From this we conclude that $|b_j - b_i| \geq |j - i|\, M_n$ for $1 \leq i,\, j \leq n$. Therefore

$$\begin{aligned} M_n^2 \sum_{1\leq i,j\leq n} (j-i)^2 &\leq \sum_{1\leq i,j\leq n} (b_j - b_i)^2 = \sum_{1\leq i,j\leq n} (a_j - a_i)^2 \\ &= \sum_{1\leq i,j\leq n} (a_j^2 + a_i^2 - 2a_i a_j) \\ &\leq 2n \sum_{k=1}^{n} a_k^2 - 2\left(\sum_{k=1}^{n} a_k\right)^2 \leq 2n, \end{aligned}$$

since $\sum_{k=1}^n a_k^2 \leq 1$ when $(a_1, \dots, a_n) \in A$. On the other hand,

$$\sum_{1\leq i,j\leq n} (j-i)^2 = 2n \sum_{k=1}^{n} k^2 - 2\left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n^2-1)}{6}.$$

It follows that $M_n^2 \leq 12/(n(n^2-1))$, so $M_n \leq \sqrt{12/(n(n^2-1))}$.

Conversely, if we consider $(a_1^{(0)}, a_2^{(0)}, \dots, a_n^{(0)})$ defined by

$$a_k^{(0)} = \sqrt{\frac{12}{n(n^2-1)}}\left(k - \frac{n+1}{2}\right), \qquad k = 1, 2, \dots, n,$$

then $(a_1^{(0)}, \dots, a_n^{(0)}) \in A$ and

$$\min_{1\leq i<j\leq n} \left|a_i^{(0)} - a_j^{(0)}\right| = \sqrt{\frac{12}{n(n^2-1)}}.$$

Thus $M_n \geq \sqrt{12/(n(n^2-1))}$.

Editorial comment. Marian Tetiva (Romania) notes that a stronger form of this problem appeared as Problem E2032, this MONTHLY **76** (1969) 691–692, proposed by D. S. Mitrinović. See also Problem 3.9.9 in Mitrinović, *Analytic Inequalities* (Springer-Verlag, 1970).

Also solved by A. Alt, R. F. de Andrade, M. R. Avidon, R. Bagby, D. Beckwith, J. Cade, R. Chapman (U.K.), L. Comerford, W. J. Cowieson, P. P. Dályay (Hungary), A. Fielbaum (Chile), D. Fleischman, O. Geupel (Germany), J.-P. Grivaux (France), E. A. Herman, A. Ilić (Serbia), T. Konstantopoulos (U.K.), J. Kuplinsky, J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, D. Ray, K. Schilling, B. Schmuland (Canada), J. Simons (U.K.), R. Stong, M. Tetiva (Romania), E. I. Verriest, GCHQ Problem Solving Group (U.K.), NSA Problems Group, and the proposer.

A Cauchy–Schwarz Puzzle

11458 [2009, 747]. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicenţiu Rădulescu, Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania.* Let $a_1, \dots, a_n$ be nonnegative and let r be a positive integer. Show that

$$\left(\sum_{1\leq i,j\leq n} \frac{i^r j^r a_i a_j}{i+j-1}\right)^2 \leq \sum_{m=1}^{n} m^{r-1} a_m \sum_{1\leq i,j,k\leq n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

Solution by Francisco Vial, student, Pontificia Universidad Católica de Chile, Santiago, Chile. Let $f(x) := \sum_{i=1}^{n} i^r a_i x^{i-1}$, so

$$\int_0^1 f(x)\,dx = \sum_{m=1}^{n} m^{r-1}\,a_m,$$

$$\int_0^1 f^2(x)\,dx = \int_0^1 \left(\sum_{1\le i,j\le n} i^r j^r\, a_i a_j\, x^{i+j-2} \right) dx = \sum_{1\le i,j\le n} \frac{i^r j^r\, a_i a_j}{i+j-1}, \qquad \text{and}$$

$$\int_0^1 f^3(x)\,dx = \int_0^1 \left(\sum_{1\le i,j,k\le n} i^r j^r k^r a_i a_j a_k x^{i+j+k-3} \right) dx = \sum_{1\le i,j,k\le n} \frac{i^r j^r k^r a_i a_j a_k}{i+j+k-2}.$$

The stated inequality is equivalent to

$$\left(\int_0^1 f^2(x)\,dx \right)^2 \le \left(\int_0^1 f(x)\,dx \right) \left(\int_0^1 f^3(x)\,dx \right),$$

which follows by applying the Cauchy–Schwarz inequality to $f(x)^{1/2}$ and $f(x)^{3/2}$.

Remarks. Because $a_1, \ldots, a_n$ are nonnegative, $f(x)$ is nonnegative and continuous on [0, 1], so $f(x)^{1/2}$ and $f(x)^{3/2}$ are real and well defined. The parameter r need not be an integer.

Also solved by M. R. Avidon, R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Simons (U.K.), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposers.

An Orthocenter Inequality

11461 [2009, 844]. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.* Let a, b, and c be the lengths of the sides opposite vertices A, B, and C of an acute triangle. Let H be the orthocenter. Let d_a be the distance from H to side BC, and similarly for d_b and d_c. Show that

$$\frac{1}{d_a + d_b + d_c} \ge \frac{2}{3} \left(\frac{3}{abc} \left(\frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} \right) \right)^{1/4}.$$

Solution by Michael Vowe, Fachhochschule Nordwestschweiz, Muttenz, Switzerland. Let R be the circumradius, r the inradius, F the area, and s the semiperimeter. From $d_a = 2R \cos B \cos C$, $d_b = 2R \cos C \cos A$, $d_c = 2R \cos A \cos B$, we obtain

$$d_a + d_b + d_c = 2R\big(\cos A \cos B + \cos B \cos C + \cos C \cos A\big) \le 2r \left(1 + \frac{r}{R}\right)$$

(see 6.10, p. 181, in D. Mitrinovic et al., *Recent Advances in Geometric Inequalities,* Dordrecht, 1989). From Jensen's inequality for concave functions (here, the square root), we have

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le 3 \cdot \sqrt{\frac{1}{3} \left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right)} = \sqrt{\frac{6s}{abc}}.$$