PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Doug Hensley, Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Itshak Borosh, Paul Bracken, Ezra A. Brown, Randall Dougherty, Dennis Eichhorn, Tamás Erdélyi, Zachary Franco, Christian Friesen, Ira M. Gessel, Jerrold R. Griggs, Jerrold Grossman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Ullman, Charles Vanden Eynden, and Fuzhen Zhang.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted solutions should arrive at that address before September 30, 2007. Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

11298. *Proposed by Jakob Jonsson, MIT, Cambridge, MA, and James Propp, University of Massachusetts Lowell, Lowell, MA.* Show that for $n \geq 3$, if a convex n-gon admits a triangulation in which every vertex is incident with an odd number of triangles, then n must be a multiple of 3. (A *triangulation* of a convex n-gon is a dissection of that n-gon into $n-2$ triangles using $n-3$ non-crossing diagonals.)

11299. *Proposed by Pablo Fernàndez Refolio, Universidad Autónoma de Madrid, Madrid, Spain.* Show that

$$\prod_{n=2}^{\infty}\left(\frac{1}{e}\left(\frac{n^2}{n^2-1}\right)^{n^2-1}\right) = \frac{e\sqrt{e}}{2\pi}.$$

11300. *Proposed by Ulrich Abel, University of Applied Sciences Giessen-Friedberg, Friedberg, Germany.* For integers k and n with $0 \leq k \leq n$, let $p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}$. Let $K_n(x,y) = \sum_{k=0}^{n}(y-k/n)p_{n,k}(x)p_{n,k}(y)$. Prove that for $0 \leq u \leq 1$ and $0 \leq y \leq 1$, $\int_{x=0}^{u} K_n(x,y)\,dx \geq 0$.

11301. *Proposed by Finbarr Holland, University College Cork, Ireland.* Find the least real number M such that, for all complex a, b, and c,

$$\left|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)\right| \leq M(|a|^2+|b|^2+|c|^2)^2.$$

11302. *Proposed by Horst Alzer, Waldbröl, Germany.* Find

$$\sum_{k=2}^{\infty}\frac{(2k+1)H_k^2}{(k-1)k(k+1)(k+2)},$$

where H_k is the kth harmonic number, defined to be $\sum_{j=1}^{k} 1/j$.

11303. *Proposed by M. Farrokhi D. G., Ferdowsi University of Mashad, Mashad, Iran.* Let A be an invertible matrix with nonnegative integer entries. Show that if the union over all n of the set of entries of A^n is finite, then A is a permutation matrix.

11304. *Proposed by Teodora-Liliana Rădulescu, Fraţii Buzeşti College, Craiova, and Vicenţiu Rădulescu, University of Craiova, Romania.*
(**a**) Find a sequence $\langle z_n \rangle$ of distinct complex numbers, and a sequence $\langle \alpha_n \rangle$ of nonzero real numbers, such that for almost all complex numbers z (excluding a set of measure zero), $\sum_{n=1}^{\infty} \alpha_n |z - z_n|^{-1}$ diverges to $+\infty$, yet not all α_n are positive.
(**b**) Let $\langle \beta_n \rangle$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} |\beta_n|$ is finite and such that, for almost all z in $\mathbb{C}$, $\sum_{n=1}^{\infty} \beta_n |z - z_n|^{-1}$ converges to a nonnegative real number. Prove that $\beta_n \geq 0$ for all n.
(**c***) Can there be a sequence $\langle \alpha_n \rangle$ of real numbers, not all positive, and a sequence $\langle z_n \rangle$ of distinct complex numbers, such that for almost all complex z, $\sum_{n=1}^{\infty} \alpha_n |z - z_n|^{-1}$ converges to a positive real number?

SOLUTIONS

Two Heads are Better than One

11173 [2005, 749]. *Proposed by M. N. Deshpande, Institute of Science, Nagpur, India, and J. P. Shiwalkar, Hislop College, Nagpur, India.* A *double-head* in a sequence of coin tosses is an occurrence of two consecutive heads (in HHHTHHTHHHH there are six double-heads). A fair coin is flipped until r double-heads are obtained, and the number X_r of flips made to this point is recorded. Show that if E_r is the expected value of X_r and V_r is the variance, then $5E_r - V_r$ is constant, independent of r.

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. Let $Y_1 = X_1$, and, for $r \geq 2$, let $Y_r = X_r - X_{r-1}$. Partitioning according to whether the sequence begins T, HT, or HH yields

$$E(Y_1) = \tfrac{1}{2}(1 + E(Y_1)) + \tfrac{1}{4}(2 + E(Y_1)) + \tfrac{1}{4} \cdot 2.$$

Hence $E(Y_1) = 6$. For $r \geq 2$, $E(Y_r) = \frac{1}{2}(1 + E(Y_1)) + \frac{1}{2} \cdot 1 = 4$. Therefore

$$E_r = \sum_{i=1}^{r} E(Y_r) = 2 + 4r.$$

Similarly,

$$E(Y_1^2) = \tfrac{1}{2}(1 + E(Y_1))^2 + \tfrac{1}{4}(2 + E(Y_1))^2 + \tfrac{1}{4} \cdot 2^2 = \tfrac{5}{2} + 2E(Y_1) + \tfrac{3}{4}E(Y_1^2).$$

This yields $E(Y_1^2) = 58$, and so $V_1 = E(X_1^2) - E(X_1)^2 = 22$. For $r \geq 2$,

$$E(Y_r^2) = \frac{1}{2}(1 + E(Y_1))^2 + \frac{1}{2} \cdot 1 = 1 + E(Y_1) + \frac{1}{2}E(Y_1^2) = 36,$$

and so $\mathrm{Var}(Y_r) = E(Y_r^2) - E(Y_r)^2 = 20$. Because the random variables Y_r are independent, $V_r = \sum_{i=1}^{r} \mathrm{Var}(Y_r) = 2 + 20r$. Therefore $5E_r - V_r = 8$.

Editorial comment. The problem was generalized by many solvers to the case of a biased coin and to the problem of runs of k heads for $k > 2$. Stephen Herschkorn gave the most general statement: Let $W_1, W_2, \ldots$ be a sequence of independent identically distributed discrete random variables, and let q be an n-tuple of values in the range of W_1. If X_r is the time of the rth appearance of q in the sequence, then $E(X_r)$ and $\operatorname{Var}(X_r)$ are both affine functions of r. Hence $cE(X_r) - \operatorname{Var}(X_r)$ is constant for some choice of c.

Solved also by M. Andreoli, D. Beckwith, R. Chapman (U. K.), S. Herschkorn, V. Karwe, G. Keselman, G. Lavau (France) R. Leitch (U. K.), J. H. Lindsey II, O. P. Lossers (Netherlands), D. Lovit, K. McInturff, E. Omey & S. Van Gulck (Belgium), R. Pratt & F. Chen, J. Resing (Netherlands) K. Schilling, B. Schmuland (Canada), D. Senft, A. Stadler (Switzerland), R. Stong, G. A. Stoops, BSI Problem Solving Group (Germany), Hope College Problem Solving Group, Microsoft Research Problems Group, and NSA Problems Group.

Learning from Experience

11178 [2005, 750]. *Proposed by Jon Bentley and Colin Mallows, Avaya Labs, Basking Ridge, NJ.* Balls are to be thrown independently into unequally likely boxes $1, 2, \dots,$ K, with $P(\text{ball lands in box } j) = q_j$, until n balls have been thrown. The *player* bets that when the next ball is thrown it will go into whichever box has received the most balls out of the first n throws. (If there are ties, she breaks the tie at random.) Prove that, whatever the values of $q_1, \dots, q_k$, her probability of winning is a strictly increasing function of n.

Solution by Richard Stong, Rice University, Houston, TX. The claim is not quite correct. The probability of winning is a nondecreasing function of n, but it is not strictly increasing. The probability of winning on the second throw is the same as the probability of winning on the third throw. If $K = 2$ (or if $K = 3$ and one of the q_i is zero), then the probability of winning on any odd toss is the same as the probability of winning on the previous even toss. If $K = 2$ and $\{q_1, q_2\} = \{0, 1\}$, then the probability of winning is constant for $n \geq 2$.

Consider player A, who follows the given strategy, and player B, who follows almost the same strategy but ignores the result of the nth throw. We are asked to show that player A wins with probability at least as great as player B.

It suffices to consider the cases where A and B bet differently—that is, either the nth throw broke a tie for first or created or enlarged a tie for first. By symmetry, we may assume that the tie involved the first r boxes and let $Q = \sum_{i=1}^{r} q_i$.

Given that boxes 1 through r are tied at step $n-1$ and the tie is broken on step n, the probability that the tie is broken in favor of box i is q_i/Q. Hence A's conditional probability of winning is $(q_1^2 + q_2^2 + \cdots + q_r^2)/Q$, and B's conditional probability of winning is simply Q/r. By the Cauchy–Schwarz inequality, $r(q_1^2 + q_2^2 + \cdots + q_r^2) \geq Q$, so A is at least as likely to win as B.

Given that boxes 1 through r are tied at step n, any orders of putting the balls into boxes that have the same totals are equally likely. Thus, given that one of the r tied boxes was not part of the tie at step $n-1$, each of the r boxes is equally likely to be the one that received the last ball. If box i received the last ball, then B's probability of winning is $(Q - q_i)/(r-1)$; hence B's conditional probability of winning is

$$\sum_{i=1}^{r} \frac{1}{r} \cdot \frac{Q - q_i}{r-1} = \frac{Q}{r-1} - \frac{Q}{r(r-1)} = \frac{Q}{r}.$$

Since A's conditional probability of winning is also Q/r, the two players have equal conditional probabilities of winning.

Finally, we analyze when equality can hold. For equality to hold in our application of the Cauchy–Schwarz inequality, we must have $q_1 = \cdots = q_r$. If any tie is possible, then this contradicts the assumption of unequal q_i. Thus the only cases where A does not do strictly better are when no tie is possible after $n-1$ tosses. This occurs when

$n = 2$, when there are two boxes with positive probability and n is even, and when there is only one box with positive probability. These correspond to the cases described in the first paragraph.

Editorial comment. The Microsoft Research Problems Group and most of the solvers pointed out that the problem as stated is not quite correct, as noted above.

Also solved by J. H. Lindsey II, R. E. Prather, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposers.

A Determinant by Möbius Inversion

11179 [2005, 750]. *Proposed by David Beckwith, Sag Harbor, NY.* For positive integers i and j let $m_{ij} = \begin{cases} -1 & j \mid (i+1) \\ 0 & j \nmid (i+1) \end{cases}$, and when $n \geq 2$ let M_n be the $(n-1) \times (n-1)$ matrix with (i, j)-entry m_{ij}. Evaluate $\det M_n$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL. By convention, set $\det M_1 = 1$ for the empty matrix. We show that $\det M_n = \mu(n)$ for $n \geq 1$, where μ is the Möbius function ($\mu(n) = (-1)^k$ if n is square-free with k prime divisors, and otherwise $\mu(n) = 0$).

For $n \geq 2$, expanding along row $n-1$ of M_n yields a contribution from each column d such that $d \mid n$ and $d < n$. In the later columns, -1 moves to the diagonal of the resulting smaller matrix. In the earlier columns, M_d remains in the upper-left corner. Hence the contribution is $(-1)^{n-1+d}(-1)^{n-d} \det M_d$. We conclude that $\det M_n = -\sum_{d \mid n \text{ and } d<n} \det M_d$ for $n \geq 2$.

Letting $f(1) = 1$ and $f(n) = 0$ for $n \geq 2$, we have $f(n) = \sum_{d \mid n} \det M_d$ for all n. The Möbius Inversion Formula immediately yields

$$\det M_n = \sum_{d \mid n} \mu(d) f(n/d) = \mu(n).$$

Editorial comment. Most solvers used Möbius inversion. Albert Stadler applied elementary column operations to reduce to a directly computable determinant.

Also solved by T. Achenbach, S. Amghibech (Canada), D. R. Bridges, R. Chapman (U. K.), W. Chu (Italy), K. David, L. M. DeAlba, Y. Dumont (France), J.-P. Grivaux (France), E. A. Herman, J. H. Lindsey II, O. P. Lossers (Netherlands), R. Martin (U. K.) M. Reyes, J. Singh (India), A. Stadler (Switzerland), A. Stenger, R. Stong, Y. Tam & M. Tam, R. Tauraso (Italy), M. Tetiva (Romania), L. Wenstrom, GCHQ Problem Solving Group (/uk), Microsoft Research Problems Group, Missouri State University Problem Solving Group, NSA Problems Group, and the proposer.

Perfect Powers in an Arithmetic Progression

11182 [2005, 839]. *Proposed by Shahin Amrahov, University of Economy and Technology, Ankara, Turkey.* Let $\langle a_n \rangle$ be an arithmetic progression of positive integers for which the common difference is prime. Given that the sequence includes both a term that is a perfect jth power and a term that is a perfect kth power, and that j and k are relatively prime, prove that there exists a term that is a perfect jkth power.

Solution by Nicholas C. Singer, Annandale, VA. If p is the common difference, then $a_n = a_0 + np$ for $n \geq 0$. The case $a_0 = 1$ is trivial, so suppose $a_0 > 1$. The integers in the progression are those at least a_0 that are congruent to a_0 modulo p. We are given positive integers α and β such that $\alpha^j \equiv a_0 \equiv \beta^k \pmod{p}$ and $\alpha^j, \beta^k \geq a_0 > 1$.

Since j and k are relatively prime, there exist integers t and u such that $jt + ku = 1$, and thus positive integers r and s such that $jr + ks \equiv 1 \pmod{(p-1)}$. By Fermat's Little Theorem, for any such r and s,

$$(\alpha^s \beta^r)^{jk} \equiv (\alpha^j)^{ks} (\beta^k)^{jr} \equiv a_0^{ks} a_0^{jr} \equiv a_0^{ks+jr} \equiv a_0 \pmod{p}.$$

All instances of $(\alpha^s \beta^r)^{jk}$ are perfect jkth powers in the progression.

Editorial comment. Singer notes that the argument given is also valid when $p \mid a_0$, but that the claim is trivial in that case, since p^{njk} then lies in the progression for sufficiently large n.

Gerry Myerson, Byron Schmuland, and Marian Tetiva each observed that the conclusion is also valid for an arithmetic progression $\langle a_n \rangle$ with composite common difference m. When a_0 is relatively prime to m or is a multiple of m, this can be proved essentially as above, using Euler's generalization of Fermat's Little Theorem. When $1 < \gcd(a_0, m) < m$, more difficult arguments are needed.

Also solved by M. R. Avidon, O. Bagdasar (Romania), P. Budney, R. Chapman (U. K.), J. Christopher, Y. Dumont (France), M. Goldenberg & M. Kaplan, L. Jones & R. Keller, G. Myerson (Australia), K. E. Schilling, B. Schmuland (Canada), A. E. Stadler (Switzerland), A. L. Stenger, R. A. Stong, R. Tauraso (Italy), M. Tetiva (Romania), the GCHQ Problem Solving Group (U. K.), the Houghton College Problem Solving Group, the Microsoft Research Problems Group, the NSA Problems Group, the Northwestern University Math Problem Solving Group, and the proposer.

Building Two Piles of Equal Height

11183 [2005, 839]. *Proposed by David Beckwith, Sag Harbor, NY.* The left and right pillars of a triumphal arch are each to be built of blocks of height 1 or 2. Blocks of height 2 may not sit upon blocks of height 1. How many designs are feasible if the lintel must sit level upon the pillars and if exactly n blocks must be used in the construction of the pillars? (Thus, if $n = 3$ there are two designs: left pillar of one height-two block and right pillar of two height-one blocks, or vice-versa.)

Solution I by Vadim Ponomarenko, Trinity University, San Antonio, TX. Let c_n be the desired quantity. Let a_n be the number of symmetric arches and let b_n be the number of asymmetric arches with more 2-blocks on the left, so $c_n = a_n + 2b_n$. Since all 2-blocks must be below all 1-blocks, we have $a_n = 1 + \frac{n}{2}$ for n even and nonnegative, and clearly $a_n = 0$ for n odd (or $n < 0$).

For $n \geq 3$, arches counted by b_n arise from symmetric or asymmetric arches with $n - 3$ blocks by inserting a 2-block at the bottom of the left pillar and two 1-blocks at the top of the right pillar. Thus $b_n = a_{n-3} + b_{n-3}$ for $n \geq 3$, with $b_n = 0$ for $n < 3$. If n is even, then $a_{n-3} = 0$, so $b_n = b_{n-3} = a_{n-6} + b_{n-6}$, and thus $b_n = a_{n-6} + a_{n-12} + a_{n-18} + \cdots$. If n is odd, then $b_n = a_{n-3} + b_{n-3} = a_{n-3} + a_{n-9} + a_{n-15} + \cdots$. Setting $n = 6k + \alpha$, where $0 \leq \alpha < 6$, we evaluate these sums to obtain

$$c_n = \begin{cases} 3k^2 + (\alpha + 2)k + \frac{1}{2}\alpha + 1 & \text{if } n \text{ is even,} \\ 3k^2 + (\alpha + 2)k + \alpha - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution II by Rob Pratt, Raleigh, NC. Each design consists of copies of the following four basic units, with all copies of unit 1 at the bottom of the pillars, all copies of unit 2 at the top, and copies of units 3 or 4 (only one type) in the middle:

unit	left pillar	right pillar
1	one height-two block	one height-two block
2	one height-one block	one height-one block
3	one height-two block	two height-one blocks
4	two height-one blocks	one height-two block

Letting n_i be the number of copies of unit i in an arch, we have a one-to-one correspondence between n-block arches and nonnegative integer solutions of $2n_1 + 2n_2 + 3n_3 + 3n_4 = n$ with $n_3 n_4 = 0$. Let c_n be the number of such solutions. Letting $C(z) = \sum_{n=0}^{\infty} c_n z^n$, we have

$$\begin{aligned} C(z) &= (1 + z^2 + z^4 + \cdots)^2(1 + 2z^3 + 2z^6 + \cdots) \\ &= \left(\frac{1}{1 - z^2}\right)^2 \left(\frac{2}{1 - z^3} - 1\right) = \frac{1 + z^3}{(1 - z^2)^2(1 - z^3)} \\ &= \frac{1/6}{1 - z^3} + \frac{1/12}{(1 - z)^2} + \frac{3/4}{1 - z^2} - \frac{2z/3}{1 - z^3}. \end{aligned}$$

Thus

$$\begin{aligned} c_n &= \frac{1}{6}\binom{n+2}{2} + \frac{1}{12}\binom{n+1}{1} + \frac{3}{4} \cdot \frac{1 + (-1)^n}{2} - \frac{2}{3} \cdot \frac{1 + \omega^{n-1} + \omega^{2(n-1)}}{3} \\ &= \frac{1}{12}n^2 + \frac{1}{3}n + \frac{29}{72} + \frac{3}{8}(-1)^n - \frac{2}{9}(\omega^{n-1} + \omega^{2n-2}), \end{aligned}$$

where ω is the principal cube root of unity.

Editorial comment. Solvers gave various formulas for c_n. Many showed that $c_n = \frac{1}{12}(n^2 + 4n + \epsilon_n)$, where $\epsilon_n = 12, -5, 12, 3, 4, 3$ according as $c_n \equiv 0, 1, 2, 3, 4, 5 \pmod 6$. Several showed that

$$c_n = \begin{cases} \left\lfloor \frac{1}{12}(n^2 + 4n + 12) \right\rfloor, & \text{if } n \text{ is even} \\ \left\lfloor \frac{1}{12}(n^2 + 4n + 3) \right\rfloor, & \text{if } n \text{ is odd.} \end{cases}$$

The sequence c_n is A008806 in the Online Encyclopedia of Integer Sequences.

Also solved by M. Avidon, J. Binz (Switzerland), M. Bowron, K. Calderhead, R. Chapman (U. K.), Y. Dumont (France), E. A. Herman, J. Hutchinson & S. Wagon, S. C. Locke, D. Lovit, O. P. Lossers (Netherlands), A. Miller, C. R. Pranesachar (India), S. Seltzer, A. Stadler (Switzerland), R. Staum, R. Stong, R. Tauraso, C. G. Wastun, GCHQ Problem Solving Group (/uk), Houghton College Problem Solving Group, NSA Problems Group, and the proposer.

When Does It Converge

11185 [2005, 840]. *Proposed by Rainer Brück, University of Dortmund, Dortmund, Germany, and Raymond Mortini, University of Metz, Metz, France.* Find all natural numbers n and positive real numbers α such that the integral

$$I(\alpha, n) = \int_0^{\infty} \log\left(1 + \frac{\sin^n x}{x^{\alpha}}\right) dx$$

converges.

Solution by Robin Chapman, University of Bristol, Bristol, U. K. The integral $I(\alpha, n)$ converges if and only if n is even and $\alpha > 1$ or n is odd and $\alpha > 1/2$. It converges absolutely if and only if $\alpha > 1$.

We first show that the integral always "converges at zero"; that is, the integral $\int_0^1 \log\left(1+x^{-\alpha}\sin^n(x)\right)\,dx$ is (absolutely) convergent. When $\alpha \le n$ the integrand extends to a continuous function on the closed interval $[0,1]$, so the integral converges. If $\alpha > n$, then

$$I(\alpha,n) = \int_0^1 \log\left(x^{\alpha-n} + \frac{\sin^n x}{x^n}\right) dx + (n-\alpha)\int_0^1 \log x\,dx.$$

The first of these integrals is certainly convergent, and an integration by parts shows that $\int_0^1 \log x\,dx = -1$. Hence $I(\alpha,n)$ is convergent at zero, so we only need to determine whether it converges at ∞.

With $L(t) = \log(1+t)$, we have $L(t) = t + O(t^2)$ in a neighborhood of zero, so there is some $a > 0$ such that $|t|/2 \le |L(t)| < 2|t|$ when $|t| < a$. Consider the integral $\int_\pi^\infty \log\left(1+x^{-\alpha}\sin^n(x)\right)dx$, which we write as a sum $\sum_{k=1}^\infty I_k(\alpha,n)$, where

$$I_k(\alpha,n) = \int_{k\pi}^{(k+1)\pi} L\left(\frac{\sin^n x}{x^\alpha}\right) dx.$$

For large enough k,

$$|I_k(\alpha,n)| \le 2\int_{k\pi}^{(k+1)\pi} \frac{|\sin^n x|}{x^\alpha}\,dx \le 2\int_{k\pi}^{(k+1)\pi} \frac{dx}{x^\alpha} \le \frac{2\pi^{\alpha-1}}{k^\alpha}.$$

When $\alpha > 1$ the sum $\sum_{k=1}^\infty |I_k(\alpha,n)|$ is convergent, so the integral $I(\alpha,n)$ is absolutely convergent.

If n is even and $\alpha \le 1$, then $I_k(\alpha,n) \ge 0$. For large enough k,

$$I_k(\alpha,n) \ge \frac{1}{2}\int_{k\pi}^{(k+1)\pi} \frac{\sin^n x}{x^\alpha}\,dx \ge \frac{1}{2(k+1)^\alpha\,\pi^\alpha}\int_{k\pi}^{(k+1)\pi} \sin^n x\,dx$$

$$= \frac{1}{2(k+1)^\alpha\,\pi^\alpha}\int_0^\pi \sin^n x\,dx.$$

It follows that the series $\sum_{k=1}^\infty I_k(\alpha,n)$ is divergent. We conclude that when n is even, $I(\alpha,n)$ converges if and only if $\alpha > 1$.

Suppose that n is odd. Setting $M(t) = t - L(t)$, we have $M(t) = t^2/2 + O(t^3)$ in a neighborhood of zero. It follows that there is some $a > 0$ such that $t^2/3 \le M(t) \le t^2$ whenever $|t| < a$. We claim that the integral $\int_\pi^\infty x^{-\alpha}\sin^n(x)\,dx$ converges. Putting $S(x) = \int_\pi^x \sin^n t\,dt$, we have

$$S(x+2\pi) - S(x) = \int_x^{x+2\pi} \sin^n t\,dt = \int_x^{x+\pi} \sin^n t\,dt + \int_{x+\pi}^{x+2\pi} \sin^n t\,dt = 0$$

because $\sin^n(t+\pi) = -\sin^n t$. Thus S is bounded. Integration by parts shows that

$$\int_\pi^N \frac{\sin^n x}{x^\alpha}\,dx = \frac{S(N)}{N^\alpha} + \alpha\int_\pi^N \frac{S(x)}{x^{\alpha+1}}\,dx,$$

which converges as $N \to \infty$. Thus $I(\alpha,n)$ is convergent if and only if $J(\alpha,n)$ is convergent, where

$$J(\alpha,n) = \int_\pi^\infty M\left(\frac{\sin^n x}{x^\alpha}\right) dx.$$

The integrand of $J(\alpha, n)$ is nonnegative. For sufficiently large b,

$$\int_b^\infty M\left(\frac{\sin^n x}{x^\alpha}\right) dx \le \int_b^\infty \frac{\sin^{2n} x}{x^{2\alpha}}\, dx \le \int_b^\infty \frac{dx}{x^{2\alpha}}.$$

Thus if $\alpha > 1/2$, then $J(\alpha, n)$ is convergent and so is $I(\alpha, n)$. If $\alpha \le 1/2$, then $J(\alpha, n) = \sum_{k=1}^\infty J_k(\alpha, n)$, where

$$J_k(\alpha, n) = \int_{k\pi}^{(k+1)\pi} M\left(\frac{\sin^n x}{x^\alpha}\right) dx.$$

For large enough k,

$$J_k(\alpha, n) \ge \frac{1}{3}\int_{k\pi}^{(k+1)\pi} \frac{\sin^{2n} x}{x^{2\alpha}}\, dx \ge \frac{1}{3(k+1)^{2\alpha}\pi^{2\alpha}} \int_0^\pi \frac{\sin^{2n} x}{x^{2\alpha}}\, dx,$$

so the series $\sum_{k=1}^\infty J_k(\alpha, n)$ is divergent. We conclude that when n is odd, the integral $I(\alpha, n)$ is convergent if and only if $\alpha > 1/2$.

Also solved by P. Bracken, Y. Dumont (France), E. A. Herman, T. L. McCoy (Taiwan), A. Stadler (Switzerland), R. Stong, GCHQ Problem Solving Group (U. K.), and the proposers.

Tiling 4-Rowed Rectangles with Dominoes

11187 [2005, 929]. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL.* Find a closed formula for the number of ways to tile a 4 by n rectangle with 1 by 2 dominoes.

Solution I by Northwestern University Math Problem Solving Group. We find and solve a recurrence for the number of tilings. Let f_n be the number of domino-tilings of a $4 \times n$ rectangle. Also let g_n be the number of domino-tilings of a defective $4 \times n$ rectangle missing the top two (or the bottom two) squares in the last column, and let h_n be the number of domino-tilings of a defective $4 \times n$ rectangle missing the top and bottom squares in the last column.

To establish a system of recurrences, consider the ways to cover the nth column of a $4 \times n$ rectangle, with $n \ge 2$. If it uses two vertical dominoes, then there are f_{n-1} completions. If it uses one vertical domino and two adjacent horizontal dominoes (two cases), then there are g_{n-1} completions. If one vertical domino and two nonadjacent horizontal dominoes, then there are h_{n-1} completions. If four horizontal dominoes, then there are f_{n-2} completions. Similar (simpler) case analysis gives recursive expressions for g_n and h_n. For $n \ge 2$, we obtain

$$\begin{aligned} f_n &= f_{n-1} + f_{n-2} + 2g_{n-1} + h_{n-1} \\ g_n &= g_{n-1} + f_{n-1} \\ h_n &= h_{n-2} + f_{n-1}, \end{aligned}$$

with the initial conditions $f(0) = f(1) = g(1) = h(1) = 1$ and $h(0) = 0$. Next, we eliminate g and h by algebraic manipulations (expand f_n and f_{n-2}, and then use $h_{n-1} - h_{n-3} = f_{n-2}$ and $g_{n-3} = g_{n-2} - f_{n-3}$ and $g_{n-1} - g_{n-2} = f_{n-2}$). The resulting recurrence

$$f_n = f_{n-1} + 5f_{n-2} + f_{n-3} - f_{n-4}$$

is valid for $n \ge 4$. Hence we also compute $f(2) = 5$ and $f(3) = 11$ from the original system.

This determines $\langle f \rangle$, but to obtain a "closed formula" we solve the recurrence using standard methods. The characteristic polynomial is

$$p(x) = x^4 - x^3 - 5x^2 - x + 1,$$

with roots

$$x_1 = \tfrac{1}{4}\left(1 - \sqrt{29} - \sqrt{14 - 2\sqrt{29}}\right), \qquad x_2 = \tfrac{1}{4}\left(1 - \sqrt{29} + \sqrt{14 - 2\sqrt{29}}\right),$$
$$x_3 = \tfrac{1}{4}\left(1 + \sqrt{29} - \sqrt{14 + 2\sqrt{29}}\right), \qquad x_4 = \tfrac{1}{4}\left(1 + \sqrt{29} + \sqrt{14 + 2\sqrt{29}}\right).$$

Since the roots are distinct, the sequence has the form $f_n = \sum_{i=1}^{4} a_i x_i^n$. We find the coefficients by evaluating at $n \in \{0, 1, 2, 3\}$ and solving the resulting system of four linear equations. After simplification, the answer is

$$f_n = \frac{1}{\sqrt{29}}(-x_1^{n+1} - x_2^{n+1} + x_3^{n+1} + x_4^{n+1}).$$

Solution II by Christopher Carl Heckman, Arizona State University, Tempe, AZ. We count the ways to tile a $4 \times n$ rectangle with a possibly defective last column. There are r_n ways with no squares missing, a_n ways lacking squares $(1, n)$ and $(4, n)$, t_n ways lacking squares $(2, n)$ and $(3, n)$, and l_n ways lacking squares $(1, n)$ and $(2, n)$. Reasoning as in Solution I, we obtain

$$r_n = r_{n-2} + 2l_{n-1} + a_{n-1} + r_{n-1}, \qquad a_n = r_{n-1} + t_{n-1}$$
$$t_n = a_{n-1}, \qquad l_n = l_{n-1} + r_{n-1}$$

for all $n \geq 2$, with initial conditions $r_0 = r_1 = a_1 = l_1 = 1$ and $t_1 = 0$.

With $\mathbf{x}_n = (r_n\ a_n\ t_n\ l_n\ r_{n-1})^T$, we write the system as the matrix equation

$$\mathbf{x}_n = \begin{pmatrix} r_n \\ a_n \\ t_n \\ l_n \\ r_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{n-1} \\ a_{n-1} \\ t_{n-1} \\ l_{n-1} \\ r_{n-2} \end{pmatrix} = \mathbf{M}\mathbf{x}_{n-1}$$

where $\mathbf{M}$ is the 5×5 matrix of coefficients (with eigenvalues 1 and the four x_j from the first solution). Thus $\mathbf{x}_n = \mathbf{M}^{n-1}\mathbf{x}_1$. The initial conditions give $\mathbf{x}_1$, and multiplying by (1 0 0 0 0) on the left extracts r_n, so we obtain the "closed formula"

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix} \mathbf{M}^{n-1} \begin{pmatrix}1 & 1 & 0 & 1 & 1\end{pmatrix}^T.$$

Solution III by Jerry Minkus, San Francisco, CA. Let m and n be positive integers with mn even. Let $k_{m,n}$ be the number of domino-tilings of an $m \times n$ rectangle. P. W. Kasteleyn proved (see "The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice", *Physica* **27** (1961) 1209–1225) that

$$k_{m,n} = \prod_{j=1}^{\lfloor m/2 \rfloor} \frac{c_j^{n+1} - \hat{c}_j^{n+1}}{2b_j},$$

where

$$b_j = \sqrt{1 + \cos^2 \frac{j\pi}{m+1}}, \qquad c_j = b_j + \cos \frac{j\pi}{m+1}, \qquad \hat{c}_j = -b_j + \cos \frac{j\pi}{m+1}.$$

When $m = 4$,

$$\cos\frac{\pi}{m+1} = \cos\frac{\pi}{5} = \frac{\tau}{2} \qquad \text{and} \qquad \cos\frac{2\pi}{m+1} = \cos\frac{2\pi}{5} = \frac{1}{2\tau} = -\frac{\tilde{\tau}}{2},$$

where $\tau = (1+\sqrt{5})/2$ and $\tilde{\tau} = (1-\sqrt{5})/2$. Thus in this case Kasteleyn's formula gives

$$k_{4,n} = \frac{c_1^{n+1} - \hat{c}_1^{n+1}}{2b_1}\,\frac{c_2^{n+1} - \hat{c}_2^{n+1}}{2b_2}, \qquad \text{where}$$

$$b_1 = \sqrt{1+\frac{\tau^2}{4}} = \frac{1}{2}\sqrt{5+\tau}, \qquad b_2 = \sqrt{1+\frac{\tilde{\tau}^2}{4}} = \frac{1}{2}\sqrt{5+\tilde{\tau}},$$

$$c_1 = \frac{1}{2}\left(\sqrt{5+\tau}+\tau\right), \qquad c_2 = \frac{1}{2}\left(\sqrt{5+\tilde{\tau}}-\tilde{\tau}\right),$$

$$\hat{c}_1 = -\frac{1}{2}\left(\sqrt{5+\tau}-\tau\right), \qquad \hat{c}_2 = -\frac{1}{2}\left(\sqrt{5+\tilde{\tau}}+\tilde{\tau}\right).$$

Since $2b_1 2b_2 = \sqrt{5+\tau}\sqrt{5+\tilde{\tau}} = \sqrt{25+5(\tau+\tilde{\tau})+\tau\tilde{\tau}} = \sqrt{25+5(1)+(-1)} = \sqrt{29}$, the "closed formula" can be rewritten

$$k_{4,n} = \frac{1}{\sqrt{29}}(c_1^{n+1} - \hat{c}_1^{n+1})(c_2^{n+1} - \hat{c}_2^{n+1}).$$

Editorial comment. The $3 \times n$ case appeared as problem E2417 (solution published in May, 1974). Douglas Rogers informed us that the $4 \times n$ problem appeared as early as N. W. Rymer, Project, problems and patience, *Math. Gaz.* **63** (1979) 1–7 (though Kasteleyn's general solution is even earlier). The sequence for the $4 \times n$ problem is A005178 in *The On-Line Encyclopedia of Integer Sequences*.

Some contributors submitted solutions that might not be considered "closed formulas" (such as a summation of n terms involving binomial coefficients). The criteria for closed formulas are far from clear. Are binomial coefficients allowed? Does expressing the generating function as a rational function suffice? (As David Beckwith and others showed, the generating function here is $(1-x^2)/(1-x-5x^2-x^3+x^4)$.) Solution II is expressed as a "closed formula", but it is really just a restatement of the recurrences (any such linear system can be written in matrix form). One can therefore argue that linear recurrence relations alone constitute a closed formula.

From a complexity viewpoint, with the four arithmetic operations as primitive, the number of domino-tilings can be computed in $O(n)$ time from a recurrence or from the formulas given above. Using successive squaring to exponentiate in each formula, the complexity can be reduced to $O(\log n)$. However, computing with irrational numbers (Solutions I and III) leads to round-off problems, and each matrix multiplication in Solution II requires over 100 primitive operations. A liberal interpretation of "closed formula" guides our list of solvers.

Also solved by M. R. Avidon, D. Beckwith, J. C. Binz (Switzerland), R. Chapman (U. K.), M. Cornick and N. Mecholsky, P. P. Dályay (Hungary), R. Ehrenborg, G. F. Feisner, E. A. Herman, O. P. Lossers (Netherlands), K. McInturff, R. Pratt, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), L. Wenstrom, Don West, the GCHQ Problem Solving Group (U. K.), the Microsoft Research Problems Group, the Missouri State University Problem Solving Group, the VMI Problem Solving Group, and the proposer.