and for $k \geq 1$,

$$I_k = \text{Re}\left(\int_0^\infty e^{-(\alpha-2ki)x} x^{m-1}\,dx\right) = \text{Re}\,\frac{\Gamma(m)}{(\alpha-2ki)^m} = \text{Re}\,\frac{\Gamma(m)\cdot(\alpha+2ki)^m}{(\alpha^2+4k^2)^m}.$$

To put this in real form, define $\theta_k = \tan^{-1}(2k/\alpha)$ so we can write $(\alpha + 2ki)^m = (\sqrt{\alpha^2+4k^2})^m[\cos(m\theta_k) + i\sin(m\theta_k)]$. The real part I_k of the integral can be calculated from this, and

$$I = \Gamma(m)\left(\frac{1}{\alpha^m} + 2\sum_{k=1}^{m}\frac{\cos\left(m\tan^{-1}\frac{2k}{\alpha}\right)}{(\alpha^2+4k^2)^{m/2}}\right).$$

Also solved by U. Abel (Germany), K. F. Andersen (Canada), R. Bagby, D. Beckwith, R. Boukharfane (France), K. N. Boyadzhiev, P.Bracken, B. Bradie, M. A. Carlton, R. Chapman (U. K.), H. Chen, D. F. Connon (U. K.), B. E. Davis, J. L. Ekstrom, C. Georghiou (Greece), M. L. Glasser, E. A. Herman, M. Hoffman, B. Karaivanov & T. S. Vassilev (U.S.A & Canada), O. Kouba (Syria), K. D. Lathrop, M. Omarjee (France), P. Perfetti (Italy), C. M. Russell, R. Sargsyan (Armenia), M. A. Shayib & M. Misaghian, A. Stenger, R. Stong, R. Tauraso (Italy), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), M. Vowe (Switzerland), H. Widmer (Switzerland), M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposer.

Growth Rate for Solution

11799 [2014, 739]. *Proposed by Vicenţiu Rădulescu, King Abdulaziz University, Jeddah, Saudi Arabia.* Let a, b, and c be positive.
(**a**) Prove that there is a unique continuously differentiable function f from $[0,\infty)$ into $\mathbb{R}$ such that $f(0) = 0$ and, for all $x \geq 0$,

$$f'(x)\left(1 + a|f(x)|^b\right)^c = 1.$$

(**b**) Find, in terms of a, b, and c, the largest θ such that $f(x) = O(x^\theta)$ as $x \to \infty$.

Solution by Kenneth F. Andersen, Edmonton, AB, Canada. (**a**) If f is a solution, then

$$f'(x)\left(1 + a|f(x)|^b\right)^c = 1 \tag{1}$$

implies $f'(x) > 0$ for all $x \geq 0$. Since $f(0) = 0$, we conclude that f is nonnegative and strictly increasing on $[0,\infty)$. We claim that f is unbounded. Indeed, if $f(x) \leq M$, then (1) shows $f'(x) \geq (1 + aM^b)^{-c} > 0$ for $x \geq 0$ so that

$$f(x) = \int_0^x f'(t)\,dt \geq \frac{x}{(1+aM^b)^c},$$

and thus, $f(x) > M$ for sufficiently large x, a contradiction. Thus, f is a bijection of $[0,\infty)$ onto itself, with a continuously differentiable inverse f^{-1} satisfying $f^{-1}(0) = 0$ and

$$\left(f^{-1}\right)'\left(f(x)\right)\, f'(x) = 1$$

for all $x \geq 0$. Combining this with (1) yields

$$\left(f^{-1}\right)'(y) = \left(1 + ay^b\right)^c$$

for all $y \in [0,\infty)$. Since $(1 + ay^b)^c$ is a continuous function of y, by the fundamental theorem of calculus,

$$f^{-1}(y) = \int_0^y \left(1 + at^b\right)^c dt.$$

Since inverses are unique, this uniquely determines f on $[0,\infty)$.

(**b**) Clearly, there is no maximal value of θ satisfying the stated requirement. If θ satisfies the requirement, then so does $\theta + 1$. We will show that the minimal value of θ satisfying the stated requirement is $(1+bc)^{-1}$. Note that

$$\lim_{y\to\infty} y^{-1-bc}\int_0^y \left(1+at^b\right)^c dt = \lim_{y\to\infty}\int_0^1 \left(y^{-b}+as^b\right)^c ds = \int_0^1 a^c s^{bc}\,ds = \frac{a^c}{bc+1}.$$

Thus,

$$\lim_{x\to\infty} x^{-\theta} f(x) = \lim_{y\to\infty}\left(\int_0^y \left(1+at^b\right)^c dt\right)^{-\theta} y$$

is finite if and only if $\theta \geq (1+bc)^{-1}$.

Editorial comment. In "simplifying" the statement, the editors mistakenly wrote "largest" instead of "smallest."

Also solved by R. Bagby, R. Chapman (U. K.), J.-P. Grivaux (France), O. Kouba (Syria), J. H. Lindsey II, I. Pinelis, A. Stenger, R. Stong, M. L. Treuden, E. I. Verriest, GCHQ Problem Solving Group (U. K.), and the proposer.

Arithmetic Minus Geometric Means

11800 [2014, 739]. *Proposed by Oleksiy Klurman, University of Montreal, Montreal, Canada.* Let f be a continuous function from $[0,1]$ into $\mathbb{R}^+$. Prove that

$$\int_0^1 f(x)\,dx - \exp\left[\int_0^1 \log f(x)\,dx\right] \leq \max_{0\leq x,y\leq 1}\left(\sqrt{f(x)}-\sqrt{f(y)}\right)^2.$$

Composite Solution by Rafik Sargsyan, Yerevan State University, Yerevan, Armenia, and Kenneth Schilling, Mathematics Department, University of Michigan–Flint, Flint, MI. We will prove a stronger result. Recall that the arithmetic, geometric, and harmonic means A, G, and H of f on $[0,1]$ satisfy

$$A = \int_0^1 f(x)\,dx \geq G = \exp\left[\int_0^1 \log f(x)\,dx\right] \geq H = \left[\int_0^1 \frac{dx}{f(x)}\right]^{-1}.$$

Let M and m be the maximum and minimum values of f, respectively. We show that $A - H \leq \left(\sqrt{M}-\sqrt{m}\right)^2$, which is stronger than the requested inequality $A - G \leq \left(\sqrt{M}-\sqrt{m}\right)^2$.

For $x \in [0,1]$, define $s(x)$ by the relation

$$f(x) = s(x)\cdot m + (1-s(x))\cdot M$$

and let $t = \int_0^1 s(x)\,dx$. We have

$$\int_0^1 f(x)\,dx = tm + (1-t)M.$$

By convexity, $\frac{1}{f(x)} \leq \frac{s(x)}{m} + \frac{1-s(x)}{M}$, so

$$\int_0^1 \frac{dx}{f(x)} \leq \frac{t}{m} + \frac{1-t}{M} = \frac{tM+(1-t)m}{mM}.$$