on $(-\infty,0)$ and on $(0,\infty)$, is very important. For any $u \in E \setminus E^-$, assume that the convex subset $\widehat{E}(u) := E^- \oplus \mathbb{R}^+ u = E^- \oplus \mathbb{R}^+ u^+$ of E, where $\mathbb{R}^+ = [0, \infty)$. Because of ($f_4'$), $\widehat{E}(u)$ intersects $\mathcal{M}$ at a unique point which is the unique global maximum of $\Phi|_{\widehat{E}(u)}$. If (f_4') is replaced by (f_4) below, that is

(f_4) $u \mapsto \dfrac{f(x,u)}{|u|}$ is non-decreasing on $(-\infty, 0)$ and on $(0, \infty)$,

an explicit example in [14] show $\widehat{E}(u)$ and $\mathcal{M}$ may intersect on a finite line segment. More precisely, $\widehat{E}(u) \cap \mathcal{M} \neq \emptyset$ and, if $w \in \widehat{E}(u) \cap \mathcal{M}$, then there exist $\sigma_w > 0$, $\tau_w \geq \sigma_w$ such that $\widehat{E}(u) \cap \mathcal{M} = [\sigma_w, \tau_w]w$. Thus, the Nehari manifold (it may not be a manifold) $\mathcal{M}$ may not be homeomorphic to the unit sphere as in [10] and the generalized Nehari manifold method [9], [10] can not be applied. To solve the problem, Tang [11], [12] developed a non-Nehari manifold method. The main idea of his approach lies on finding a minimizing Cerami sequence for the energy functional outside the Nehari–Pankov manifold by using the diagonal method. We also observed that de Paiva, Kryszewski and Szulkin [1] introduced a nonsmooth approach to overcome this difficulty. Their results generalize results of Szulkin and Weth [10]. Recently, using the Schrödinger equation with the zero mass as a model, Mederski, Schino and Szulkin [7] introduced a smooth method to show the existence of ground state solutions.

In recent years, the various mathematical problems on graphs have been extensively investigated (see [2]–[6] and references therein). In particular, in the monograph [2], Grigor'yan introduced the discrete Laplace operator on finite and infinite graphs; in [5], Hua and Xu studied the nonlinear Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ on the lattice graph $\mathbb{Z}^N$. They proved the existence of ground state solutions when the potential function V is periodic or bounded via the method of Nehari manifold. In order to apply the method in [10], the authors in [5] assume that (f_4') holds. A natural question is, if the assumption (f_4') is replaced by (f_4), does the results in [5] still hold? Due to our scope, we would like to mention the recent contribution [3] where Han, Shao and Zhao studied the following nonlinear biharmonic equation

$$\Delta^2 u - \Delta u + (\lambda a(x) + 1)u = |u|^{p-2}u \tag{1.3}$$

on a locally finite graph $G = (\mathbb{V}, E)$. Here $\lambda > 1$ and $p > 2$ are constants and $a(x) : V \to \mathbb{R}$ is a potential satisfying:

(A_1) $a(x) \geq 0$ and the potential well $\Omega = \{x \in \mathbb{V} : a(x) = 0\}$ is a non-empty, connected and bounded domain in V.

(A_2) There exists a vertex $x_0 \in \mathbb{V}$ such that $a(x) \to +\infty$ as $d(x, x_0) \to +\infty$.

One of the significant challenges in studying problem (1.3) is the absence of compactness. However, under the assumption (A_2) (corresponding to the coercive potential in the continuous case), they have the compact embedding theorem they need. Therefore, a natural question is to investigate problem (1.3) without the coercive potential, that is, the assumption (A_2) does not hold. In this case, it is more difficult to overcome the lack of compactness. In this paper, we will make some crucial attempts to study problem (1.3) on the lattice graph, we also believe that the research method in this paper can be applied to the exploration of other related problems. On the other hand, $|u|^{p-2}u/|u|$ is increasing on $(-\infty, 0)$ and on $(0, \infty)$. If the function $|u|^{p-2}u$ is replaced by a general nonlinearity $f(x, u)$, and the weaker assumption (f_4) (compared to (f_4')) is imposed, then studying the ground state solutions of problem (1.3) become a very challenging task.

In this paper, we are concerned with the biharmonic equation with potentials on graphs

$$\Delta^2 u - \Delta u + V(x)u = f(x, u), \quad x \in \mathbb{V}. \tag{1.4}$$

Let $G = (\mathbb{V}, \mathbb{E})$ be a simple, undirected, locally finite graph, where $\mathbb{V}$ denotes a set of vertices and $\mathbb{E}$ denotes a set of edges. Let us write $x \sim y$(x is a neighbour of y) if $\{x, y\} \in \mathbb{E}$. The graph G is undirected means that each $\{x, y\} \in \mathbb{E}$ is unordered. A graph is called locally finite if each vertex has finitely many neighbours. Denote the set of functions on G by $C(\mathbb{V})$. For any functions $u, v \in C(\mathbb{V})$, define the associated gradient form as

$$\Gamma(u, v)(x) := \sum_{y\sim x} \frac{1}{2}\,(u(y) - u(x))(v(y) - v(x)).$$

Write $\Gamma(u) = \Gamma(u, u)$ and

$$|\nabla u\,|(x) := \sqrt{\Gamma(u)(x)} = \left(\sum_{y\sim x} \frac{1}{2}\,(u(y) - u(x))^2\right)^{1/2}.$$

The Laplacian on $G = (\mathbb{V}, \mathbb{E})$ is defined for any function $u \in C(\mathbb{V})$ and $x \in \mathbb{V}$ as

$$\Delta u(x) := \sum_{y\in\mathbb{V},\, y\sim x} (u(y) - u(x))$$

and $\Delta^2 u = \Delta(\Delta u)$. For the more details, we refer to [4].

Let μ be the counting measure on $\mathbb{V}$, i.e. for any subset $A \subset \mathbb{V}$, $\mu(A) := \#\{x : x \in A\}$. For any function f on $\mathbb{V}$, we write

$$\int_{\mathbb{V}} f\, d\mu := \sum_{x\in\mathbb{V}} f(x)$$

whenever it makes sense.

Let $C_c(\mathbb{V})$ be the set of all functions of finite support, from [3, Proposition 2.2], we know that $H^2(\mathbb{V})$ be the completion of $C_c(\mathbb{V})$ under the norm

$$\|u\|_{H^2} = \left(\int_{\mathbb{V}} \big(|\Delta u|^2 + |\nabla u|^2 + u^2\big) d\mu\right)^{1/2}.$$

Clearly, $H^2(\mathbb{V})$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{H^2} = \int_{\mathbb{V}} (\Delta u \Delta v + \nabla u \nabla v + uv)\, d\mu, \quad \text{for all } u, v \in H^2(\mathbb{V}).$$

Let $V(x) \geq V_0 > 0$ for all $x \in \mathbb{V}$. We define a space of functions

$$H_V = \left\{ u \in H^2(\mathbb{V}) : \int_{\mathbb{V}} V(x) u^2\, d\mu < +\infty \right\}$$

with a norm

$$\|u\| = \left(\int_{\mathbb{V}} \left(|\Delta u|^2 + |\nabla u|^2 + V(x) u^2 \right) d\mu \right)^{1/2}.$$

It is clear that for a bounded uniformly positive function $V : \mathbb{V} \to \mathbb{R}$, i.e. there exist $V_0, V_1 > 0$, such that

$$V_0 \leq V(x) \leq V_1, \quad \text{for all } x \in \mathbb{V},$$

the norms are equivalent between $\| \cdot \|_{H^2}$ and $\| \cdot \|$.

For any $p \in [1, \infty]$, let $\ell^p(\mathbb{V})$ be the space of ℓ^p summable functions on $\mathbb{V}$ and we write $\| \cdot \|_p$ as the $\ell^p(\mathbb{V})$ norm, i.e.

$$\|u\|_p := \begin{cases} \left(\sum_{x \in \mathbb{V}} |u(x)|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{x \in \mathbb{V}} |u(x)|, & p = \infty. \end{cases}$$

Let us denote by $\mathbb{Z}^N$ the standard lattice graph with the set of vertices

$$\{x = (x_1, \ldots, x_N) : x_i \in \mathbb{Z},\ 1 \leq i \leq N\}$$

and the set of edges

$$\left\{ \{x, y\} : x, y \in \mathbb{Z}^N, \sum_{i=1}^{N} |x_i - y_i| = 1 \right\}.$$

For $T \in \mathbb{N}$, a function g on $\mathbb{Z}^N$ is called T-periodic if $g\,(x + Te_i) = g(x)$, for all $x \in \mathbb{Z}^N$, $1 \leq i \leq N$, where e_i is the unit vector in the i-th coordinate.

Throughout this paper, we always assume that $\inf_{x \in \mathbb{Z}^N} V(x) = V_0 > 0$. Motivated by [7], [5], in this present paper, we aim to consider the existence of ground states to the biharmonic equation (1.4) on the lattice graph. More precisely, we study two cases of the potentials, one is periodic; nd the other corresponds to the case where V has a bounded potential well.

The first result of this paper is as follows.

THEOREM 1.1. *If $\mathbb{V}$ is the lattice graph $\mathbb{Z}^N$, suppose the following assumptions hold*:

(V_1) *V is T-periodic in $x_1, \ldots, x_N$.*

and

(f$_1$) *$f(x,u)$ is continuous in $u\in\mathbb{R}$, and T-periodic in $x_1,\dots,x_N$ and*

$$|f(x,u)|\le a\left(1+|u|^{p-1}\right)\quad \text{for some } a>0 \text{ and } p>2;$$

(f$_2$) *$f(x,u)=o(u)$ uniformly in x as $|u|\to 0$;*

(f$_3$) *$\dfrac{F(x,u)}{u^2}\to\infty$ uniformly in x as $|u|\to\infty$, where $F(x,u):=\displaystyle\int_0^u f(x,s)ds$;*

(f$_4$) *$u\mapsto \dfrac{f(x,u)}{|u|}$ is non-decreasing on $(-\infty,0)$ and on $(0,\infty)$.*

Then problem (1.4) *has a ground state solution $u\in H_V$ with $J(u)=c>0$, c is defined as*

$$c=\inf_{\mathcal{N}} J>0\quad \text{where } \mathcal{N}:=\left\{u\in H_V\setminus\{0\}: J'(u)u=0\right\}.$$

Moreover, we will consider the case that the potential V has a bounded potential well in the sense that $\lim\limits_{|x|\to\infty} V(x)$ exists and is equal to $\sup\limits_{x\in\mathbb{Z}^N} V$. In this case, we consider that the nonlinearity is autonomous, that is, $f(x,u)=f(u)$. More precisely, we have the result as follows.

Theorem 1.2. *If $\mathbb{V}$ is the lattice graph $\mathbb{Z}^N$, and the following assumptions hold*:

(V$_2$) $0<\inf\limits_{x\in\mathbb{Z}^N} V(x)\le \lim\limits_{|x|\to\infty} V(x)=\sup\limits_{x\in\mathbb{Z}^N} V(x)<\infty$.

(f$_5$) *$f(u)$ is continuous in $u\in\mathbb{R}$, and $|f(u)|\le a(1+|u|^{p-1})$ for some $a>0$ and $p>2$;*

(f$_6$) *$f(u)=o(u)$ as $|u|\to 0$;*

(f$_7$) *$\dfrac{F(u)}{u^2}\to\infty$ as $|u|\to\infty$;*

(f$_8$) *$u\mapsto\dfrac{f(u)}{|u|}$ is non-decreasing on $(-\infty,0)$ and on $(0,\infty)$.*

Then problem (1.4) *has a ground state solution $u\in H_V$ with $J(u)=c>0$, c is defined as*

$$c_{\mathcal{N}}=\inf_{\mathcal{N}} J>0\quad \text{where } \mathcal{N}:=\left\{u\in H_V\setminus\{0\}: J'(u)u=0\right\}.$$

In our study, $u\mapsto f(x,u)/|u|$ or $u\mapsto f(u)/|u|$ is only non-decreasing on $\mathbb{R}\setminus\{0\}$, which allows to consider larger classes of nonlinearities in the reaction. Finally, we think that the method in this paper is also applicable to the nonlinear Schrödinger equation $-\Delta u+V(x)u=f(x,u)$ on the lattice graph $\mathbb{Z}^N$ if $u\mapsto f(x,u)/|u|$ is non-decreasing on $\mathbb{R}\setminus\{0\}$.

Finally, we shall extend the results on lattice graph $\mathbb{Z}^N$ to quasi-transitive graphs. We call G is a quasi-transitive graph if there are finitely many orbits for the action of $\mathrm{Aut}(G)$ on G where $\mathrm{Aut}(G)$ is the set of automorphisms of G. For more details on quasi-transitive graphs, see [13].

Theorem 1.3. *Let $G = (\mathbb{V}, \mathbb{E})$ be a quasi-transitive graph, $\Gamma \leq \mathrm{Aut}(G)$. Assume that the action of Γ on G has finitely many orbits. Then Theorem* 1.2 *holds on G. By substituting T-periodicity of f and V with Γ-invariance, Theorem* 1.1 *holds on G.*

The paper is organized as follows. In Section 2 we present some preliminaries on graphs. In Section 3, we give the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2. In the last section, we complete the proof of Theorem 1.3.

2. The variational framework and some preliminaries

Throughout this paper, we always assume that

$$\#\{y \in \mathbb{V} : y \sim x\} \leq C, \quad \text{for all } x \in \mathbb{V},$$

where C is a uniform constant which is independent of $x \in \mathbb{V}$. Under this assumption, we may show that $H^2(\mathbb{V})$ and $\ell^2(\mathbb{V})$ are equivalent. In fact, for any $u \in \ell^2(\mathbb{V})$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\Delta u(x)|^2 &= \Big| \sum_{y\sim x} (u(y) - u(x)) \Big|^2 \\ &\leq C \sum_{y \sim x} |u(y) - u(x)|^2 \leq 2C^2 |u(x)|^2 + 2C \sum_{y\sim x} |u(y)|^2. \end{aligned}$$

Thus

$$\int_{\mathbb{V}} |\Delta u|^2 \, d\mu = \sum_{x \in \mathbb{V}} |\Delta u(x)|^2 \leqslant 4C^2 \sum_{x\in\mathbb{V}} |u(x)|^2 = 4C^2 \|u\|_2^2.$$

Similarly, we have

$$\int_{\mathbb{V}} |\nabla u|^2 \, d\mu = \frac{1}{2} \sum_{x\in\mathbb{V}} \sum_{y\sim x} (u(y) - u(x))^2 \leq \sum_{x\in\mathbb{V}} \sum_{y \sim x} \left(u^2(y) + u^2(x)\right) \leq 2C \|u\|_2^2.$$

Hence, we conclude that $u \in H^2(\mathbb{V})$ and

$$\|u\|_2 \leq \|u\|_{H^2} \leq \left(4C^2 + 2C + 1\right)^{1/2} \|u\|_2.$$

From [5], we know that $H^1(\mathbb{V})$ and $\ell^2(\mathbb{V})$ are equivalent. Thus, $H^1(\mathbb{V})$, $H^2(\mathbb{V})$ and $\ell^2(\mathbb{V})$ are mutually equivalent.

We now provide a Sobolev embedding theorem, which will be very useful for our problems.

Lemma 2.1. *$H^2(\mathbb{V})$ is continuously embedded into $\ell^q(\mathbb{V})$ for any $q \in [2, +\infty]$. Namely, there exists a constant C_q depending only on q such that*

$$\|u\|_q \leq C_q \|u\|_{H^2} \quad \text{for any } u \in H^2(\mathbb{V}).$$

Moreover, for any bounded sequence $\{u_n\} \subset H^2(\mathbb{V})$, there exists $u \in H^2(\mathbb{V})$ such that, up to a subsequence, if necessary,

$$\begin{cases} u_n \rightharpoonup u & \text{in } H^2(\mathbb{V}); \\ u_n(x) \to u(x) & \text{pointwise}\, x \in \mathbb{V}. \end{cases}$$

Proof. For any $u \in H^2(\mathbb{V})$ and any vertex $x_0 \in \mathbb{V}$, we have

$$\|u\|_{H^2}^2 = \int_{\mathbb{V}} \left(|\Delta u|^2 + |\nabla u|^2 + u^2\right) d\mu \geq \int_{\mathbb{V}} u^2\, d\mu \geq u^2(x_0)$$

which gives

$$|u(x_0)| \leq \|u\|_{H^2}.$$

Therefore, $H^2(\mathbb{V}) \hookrightarrow \ell^\infty(\mathbb{V})$ continuously. Thus $H^2(\mathbb{V}) \hookrightarrow \ell^q(\mathbb{V})$ continuously for any $2 \leq q < \infty$ by the interpolation inequality. In fact, for any $u \in H^2(\mathbb{V})$, we have $u \in \ell^2(\mathbb{V})$. Then, for any $2 \leq q < \infty$,

$$\int_{\mathbb{V}} |u|^q\, d\mu = \int_{\mathbb{V}} |u|^2 |u|^{q-2}\, d\mu \leq \|u\|_\infty^{q-2} \int_{\mathbb{V}} |u|^2\, d\mu < +\infty,$$

which implies that $u \in \ell^q(\mathbb{V}), 2 \leq q < \infty$.

Since $H^2(\mathbb{V})$ is a Hilbert space, it is reflexive. Thus, for any bounded sequence $\{u_n\} \subset H^2(\mathbb{V})$, up to a subsequence if necessary, $u_n \rightharpoonup u$ in $H^2(\mathbb{V})$. From the previous arguments, we know that $\{u_n\} \subset H^2(\mathbb{V})$ is also bounded in $\ell^2(\mathbb{V})$ and we have $u_n \rightharpoonup u$ in $\ell^2(\mathbb{V})$, which shows that, for any $v \in \ell^2(\mathbb{V})$,

$$\lim_{n\to\infty} \int_{\mathbb{V}} (u_n - u)\, v\, d\mu = \lim_{n\to\infty} \sum_{x \in \mathbb{V}} (u_n(x) - u(x))\, v(x) = 0. \tag{2.1}$$

Take any $x_0 \in \mathbb{V}$ and let

$$v_0(x) = \begin{cases} 1 & x = x_0, \\ 0 & x \neq x_0. \end{cases}$$

Obviously, $v_0 \in \ell^2(\mathbb{V})$. By substituting v_0 into (2.1), one has

$$\lim_{n\to\infty} \left(u_n\left(x_0\right) - u\left(x_0\right)\right) = 0.$$

Since $x_0 \in \mathbb{V}$ is arbitrary, we complete the proof of theorem. □

Now, we introduce a formula of integration by parts on graphs, which is fundamental when applying methods in the calculus of variations. For the detail of proof, please refer to [3, Lemma 2.3]

Lemma 2.2. *Suppose that $u \in H^2(\mathbb{V})$. Then, for any $v \in C_c(\mathbb{V})$, we have*

$$\int_{\mathbb{V}} (\Delta^2 u) v\, d\mu = \int_{\mathbb{V}} \Delta u\, \Delta v\, d\mu \quad \text{where } \Delta^2 u = \Delta(\Delta u).$$

The functional associated with (1.4) is

$$J(u)=\frac{1}{2}\int_{\mathbb{V}}\left(|\Delta u|^2+|\nabla u|^2+V(x)u^2\right)d\mu-\int_{\mathbb{V}}F(x,u)\,d\mu.$$

From (f_1), (f_2) and Lemma 2.1, it is easy to verify that $J\in C^1(H_V,\mathbb{R})$ and

$$J'(u)v=\int_{\mathbb{V}}(\Delta u\,\Delta v+\nabla u\nabla v+V(x)uv)\,d\mu-\int_{\mathbb{V}}f(x,u)\,v\,d\mu,\quad \text{for all } v\in H_V.$$

By Lemma 2.2, for any given $y\in\mathbb{V}$, one can take the test function $\phi=\delta_y(x)$ and obtain

$$\Delta^2u(y)-\Delta u(y)+V(y)\,u(y)=f(y,u(y))$$

Since y is arbitrary, we conclude that u is a pointwise solution of problem (1.4).

3. The periodic case

Lemma 3.1. *For all* $t\in\mathbb{R}$, $\frac{1}{2}f(x,t)\,t-F(x,t)\geq 0$.

Proof. If $t=0$, it is clear that $f(x,0)=F(x,0)=0$ and $f(x,0)t/2-F(x,0)=0$. Now we only need to prove the case $t>0$, because the proof for $t<0$ is similar. By (f_2) and (f_4), $f(x,t)\geq 0$ for any $t>0$. Thus, we have $F(x,t)\geq 0$ for any $t>0$, where $F(x,t)=\int_0^t f(x,s)\,ds$. Moreover, for $t>0$, using (f_4) again, one has

$$F(x,t)=\int_0^t f(x,s)\,ds=\int_0^t\frac{f(x,s)}{s}\,s\,ds\leq\frac{f(x,t)}{t}\int_0^1 s\,ds\leq\frac{1}{2}f(x,t)t. \qquad \square$$

Lemma 3.2.

(a) *There exists* $\alpha>0$ *such that* $J(u)\geq\alpha$ *for* $\|u\|=\delta$ *small enough.*
(b) *Fix* $u\in H_{\mathbb{V}}\setminus\{0\}$. $J(tu)\to-\infty$ *as* $t\to+\infty$.

Proof. (a) First of all, by $(f_1),(f_2),(f_4)$, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that, for any $x\in\mathbb{V}$ and $u\in\mathbb{R}$, we have

$$|f(x,u)|\leq\varepsilon|u|+C_\varepsilon|u|^{p-1}, \tag{3.1}$$

$$|F(x,u)|\leq\frac{\varepsilon}{2}|u|^2+\frac{C_\varepsilon}{p}|u|^p. \tag{3.2}$$

Since $V(x)\geq V_0>0$ for all $x\in\mathbb{V}$, for $\varepsilon>0$ small, by (3.2) and Lemma 2.1, we have

$$J(u)\geq\frac{1}{2}\|u\|^2-\frac{\varepsilon}{2}|u|_2^2-C_\varepsilon|u|_p^p\geq\frac{1}{4}\|u\|^2-C_\varepsilon c_2\|u\|^p\geq\alpha,$$

if $\|u\|=\delta$ is small enough.

(b) Fix $u\in H_{\mathbb{V}}\setminus\{0\}$. Then

$$\frac{J(tu)}{t^2}=\frac{1}{2}\|u\|^2-\int_{\{u(x)\neq 0\}}\frac{F(x,tu)}{(tu)^2}u^2\,d\mu.$$

By (f_3), we have

$$\lim_{t\to+\infty}\left(\frac{1}{2}\|u\|^2-\int_{\{u(x)\neq 0\}}\frac{F(x,tu)}{(tu)^2}u^2\,d\mu\right)=-\infty,$$

thus $\lim\limits_{t\to+\infty} J(tu)=-\infty$. This completes the proof. □

From Lemma 3.2, we can define

$$c_{\mathcal{M}}:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t))$$

where $\Gamma:=\{\gamma\in\mathcal{C}([0,1],\ H_V):\gamma(0)=0,\ J(\gamma(1))<0\}$.

It is easy to see that $c_{\mathcal{M}}\geq\alpha$. Moreover, there exists a (PS) sequence $\{u_n\}$ of J with the level $c_{\mathcal{M}}$, that is

$$J(u_n)\to c_{\mathcal{M}},\quad J'(u_n)\to 0,\quad \text{as}\ \ n\to\infty. \tag{3.3}$$

Lemma 3.3. *If $u\in\mathcal{N}$, then u is a (not necessarily unique) maximizer of J on $\mathbb{R}^+u$.*

Proof. Let $u\in\mathcal{N}$. For any $t\geq 0$, by an explicit computation, one has

$$J(tu)=J(tu)-J'(u)\left(\frac{t^2-1}{2}\,u\right)\leq J(u).$$

In fact, we only need to prove the following inequality

$$\frac{t^2-1}{2}\int_{\mathbb{V}}f(x,u)\,u\,d\mu-\int_{\mathbb{V}}F(x,tu)\,d\mu\leq-\int_{\mathbb{V}}F(x,u)\,dx. \tag{3.4}$$

From (f_2), (f_4), it is easy to verify (3.4) and this concludes the proof. □

From Lemma 3.3, we can define

$$c_{\mathcal{N}}:=\inf_{\mathcal{N}}J\quad\text{where}\ \mathcal{N}:=\{u\in H_V\setminus\{0\}:J'(u)u=0\}.$$

Now we reveal that the relation between $c_{\mathcal{N}}$ and $c_{\mathcal{M}}$.

Lemma 3.4. *Under the assumptions (f_1)–(f_4), we have $c_{\mathcal{N}}=c_{\mathcal{M}}>0$.*

Proof. For any $u\in H_V\setminus\{0\}$ and let $g(t)=J(tu)$ for $t>0$. By Lemma 3.2 (a), 0 is a strict local minimizer of J in H_V and $J(tu)\to-\infty$ as $t\to-\infty$. Hence $\max\limits_{t>0}J(tu)\geq\alpha$. By Lemma 3.3, it is possible that there exist $t_1,t_2>0$ with $t_1\neq t_2$, $t_1u,t_2u\in\mathcal{N}$, and $J(t_1u)=J(t_2u)$. Consequently, there exist $0<t_{\min}\leq t_{\max}$ such that $tu\in\mathcal{N}$ if and only if $t\in[t_{\min},t_{\max}]$ and $J(tu)$ has the same value for any $t\in[t_{\min},t_{\max}]$. Hence $g'(t)>0$ for $0<t<t_{\min}$ and $g'(t)<0$ for $t>t_{\max}$. It follows that $H_V\backslash\mathcal{N}$ consists of two connected components and therefore each path in Γ must intersect $\mathcal{N}$. Therefore $c_{\mathcal{M}}\geq c_{\mathcal{N}}$. Since $c_{\mathcal{N}}=\inf\limits_{u\in H_V\setminus\{0\}}\max\limits_{t>0}J(tu)$, $c_{\mathcal{M}}\leq c_{\mathcal{N}}$, so $c_{\mathcal{M}}=c_{\mathcal{N}}$. □

Now we prove the boundedness of the (PS) sequence $\{u_n\}$ of J with the level $c_{\mathcal{M}}$.

LEMMA 3.5. *Under the assumptions* (V_1)*and* (f_1)–(f_4)*, If* (u_n) *is a* (PS) *sequence of* J *with the level* $c_{\mathcal{M}}$*, then* (u_n) *is bounded in* H_V*.*

PROOF. Assume that (u_n) is a (PS) sequence of J with the level $c_{\mathcal{M}}$, that is

$$J(u_n) \to c_{\mathcal{M}} \quad \text{and} \quad J(u_n) \to 0 \quad \text{as } n \to \infty.$$

Now we show that the sequence $\{u_n\}$ is bounded in H_V. Argue by contradiction, assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = u_n / \|u_n\|$, up to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ in H_V. By Lemma 2.1, one has $\sup_{n\in\mathbb{N}} \|v_n\|_p < \infty$. Suppose that $v_n \to 0$ in $\ell^p(\mathbb{Z}^{\mathbb{N}})$. By (3.2), for any $R > 0$, one has

$$|F(x, R\, v_n)| \le \frac{\varepsilon}{2} |Rv_n|^2 + \frac{C_\varepsilon}{p} |R\, v_n|^p$$

and, for n large enough,

$$\left| \int_{\mathbb{Z}^N} F(x, R\, v_n)\, d\mu \right| \le \frac{\varepsilon}{2} \int_{\mathbb{Z}^N} |R\, v_n|^2\, d\mu + \frac{C_\varepsilon}{p} \int_{\mathbb{Z}^N} |R\, v_n|^p d\mu \le C\varepsilon.$$

Now, fix an $R > \sqrt{2c_{\mathcal{M}}}$, by Lemmas 3.3, 3.4 and definition of $c_{\mathcal{M}}$, for any $t_n \in \mathbb{R}^+$ such that $t_n u_n \in \mathcal{N}$, we have for n large enough

$$J(u_n) + o_n(1) = J(t_n u_n) \ge J(R\, v_n) = \frac{1}{2} R^2 - \int_{\mathbb{Z}^N} F(x, R\, v_n)\, d\mu.$$

Let $n \to \infty$. One has $c_{\mathcal{M}} \ge R^2/2$ which is a contradiction. Thus, $v_n \not\to 0$ in $\ell^p(\mathbb{Z}^N)$. From the boundedness of the sequence $\{v_n\}$ in $\ell^p(\mathbb{Z}^N)$, we have

$$\overline{\lim_n} \|v_n\|_p = c_1 > 0 \tag{3.5}$$

for some positive constant c_1. By the interpolation inequality, we have

$$\|v_n\|_p \le \|v_n\|_2^{2/p} \|v_n\|_\infty^{(p-2)/p} \le c_2^{2/p} \|v_n\|_\infty^{(p-2)/p},$$

where $c_2 > 0$ is a constant. Thus, there exists $\eta > 0$ such that

$$\lim_{n\to\infty} \|v_n\|_\infty \ge \eta.$$

Moreover, there exists a sequence of $(y_n) \in \mathbb{Z}^N$ such that

$$|v_n(y_n)| \ge \frac{\eta}{2}$$

for all $n \in \mathbb{N}$. Now, we define $\widetilde{v}_n(x) = v_n(x + k_n T)$ with $k_n = (k_n^1, \dots, k_n^N)$ to ensure that $\{y_n - k_n T\} \subset \Omega$ where $\Omega = [0, T)^N \cap \mathbb{Z}^N$ is a bounded domain in $\mathbb{Z}^N$ and $\widetilde{v}_n(x) = \widetilde{u}_n(x)/\|\widetilde{u}_n\|$. Since $V(x)$ and $f(x,t)$ are T-periodic in x, J and J' are invariant under the translation. Thus, we have

$$J(\widetilde{u}_n) \to c, \quad J'(\widetilde{u}_n) \to 0 \quad \text{and} \quad \widetilde{v}_n \to \widetilde{v} \neq 0 \quad \text{pointwise in } \mathbb{Z}^N.$$

Without loss of generality, let $x_0 \in \mathbb{Z}^N$ with $\widetilde{v}(x_0) \neq 0$ and

$$\widetilde{u}_n(x_0) = \|\widetilde{u}_n\|\, \widetilde{v}_n(x_0) \to \infty \quad \text{as } n \to \infty.$$

Moreover, from (f_3), we have

$$0<\frac{J\left(\widetilde{u}_n\right)}{\left\|\widetilde{u}_n\right\|^2}=\frac{1}{2}-\int_{\mathbb{Z}^N} \frac{F(x, \widetilde{u}_n)}{\left\|\widetilde{u}_n\right\|^2}\, d\mu \leq \frac{1}{2}-\frac{F(x_0, \widetilde{u}_n\left(x_0\right))}{\left|\widetilde{u}_n\left(x_0\right)\right|^2}\left|\widetilde{v}_n\left(x_0\right)\right|^2 \rightarrow-\infty,$$

as $n \rightarrow \infty$, which is a contradiction. □

It is the position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 3.2 and 3.5, there exists a bounded (PS) sequence $\{u_n\}$ for J with the level $c_{\mathcal{M}}$. We may assume that $u_n \rightharpoonup u$ in H_V, and $u_n \rightarrow u$ pointwise in $\mathbb{Z}^N$ as $n \rightarrow \infty$, thus u is a weak solution of problem (1.4). If $u_n \rightarrow 0$ in $\ell^p(\mathbb{Z}^{\mathbb{N}})$, for $p>2$, by (3.1), we have

$$\int_{\mathbb{Z}^N} f\left(x, u_n\right) u_n \, d\mu=o_n(1). \tag{3.6}$$

By (3.6), we have

$$o_n(1)=J'\left(u_n\right) u_n=\left\|u_n\right\|^2-\int_{\mathbb{Z}^N} f\left(x, u_n\right) u_n \, d\mu=\left\|u_n\right\|^2+o_n(1)$$

and therefore $u_n \rightarrow 0$ in H_V. However, this contradicts $c_{\mathcal{M}} \geq \alpha>0$. Thus, $u_n \nrightarrow 0$ in $\ell^p(\mathbb{Z}^{\mathbb{N}})$, for $p>2$. From the boundedness of the sequence $\{u_n\}$ in $\ell^p(\mathbb{Z}^{\mathbb{N}})$, we have

$$\varlimsup_n \left\|u_n\right\|_p=c_1>0 \tag{3.7}$$

for some positive constant c_1. By the interpolation inequality, we have

$$\left\|u_n\right\|_p \leq\left\|u_n\right\|_2^{2/p}\left\|u_n\right\|_{\infty}^{(p-2)/p} \leq c_2^{2/p}\left\|u_n\right\|_{\infty}^{(p-2)/p},$$

where $c_2>0$ is a constant. Thus, there exists $\eta>0$ such that

$$\lim_{n \rightarrow \infty}\left\|u_n\right\|_{\infty} \geq \eta.$$

Moreover, there exists a sequence of $(y_n) \in \mathbb{Z}^N$ such that $|u_n(y_n)| \geq \eta/2$ for all $n \in \mathbb{N}$. Now, we define $\widetilde{u}_n(x)=u_n\left(x+k_n T\right)$ with $k_n=\left(k_n^1, \ldots, k_n^N\right)$ to ensure that $\{y_n-k_n T\} \subset \Omega$ where $\Omega=[0, T)^N \cap \mathbb{Z}^N$ is a bounded domain in $\mathbb{Z}^N$. Since $V(x)$, $f(x, t)$ are T-periodic in x, J and J' are invariant under the translation. Thus, we have

$$J\left(\widetilde{u}_n\right) \rightarrow c, \quad J'\left(\widetilde{u}_n\right) \rightarrow 0 \quad \text{and} \quad \widetilde{u}_n \rightarrow \widetilde{u} \neq 0 \quad \text{pointwise in } \mathbb{Z}^N$$

and $\widetilde{u}$ is a nontrivial solution of problem (1.4).

Finally, we show that $J(\widetilde{u}) = c_{\mathcal{N}} = \inf_{\mathcal{N}} J$. By Lemma 3.4, Fatou's lemma and Lemma 3.1, one has

$$\begin{aligned} c_{\mathcal{N}} + o(1) &= J(\widetilde{u}_n) - \frac{1}{2} J'(\widetilde{u}_n)\widetilde{u}_n \\ &= \int_{\mathbb{Z}^N} \left(\frac{1}{2} f(x, \widetilde{u}_n)\widetilde{u}_n - F(x, \widetilde{u}_n) \right) d\mu \\ &\geq \int_{\mathbb{Z}^N} \left(\frac{1}{2} f\big(x, \widetilde{u}\big)\widetilde{u} - F\big(x, \widetilde{u}\big) \right) d\mu + o_n(1) \\ &= J\big(\widetilde{u}\big) - \frac{1}{2} J'\big(\widetilde{u}\big)\widetilde{u} + o(1) = J\big(\widetilde{u}\big) + o_n(1). \end{aligned}$$

Hence $J(\widetilde{u}) \leq c_{\mathcal{N}}$. The reverse inequality is obvious, and we complete the proof of Theorem 1.1. □

4. The potential well case

In this section, we need to consider the limit equation

$$\Delta^2 u - \Delta u + V_\infty u = f(u), \quad x \in \mathbb{Z}^N$$

and define its associated energy functional as

$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{Z}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V_\infty u^2 \right) d\mu - \int_{\mathbb{Z}^N} F(u)\, d\mu.$$

Note that the ground state energy of J_∞ can be characterized as

$$c_\infty = \inf_{\mathcal{N}_\infty} J_\infty(u) = \inf_{w \in H^2(\mathbb{Z}^N)\setminus\{0\}} \max_{s>0} J_\infty(sw)$$

where

$$\mathcal{N}_\infty := \left\{ u \in H^2(\mathbb{Z}^N) \setminus \{0\} : J'_\infty(u)u = 0 \right\}.$$

Similar to Section (3), under the assumptions (V_2) and (f_5)–(f_8), we can show that

$$c_{\mathcal{N}} = \inf_{\mathcal{N}} J = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\begin{aligned} \mathcal{N} &:= \{ u \in H_V \backslash \{0\} : J'(u)u = 0 \}, \\ \Gamma &:= \{ \gamma \in \mathcal{C}([0,1], H_V) : \gamma(0) = 0, J(\gamma(1)) < 0 \}. \end{aligned}$$

Now we show relation between c_∞ and $c_{\mathcal{N}}$.

LEMMA 4.1. *Under the assumptions*(V_2) *and* (f_5)–(f_8), *we have* $c_\infty \geq c_{\mathcal{N}}$.

PROOF. From (V_2), we know that $V_\infty = \sup_{x \in \mathbb{Z}^N} V(x)$. It is clear that $c_\infty \geq c_{\mathcal{N}}$. If $V(x) \equiv V_\infty$, then this is a special case of periodic potential and $c_\infty \geq c_{\mathcal{N}}$. Otherwise, $V(x) < V_\infty$ on some subset of $\mathbb{Z}^N$. By Theorem 1.1, we know that

c_∞ can be attained at some point $u \in \mathcal{N}_\infty$, i.e. $J_\infty(u) = c_\infty$. Then, for any $s > 0$,

$$c_\infty = J_\infty(u) \geq J_\infty(su) > J(su),$$

i.e.

$$c_\infty > \max_{s>0} J(su) \geq \inf_{u \in H_V \setminus \{0\}} \max_{s>0} J(su) = c_{\mathcal{N}}. \qquad \square$$

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Similar to Section 3, we can show that J satisfies the mountain pass geometry. Thus, there exists a (PS) sequence of $\{u_n\}$ for J with $J(u_n) \to c_{\mathcal{N}}$ and $J'(u_n) \to 0$ as $n \to \infty$. First of all, we show the boundedness of $\{u_n\}$ in H_V. Suppose $\|u_n\| \to \infty$ as $n \to \infty$. Letting $v_n = u_n/\|u_n\|$, up to a subsequence if necessary, we have $v_n \rightharpoonup v$ in H_V and $v_n(x) \to v(x)$ for all $x \in \mathbb{Z}^{\mathbb{N}}$. Since $\sup_{n\in\mathbb{N}} \|v_n\|_p < \infty$. By a similar argument in the proof of Lemma 3.5, $v_n \nrightarrow 0$ in $\ell^q(\mathbb{Z}^N)$. Again, by the interpolation inequality, one has a subsequence $\{v_n\}$ and corresponding $\{y_n\} \subset \mathbb{Z}^N$ such that $|v_n(y_n)| \geq \delta > 0$ for all n. If $\{y_n\}$ is bounded, we get a contradiction similar to the proof of Lemma 3.5. Otherwise, there exists a subsequence $|y_n| \to \infty$ as $n \to \infty$. Let $\widetilde{v}_n(x) := v_n(x - y_n)$. Then $\{\widetilde{v}_n\}$ is bounded in H_V, for $\|v_n\| = 1$. By passing to a subsequence, one can assume that $\widetilde{v}_n \rightharpoonup \widetilde{v}$ in H_V and $\widetilde{v}_n(x) \to \widetilde{v}(x)$ for all $x \in \mathbb{Z}^{\mathbb{N}}$ with $\widetilde{v} \neq 0$. Moreover, from (f$_3$), we have

$$0 < \frac{J(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_{\mathbb{Z}^N} \frac{F(u_n)}{u_n^2} v_n^2 \, d\mu = \frac{1}{2} - \int_{\mathbb{Z}^N} \frac{F(\widetilde{u}_n)}{\widetilde{u}_n^2} \widetilde{v}_n^2 \, d\mu \to -\infty,$$

as $n \to \infty$. This is a contradiction, so that (u_n) is bounded in H_V.

By similar arguments, we obtain a subsequence $\{u_n\}$ and corresponding sequence $\{y_n\} \subset \mathbb{Z}^N$ such that $|u_n(y_n)| \geq \eta > 0$ for all n. Therefore, $\widehat{u}_n \rightharpoonup \widehat{u} \neq 0$ with $\widehat{u}_n(x) := u_n(x - y_n)$. It suffices to prove that $\{y_n\}$ is bounded. Suppose $|y_n| \to \infty$ as $n \to \infty$, we claim that $J'_\infty(\widehat{u}) = 0$. In fact, for any $w \in C_c(\mathbb{Z}^N)$, let $w_n(x) := w(x - y_n)$, we have

$$|J'(u_n) w_n| \leq \|J'(u_n)\|_{(H_V)^*} \|w_n\| = \|J'(u_n)\|_{(H_V)^*} \|w\| \to 0, \qquad \text{as } n \to \infty.$$

Moreover,

$$\begin{aligned} J'(u_n) w_n &= \int_{\mathbb{Z}^N} \big(\Delta u_n \, \Delta w_n + \nabla u_n \nabla w_n + V(x) u_n w_n\big) \, d\mu \\ &\quad - \int_{\mathbb{Z}^N} f(x, u_n) w_n \, d\mu, \\ &= \int_{\mathbb{Z}^N} \big(\Delta \widehat{u}_n \, \Delta w + \nabla \widehat{u}_n \nabla w + V(x - y_n) \widehat{u}_n(x) \, w\big) \, d\mu \\ &\quad - \int_{\mathbb{Z}^N} f(\widehat{u}_n) \, w(x) \, d\mu \end{aligned}$$

$$\to \int_{\mathbb{Z}^N} \big(\Delta\widehat{u}\,\Delta w + \nabla\widehat{u}\,\nabla w + V_\infty(x)\,\widehat{u}(x)\,w(x)\big)\,d\mu$$
$$- \int_{\mathbb{Z}^N} f(\widehat{u})\,w(x)\,d\mu = J'_\infty(\widehat{u})w.$$

Hence

$$\begin{aligned} c_{\mathcal{N}} + o(1) &= J\big(u_n\big) - \frac{1}{2}\,J'\big(u_n\big)u_n = \int_{\mathbb{Z}^N}\left(\frac{1}{2}\,f\big(u_n\big)u_n - F\big(u_n\big)\right)d\mu \\ &= \int_{\mathbb{Z}^N}\left(\frac{1}{2}\,f\big(\widehat{u}_n\big)\widehat{u}_n - F\big(\widehat{u}_n\big)\right)d\mu \ge \int_{\mathbb{Z}^N}\left(\frac{1}{2}\,f\big(\widehat{u}\big)\widehat{u} - F\big(\widehat{u}\big)\right)d\mu + o(1) \\ &= J_\infty(\widehat{u}) - \frac{1}{2}\,J'_\infty(\widehat{u})\widehat{u} + o(1) = J_\infty(\widehat{u}) + o(1) \ge c_\infty + o(1), \end{aligned}$$

as $n \to \infty$. This is a contradiction, so that $\{y_n\}$ is bounded. We conclude the proof of Theorem 1.2. □

5. The quasi-transitive graphs

In this section, we prove the main results on quasi-transitive graphs.

Proof of Theorem 1.3. Let $G/\Gamma = \{\rho_1, \dots, \rho_m\}$ denote the set of finitely many orbits, $\Omega = \{z_1, \dots, z_m\} \subset \mathbb{V}$ where $z_i \in \rho_i$, $1 \le i \le m$. Replace the translations in the proof of Theorems 1.1 and 1.2 with Γ-action on functions, respectively. For instance, suppose $|v_n(y_n)| \ge \delta > 0$. For each y_n and $n \in \mathbb{N}$, there exists $g_n \in \Gamma$ such that $g_n y_n = z_i \in \Omega$. Set $\widetilde{v}_n := v_n \circ g_n^{-1}$, then we obtain a subsequence $\{\widetilde{v}_n\}$ such that $\|\widetilde{v}_n\|_{\ell^\infty(\Omega)} \ge \delta$ with Ω being a finite subset. By similar arguments in Theorems 1.1 and 1.2, we can complete the proof of Theorem 1.3 on quasi-transitive graph G. □

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