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Research announcement

Resonant Neumann problems with indefinite and unbounded potential

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ABSTRACT

In this paper, we report on some recent results obtained in our joint paper Papageorgiou and Rădulescu (in press). We establish multiplicity properties for a class of semilinear Neumann problems driven by the Laplacian plus on unbounded and indefinite potential. The reaction is a Carathéodory function which exhibits linear growth near $\pm\infty$. We allow for resonance to occur with respect to a nonprincipal nonnegative eigenvalue. The approach combines critical point theory, Morse theory and the Lyapunov–Schmidt method.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. Consider the following semilinear Neumann problem:

$$-\Delta u(z) + \beta(z)u(z) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. The potential function $\beta(\cdot)$ is in general unbounded and sign changing. More precisely, we assume that $\beta \in L^s(\Omega)$ with $s > N$. Also, the reaction $f(z, x)$ is a Carathéodory function that exhibits linear growth near $\pm\infty$. We allow for resonance to occur with respect to any nonnegative nonprincipal eigenvalue of $(-\Delta + \beta(\cdot),\ H^1(\Omega))$. So, we assume that asymptotically at $\pm\infty$ the quotient $\frac{f(z,x)}{x}$ is located in the spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$ with $m \geqslant \max\{m_0, 2\}$, where $\hat{\lambda}_{m_0}$ is the first nonnegative eigenvalue of $(-\Delta + \beta(\cdot),\ H^1(\Omega))$. Hence, if $\beta \equiv 0$, then $m_0 = 2$ and so $m \geqslant 2$. We allow resonance with respect to the left end $\hat{\lambda}_m$ and nonuniform nonresonance with respect to the right end $\hat{\lambda}_{m+1}$. Problems with double resonance (that is, possible resonance at both ends of the spectral interval), were studied by O'Regan, Papageorgiou and Smyrlis [1], with $\beta \equiv 0$ (see also Hu and Papageorgiou [2] for Dirichlet problems with $\beta \neq 0$).

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The following linear eigenvalue problem has a central role in the analysis of problem (1):

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

This eigenvalue problem was studied by Papageorgiou and Smyrlis [3]. So, suppose that $\beta \in L^{N/2}(\Omega)$ if $N \geqslant 3$, $\beta \in L^r(\Omega)$ with $r > 1$ if $N = 2$ and $\beta \in L^1(\Omega)$ if $N = 1$. Let $\tau : H^1(\Omega) \to \mathbb{R}$ be the energy functional defined by

$$\tau(u) = \|Du\|_2^2 + \int_\Omega \beta(z)u(z)^2 dz \quad \text{for all } u \in H^1(\Omega).$$

Then the eigenvalue problem (2) has a smallest eigenvalue $\hat{\lambda}_1 > -\infty$ given by

$$\hat{\lambda}_1 = \inf\left[\frac{\tau(u)}{\|u\|_2^2} : u \in H^1(\Omega),\ u \neq 0\right]. \tag{3}$$

From (3) it follows that we can find $\xi_0 > \max\{-\hat{\lambda}_1, 0\}$ such that

$$\tau(u) + \xi_0\|u\|_2^2 \geqslant c_1\|u\|^2 \quad \text{for all } u \in H^1(\Omega) \text{ and some } c_1 > 0. \tag{4}$$

Using (4) and the spectral theorem for compact self-adjoint operators (see, for example, Gasinski and Papageorgiou [4, p. 297]), we obtain a sequence $\{\hat{\lambda}_k\}_{k\geqslant 1}$ consisting of all the eigenvalues of (2) such that $\hat{\lambda}_k \to +\infty$ when $k \to \infty$. To these eigenvalues corresponds a sequence $\{\hat{u}_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ of eigenfunctions which form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Moreover, if $\beta \in L^s(\Omega)$ with $s > N$, then the regularity results of Wang [5], imply that $\{\hat{u}_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$. These eigenvalues admit variational characterizations in terms of the Rayleigh quotient $\frac{\tau(u)}{\|u\|_2^2}$ for all $u \in H^1(\Omega) \setminus \{0\}$. In what follows, by $E(\hat{\lambda}_k)$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$, $k \geqslant 1$.

Throughout this paper, our hypotheses on the potential function $\beta(\cdot)$ are the following:

$H_0 : \beta \in L^s(\Omega)$ with $s > N$ and $\beta^+ \in L^\infty(\Omega)$.

2. Existence of multiple solutions

We assume that the resonance occurs at $\pm\infty$ with respect to any nonnegative nonprincipal eigenvalue of $(-\Delta - \beta, H^1(\Omega))$. So, in what follows, $\hat{\lambda}_{m_0}$ denotes the first nonnegative eigenvalue of this operator.

The hypotheses on the reaction term $f(z, x)$ are the following:

$H_1 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist an integer $m \geqslant \max\{m_0, 2\}$ and a function $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z) \leqslant \hat{\lambda}_{m+1} \quad \text{a.e. in } \Omega,\ \eta \neq \hat{\lambda}_{m+1}$$
$$(f(z,x) - f(z,y))(x-y) \leqslant \eta(z)(x-y)^2 \quad \text{for a.a } z \in \Omega,\ \text{all } x, y \in \mathbb{R};$$

(ii) $\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then we have

$$\lim_{x\to\pm\infty} [f(z,x)x - 2F(z,x)] = -\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exists a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \leqslant \hat{\lambda}_1 \quad \text{a.e. in } \Omega,\ \vartheta \neq \hat{\lambda}_1$$
$$\limsup_{x\to 0} \frac{2F(z,x)}{x^2} \leqslant \vartheta(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that

$$f(z,x)x + \xi_\varrho x^2 \geqslant 0 \quad \text{for a.a } z \in \Omega,\ \text{all } |x| \leqslant \varrho.$$

We observe that hypotheses H_1(i), (ii) imply that asymptotically at $\pm\infty$, the quotient $\frac{f(z,x)}{x}$ is in the spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$ with possible resonance with respect to $\hat{\lambda}_m$ (see H_1(ii)), while at the other end we have nonuniform nonresonance (see H_1(i)).

The following function satisfies hypotheses H_1 above. For the sake of simplicity, we drop the z-dependence:

$$f(x) = \begin{cases} \vartheta x + \xi|x|^{p-2}x & \text{if } |x| \leqslant 1 \\ \lambda x + \dfrac{c}{x} & \text{if } 1 < |x|, \end{cases}$$

with $\vartheta < \hat{\lambda}_1$, $p > 2$, $\xi = \lambda + c - \vartheta$, $\lambda \in [\hat{\lambda}_m, \hat{\lambda}_{m+1})$ for some integer $m \geqslant \max\{m_0, 2\}$, $c > 0$, $2c < \lambda$.

Our first multiplicity property is the following "three solutions" theorem.

Theorem 1. *Assume that hypotheses H_0 and H_1 hold. Then problem (1) has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C^1(\overline{\Omega}).$$

Sketch of the proof. We set $\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)ds$ and introduce the C^1- functionals $\varphi, \hat{\varphi}_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{2}\tau(u) - \int_\Omega F(z,u(z))dz$$
$$\hat{\varphi}_\pm(u) = \frac{1}{2}\tau(u) + \frac{\xi_0}{2}\|u\|_2^2 - \int_\Omega \hat{F}_\pm(z,u(z))dz \quad \text{for all } u \in H^1(\Omega).$$

The main steps of the proof are the following:
(i) If hypotheses H_0 and H_1 hold, then the functionals $\hat{\varphi}_\pm$ satisfy the Cerami compactness condition.
(ii) If hypotheses H_0 and H_1 hold, then the functional φ satisfies the Cerami compactness condition.
(iii) If hypotheses H_0 and H_1 hold, then $u = 0$ is a minimizer for the functionals $\hat{\varphi}_\pm$ and φ.
(iv) If hypotheses H_0 and H_1 hold, then $\hat{\varphi}_\pm(t\hat{u}_1) \to -\infty$ as $t \to \pm\infty$.
(v) If hypotheses H_0 and H_1 hold, then problem (1) admits at least two nontrivial constant sign solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

To produce a third nontrivial solution, we will employ the so-called Lyapunov–Schmidt reduction technique as this was formulated for elliptic equations by Amann [6], Castro and Lazer [7], and Thews [8]. To this end, we introduce the following subspaces of $H^1(\Omega)$:

$$Y = \oplus_{i=1}^m E(\hat{\lambda}_i) \quad \text{and} \quad \hat{H} = Y^\perp = \overline{\oplus_{i \geqslant m+1} E(\hat{\lambda}_i)}.$$

We have the following orthogonal direct sum decomposition: $H^1(\Omega) = Y \oplus \hat{H}$.

(vi) If hypotheses H_0 and H_1 hold, then there exists a continuous map $\gamma_0 : Y \to \hat{H}$ such that

$$\varphi(y + \gamma_0(y)) = \inf\left[\varphi(y + \hat{u}) : \hat{u} \in \hat{H}\right] \quad \text{for all } y \in Y.$$

(vii) Let $\psi(y) = \varphi(y + \gamma_0(y))$ for all $y \in Y$. If hypotheses H_0 and H_1 hold, then ψ is anticoercive (that is, if $\|y\| \to \infty$, then $\psi(y) \to -\infty$).

We refer to Papageorgiou and Rădulescu [9] for detailed arguments of the proof.

By strengthening the regularity of the reaction $f(z,x)$ we can improve Theorem 1 and produce four distinct solutions. In what follows we assume that f satisfies the following hypotheses:

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) there exist an integer $m \geqslant \max\{m_0, 2\}$ and a function $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z) \leqslant \hat{\lambda}_{m+1} \quad \text{a.e. in } \Omega,\ \eta \neq \hat{\lambda}_{m+1},$$
$$|f'_x(z,x)| \leqslant \eta(z) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R};$$

(ii) $\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$;
(iii) $\lim_{x\to\pm\infty}[f(z,x)x - 2F(z,x)] = -\infty$ uniformly for a.a. $z \in \Omega$;
(iv) there exists a function $\vartheta \in L^\infty(\Omega)_+$ such that

$$\vartheta(z) \leqslant \hat{\lambda}_1 \quad \text{a.e. in } \Omega,\ \vartheta \neq \hat{\lambda}_1,$$
$$\limsup_{x\to 0} \frac{2F(z,x)}{x^2} \leqslant \vartheta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Note that hypothesis $H_2(i)$ and the mean value theorem imply that

$$|f(z,x)| \leqslant \eta(z)|x| \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R}.$$

Also, for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that for a.a. $z \in \Omega$, the mapping $x \mapsto f(z,x) + \xi_\varrho x$ is nondecreasing on $[-\varrho, \varrho]$.

A straightforward computation shows that the following function satisfies hypotheses H_2 (for the sake of simplicity we drop the z-dependence):

$$f(x) = \begin{cases} \vartheta x + \dfrac{\xi x}{1+|x|} & \text{if } |x| \leqslant 1 \\ \lambda x + \dfrac{c}{x} & \text{if } 1 < |x|, \end{cases}$$

with $\vartheta < \hat{\lambda}_1$, $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ for some integer $m \geqslant \max\{m_0, 2\}$, ξ, $c > 0$, $\vartheta + \xi < \hat{\lambda}_{m+1}$.

Our second multiplicity result is the following.

Theorem 2. *Assume that hypotheses H_0 and H_2 hold. Then problem (1) admits at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \qquad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0, \hat{y} \in C^1(\overline{\Omega}).$$

We refer to [9] for the proof of Theorem 2, as well as for related results on coercive or anticoercive problems, or for the case when the Lyapunov–Schmidt reduction method is implemented on an infinite dimensional subspace of $H^1(\Omega)$.

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