PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**
with the collaboration of Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, László Lipták, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Bogdan Petrenko, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, Sam Vandervelde, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the back of the title page. Submitted solutions should arrive at that address before March 31, 2011. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11501. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* (Correction) Let

$$g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of g about 0 are all nonnegative if and only if $a \geq \sqrt{3}$.

11530. *Proposed by Pál Peter Dályay, Szeged, Hungary.* Let A be an $m \times m$ matrix with nonnegative entries $a_{i,j}$ and with the property that there exists a permutation σ of $\{1, \dots, m\}$ for which $\prod_{i=1}^{m} a_{i,\sigma(i)} \geq 1$. Show that the union over $n \geq 1$ of the set of entries of A^n is bounded if and only if some positive power of A is the identity matrix.

11531. *Proposed by Nicuşor Minculete, "Dimitrie Cantemir" University, Brasov, Romania.* Let M be a point in the interior of triangle ABC and let $\lambda_1, \lambda_2, \lambda_3$ be positive real numbers. Let R_a, R_b, and R_c be the circumradii of triangles MBC, MCA, and MAB, respectively. Show that

$$\lambda_1^2 R_a + \lambda_2^2 R_b + \lambda_3^2 R_c \geq \lambda_1\lambda_2\lambda_3 \left(\frac{|MA|}{\lambda_1} + \frac{|MB|}{\lambda_2} + \frac{|MC|}{\lambda_3} \right).$$

(Here, for $V = A, B, C$, $|MV|$ denotes the length of the line segment MV.)

11532. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Find all prime numbers p such that there exists a 2×2 matrix A with integer entries, other than the identity matrix I, for which $A^p + A^{p-1} + \cdots + A = pI$.

doi:10.4169/000298910X521724

11533. *Proposed by Erwin Just (emeritus), Bronx Community College of the City College of New York, Bronx, NY.* Let t be a positive integer and let R be a ring, not necessarily having an identity element, such that $x + x^{2t+1} = x^{2t} + x^{10t+1}$ for each x in R. Prove that R is a Boolean ring, that is, $x = x^2$ for all x in R.

11534. *Proposed by Christopher Hillar, Mathematical Sciences Research Institute, Berkeley, CA.* Let k and n be positive integers with $k < n$. Characterize the $n \times n$ real matrices M with the property that for all $v \in \mathbb{R}^n$ with at most k nonzero entries, Mv also has at most k nonzero entries.

11535. *Proposed by Marian Tetiva, Bîrlad, Romania.* Let f be a continuously differentiable function on $[0, 1]$. Let $A = f(1)$ and let $B = \int_0^1 x^{-1/2} f(x)\,dx$. Evaluate

$$\lim_{n\to\infty} n \left(\int_0^1 f(x)\,dx - \sum_{k=1}^{n} \left(\frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} \right) f\left(\frac{(k-1)^2}{n^2} \right) \right)$$

in terms of A and B.

11536. *Proposed by Mihaly Bencze, Brasov, Romania.* Let K, L, and M denote the respective midpoints of sides AB, BC, and CA in triangle ABC, and let P be a point in the plane of ABC other than K, L, or M. Show that

$$\frac{|AB|}{|PK|} + \frac{|BC|}{|PL|} + \frac{|CA|}{|PM|} \ge \frac{|AB| \cdot |BC| \cdot |CA|}{4|PK| \cdot |PL| \cdot |PM|},$$

where $|UV|$ denotes the length of segment UV.

SOLUTIONS

The Number of *k*-cycles in a Random Permutation

11378 [2008, 664]. *Proposed by Daniel Troy (Emeritus), Purdue University–Calumet, Hammond, IN.* Let n be a positive integer, and let $U_1, \dots, U_n$ be random variables defined by one of the following two processes:

- **A:** Select a permutation of $\{1, \dots, n\}$ at random, with each permutation of equal probability. Then take U_k to be the number of k-cycles in the chosen permutation.
- **B:** Repeatedly select an integer at random from $\{1, \dots, M\}$ with uniform distribution, where M starts at n and at each stage in the process decreases by the value of the last number selected, until the sum of the selected numbers is n. Then take U_k to be the number of times the randomly chosen integer took the value k.

Show that the probability distribution of $(U_1, \dots, U_n)$ is the same for both processes.

Solution by O.P. Lossers, Eindhoven University of Technology, Netherlands. First we introduce a standard notation for the permutations: in each cycle put the lowest number in front, and list the cycles with the first elements in decreasing order. Next we count the permutations of n objects where the last cycle has length k. The last cycle starts with 1, and the other $k - 1$ elements are arbitrary, in any order. Hence there are $(n-1)!/(n-k)!$ ways to fill the last cycle, and then the permutation can be completed in $(n-k)!$ ways. Hence the number of permutations in which the last cycle has length k is $(n-1)!$, independent of k. It follows that the length of the last cycle is uniformly distributed, and the remaining cycles are produced by the same process on the remaining $n - k$ elements. Hence the production of cycle lengths from back to front under process A emulates process B.

Editorial comment. Erich Bach noted that the use of process B to generate the cycle lengths of random permutations has appeared before, such as in E. Bach, Exact Analysis of a Priority Queue Algorithm for Random Variate Generation, *Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 1994, 48–56.

Also solved by E. Bach, D. Beckwith, R. Chapman (U. K.), S. J. Herschkorn, J. H. Lindsey II, R. Martin (Germany), J. H. Nieto (Venezuela), M. A. Prasad (India), K. Schilling, J. H. Smith, P. Spanoudakis (U. K.), R. Stong, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

When $HI = IO$

11398 [2008, 948]. *Proposed by Stanley Huang, Jiangzhen Middle School, Huaining, China.* Assume acute triangle ABC has its middle-sized angle at A. Suppose further that the incenter I is equidistant from the circumcenter O and the orthocenter H. Show that angle A has measure 60 degrees and that the circumradius of IBC is the same as that of ABC.

Composite solution by the Editors. The restriction to acute triangles appears to be unnecessary.

V. V. Garcia (Huelva, Spain) pointed to Problem E2282, this MONTHLY, April 1972, pp. 397–8, where it is shown that (excepting only equilateral triangles, for which $IO = 0$, and not excluding right or obtuse triangles) HI/IO is (1) less than 1, (2) equal to 1, or (3) greater than 1, according as the middle-sized angle of the triangle is (1) greater than, (2) equal to, or (3) less than 60°. Geometrically, this means that with respect to the perpendicular bisector λ of the Euler segment, I is (1) on the H side of λ, (2) on λ, or (3) on the O side of λ. Thus when I is equidistant from O and H, i.e., on λ, the middle-sized angle must be 60°.

The second claim of this problem is too humble. Actually, when angle A has measure 60°, the reflection $\mathcal{C}'$ of the circumcircle $\mathcal{C}$ of ABC across BC, which of course has the same radius, contains not only I (making it the circumcircle of BIC) but also O and H. A proof of this expanded claim was submitted to this MONTHLY in 1998 by W. W. Meyer as part of a solution to Problem 10547. Here, we will give a proof based on the solution by Jerry Minkus (San Francisco, CA): Let the angles at A, B, and C be α, β, and γ, respectively. We have shown that $\alpha = 60°$.

Claim. I lies on $\mathcal{C}'$. *Proof.* Designate the midpoint of BC as M. Let P be the point on the opposite side of BC from A at which the perpendicular bisector of BC meets $\mathcal{C}$. Triangles BPM and CPM are congruent, so arcs BP and CP are congruent. Therefore angles BAP and CAP are congruent. Thus AP is the angle bisector of BAC, and therefore AP contains I.

It is known that $R^2 - IO^2 = 2Rr$, which may also be observed by constructing the diameter of $\mathcal{C}$ through I. Thus $IA \cdot IP = (R + OI) \cdot (R - OI) = R^2 - OI^2 = 2Rr$. Since $IA = r/\sin(\alpha/2)$, we have $IP = 2R\sin(\alpha/2)$. Similarly, BP and CP are also equal to $2R\sin(\alpha/2)$. Hence B, C, and I all lie on a circle about P. When $E\alpha = 60°$, the radius of that circle is R, because $\sin(60°/2) = 1/2$. Hence P is the reflection in BC of O, and the circle just referenced containing B, C, and I is the circle $\mathcal{C}'$.

Claim. O lies on $\mathcal{C}'$. *Proof.* O and P are reflections of each other in BC.

Claim. H lies on $\mathcal{C}'$. *Proof.* Note that $AH = 2R\cos\alpha$. This may be seen by extending ray CO to meet $\mathcal{C}$, say at Q. Then since CQ is a diameter, its length is $2R$, angle CBQ is right, and $\angle BQC = \angle BAC = \alpha$, so $BQ = 2R\cos\alpha$. Now BQ is parallel to AH, and similarly, AQ is parallel to BH. Thus $AHBQ$ is a parallelogram and $AH = BQ = 2R\cos\alpha$. Here we have $\alpha = 60°$ and $\cos 60° = 1/2$, so $AH = R$. We may conclude that $AOPH$ is a parallelogram, since AH is parallel to OP and of the same length. (It is in fact a rhombus.) It follows that $PH = AO = R$. Thus as claimed H lies on $\mathcal{C}'$.

Editorial comment. The Blundon result from E2282 may be strengthened in an interesting way due to Francisco Bellot Rosado (Spain), who submitted it to this MONTHLY in 1998 as part of a solution to Problem 10547: Let G denote the centroid of the triangle. The incenter I always lies inside the circle whose diameter is GH, because the angle GIH is always obtuse. Since the perpendicular bisector λ of the Euler segment OH divides the circle of Bellot Rosado into a larger and a smaller piece, I is (1) in the larger piece, (2) on line λ, or (3) in the smaller piece, according as the middle-sized angle of ABC is (1) greater than, (2) equal to, or (3) less than 60°.

Also solved by M. Bataille (France), R. Chapman (U. K.), C. Curtis, Y. Dumont (France), D. Fleischman, V. V. Garcia (Spain), D. Grinberg, J.-P. Grivaux (France), E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), O. Kouba (Syria), J. H. Lindsey II, J. Minkus, R. Stong, M. Tetiva (Romania), D. Vacaru (Romania), Z. Vörös (Hungary), M. Vowe (Switzerland), J. B. Zacharias & K. Greeson, GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Alternating Series

11409 [2009, 83]. *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.* For positive real α and β, let

$$S(\alpha, \beta, N) = \sum_{n=2}^{N} n \log(n)(-1)^n \prod_{k=2}^{n} \frac{\alpha + k \log k}{\beta + (k+1)\log(k+1)}.$$

Show that if $\beta > \alpha$, then $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. Let $\omega_k = k \log k$. Write

$$a_n = \omega_n \prod_{k=2}^{n} \frac{\alpha + \omega_k}{\beta + \omega_{k+1}} = b_n \prod_{k=3}^{n} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right), \quad \text{where} \quad b_n = \frac{(\alpha + \omega_2)\omega_n}{\beta + \omega_{n+1}}, \tag{1}$$

and suppose $\beta > \alpha$. We will prove that

$$\sum_{n=2}^{\infty} (-1)^n a_n \quad \text{converges,}$$

so $\lim_{N\to\infty} S(\alpha, \beta, N)$ exists. By the alternating series test of Leibniz, and noting $a_n > 0$, it suffices to prove

(i) $a_{n+1}/a_n < 1$ for all sufficiently large n, and

(ii) $a_n \to 0$ as $n \to \infty$.

(i) From the definition of a_n in (1),

$$\frac{a_{n+1}}{a_n} = \frac{\omega_{n+1}(\alpha + \omega_{n+1})}{\omega_n(\beta + \omega_{n+2})},$$

so $a_{n+1}/a_n < 1$ is equivalent to $\omega_{n+1}\,\alpha + (\omega_{n+1}^2 - \omega_n\,\omega_{n+2}) < \omega_n\,\beta$. Calculation shows $\omega_{n+1}^2 - \omega_n\,\omega_{n+2} = (\log n)^2 + \log n + 1 + o(1)$. Because $\beta > \alpha$ and $\omega_{n+1} \sim \omega_n = n \log n$, the required result follows.

(ii) Because $\lim_{n\to\infty} b_n$ exists, to show $\lim_{n\to\infty} a_n = 0$ it suffices to show that the infinite product

$$\prod_{k=3}^{\infty} \left(1 - \frac{\beta - \alpha}{\beta + \omega_k}\right) \tag{2}$$

diverges to zero. Recall that if $0 < c_k < 1$ for all k and $\sum_{k=1}^{\infty} c_k$ diverges, then $\prod_{k=1}^{\infty}(1 - c_k)$ diverges to 0. In the present case, the divergence of

$$\sum_{k=3}^{\infty} \frac{1}{\omega_k} = \sum_{k=3}^{\infty} \frac{1}{k \log k}$$

shows that the infinite product in (2) diverges to 0. (That the sum diverges is well known, as it follows from the integral test or Cauchy condensation test.)

Also solved by S. Amghibech (Canada), P. Bracken, R. Chapman (U. K.), P. P. Dályay (Hungary), D. Grinberg, J. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), and the proposer.

A Fix for a Triangle Inequality

11413 [2009, 179]. *Proposed by Mihály Bencze, Brasov, Romania.* Let θ_i for $1 \le i \le 5$ be nonnegative, with $\sum_1^3 \theta_i = \pi$, $\theta_4 = \theta_1$, and $\theta_5 = \theta_2$. Let $S = \sum_{i=1}^3 \sin\theta_i$. Show that

$$S \le \frac{3\sqrt{3}}{2} - 4 \max_{1 \le i \le 3}\left(\sin^2\left(\frac{1}{(4)}(\theta_i - \theta_{i+1})\right)\cos\left(\frac{1}{2}\theta_{i+2}\right) + \sqrt{3}\sin^2\left(\frac{1}{12}(\pi - 3\theta_{i+2})\right)\right).$$

Solution by Richard Stong, San Diego, CA. (The originally published statement had a misprint, with "2" where "(4)" now stands.) If $A, B, C \ge 0$ with $A + B + C = \pi$, then

$$S = \sin A + \sin B + \sin C = 4\cos(A/2)\cos(B/2)\cos(C/2).$$

Hence

$$\begin{aligned} S + 4\sin^2((A - B)/4)\cos(C/2) &= 4\cos^2((A + B)/4)\cos(C/2) \\ &= 4\cos^2(\pi - C)/4)\cos(C/2). \end{aligned}$$

Applying the identity

$$4\cos(x + 2y)\cos^2(x - y) + 8\sin^2 y\cos x = 4\cos^3 x - 4\sin^2 y\cos(x - 2y)$$

with $x = \pi/6$ and $y = (\pi - 3C)/12$, we have

$$4\cos\frac{C}{2}\cos^2\frac{\pi - C}{4} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}$$

or, combined with the above,

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} = \frac{3\sqrt{3}}{2} - 4\sin^2\frac{\pi - 3C}{12}\cos\frac{2\pi - 3C}{6}.$$

Since $0 \le C \le \pi$, the last cosine is nonnegative, and hence

$$S + 4\sin^2\frac{A - B}{4}\cos\frac{C}{2} + 4\sqrt{3}\sin^2\frac{\pi - 3C}{12} \le \frac{3\sqrt{3}}{2}.$$

Apply this result three times, taking (A, B, C) to be $(\theta_1, \theta_2, \theta_3)$, then $(\theta_2, \theta_3, \theta_1)$, and finally $(\theta_3, \theta_1, \theta_2)$, to obtain the desired result.

Editorial comment. Some solvers corrected the problem by showing that it holds as originally printed but with the inequality reversed.

Also solved by R. Bagby, P. P. Dályay (Hungary), J. H. Lindsey II, GCHQ Problem Solving Group (U. K.), and Microsoft Research Problems Group.

Blundon's Inequality Improved

11414 [2009, 179]. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let ABC be a triangle with largest angle at A, let A also denote the measure of that angle, let $c = \cot(A/2)$, and let s, r, and R be the semi-perimeter, inradius, and circumradius of the triangle, respectively. Show that Blundon's Inequality $s \le 2R + r(3\sqrt{3} - 4)$ can be strengthened to

$$s \le 2R + r\left(3\sqrt{3} - 4 - \frac{(\sqrt{3} - c)^3}{4c}\right).$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Lemma. *If a, b, c are positive real numbers such that $a + b + c = abc$ and $c = \min\{a, b, c\}$, then $(a - 1)(b - 1)(c - 1) \le 6\sqrt{3} - 10 - (\sqrt{3} - c)^3/(2c)$.*

Proof. Note that $ba = (abc)/c > (abc)/(a + b + c) = 1$, and similarly $bc > 1$ and $ca > 1$. Thus at most one of the numbers a, b, c can be less than 1. Hence $a \ge 1$ and $b \ge 1$. The equality $a + b + c = abc$ yields $c = (a + b)/(ab - 1)$. We must show that if $a, b \ge 1$ and $ab > 1$, then $f(a, b) \le 0$, where

$$f(a, b) = (a - 1)(b - 1)\left(\frac{a + b}{ab - 1} - 1\right) - (6\sqrt{3} - 10) - \frac{\left(\sqrt{3}(ab - 1) - (a + b)\right)^3}{2(ab - 1)^2(a + b)}.$$

Put $a = 1 + x$ and $b = 1 + y$ with $x, y \ge 0$, and rewrite the function as

$$\begin{aligned} f(1 + x, 1 + y) = -2(x + y + 2)(x + y + xy)\big(&x^2y^2 \\ &+ (6\sqrt{3} - 12)xy + (6\sqrt{3} - 10)(x + y)\big). \end{aligned}$$

Observe $x + y \ge 2\sqrt{xy}$ and substitute $t = \sqrt{xy}$ to reduce the inequality to $p(t) \ge 0$ for all $t \ge 0$, where $p(t) = t^4 + (6\sqrt{3} - 12)t^2 + (12\sqrt{3} - 20)t$. This follows from the factorization $p(t) = t(t - (\sqrt{3} - 1))^2(t + 2\sqrt{3} - 2)$. ∎

In triangle ABC, the numbers $a = \cot(A/2)$, $b = \cot(B/2)$, and $c = \cot(C/2)$ satisfy $a + b + c = s/r = abc$, $ab + bc + ca = (4R + r)/r$, and $c = \min\{a, b, c\}$. By the lemma,

$$\begin{aligned} s &= \frac{r}{2}\big[(a - 1)(b - 1)(c - 1) + ab + bc + ca + 1\big] \\ &\ge \frac{r}{2}\left(6\sqrt{3} - 10 - \frac{(\sqrt{3} - c)^3}{2c} + \frac{4R + r}{r} + 1\right) \\ &= 2R + r\left(3\sqrt{3} - 4 - \frac{(\sqrt{3} - c)^3}{4c}\right). \end{aligned}$$

Equality holds if and only if the triangle is equilateral.

Editorial comment. Richard Stong proved the stronger inequality

$$s \leq 2R + r\left(3\sqrt{3} - 4 - \frac{9(2-\sqrt{3})}{8}\,\frac{(\sqrt{3}-c)^2}{c^2}\right).$$

Also solved by J. H. Lindsey II, C. R. Pranesachar (India), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Closed-Form Definite Integral

11416 [2009, 180]. *Proposed by Yaming Yu, University of California Irvine, Irvine, CA.* Let f be the decreasing function on $(0, \infty)$ that satisfies

$$f(x)e^{-f(x)} = xe^{-x}.$$

(To visualize, draw a graph of the function xe^{-x} and a horizontal line that is tangent to it or crosses it at two points; if one of these points is x, then the other is $f(x)$.) Show that

$$\int_0^\infty x^{-1/6}(f(x))^{1/6}\,dx = \frac{2\pi^2}{3}.$$

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. From the definition of f, we have $f(x)/x = e^{f(x)-x}$. Writing $u = f(x)/x$ and eliminating $f(x)$ gives $x = \log u/(u-1)$, so that as x increases from 0 to ∞, u decreases from ∞ to 0. The integral to be computed, call it A, can then be written as $A = \int_0^\infty u^\alpha(x)\,dx$ (with $\alpha = 1/6$). Integrating first by parts and then changing variables from x to u in the resulting integral gives

$$A = \int_{x=0}^\infty u^\alpha(x)\,dx = xu^\alpha(x)\Big|_{x=0}^\infty + \alpha\int_{u=0}^\infty \frac{u^{\alpha-1}\log u}{u-1}\,du.$$

Here we could refer to Gradshteyn & Ryzhik (formula 4.254.1) and Abramowitz & Stegun (formula 6.4.7). In this special case, though, there is a simpler solution. For $0 < \alpha < 1$ the integral converges. The first term on the right-hand side is zero because it is equal to $u^\alpha \log(u)/(u-1)|_{u=\infty}^0$. Split the second term into two parts:

$$-\alpha\int_0^1 \frac{u^{\alpha-1}\log u}{1-u}\,du + \alpha\int_1^\infty \frac{u^{\alpha-2}\log u}{1-1/u}\,du.$$

Expand $(1-u)^{-1}$ and $(1-1/u)^{-1}$ as geometric series, then integrate:

$$\alpha\sum_{n=0}^\infty \frac{1}{(n+\alpha)^2} + \alpha\sum_{n=1}^\infty \frac{1}{(n+1-\alpha)^2}.$$

Using the Hurwitz zeta function notation $\zeta(s,a) = \sum_{n=0}^\infty (n+a)^{-s}$, for arbitrary α in $(0,1)$ this can be written as $\alpha(\zeta(2,\alpha) + \zeta(2,1-\alpha))$. Starting with the known fact that $\zeta(2) = \zeta(2,1) = \pi^2/6$, elementary calculations give $\zeta(2,1/2) = 3\zeta(2)$ and $\zeta(2,1/3) + \zeta(2,2/3) = 8\zeta(2)$, so that $\zeta(2,1/6) + \zeta(2,5/6)$ is given by

$$\sum_{k=1}^6 \zeta(2,k/6) - \sum_{k=1}^2 \zeta(2,k/3) - \zeta(2,1/2) - \zeta(2,1) = (36-8-3-1)\zeta(2) = 4\pi^2.$$

The required sum is thus $4\alpha\pi^2 = 2\pi^2/3$.

Also solved by R. Bagby, D. Beckwith, P. Bracken, B. S. Burdick, P. Corn, L. Gerber, M. L. Glasser, J. Grivaux (France), E. A. Herman, F. Holland & T. Carroll (Ireland), K. McInturff, O. G. Ruehr, V. Rutherfoord, R. Stong, J. B. Zacharias, BSI Problems Group (Germany), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.

An Integral-Derivative Inequality

11417 [2009, 180]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Let f be a continuously differentiable real-valued function on [0, 1] such that $\int_{1/3}^{2/3} f(x)\,dx = 0$. Show that $\int_0^1 (f'(x))^2\,dx \geq 27\left(\int_0^1 f(x)\,dx\right)^2$.

Solution by Moubinool Omarjee, Paris, France. Let $h(x)$ be the continuous, piecewise linear function given by

$$h(x) = \begin{cases} -x, & 0 \leq x \leq 1/3, \\ 2x - 1, & 1/3 \leq x \leq 2/3, \\ 1 - x, & 2/3 \leq x \leq 1. \end{cases}$$

Integrating by parts gives

$$\int_0^1 h(x) f'(x)\,dx = \int_0^1 f(x)\,dx - 3\int_{1/3}^{2/3} f(x)\,dx = \int_0^1 f(x)\,dx,$$

and we compute that

$$\int_0^1 h(x)^2\,dx = \frac{1}{27}.$$

Hence the Cauchy-Schwarz inequality applied to h and f' reads

$$\int_0^1 (f'(x))^2\,dx \geq 27\left(\int_0^1 f(x)\,dx\right)^2,$$

as desired.

Editorial comment. Several solvers remarked that this problem generalizes with essentially the same proof. In the simplest form, suppose that $\phi(x)$ is an integrable function with $\int_0^1 \phi(x)\,dx = 1$, and define $h(x) = -x + \int_0^x \phi(t)\,dt$ and $C = \int_0^1 h(x)^2\,dx$. For any continuously differentiable real-valued function f on [0, 1] such that $\int_0^1 f(x)\phi(x)\,dx = 0$, one has

$$C\int_0^1 (f'(x))^2\,dx \geq \left(\int_0^1 f(x)\,dx\right)^2.$$

More generally, this holds with $\phi(x)\,dx$ replaced by a signed Borel measure.

Also solved by K. F. Andersen (Canada), R. Chapman (U. K.), P. P. Dályay (Hungary), P. J. Fitzsimmons, O. Geupel (Germany), J. Grivaux (France), G. Keselman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), D. S. Ross, R. Tauraso (Italy), P. Venkataramana, E. I. Verriest, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), St. John's University Problem Solving Group, and the proposers.

11426 [2009, 365]. *Proposed by M. L. Glasser, Clarkson University, Potsdam, NY.* Find

$$\frac{\Gamma(1/14)\Gamma(9/14)\Gamma(11/14)}{\Gamma(3/14)\Gamma(5/14)\Gamma(13/14)},$$

where Γ denotes the usual gamma function, given by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$.

Solution by Matthew A. Carlton, Cal Poly State University, San Luis Obispo, CA. The multiplication formula for the gamma function may be written as

$$\Gamma(z) = 2\sqrt{\pi}\cdot 2^{-2z}\cdot\frac{\Gamma(2z)}{\Gamma(z+1/2)}.$$

Apply this with z equal to each of the six values in the original expression, e.g.

$$\Gamma(1/14) = 2\sqrt{\pi}\cdot 2^{-1/7}\cdot\frac{\Gamma(1/7)}{\Gamma(4/7)}.$$

The numerator of the original expression can then be written

$$\begin{aligned}&(2\sqrt{\pi}\,)^3\cdot 2^{-1/7-9/7-11/7}\cdot\frac{\Gamma(1/7)\Gamma(9/7)\Gamma((11/7)}{\Gamma(4/7)\Gamma(8/7)\Gamma(9/7)}\\ &= 8\pi^{3/2}\cdot\frac{1}{8}\cdot\frac{\Gamma(1/7)\cdot 4/7\Gamma(4/7)}{\Gamma(4/7)\cdot 1/7\Gamma(1/7)} = 4\pi^{3/2}.\end{aligned}$$

Similarly, the denominator simplifies to $2\pi^{3/2}$. Thus the quotient is 2.

Editorial comment. Some solvers provided generalizations. The most interesting and complete was from Albert Stadler (Switzerland). Let p be an odd prime, and denote the Legendre symbol by $\left(\frac{k}{p}\right)$. Then

$$\prod_{k=1}^{p}\Gamma\left(\frac{2k-1}{2p}\right)^{\left(\frac{2k-1}{p}\right)} = \begin{cases}1, & \text{if } p\equiv 1 \pmod 8,\\ \varepsilon(p)^{h(p)}, & \text{if } p\equiv 5 \pmod 8,\\ 2^{-\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\frac{k}{p}}, & \text{if } p\equiv 7 \pmod 8,\end{cases} \qquad (*)$$

where $\varepsilon(p)$ denotes the fundamental unit and $h(p)$ the class number of the real quadratic field $\mathbb{Q}(\sqrt{p}\,)$. The case $p\equiv 3 \pmod 8$ was not resolved. The fundamental unit $\varepsilon(p) = (x+y\sqrt{p}\,)/2$ is a solution of Pell's equation $x^2 - py^2 = 4$ with the property that both x and y are positive and y is minimal. The result asked for here is the case $p = 7$. Other examples ($p = 5$, 13, 17):

$$\frac{\Gamma(1/10)\Gamma(9/10)}{\Gamma(3/10)\Gamma(7/10)} = \frac{3+\sqrt{5}}{2},$$

$$\frac{\Gamma(1/26)\Gamma(3/26)\Gamma(9/26)\Gamma(17/26)\Gamma(23/26)\Gamma(25/26)}{\Gamma(5/26)\Gamma(7/26)\Gamma(11/26)\Gamma(15/26)\Gamma(19/26)\Gamma(21/26)} = \frac{11+3\sqrt{13}}{2},$$

$$\frac{\Gamma(1/34)\Gamma(9/34)\Gamma(13/34)\Gamma(15/34)\Gamma(19/34)\Gamma(21/34)\Gamma(25/34)\Gamma(33/34)}{\Gamma(3/34)\Gamma(5/34)\Gamma(7/34)\Gamma(11/34)\Gamma(23/34)\Gamma(27/34)\Gamma(29/34)\Gamma(31/34)} = 1.$$

Since the values in $(*)$ are algebraic numbers, we have a corollary: If p is an odd prime $\not\equiv 3 \pmod 8$, then the $p-1$ numbers $\Gamma((2k-1)/(2p))$, $1\le k\le p$, $k\ne(p-1)/2$, are algebraically dependent.

Also solved by Z. Ahmed & M. A. Prasad (India), K. F. Andersen (Canada), R. Bagby, B. Bauldry, D. Beckwith, P. Bracken, M. A. Carlton, R. Chapman (U. K.), H. Chen, C. K. Cook, P. Costello, P. P. Dályay (Hungary), F. Flores & F. Mawyer, M. R. Gopal, D. Gove, G. C. Greubel, D. Grinberg, J. Grivaux (France), J. A. Grzesik, C. C. Heckman, E. A. Herman, D. Hou, R. Howard, E. Hysnelaj (Australia) & E. Bojaxhiu (Albania), G. Keselman, T. Konstantopoulis (U. K.), O. Kouba (Syria), V. Krasniqi (Kosova), H. Kwong, G. Lamb, O. P. Lossers (Netherlands), R. Martin (Germany), K. McInturff, A. Nijenhuis, O. Padé (Israel), R. Padma (India), C. R. Pranesachar (India), H. Riesel (Sweden), I. Rusodimos, O. A. Saleh & S. Byrd, A. S. Shabani (Kosova), M. A. Shayib, N. C. Singer, A. Stadler (Switzerland), R. Stong, T. Tam, R. Tauraso (Italy), Z. Vörös (Hungary), M. Vowe (Switzerland), Z. Wenlong (China), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

An Equilateral Condition

11427 [2009, 365]. *Proposed by Viorel Băndilă, C.A. Rosetti High School, Bucharest, Romania.* In a triangle ABC, let m be the length of the median from A, l the length of the angle bisector from B, and h the length of the altitude from C. Let a, b, and c be the lengths of the edges opposite A, B, and C, respectively. Show that ABC is equilateral if and only if $a^2 + m^2 = b^2 + l^2 = c^2 + h^2$.

Solution by Bianca-Teodora Iordache, student, "Carol I" High School, Craiova, Romania. If ABC is equilateral, then $a = b = c$ and $m = l = h$, so the equations hold. We must prove the converse. Let m_a, l_a, and h_a denote the lengths of the median, angle bisector, and altitude, respectively, corresponding to the edge a, and define similar notation for edges b and c. We must prove that

$$a^2 + m_a^2 = b^2 + l_b^2 = c^2 + h_c^2 \quad \Longrightarrow \quad a = b = c.$$

Claim 1. $a^2 + m_a^2 \le b^2 + m_b^2 \iff a \le b$. Indeed,

$$a^2 + m_a^2 = a^2 + \frac{2(b^2 + c^2) - a^2}{4} = \frac{3a^2 + 2b^2 + 2c^2}{4}.$$

Hence $a^2 + m_a^2 \le b^2 + m_b^2 \iff 3a^2 + 2b^2 + 2c^2 \le 3b^2 + 2a^2 + 2c^2 \iff a^2 \le b^2 \iff a \le b$.

Claim 2. $a^2 + h_a^2 \le b^2 + h_b^2 \iff a \le b$. Using $h_a = 2S/a$, where S is the area of ABC, we have $a^2 + h_a^2 = a^2 + 4S^2/a^2$, so

$$a^2 + h_a^2 \le b^2 + h_b^2 \iff (b^2 - a^2)\frac{a^2b^2 - 4S^2}{a^2b^2} \ge 0 \iff b \ge a.$$

Also recall that $h_a \le l_a \le m_a$ and similarly for b, c. Next suppose that $a^2 + m_a^2 = b^2 + l_b^2 = c^2 + h_c^2$. We have $a^2 + m_a^2 = c^2 + h_c^2 \le c^2 + m_c^2$, so $a \le c$ from Claim 1. We have $c^2 + h_c^2 = b^2 + l_b^2 \ge b^2 + h_b^2$, so $b \le c$ by Claim 2. From the Heron formula, $16S^2 = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 2\sum a^2b^2 - \sum a^4$, using $\sum$ for sums over cyclic permutations of the triangle. Now $a^2 + m_a^2 = c^2 + h_c^2$ so

$$\frac{3a^2 + 2b^2 + 2c^2}{4} = c^2 + \frac{2\sum a^2b^2 - \sum a^4}{4c^2},$$

so $c^2(3a^2 + 2b^2 - 2c^2) = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$ and thus

$$c^2(c^2 - a^2) = (b^2 - a^2)^2. \tag{1}$$

Since $c^2 \ge b^2 > b^2 - a^2$ and $c^2 - a^2 \ge b^2 - a^2$, for equality in (1) we must have $c^2 - a^2 = b^2 - a^2 = 0$. This shows $a = c$ and $a = b$ as required.

Also solved by R. Bagby, M. Bataille (France), H. Caerols (Chile), R. Chapman (U. K.), G. Crandall, P. P. Dályay (Hungary), D. Fleischman, D. Gove, J. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, J. McHugh, J. Minkus, M. A. Prasad (India), A. Stadler (Switzerland), R. Stong, M. Tetiva (Romania), Z. Vörös (Hungary), GCHQ Problem Solving Group (U. K.), Microsoft Research Problems Group, and the proposer.