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Bifurcation analysis for nonhomogeneous Robin problems with competing nonlinearities

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ABSTRACT

In this paper, we report on some recent results obtained in our joint paper Papageorgiou and Rădulescu (2015). We consider a Robin problem driven by a nonhomogeneous differential operator and with a reaction that exhibits competing effects of concave (that is, sublinear) and convex (that is, superlinear) nonlinearities. Without employing the Ambrosetti–Rabinowitz condition, we establish a bifurcation property of the positive solutions near the origin. The approach relies on variational methods and elliptic estimates.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. Let $a : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous strictly monotone map. Let $\partial u/\partial n_a$ denote the conormal derivative defined by $\partial u/\partial n_a := (a(Du), n)_{\mathbb{R}^N}$, where $n(z)$ is the outward unit normal at $z \in \partial\Omega$.

In this paper we study the following nonlinear Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z), \lambda) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a}(z) + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0, \quad 1 < p < \infty. \end{cases} \tag{P_λ}$$

The reaction $f(z, x, \lambda)$ is a parametric function with $\lambda > 0$ being the parameter and $(z, x) \to f(z, x, \lambda)$ is a Carathéodory function. We assume that $f(z, \cdot, \lambda)$ exhibits competing nonlinearities, namely near the

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origin it has a "concave" term (that is, a strictly $(p-1)$-sublinear term), while near $+\infty$ the reaction is a "convex" term (that is, $x \longmapsto f(z,x,\lambda)$ is $(p-1)$-superlinear). A special case of our reaction is the function $f(z,x,\lambda) = f(x,\lambda) = \lambda x^{q-1} + x^{r-1}$, for all $x \geqslant 0$ with

$$1 < q < p < r < p^* := \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

The first work concerning positive solutions for problems with concave and convex nonlinearities, was that of Ambrosetti, Brezis and Cerami [1]. They studied semilinear equations driven by the Dirichlet Laplacian and with a reaction of the form (1). Their work was extended to equations driven by the Dirichlet p-Laplacian by Garcia Azorero, Manfredi and Peral Alonso [2] and by Guo and Zhang [3]. We also refer to the contributions of de Figueiredo, Gossez and Ubilla [4,5] to concave–convex type problems and general nonlinearities for the Laplacian, resp. p-Laplacian case. Extensions to equations involving more general reactions were obtained by Gasinski and Papageorgiou [6], Hu and Papageorgiou [7] and Rădulescu and Repovš [8].

Let $\eta \in C^1(0,\infty)$ and assume that

$$0 < \hat{c} \leqslant \frac{t\eta'(t)}{\eta(t)} \leqslant c_0 \quad \text{and} \quad c_1 t^{p-1} \leqslant \eta(t) \leqslant c_2(1+t^{p-1}) \quad \text{for all } t > 0 \text{ with } c_1, c_2 > 0,\ 1 < p < \infty. \tag{1}$$

The hypotheses on the map $a(\cdot)$ are the following:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0,\infty)$, $a_0(t)t \to 0$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(y)| \leqslant c_3 \frac{\eta(|y|)}{|y|}$ for some $c_3 > 0$, all $y \in \mathbb{R}^N \backslash \{0\}$;

(iii) $\frac{\eta(|y|)}{|y|}|\xi|^2 \leqslant (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \backslash \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)s ds$ for all $t \geqslant 0$, then $pG_0(t) - a_0(t)t^2 \geqslant -\hat{\xi}$ for all $t \geqslant 0$, some $\hat{\xi} > 0$;

(v) there exists $\tau \in (1,p)$ such that $t \longmapsto G_0(t^{1/\tau})$ is convex on $(0,\infty)$, $\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0$ and

$$a_0(t)t^2 - \tau G_0(t) \geqslant \tilde{c}t^p \quad \text{for some } \tilde{c} > 0,\ \text{all } t > 0.$$

According to the above conditions, the potential function $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then the function $y \longmapsto G(y)$ is convex and differentiable on $\mathbb{R}^N \backslash \{0\}$. We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \backslash \{0\},\ \nabla G(0) = 0.$$

So, $G(\cdot)$ is the primitive of the map $a(\cdot)$. Because $G(0) = 0$ and $y \longmapsto G(y)$ is convex, from the properties of convex functions, we have $G(y) \leqslant (a(y), y)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N$.

The following properties follow by straightforward arguments.

Lemma 1. *Assume that hypotheses $H(a)$* (i)–(iii) *hold. Then*

(a) *the mapping $y \longmapsto a(y)$ is continuous and strictly monotone, hence maximal monotone too;*

(b) $|a(y)| \leqslant c_4(1+|y|^{p-1})$ *for some $c_4 > 0$, all $y \in \mathbb{R}^N$;*

(c) $(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$;
(d) *for all* $y \in \mathbb{R}^N$ *we have* $\frac{c_1}{p(p-1)}|y|^p \leqslant G(y) \leqslant c_5(1+|y|^p)$ *with* $c_5 > 0$.

The hypotheses on the boundary weight map $\beta(\cdot)$ are the following:
$H(\beta)$: $\beta \in C^{1,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.
Throughout this paper we assume that the reaction f satisfies the following hypotheses.
$H(f)$: $f : \Omega \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is a function such that for a.a. $z \in \Omega$ and all $\lambda > 0 f(z,0,\lambda) = 0$ and

(i) for all $(x,\lambda) \in \mathbb{R} \times (0,\infty)$, $z \longmapsto f(z,x,\lambda)$ is measurable, while for a.a. $z \in \Omega$, $(x,\lambda) \longmapsto f(z,x,\lambda)$ is continuous;
(ii) $|f(z,x,\lambda)| \leqslant a_\lambda(z)(1+x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, all $\lambda > 0$, with $a_\lambda \in L^\infty(\Omega)$, $\lambda \longmapsto \|a_\lambda\|_\infty$ bounded on bounded sets in $(0,\infty)$ and $p < r < p^*$;
(iii) if $F(z,x,\lambda) = \int_0^x f(z,s,\lambda)ds$, then $\lim_{x\to+\infty} \frac{F(z,x,\lambda)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iv) there exists $\vartheta = \vartheta(\lambda) \in \left((r-p)\max\left\{\frac{N}{p},1\right\}, p^*\right)$ such that

$$0 < \gamma_0 \leqslant \liminf_{x\to+\infty} \frac{f(z,x,\lambda)x - pF(z,x,\lambda)}{x^\vartheta} \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) there exist $1 < \mu = \mu(\lambda) < q = q(\lambda) < \tau$ (see hypothesis $H(a)$ (v)) and $\gamma = \gamma(\lambda) > \mu$, $\delta_0 = \delta_0(\lambda) \in (0,1)$ such that

$$c_6 x^q \leqslant f(z,x,\lambda)x \leqslant qF(z,x,\lambda) \leqslant \xi_\lambda(z)x^\mu + \tau x^\gamma \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta_0$$

with $c_6 = c_6(\lambda) > 0$, $c_6(\lambda) \to +\infty$ as $\lambda \to +\infty$, $\bar{c} = \bar{c}(\lambda) > 0$, $\xi_\lambda \in L^\infty(\Omega)_+$ with $\|\xi_\lambda\|_\infty \to 0$ as $\lambda \to 0^+$;
(vi) for every $\rho > 0$, there exists $\xi_\rho = \xi_\rho(\lambda) > 0$ such that for a.a. $z \in \Omega$, $x \longmapsto f(z,x,\lambda) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$;
(vii) for every interval $K = [x_0, \hat{x}]$ with $x_0 > 0$ and every $\lambda > \lambda' > 0$, there exists $d_K(x_0,\lambda)$ nondecreasing in λ with $d_K(x_0,\lambda) \to +\infty$ as $\lambda \to +\infty$ and $\hat{d}_K(x_0,\lambda,\lambda')$ such that

$$f(z,x,\lambda) \geqslant d_K(x_0,\lambda) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in K$$
$$f(z,x,\lambda) - f(z,x,\lambda') \geqslant \hat{d}_K(x_0,\lambda,\lambda') \quad \text{for a.a. } z \in \Omega, \text{ all } x \in K.$$

The following functions satisfy hypotheses $H(f)$. For the sake of simplicity, we drop the z-dependence:

$$f_1(x,\lambda) = \lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geqslant 0, \text{ with } 1 < q < p < r < p^*$$

$$f_2(x,\lambda) = \begin{cases} \lambda x^{q-1} - x^{\eta-1} & \text{if } x \in [0,1] \\ x^{p-1}\left(\ln x + \frac{1}{p}\right) + \left(\lambda - \frac{1}{p}\right)x^{\nu-1} & \text{if } x > 1 \end{cases}$$

with $q, \nu \in (1,p)$ and $\eta > p$

$$f_3(x,\lambda) = \begin{cases} x^{q-1} & \text{if } x \in [0,\rho(\lambda)] \\ x^{r-1} + \eta(\lambda) & \text{if } x > \rho(\lambda) \end{cases}$$

with $1 < q < p < r < p^*$, $\eta(\lambda) = \rho(\lambda)^{p-1} - \rho(\lambda)^{r-1}$
and $\rho(\lambda) \to 0^+$ as $\lambda \to 0^+$.

Since we are interested to find positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(z, x, \lambda) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$ and all $\lambda > 0$. Note that hypotheses $H(f)$ (ii), (iii) imply that

$$\lim_{x\to+\infty} \frac{f(z,x,\lambda)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Thus $f(z, \cdot, \lambda)$ is $(p-1)$-superlinear near $+\infty$. However, we do not employ the Ambrosetti–Rabinowitz (AR) condition (unilateral version) (Cf. [9]). We say that $f(z, \cdot, \lambda)$ satisfies the (unilateral) (AR)-condition, if there exist $\eta = \eta(\lambda) > p$ and $M = M(\lambda) > 0$ such that

$$\begin{aligned} &\text{(a)} \quad 0 < \eta F(z,x,\lambda) \leqslant f(z,x,\lambda)x \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant M, \\ &\text{(b)} \quad \operatorname{ess\,inf}_{\Omega} F(\cdot, M, \lambda) > 0. \end{aligned} \tag{2}$$

Integrating (2)a and using (2)b, we obtain a weaker condition, namely that

$$c_7 x^\eta \leqslant F(z,x,\lambda) \quad \text{for a.a. } z \in \Omega, \text{ all } z \geqslant M \text{ and some } c_7 > 0. \tag{3}$$

Evidently (3) implies the much weaker hypothesis $H(f)$ (iii). In (2) we may assume that $\eta > (r-p)$ max $\left\{\frac{N}{p}, 1\right\}$. Then we have

$$\begin{aligned} \frac{f(z,x,\lambda)x - pF(z,x,\lambda)}{x^\eta} &= \frac{f(z,x,\lambda)x - \eta F(z,x,\lambda)}{x^\eta} + \frac{(\eta-p)F(z,x,\lambda)}{x^\eta} \\ &\geqslant (\eta-p)c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant M \text{ (see (2)a and (3))}. \end{aligned}$$

So, we see that the (AR)-condition implies hypothesis H_1 (iv). This weaker "superlinearity" condition incorporates in our setting $(p-1)$-superlinear nonlinearities with "slower" growth near $+\infty$, which fail to satisfy the (AR)-condition (see the function $f_2(\cdot, \lambda)$ defined above). Finally note that hypothesis $H(f)$ (v) implies the presence of a concave nonlinearity near zero.

The main result of this paper establishes the following bifurcation property.

Theorem 2. *Assume that hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold. Then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least two positive solutions* $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \leqslant \hat{u}$, $u_0 \neq \hat{u}$;
(b) *for $\lambda = \lambda^*$ problem (P_{λ^*}) has at least one positive solution $u_* \in \operatorname{int} C_+$;*
(c) *for all $\lambda > \lambda^*$ problem (P_λ) has no positive solution.*

Sketch of the Proof. We introduce the following Carathéodory function

$$\hat{f}(z,x,\lambda) = f(z,x,\lambda) + (x^+)^{p-1} \quad \text{for all } (z,x,\lambda) \in \Omega \times \mathbb{R} \times (0, +\infty).$$

Let $\hat{F}(z,x,\lambda) = \int_0^x \hat{f}(z,s,\lambda)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}(z,u,\lambda)dz.$$

We split the proof into several steps.

Step 1. For all $\lambda > 0$, the energy functional $\hat{\varphi}_\lambda$ satisfies the Cerami compactness condition.

Step 2. There is some $\lambda_+ > 0$ such that for all $\lambda \in (0, \lambda_+)$ there exists $\rho_\lambda > 0$ for which we have

$$\inf \{\hat{\varphi}_\lambda(u) : \|u\| = \rho_\lambda\} = \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0).$$

Step 3. If $\lambda > 0$ and $u \in \operatorname{int} C_+ := \{v \in C^1(\overline{\Omega}) : v(z) > 0 \text{ for all } z \in \overline{\Omega}\}$, then $\hat{\varphi}_\lambda(tu) \to -\infty$ as $t \to \infty$. This property is a direct consequence of hypothesis $H(f)$ (iii).

Next, we consider the following sets:

$$\mathcal{S} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$
$$S(\lambda) = \text{the set of positive solutions of } (P_\lambda).$$

Step 4. We have $\mathcal{S} \neq \emptyset$ and for every $\lambda \in \mathcal{S}$ we have $\emptyset \neq S(\lambda) \subseteq \operatorname{int} C_+$.

Step 5. If $\lambda \in \mathcal{S}$, then $(0, \lambda] \subseteq \mathcal{S}$.

Step 6. Set $\lambda^* = \sup \mathcal{S}$. We have $\lambda^* < \infty$.

Step 7. For all $\eta \in (0, \lambda^*)$, problem (P_η) admits at least two distinct positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$ with $u_0 \leqslant \hat{u}$.

Next we examine what happens in the critical case $\lambda = \lambda^*$. To this end, note that hypotheses $H(f)$ (ii), (v) imply that we can find $c_8 = c_8(\lambda) > 0$ such that

$$f(z, x, \lambda) \geqslant c_6 x^{q-1} - c_8 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } z \geqslant 0. \tag{4}$$

This unilateral growth estimate on the reaction $f(z, \cdot, \lambda)$ leads to the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = c_6 u(z)^{q-1} - c_8 u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_0}(z) + \beta(z) u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{5}$$

Step 8. Problem (5) admits a unique positive solution $\bar{u} \in \operatorname{int} C_+$.

Step 9. If $\lambda \in \mathcal{S}$, then $\bar{u} \leqslant u$ for all $u \in S(\lambda)$.

Step 10. We have $\lambda^* \in \mathcal{S}$ and so $\mathcal{S} = (0, \lambda^*]$.

We refer to [10] for detailed arguments of the proof, as well as for related results on Neumann problems with competing nonlinearities.

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