Let $g : (0, +\infty) \to (0, +\infty)$ be a continuous function such that
\[ \lim_{x \to +\infty} \frac{g(x)}{x^{1+\alpha}} = +\infty, \quad (1) \]
for some $\alpha > 0$. Let $f : \mathbb{R} \to (0, +\infty)$ be a twice differentiable function. Assume that there exists $a > 0$ and $x_0 \in \mathbb{R}$ such that
\[ f''(x) + f'(x) > af(f(x)), \quad \text{for all } x \geq x_0. \quad (2) \]
Prove that $\lim_{x \to +\infty} f(x)$ exists, is finite and compute its value.

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**Solution.** If $x_1 > x_0$ is a critical point of $f$ then, by (2), $f''(x_1) > 0$, so $x_1$ is a relative minimum point of $f$. This implies that $f'(x)$ does not change sign if $x$ is sufficiently large. Consequently, we can assume that $f$ is monotone on $(x_0, +\infty)$, hence $\ell := \lim_{x \to +\infty} f(x)$ exists.

The difficult part of the proof is to show that $\ell$ is finite. This will be deduced after applying in a decisive manner our superlinear growth assumption (1). Arguing by contradiction, let us assume that $\ell = +\infty$. In particular, it follows that $f$ is monotone increasing on $(x_0, +\infty)$. Define the function
\[ u(x) = e^{x/2} f(x), \quad x \geq x_0. \]
Then $u$ is increasing and, for any $x \geq x_0$,
\[ u''(x) = \frac{1}{4} u(x) + e^{x/2} \left( f''(x) + f'(x) \right) > \frac{1}{4} u(x) + ae^{x/2} g(f(x)). \quad (3) \]
Our hypothesis (1) and the assumption $\ell = +\infty$ yield some $x_1 > x_0$ such that
\[ g(f(x)) \geq f^{1+\alpha}(x), \quad \forall x \geq x_1. \quad (4) \]
So, by (3) and (4),
\[ u''(x) > \frac{1}{4} u(x) + Cu(x)f^\alpha(x) > Cu(x)f^\alpha(x), \quad \forall x \geq x_1, \quad (5) \]
for some $C > 0$. In particular, since $\ell = +\infty$, there exists $x_2 > x_1$ such that
\[ u''(x) > u(x), \quad \forall x \geq x_2. \quad (6) \]
We claim a little more, namely that there exists $C_0 > 0$ such that
\[ u''(x) > C_0 u^{1+\alpha/2}(x), \quad \forall x \geq x_2. \quad (7) \]
Indeed, let us first choose $0 < \delta < \min\{e^{-x_2} u(x_2), e^{-x_2} u'(x_2)\}$. We prove that
\[ u(x) > \delta e^x, \quad \forall x \geq x_2. \quad (8) \]
For this purpose, consider the function \( v(x) = u(x) - \delta e^x \). Arguing by contradiction and using \( v(x_2) > 0 \) and \( v'(x_2) > 0 \), we deduce the existence of a relative maximum point \( x_3 > x_2 \) of \( v \). So, \( v(x_3) > 0 \), \( v'(x_3) = 0 \) and \( v''(x_3) \leq 0 \). Hence \( \delta e^{x_3} = u'(x_3) < u(x_3) \). But, by (6), \( u''(x_3) > u(x_3) \), which yields \( v''(x_3) > 0 \), a contradiction. This concludes the proof of (8).

Returning to (5) and using (8) we find
\[
\frac{d^2}{dx^2}f(x) > C u^{1+\alpha/2}(x) u^{\alpha/2}(x) e^{-\alpha x/2} > C_0 u^{1+\alpha/2}(x), \quad \forall x > x_2,
\]
where \( C_0 = C \delta^{\alpha/2} \). This proves our claim (7). So
\[
u'(x)\frac{d^2}{dx^2}f(x) > C_0 u^{1+\alpha/2}(x) u'(x), \quad \forall x > x_2.
\]
Hence
\[
\frac{1}{2} u'^2(x) - C_1 u^{2+\beta}(x) > 0, \quad \forall x > x_2,
\]
where \( C_1 = 2C_0/(4 + \alpha) \) and \( \beta = \alpha/2 > 0 \). Therefore
\[
u'^2(x) \geq C_2 + C_3 u^{2+\beta}(x), \quad \forall x > x_2,
\]
for some positive constants \( C_2 \) and \( C_3 \). So, since \( u \) is unbounded, there exists \( x_3 > x_2 \) and \( C_4 > 0 \) such that
\[
u'(x) \geq C_4 u^{1+\gamma}(x), \quad \forall x > x_3,
\]
where \( \gamma = \beta/2 > 0 \).

Applying the mean value theorem we find
\[
u^{-\gamma}(x_3) - \nu^{-\gamma}(x) = \gamma(x - x_3)\nu^{-\gamma-1}(\xi_x)u'(\xi_x) \geq C_4 \gamma(x - x_3), \quad \forall x > x_3,
\]
where \( \xi_x \in (x_3, x) \). Taking \( x \to +\infty \) in the above inequality we obtain a contradiction since the left hand-side converges to \( \nu^{-\gamma}(x_3) \) (because \( \ell = +\infty \)) while the right hand-side diverges to \(+\infty\). This contradiction shows that \( \ell = \lim_{x \to +\infty} f(x) \) must be finite.

We prove in what follows that \( \ell = 0 \). Arguing by contradiction, let us assume that \( \ell > 0 \). We first observe that relation (2) yields, by integration,
\[
f'(x) - f'(x_0) + f(x) - f(x_0) \geq a \int_{x_0}^x g(f(t))dt.
\]
Since \( \ell \) is finite, it follows by (9) that \( \lim_{x \to +\infty} f'(x) = +\infty \). But this contradicts the fact that \( \lim_{x \to +\infty} f(x) \) is finite.

**Remark.** The result stated in our problem does not remain true if \( g \) has a linear growth at \(+\infty\), so if (1) fails. Indeed, it is enough to choose \( f(x) = e^x \) and \( g \) the identity map. We also remark that “\( \ell \) is finite” does not follow if the growth hypothesis (1) is replaced by the weaker one \( \lim_{x \to -\infty} g(x)/x = +\infty \). Indeed, if \( g(x) = x \ln(1 + x) \) and \( f(x) = e^{x^2} \), then \( \ell = +\infty \).