



Schrödinger-Bopp-Podolsky Systems with Double Criticality

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Abstract

We are concerned with the bifurcation analysis of the following Schrödinger-Bopp-Podolsky system with double criticality:

$$\begin{cases} -\Delta u + u + \phi|u|^3u = \lambda h(x)u + |u|^4u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi |u|^5, & \text{in } \mathbb{R}^3, \end{cases}$$

where $h \geq 0$ is a weight function and $a > 0$ is the Bopp-Podolsky constant. The value of the real parameter λ associated with the subcritical term can determine whether a ground state solution exists. A feature of the analysis developed in this paper is given by the combined effects of the critical nonlocal term and the critical reaction term. The main result establishes a bifurcation alternative with respect to the eigenvalues of a related linear eigenvalue problem on the entire space.

Keywords Schrödinger-Bopp-Podolsky system · Doubly critical growth · Global bifurcation theorem · Weak solutions

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1 Introduction and main results

In this paper, we study the bifurcation and existence of solutions for the following Schrödinger-Bopp-Podolsky system involving doubly critical growth:

$$\begin{cases} -\Delta u + u + \phi|u|^3u = \lambda h(x)u + |u|^4u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi|u|^5, & \text{in } \mathbb{R}^3, \end{cases} \quad (SBP)$$

where $\lambda \in \mathbb{R}$ and $h \geq 0$ is a weight function and satisfies the following conditions:

$$(\mathcal{H}_1) \quad h \in L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \quad h \geq 0 \text{ a.e. in } \mathbb{R}^3 \text{ and } h \not\equiv 0.$$

Schrödinger-Bopp-Podolsky systems are first proposed by D'Avenia and Siciliano in [9]. Such systems can be obtained when we couple the Schrödinger field $\psi = \psi(t, x)$ with its electromagnetic field in the Bopp-Podolsky electromagnetic theory, especially in the electrostatic case of the standing wave $\psi(t, x) = e^{i\omega t}u(x)$. It is worth noting that the Bopp-Podolsky theory introduced by Bopp [2] and Podolsky [27] is a second-order gauge theory of the electromagnetic field, which can be regarded as a theory valid in the short distance range (see Frenkel [13]). For larger distances, it is experimentally indistinguishable from Maxwell's theory. Therefore, the Bopp-Podolsky parameter $a > 0$ (whose dimension is the inverse of mass) can be interpreted as a cut-off distance or associated with a effective radius of the electron. Moreover, this theory solves the so called infinity problem that appears in the classical Maxwell's theory. Studying these systems with double criticality, is therefore a way to understand a more fundamental theory of electromagnetism interacting with matter. For more detailed physical content, please refer to [1, 4, 5, 26].

D'Avenia and Siciliano [9] made the first contribution to considering the existence and nonexistence of solutions for Schrödinger-Bopp-Podolsky systems:

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $a > 0$ is the Bopp-Podolsky parameter, $\omega > 0$ is known as the frequency of the standing wave and is fixed. They obtained that the existence and nonexistence of solutions for Eq. (1.1) strongly depend on the ranges of p and q . More precisely, in the cases $p \in (2, 6)$ and $q \in (-q_*, q_*)$, and $p \in (3, 6)$ and $q \neq 0$, they obtained that Eq. (1.1) admits a non-trivial solution. In addition, they also proved the solutions of Eq. (1.1) they found are actually also the solutions of Schrödinger-Poisson systems as $a \rightarrow 0$. Finally, they verified the non-existence of solutions for Eq. (1.1) as $p \geq 6$ and $p \leq 2$. In [38], Siciliano and Silva were interested in Eq. (1.1), where $p \in (2, 6)$, $\omega > 0$ and $a > 0$. They obtained the desired results of Eq. (1.1) depending on the value of q . In fact, by the fibering approach, they obtained the nonexistence of solutions for Eq. (1.1) as q sufficiently large and possesses two radial solutions as small q . They also showed some qualitative properties of the energy level for solutions and a variational characterization of these extremal values of q .

Once we turn our attention to the relevant results of critical Schrödinger-Bopp-Podolsky systems, there are some scholars focusing on this topic. We only list some

representative and interesting results. Chen and Tang [7] considered the following critical Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \mu f(u) + u^5 & \text{in } \mathbb{R}^3, \\ -2\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.2)$$

where $a > 0$, $V \in C(\mathbb{R}^3, [0, \infty))$ with $V_\infty := \lim_{|y| \rightarrow \infty} V(y) \geq \sup_{x \in \mathbb{R}^3} V(x) > 0$ and the nonlinearity f satisfies subcritical growth. By Monotonicity trick, Mountain pass theorem and Splitting lemma, they obtained the existence of ground state solutions in the cases of constant potential and non-constant potential. In [16], Lin et al. also proved the existence of ground states solutions for Eq. (1.2) with $f(u) = \mu|u|^{p-2}u$ by Pohozaev-Nehari manifold, the arguments of Brézis-Nirenberg, Monotonicity trick and Global compactness lemma. By the means of a perturbation approach and the method of invariant sets of descending flow incorporated with minimax arguments, Wang et al. [40] explored the existence and multiplicity of sign-changing solutions for Eq. (1.2) with the potential $V(x) = 1$. In [35], Santos Damian and Siciliano were interested in the following critical Schrödinger-Bopp-Podolsky system:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + Q(x)\phi u = h(x, u) + K(x)|u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi Q(x)\phi u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.3)$$

where $\varepsilon, a > 0$ and the nonlinearity h satisfies subcritical growth. Under appropriate assumptions on the potentials $V(x)$, $Q(x)$ and $K(x)$, they proved the existence of mountain-pass solutions for Eq. (1.3) as $\varepsilon > 0$ sufficiently small. They also got some interesting vanishing rates of these solutions as $\varepsilon \rightarrow 0$. In addition, for a fixed small ε , they proved solutions of Eq. (1.3) strongly converges to those of Schrödinger-Poisson systems as $a \rightarrow 0$. For more results about Schrödinger-Poisson systems, please see [21, 24, 25, 36, 37, 42, 45].

Very recently, many researchers focus on the study of normalized solutions for Schrödinger-Bopp-Podolsky systems. In [41], Wei et al. considered the following Schrödinger-Bopp-Podolsky with doubly critical growth:

$$\begin{cases} -\Delta u - \phi|u|^3u = \lambda u + \mu|u|^{q-2}u + |u|^4u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi\phi|u|^5 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.4)$$

on the constraint $\int_{\mathbb{R}^3} |u|^2 dx = c^2$, where $a, \mu > 0$, $\lambda \in \mathbb{R}$ is the Lagrange multiplier and $q \in (2, 6)$. By variational methods, they obtained the existence and multiplicity of normalized solutions for Eq. (1.4) in the cases of L^2 -mass subcritical growth and L^2 -mass supercritical growth. In fact, in the L^2 -mass subcritical case, they proved the existence and multiplicity of normalized solutions for the above equations by truncation technique and genus theory. In the L^2 -mass supercritical case, they obtained a couple of normalized solutions by establishing a auxiliary functional. Together with concentration-compactness principle, they overcome the difficulty of lack of compactness encountered in Eq. (1.4) due to the appearance of critical nonlinearity. For some

interesting and meaningful results of Schrödinger-Bopp-Podolsky systems involving Sobolev subcritical or Sobolev critical growth, please refer to [8, 14, 15, 18, 19].

As far as we know, the above mentioned results are all obtained by variational methods to study Schrödinger-Bopp-Podolsky systems, and few studies use bifurcation theory to obtain the existence of solutions. In this paper, bifurcation studies and topological degree analyses are regularly employed to prove the existence of multiple solutions for nonlinear Laplace equations, which play crucial roles in mathematical research. Some of bifurcation theories for nonlinear eigenvalue problems commonly used here can be traced back to the relevant results of Rabinowitz [33, 34]. Recently, we note that Pucci et al. [31] considered the bifurcation and existence of solutions for doubly critical Schrödinger-Poisson systems:

$$\begin{cases} -\Delta u + u + \phi u = \lambda g(x)u + (I_\alpha * |u|^{2_\alpha^\sharp})|u|^{2_\alpha^\sharp-2}u + (I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u & \text{in } \mathbb{R}^N, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.5)$$

where $N \in [3, 6]$, $\alpha \in (0, N)$, $\lambda \in \mathbb{R}$ and $2_\alpha^\sharp$ and 2_α^* are the lower and upper critical exponents in the sense of Hardy-Littlewood-Sobolev inequality, respectively. By global bifurcation theorem [33], they obtained the existence and bifurcation of solutions for Eq. (1.5). Different from [31], Liang et al. [20] also considered the existence and bifurcation of solutions for Schrödinger-Poisson systems with doubly critical growth. However, the nonlocal terms in Eq. (1.5) satisfies the critical growth, which is the innovation of [20]. Pucci et al. [32] considered bifurcation results of (weak) solutions for Schrödinger-Poisson systems when $f(u)$ satisfies sub-critical and critical growth, using the global bifurcation theorem. In addition, they also proved the existence of unbounded components of (weak) solutions, which bifurcate from the trivial solution and infinity, respectively. The feature of [32] lies in the appearance of indefinite and critical nonlinearity. Finally, they also analyzed the regularity result, which guarantees the L^∞ -boundedness of (weak) solutions for Schrödinger-Poisson systems. For more application of Global bifurcation theorem, please refer to [17, 28, 29, 44]. On the other hand, as the Bopp-Podolsky constant $a \rightarrow 0$, Schrödinger-Bopp-Podolsky systems can be reduced to Schrödinger-Poisson systems. Please refer to [11, 23] for more results. It is quite natural to wonder whether there exist the existence and bifurcation of solutions for Schrödinger-Bopp-Podolsky systems involving doubly critical growth, i.e., the second equation in (SBP) contains $|u|^5$ satisfying critical growth. In this paper, we intend to give affirmative answers to these questions.

Now, we are ready to state the main results in this paper.

Theorem 1.1 (Bifurcation of solutions) *Assume that $(\mathcal{H}_1)$ holds, then (SBP) admits a bifurcation point $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of*

$$-\Delta u + u = \lambda h(x)u \text{ in } H^1(\mathbb{R}^3). \quad (\mathcal{P})$$

Furthermore, there is a component $\mathcal{C}$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ of nontrivial (weak) solutions for (SBP), such that $\bar{\mathcal{C}}$ contains $(\lambda_1, 0)$ and either $\mathcal{C}$ is unbounded, or contains a pair $(\lambda, 0)$, where λ is an eigenvalue of $(\mathcal{P})$ with $\lambda > \lambda_1$ and $\bar{\mathcal{C}}$ is the closure of $\mathcal{C}$.

Theorem 1.2 (Existence of solutions) *Let $(\mathcal{H}_1)$ be satisfied. Then (SBP) has at least one nontrivial (weak) solution.*

Theorem 1.3 (Asymptotic behavior of solutions) *Let $\{u_*^a\}_{a>0}$ be a family of weak solutions of (SBP) obtained in Theorem 1.2. Then, up to a subsequence,*

$$\lim_{a \rightarrow 0} u_*^a = u_*^0 \text{ in } H^1(\mathbb{R}^3),$$

where u_*^0 solves the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + \phi|u|^3 u = \lambda h(x)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi|u|^5, & \text{in } \mathbb{R}^3. \end{cases} \quad (SP)$$

Remark 1.1 Compared with the existing literature, the research of (SBP) is more interesting and challenging due to the combined effects of the appearance of $|u|^5$ in the second equation of (SBP) and the critical nonlinearity. In order to make clearer our contributions compared with the previous results, we emphasize three aspects.

- (a) It seems that the results of this paper are the first to consider the doubly critical Schrödinger-Bopp-Podolsky systems using global bifurcation theory [33]. Indeed, we use an useful technique to overcome the obstacle caused by (SBP) satisfying doubly critical growth. We first show the vanishing behavior of the nonlinearity for (SBP) at zero to establish the bifurcated result, which is controlled by the norm in $H^1(\mathbb{R}^3)$. Together with the variational characterization of the first eigenvalue for (P) , we obtain the homotopy invariance of the topological degree. Furthermore, we prove that global bifurcation theorem is satisfied.
- (b) Compared with [9], Schrödinger-Bopp-Podolsky systems in [9] merely contain subcritical nonlinearity, and even does not satisfy the critical growth in the non-local term. In this paper, (SBP) contains both $|u|^5$ in the second equation of (SBP) and the critical nonlinearity. Therefore, in terms of the form of the equations, the equations we consider are completely new. In addition, the authors in [9] considered the existence and non-existence of solutions for Schrödinger-Bopp-Podolsky systems with subcritical growth via truncation arguments and Mountain pass theorem. However, we take advantage of Global bifurcation theorem [33] to obtain the bifurcation and existence of solutions for (SBP) . Therefore, we adopt completely different methods from that in [9].
- (c) In fact, Schrödinger-Bopp-Podolsky systems can be reduced Schrödinger-Poisson systems as $a \rightarrow 0$. In [20], Liang et al. obtained the bifurcation and existence of solutions for Schrödinger-Poisson systems involving doubly critical growth. Theorems 1.1, 1.2 extend and supplement the results in [20] to some extent. On the other hand, Wei et al. [41] considered the existence and multiplicity of normalized solutions for Schrödinger-Bopp-Podolsky systems involving doubly critical nonlinearity. Due to the nature of normalized solutions, the methods in [41] cannot be applied in this paper. In the existing literature, scholars generally use variational methods to consider the existence of multiple solutions for Schrödinger-Bopp-Podolsky systems. To the best of our knowledge, there is no result in using global

bifurcation theorem [33] to consider the bifurcation and existence of solutions for (SBP). Therefore, our results are completely new.

The framework of this paper is as follows. In Sect. 2, we give some preliminary knowledge. In Sect. 3, we are devoted to showing the proof of Theorem 1.1. In Sect. 4, we prove that Theorems 1.2, 1.3 are satisfied.

2 Preliminaries

In this section, we recall some preliminary results that will be useful for our arguments. Let $\|\cdot\|_\kappa$ denotes $L^\kappa(\mathbb{R}^3)$ -norm, where $\kappa \in [1, \infty)$. Let $\mathcal{D}^{1,2}(\mathbb{R}^3)$ be the completion of $C_c^\infty(\mathbb{R}^3)$, endowed with the following norm

$$\|\nabla u\|_2^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Especially, we define the best Sobolev constant by

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3), u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}. \quad (2.1)$$

We also define that the Hilbert space $H^1(\mathbb{R}^3)$ equip with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx$$

for any $u \in H^1(\mathbb{R}^3)$. We denote its dual space by $H^{-1}(\mathbb{R}^3)$. Moreover, the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^\vartheta(\mathbb{R}^3)$ is continuous for all $\vartheta \in [2, 6]$, and the embedding $H^1(\mathbb{R}^3) \hookrightarrow L_{loc}^\vartheta(\mathbb{R}^3)$ is compact for all $\vartheta \in [2, 6)$.

In addition, we define that the Hilbert space $\mathcal{D}$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the following norm

$$\|u\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx + a^2 \int_{\mathbb{R}^3} |\Delta u|^2 dx.$$

Then there exists the continuous embeddings $\mathcal{D} \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and the space $\mathcal{D}$ is continuously embedded into $L^\infty(\mathbb{R}^3)$, see [9, Lemma 3.1].

For each fixed $u \in H^1(\mathbb{R}^3)$, the second equation in (SBP) has a unique solution ϕ_u^a . To explicitly write such a solution (see [27, Formula (2.6)]), we consider

$$\mathcal{K}(x) = \frac{1 - e^{-\frac{|x|}{a}}}{|x|}$$

and recall the following properties of the the kernel $\mathcal{K}$.

Proposition 2.1 [9, Lemma 3.3] *For $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y.$$

Moreover,

- (i) if $f \in L^1_{loc}(\mathbb{R}^3)$ and, for a.e. $x \in \mathbb{R}^3$, the map $y \mapsto \frac{f(y)}{|x-y|}$ is summable, then $\mathcal{K} * f \in L^1_{loc}(\mathbb{R}^3)$;
- (ii) if $f \in L^p(\mathbb{R}^3)$ with $p \in [1, \frac{3}{2})$, then $\mathcal{K} * f \in L^q$ for all $q \in (\frac{3p}{3-2p}, +\infty]$.

In both cases $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi f. \quad (2.2)$$

By the above results, we fix $u \in H^1(\mathbb{R}^3)$, the unique solution in $\mathcal{D}$ of the second equation in (SBP) has the explicit form

$$\phi_u^a = \mathcal{K} * |u|^5 = \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} |u(y)|^5 dy. \quad (2.3)$$

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [22]) *Given $s, r > 1$ with $\frac{1}{s} + \frac{1}{r} + \frac{1}{3} = 2$, there exists a constant $H_s > 0$ such that for $f \in L^s(\mathbb{R}^3)$, $g \in L^r(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3} \left(\frac{1}{|\cdot|} * f \right)(x) g(x) dx \leq H_s \|f\|_s \|g\|_r.$$

In the following, we recall some nice properties of ϕ_u^a .

Lemma 2.2 [41, Lemma 2.5] *For every $u \in H^1(\mathbb{R}^3)$, we have the following properties.*

- (i) For every $y \in \mathbb{R}^3$, $\phi_{u(+y)}^a = \phi_u^a(\cdot + y)$.
- (ii) $\phi_u^a \geq 0$ for all $u \in H^1(\mathbb{R}^3)$.
- (iii) For each $t > 0$, $\phi_{tu}^a = t^5 \phi_u^a$.
- (iv) $\|\phi_u^a\|_{\mathcal{D}} \leq S^{-\frac{1}{2}} \|u\|^5$ and $\int_{\mathbb{R}^3} \phi_u^a |u|^3 \varphi dx \leq S^{-1} \|u\|^9 \|\varphi\|$ for all $\varphi \in H^1(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \phi_u^a |u|^5 dx \leq S^{-5} \|u\|_6^{10}.$$

- (v) ϕ_u^a is the unique minimizer of the functional

$$E(\phi) = \frac{1}{2} \|\nabla\phi\|^2 + \frac{a^2}{2} \|\Delta\phi\|_2^2 - \int_{\mathbb{R}^3} \phi |u|^5 dx \quad \text{for } \phi \in \mathcal{D}.$$

- (vi) If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$, and

$$\int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n \varphi dx \rightarrow \int_{\mathbb{R}^3} \phi_u^a |u|^3 u \varphi dx \quad \text{for each } \varphi \in H^1(\mathbb{R}^3).$$

In order to study (SBP) involving doubly critical growth, we apply the reduction arguments introduced in D'Avenia and Siciliano [9], and by the well-known Lax-Milgram theorem, we know that (SBP) can be reduced single equation of the form:

$$-\Delta u + u + \phi_u^a |u|^3 u = \lambda g(x)u + |u|^4 u \text{ in } \mathbb{R}^3, \quad (SBP_*)$$

where ϕ_u^a will be given in (2.3). This implies that the solutions of (SBP_*) are those of (SBP) .

In order to obtain the main results in this paper, we give the definitions of (weak) solutions and bifurcations for (SBP_*) . For more details, please refer to [34].

Definition 2.1 Let $\mathcal{T}, \mathcal{E}, \mathcal{H}, \mathcal{F} : H^1(\mathbb{R}^3) \rightarrow H^{-1}(\mathbb{R}^3)$, be defined pointwise for any $u, \varphi \in H^1(\mathbb{R}^3)$ by

$$\begin{aligned} \langle \mathcal{T}(u), \varphi \rangle &= \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u \varphi) dx, & \langle \mathcal{E}(u), \varphi \rangle &= \int_{\mathbb{R}^3} \phi_u^a |u|^3 u \varphi dx, \\ \langle \mathcal{H}(u), \varphi \rangle &= \int_{\mathbb{R}^3} h(x) u \varphi dx, & \langle \mathcal{F}(u), \varphi \rangle &= \int_{\mathbb{R}^3} |u|^4 u \varphi dx. \end{aligned}$$

Definition 2.2 A function $u \in H^1(\mathbb{R}^3)$ is called a (weak) solution of (SBP_*) if

$$\mathcal{T}(u) + \mathcal{E}(u) - \lambda \mathcal{H}(u) - \mathcal{F}(u) = 0 \text{ in } H^{-1}(\mathbb{R}^3). \quad (2.4)$$

Moreover, if there exist $\{\lambda_n\} \subset \mathbb{R}$ and a sequence of nontrivial solutions $\{u_n\} \subset H^1(\mathbb{R}^3)$ of (SBP_*) , such that $(\lambda_n, u_n) \rightarrow (\lambda, 0)$ as $n \rightarrow \infty$. Therefore, $(\lambda, 0)$ is called a bifurcation point of (SBP_*) . Moreover, if

$$\mathcal{C} = \{(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^3) : u \neq 0 \text{ and } (\lambda, 0) \text{ solves } (SBP_*)\}$$

is a connected set in $\mathbb{R} \times H^1(\mathbb{R}^3)$, then $\mathcal{C}$ is called a component of nontrivial solutions for (SBP_*) .

It follows from [43, Lemma 2.13] that the following fact, please refer to [6] for more information.

Lemma 2.3 Let $(\mathcal{H}_1)$ be satisfied, then the functionals $u \rightarrow \int_{\mathbb{R}^3} h(x) u^2 dx$ and $u \mapsto \int_{\mathbb{R}^3} h(x) u v dx$ are weakly continuous in $H^1(\mathbb{R}^3)$ for any $u, v \in H^1(\mathbb{R}^3)$. Furthermore, by the spectral theory of compact symmetric operators on Hilbert spaces, it can be shown that the eigenvalue of (SBP_*) admits a sequence of eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots$. In particular, it is worthy that the first eigenvalue, denoted as $\lambda_1 > 0$, is simple and isolated, the first eigenfunction $0 < \tilde{u}_1 \in H^1(\mathbb{R}^3)$. Finally,

$$\lambda_1 = \inf_{v \in H^1(\mathbb{R}^3), v \neq 0} \frac{\|v\|^2}{\int_{\mathbb{R}^3} h(x) |v|^2 dx}.$$

Now we prove the properties of the operators $\mathcal{T}, \mathcal{E}, \mathcal{H}$ and $\mathcal{F}$, given in Definition 2.1.

Lemma 2.4 *The continuity of $\mathcal{T}$ is satisfied in $H^1(\mathbb{R}^3)$, while $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{F}$ are compact operators in $H^1(\mathbb{R}^3)$. Moreover, the operator $\mathcal{F}$ satisfies*

$$\lim_{\|u\| \rightarrow 0} \frac{\|\mathcal{F}(u)\|_{H^{-1}(\mathbb{R}^3)}}{\|u\|} = 0. \quad (2.5)$$

Proof We shall verify that the following five claims are satisfied.

Claim 1. We claim that $\mathcal{T}$ is the continuous operator in $H^1(\mathbb{R}^3)$.

Let $\{\varpi_n\}$ be a sequence and $\varpi_n \rightarrow \varpi$ in $H^1(\mathbb{R}^3)$. Therefore, for all $\mathcal{S} \in H^1(\mathbb{R}^3)$, we obtain that

$$\begin{aligned} & \sup_{\|\mathcal{S}\| \leq 1} \langle \mathcal{T}(\varpi_n) - \mathcal{T}(\varpi), \mathcal{S} \rangle \\ &= \sup_{\|\mathcal{S}\| \leq 1} \int_{\mathbb{R}^3} (\nabla \varpi_n \nabla \mathcal{S} - \nabla \varpi \nabla \mathcal{S} + \varpi_n \mathcal{S} - \varpi \mathcal{S}) dx \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, we obtain that $\mathcal{T}$ is a continuous operator.

Claim 2. We claim that $\mathcal{E}$ is the compact operator in $H^1(\mathbb{R}^3)$.

Let $\{v_n\}$ be a bounded sequence in $H^1(\mathbb{R}^3)$. By reflexivity of $H^1(\mathbb{R}^3)$ and the continuous embedding $H^1(\mathbb{R}^3) \hookrightarrow L^\vartheta(\mathbb{R}^3)$ for all $\vartheta \in [2, 6]$, then there exists $v \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$(a) v_n \rightarrow v \text{ a.e. in } \mathbb{R}^3, (b) v_n \rightharpoonup v \text{ in } L^\vartheta(\mathbb{R}^3) \text{ for all } \vartheta \in [2, 6], (c) v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^3), \quad (2.6)$$

as $n \rightarrow \infty$. Furthermore, by (2.6)-(a), (c) and Lemma 2.2-(vi), we obtain

$$\sup_{\|\mathcal{S}\| \leq 1} \int_{\mathbb{R}^3} (\phi_{v_n}^a |v_n|^3 v_n - \phi_v^a |v|^3 v) \mathcal{S} dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This yields the compactness of $\mathcal{E}$ in $H^1(\mathbb{R}^3)$.

Claim 3. We claim that the operator $\mathcal{H}$ is compact in $H^1(\mathbb{R}^3)$.

Since $h \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $\mathcal{S} \in H^1(\mathbb{R}^3)$, by (2.1), Hölder inequality and $(\mathcal{H}_1)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |h(x) \mathcal{S}|^{\frac{6}{5}} dx &\leq \left(\int_{\mathbb{R}^3} |h(x)|^{\frac{3}{2}} dx \right)^{\frac{4}{5}} \left(\int_{\mathbb{R}^3} |\mathcal{S}|^6 dx \right)^{\frac{1}{5}} \\ &\leq S^{-3/5} \|h\|_{3/2}^{6/5} \|\mathcal{S}\|^{6/5} < \infty. \end{aligned}$$

Therefore, $h\mathcal{S} \in L^{\frac{6}{5}}(\mathbb{R}^3)$. Together with (2.6), we have

$$\sup_{\|\mathcal{S}\| \leq 1} \int_{\mathbb{R}^3} h(x)(v_n - v) \mathcal{S} dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

for all $v_n \in H^1(\mathbb{R}^3)$. This implies that Claim 3 is true.

Claim 4. We prove that the operator $\mathcal{F}$ is compact in $H^1(\mathbb{R}^3)$.

By (2.6)-(a), we obtain

$$|v_n|^3 v_n \mathcal{S} \rightarrow |v|^3 v \mathcal{S} \text{ a.e. in } \mathbb{R}^3.$$

Using Hölder inequality, the boundedness of $\{v_n\}$ in $H^1(\mathbb{R}^3)$, then there exist a constant $\tilde{C} > 0$ such that

$$\int_{\mathbb{R}^3} |v_n|^3 v_n \mathcal{S} dx \leq \left(\int_{\mathbb{R}^3} |v_n|^{\frac{24}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |\mathcal{S}|^6 dx \right)^{\frac{1}{6}} \leq S^{-\frac{1}{2}} \|v_n\|^4 \|\mathcal{S}\| \leq \tilde{C}.$$

Hence, by Lebesgue dominated convergence theorem, we obtain

$$\sup_{\|\mathcal{S}\| \leq 1} \int_{\mathbb{R}^3} (|v_n|^3 v_n - |v|^3 v) \mathcal{S} dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, we obtain the compactness of $\mathcal{F}$.

Claim 5. We verify that (2.5) is true.

Fix $\{u_n\} \subset H^1(\mathbb{R}^3) \setminus \{0\}$, with $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$. Taking $\hat{u}_n = \frac{u_n}{\|u_n\|}$, by (2.1) and Hölder inequality, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\langle \mathcal{F}(u_n), \mathcal{S} \rangle}{\|u_n\|} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{\|u_n\|} \int_{\mathbb{R}^3} |u_n|^3 u_n \mathcal{S} dx \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{\|u_n\|} \left(\int_{\mathbb{R}^3} |u_n|^{\frac{24}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |\mathcal{S}|^6 dx \right)^{\frac{1}{6}} \\ &\leq S^{-\frac{1}{2}} \lim_{n \rightarrow \infty} \|u_n\|^4 \|\mathcal{S}\| \\ &\leq S^{-\frac{1}{2}} \lim_{n \rightarrow \infty} \|u_n\|^4 = 0 \end{aligned} \quad (2.7)$$

for any $\mathcal{S} \in H^1(\mathbb{R}^3)$, with $\|\mathcal{S}\| \leq 1$. Therefore, we obtain that (2.5) is true and complete the proof of Lemma 2.4. $\square$

3 Bifurcation Analysis

In this section, we recall some useful properties of the degree theory for operators from $\mathbb{X}$, which is a separable Hilbert space, into its dual space $\mathbb{X}^*$. They play crucial roles in obtaining bifurcation results.

Let $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}^*$. We call that $\mathcal{Q}$ is bounded, if $\mathcal{Q}$ takes any bounded set of $\mathbb{X}$ into a bounded set of $\mathbb{X}^*$. The operator $\mathcal{Q}$ is called demi-continuous in $u \in \mathbb{X}$ if any sequence $\{u_n\} \subset \mathbb{X}$, with $u_n \rightarrow u$ in $\mathbb{X}$, is such that $u_n \xrightarrow{*} u$. The operator $\mathcal{Q}$ satisfies

condition $(S)_+$ if any sequence $\{u_n\} \subset \mathbb{X}$ with $u_n \rightarrow u$ in $\mathbb{X}$ and

$$\limsup_{n \rightarrow \infty} \langle \mathcal{Q}(u_n), u_n - u \rangle \leq 0$$

such that $u_n \rightarrow u$ in $\mathbb{X}$.

Let $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}^*$ be a fixed operator, it is bounded, demi-continuous and satisfies condition $(S)_+$. Let $E \subset \mathbb{X}$ be a bounded open set such that $\mathcal{Q}(u) \neq 0$ for any $u \in \partial E$. Then there exists a unique integer $\deg(\mathcal{Q}, E, 0)$, called the topological degree of $\mathcal{Q}$ on the set $\overline{E}$ with respect to the origin of $\mathbb{X}^*$, satisfying the four properties for every fixed bounded open set E , i.e., existence, additivity, homotopy invariance and normalization (see [31, p. 9]). These properties have significant similarities with the properties of the Leray-Schauder degree, see [12, 39]. For the definition of a isolated zero of $\mathcal{Q}$, please refer to [31, p. 9].

Remark 3.1 Note that any continuous operator $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}^*$ is also demi-continuous. Additionally, if $\mathcal{Q}$ satisfies the condition $(S)_+$ and the operator $\mathcal{U} : \mathcal{Q} \rightarrow \mathcal{Q}^*$ is compact, then $\mathcal{Q} + \mathcal{U}$ also satisfies the condition $(S)_+$.

Now we introduce some useful facts in a separable real Hilbert space $\mathbb{X}$.

Lemma 3.1 (Theorem 3.1 of Chapter 1 and Theorem 6.1 of Chapter 2 in [39]) *Let $\mathbb{X}$ be a real Hilbert space and $\Psi : \mathbb{X} \rightarrow \mathbb{R}$ be a coercive weakly lower semi-continuous functional. Then Ψ admits a local minimum point.*

Let $\Psi : \mathbb{X} \rightarrow \mathbb{R}$ be of class $C^1(\mathbb{X})$ and let $\mathcal{Q} = \Psi'$. If $\mathcal{Q}$ satisfies the condition $(S)_+$ and u_0 is a local minimum point of Ψ and an isolated critical point of Ψ , that is an isolated zero point of $\mathcal{Q} = \Psi'$, then

$$\text{ind}(\mathcal{Q}, u_0) = 1.$$

Lemma 3.2 (Theorem 4.4 of Chapter 2 in [39]) *Let $\mathbb{X}$ be a real Hilbert space. If the operator $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}^*$ satisfies the condition $(S)_+$ and $\langle \mathcal{Q}(u), u \rangle > 0$ for all $u \in \mathbb{X}$, with $\|u\| = r > 0$, then*

$$\deg(\mathcal{Q}, B_r(0), 0) = 1.$$

Lemma 3.3 (Theorem 5.1 of Chapter 2 in [39]) *Suppose that $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}^*$ has only isolated zero points in $\overline{E} \subset \mathbb{X}$ and $\mathcal{Q}u \neq 0$ for any $u \in \partial E$, where $E \subset \mathbb{X}$ is a bounded open set. Then $\mathcal{Q}$ admits only a finite number K of zero points $u_k, k = 1, 2, \dots, K$, in E and*

$$\deg(\mathcal{Q}, \overline{E}, 0) = \sum_{k=1}^K \text{ind}(\mathcal{Q}, u_k).$$

In what follows, we take $\mathbb{X} = H^1(\mathbb{R}^3)$ and its dual space $\mathbb{X}^* = H^{-1}(\mathbb{R}^3)$. We fix $\lambda \in \mathbb{R}$ and define the operator $\mathcal{Q}_\lambda : H^1(\mathbb{R}^3) \rightarrow H^{-1}(\mathbb{R}^3)$ related to (SBP) by

$$\mathcal{Q}_\lambda = \mathfrak{A}_\lambda(u) + \mathcal{E}(u) - \mathcal{F}(u), \quad (3.1)$$

where $\mathfrak{A}_\lambda(u) = \mathcal{T}(u) - \lambda\mathcal{H}(u)$, $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$ and $\mathcal{T}, \mathcal{E}, \mathcal{H}, \mathcal{F}$ are given in Definition 2.1. Note that $H^1(\mathbb{R}^3)$ is a Hilbert space and the operators $\mathcal{E}, \mathcal{H}, \mathcal{F}$ are compact, see Lemma 2.4. In order to obtain the topological degree of the operator $\mathcal{Q}_\lambda$, together with Remark 3.1, we prove that the following result is satisfied.

Lemma 3.4 *The operator $\mathcal{T} : H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R}^3)$ satisfies the condition $(S)_+$.*

Proof Let $\{u_n\} \subset H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ such that

$$\limsup_{n \rightarrow \infty} \langle \mathcal{T}(u_n), u_n - u \rangle \leq 0.$$

Then $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ as $n \rightarrow \infty$. Moreover, since $\|\nabla u\|_2$ and $\|u\|_2$ are weak semi-continuous, we obtain

$$\begin{aligned} 0 &\geq \limsup_{n \rightarrow \infty} \langle \mathcal{T}(u_n), u_n - u \rangle = \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (\nabla u_n (\nabla u_n - \nabla u) + u_n (u_n - u)) dx \\ &= \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2 + |u_n|^2 - |u|^2) dx. \end{aligned}$$

Therefore, $\|\nabla u_n - \nabla u\|_2 \rightarrow 0$ and $\|u_n - u\|_2 \rightarrow 0$ as $n \rightarrow \infty$ by [3, Proposition 3.32]. Moreover, we have that $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, we complete the proof of Lemma 3.4. $\square$

In the following, we shall consider the bifurcation results of (SBP_*) .

Proof of Theorem 1.1 In order to obtain the desired result, we divide its proof into three steps.

Step 1. We claim that for $\delta > 0$ small enough

$$\text{ind}(\mathfrak{A}_\lambda, 0) = 1, \text{ for } \lambda \in (0, \lambda_1) \text{ and } \text{ind}(\mathfrak{A}_\lambda, 0) = -1, \text{ for } \lambda \in (\lambda_1, \lambda_1 + \delta). \quad (3.2)$$

Fix $\lambda \in (0, \lambda_1)$. By the variational characterization of the first eigenvalue λ_1 of $(\mathcal{P})$ in Lemma 2.3, we have

$$\langle \mathfrak{A}_\lambda(u), u \rangle > 0$$

for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. For every $B_r(0) \subset H^1(\mathbb{R}^3)$, so we obtain that $\deg(\mathfrak{A}_\lambda, B_r(0), 0)$ is well defined. Furthermore, it follows from Lemma 3.2 that

$$\deg(\mathfrak{A}_\lambda, B_r(0), 0) = 1. \quad (3.3)$$

Moreover, since λ_1 is isolated by Lemma 2.3, there is δ such that $(\mathcal{P})$ has no eigenvalue in the interval $(\lambda_1, \lambda_1 + \delta)$. Now, we assume that $\tau > 1$ is a constant and $\wp : \mathbb{R} \rightarrow \mathbb{R}_0^+$ is a nonnegative C^1 function, such that

$$\wp(s) = \begin{cases} 0 & \text{for } s \leq \tau, \\ \frac{2\delta}{\lambda_1}(s - 2\tau) & \text{for } s \geq 3\tau, \end{cases}$$

and $\wp > 0$ and strictly convex in $(\tau, 3\tau)$.

Fix $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and define the functional $\mathcal{I}_\lambda : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$\mathcal{I}_\lambda(u) = \frac{1}{2} \langle \mathcal{T}(u), u \rangle - \frac{\lambda}{2} \langle \mathcal{H}(u), u \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \quad (3.4)$$

for all $u \in H^1(\mathbb{R}^3)$. In the following, we claim $\mathcal{I}_\lambda \in C^1(H^1(\mathbb{R}^3))$. Indeed, by $\|\mathcal{T}(v_n) - \mathcal{T}(v)\|_* \rightarrow 0$ and $\|\mathcal{H}(v_n) - \mathcal{H}(v)\|_* \rightarrow 0$, if $v_n \rightarrow v$ in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$ via Lemma 2.4, then we obtain $\langle \mathcal{T}(u), u \rangle$ and $\langle \mathcal{H}(u), u \rangle$ are $C^1(\mathbb{R}^3)$. Moreover, a critical point $u \in H^1(\mathbb{R}^3)$ of $\mathcal{I}_\lambda$ satisfies

$$\begin{aligned} \langle \mathcal{I}'_\lambda(u), \varphi \rangle &= \langle \mathcal{T}(u), \varphi \rangle - \lambda \langle \mathcal{H}(u), \varphi \rangle + \wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \langle \mathcal{T}(u), \varphi \rangle \\ &= \left(1 + \wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \right) \langle \mathcal{T}(u), \varphi \rangle - \lambda \langle \mathcal{H}(u), \varphi \rangle = 0 \text{ for all } \varphi \in H^1(\mathbb{R}^3). \end{aligned}$$

Therefore, the solution u solves the following equation

$$\mathcal{T}(u) = \frac{\lambda}{1 + \wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right)} \mathcal{H}(u).$$

On the other hand, by the definition of $\wp$, we obtain that $1 + \wp' \geq 1$ in $H^1(\mathbb{R}^3)$. By the fact that $\lambda \in (\lambda_1, \lambda_1 + \delta)$, we obtain

$$\frac{\lambda}{1 + \wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right)} < \lambda_i,$$

where λ_i ($i = 2, 3, \dots$) is the eigenvalue of $(\mathcal{P})$ given in Lemma 2.3. Hence, the only nontrivial critical point u of $\mathcal{I}_\lambda$ is actually a solution of the following equation

$$\wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) = \frac{\lambda}{\lambda_1} - 1.$$

Note that $\wp'(s) = 0$ for $s \leq \tau$ and $\wp'(s) = \frac{2\delta}{\lambda_1}$ for $s \geq 3\tau$. Furthermore, since $\lambda \in (\lambda_1, \lambda_1 + \delta)$, we see that $\frac{\lambda}{\lambda_1} - 1 \in (0, \frac{\delta}{\lambda_1})$. Together with the definition of $\wp$, we obtain

$$\frac{1}{2} \langle \mathcal{T}(u), u \rangle \in (\tau, 3\tau).$$

Consequently, the critical point of $\mathcal{I}_\lambda$ can only be one of two possibilities: either $\tilde{u}_1$ or $-\tilde{u}_1$, where $\tilde{u}_1$ is the first eigenfunction of $(\mathcal{P})$. Using Lemma 2.3, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, $\mathcal{I}_\lambda$ exactly has three isolated critical points, i.e., $\tilde{u}_1, 0, -\tilde{u}_1$.

In the following, we need to prove that the following claims are true.

Claim 1. We verify that $\mathcal{I}_\lambda$ is weakly lower semi-continuous in $H^1(\mathbb{R}^3)$.

Indeed, we suppose that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$. It follows from the weak lower semi-continuity of $\|\cdot\|$ that

$$\liminf_{n \rightarrow \infty} \langle \mathcal{T}(v_n), v_n \rangle \geq \langle \mathcal{T}(v), v \rangle. \quad (3.5)$$

Since the compactness of $\mathcal{H}$ by Lemma 2.4 and $\wp$ is non-decreasing, together with (3.5), we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(\frac{1}{2} \langle \mathcal{T}(v_n), v_n \rangle - \lambda \langle \mathcal{H}(v_n), v_n \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(v_n), v_n \rangle \right) \right) \\ & \geq \frac{1}{2} \langle \mathcal{T}(v), v \rangle - \lambda \langle \mathcal{H}(v), v \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(v), v \rangle \right). \end{aligned} \quad (3.6)$$

Consequently, we have

$$\liminf_{n \rightarrow \infty} \mathcal{I}_\lambda(v_n) \geq \mathcal{I}_\lambda(v).$$

Claim 2. We prove that $\mathcal{I}_\lambda(u)$ is coercive.

By Lemma 2.3 and the definition of $\wp$, we obtain

$$\begin{aligned} \mathcal{I}_\lambda &= \frac{1}{2} \langle \mathcal{T}(u), u \rangle - \frac{\lambda}{2} \langle \mathcal{H}(u), u \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \\ &\geq \frac{1}{2} \langle \mathcal{T}(u), u \rangle - \frac{\lambda}{2\lambda_1} \langle \mathcal{H}(u), u \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \\ &\geq \frac{\lambda_1 - \lambda}{2\lambda_1} \langle \mathcal{T}(u), u \rangle + \wp \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \\ &\geq -\frac{\delta}{2\lambda_1} \langle \mathcal{T}(u), u \rangle + \frac{\delta}{2\lambda_1} \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle - 2\tau \right) \\ &= \frac{\delta}{2\lambda_1} \langle \mathcal{T}(u), u \rangle - \frac{4}{\lambda_1} \tau \delta = \frac{\delta}{2\lambda_1} \|u\|^2 - \frac{4}{\lambda_1} \tau \delta \rightarrow \infty \text{ as } \|u\| \rightarrow \infty. \end{aligned}$$

It is obvious to see that 0 is an isolated critical point of the “saddle-type” functional $\mathcal{I}_\lambda$. Furthermore, since $\mathcal{I}_\lambda$ is even, according to Lemma 3.1, we deduce that the local minimum of $\mathcal{I}_\lambda$ is precisely achieved at two points, i.e., $\tilde{u}_1$ and $-\tilde{u}_1$. Therefore, we have

$$\text{ind}(\mathcal{I}'_\lambda, -\tilde{u}_1) = \text{ind}(\mathcal{I}'_\lambda, \tilde{u}_1) = 1. \quad (3.7)$$

Furthermore, we claim that

$$\langle \mathcal{I}'_\lambda(u), u \rangle > 0 \quad (3.8)$$

for every $u \in H^1(\mathbb{R}^3)$, and $\|u\| = \gamma$ with $\gamma > 0$ large enough. In fact, taking the same arguments to Claim 2, we get

$$\begin{aligned} \langle \mathcal{I}'_\lambda(u), u \rangle &= \langle \mathcal{T}(u), u \rangle - \lambda \langle \mathcal{H}(u), u \rangle + \wp' \left(\frac{1}{2} \langle \mathcal{T}(u), u \rangle \right) \langle \mathcal{T}(u), u \rangle \\ &\geq \left(1 - \frac{\lambda}{\lambda_1} + \frac{2\delta}{\lambda_1} \right) \langle \mathcal{T}(u), u \rangle \geq \frac{\delta}{\lambda_1} \langle \mathcal{T}(u), u \rangle \rightarrow \infty \text{ as } \|u\| \rightarrow \infty. \end{aligned}$$

Consequently, (3.8) is true. By Lemma 3.2, we obtain

$$\deg(\mathcal{I}'_\lambda, B_\gamma(0), 0) = 1. \quad (3.9)$$

Taking large $\gamma > 0$ and using the additivity of the degree in Lemma 3.3, we obtain $\pm \tilde{u}_1 \in B_\gamma(0)$. Together with (3.7) and (3.9), we obtain

$$\text{ind}(\mathcal{I}'_\lambda, 0) = -1.$$

Moreover, from the definition of $\wp$, we have

$$\deg(\mathfrak{A}_\lambda, B_r(0), 0) = \text{ind}(\mathcal{I}'_\lambda, 0) = -1 \quad (3.10)$$

for $r > 0$ small enough. Together with (3.3) and (3.10), we have that (3.2) is satisfied.

Step 2. We prove

$$\text{ind}(\mathcal{Q}_\lambda, 0) = 1 \text{ for } \lambda \in (0, \lambda_1) \text{ and } \text{ind}(\mathcal{Q}_\lambda, 0) = -1 \text{ for } \lambda \in (\lambda_1, \lambda_1 + \delta). \quad (3.11)$$

Taking the similar arguments to Lemma 2.4, we deduce the compactness of operators $\mathcal{E}$ and $\mathcal{F}$ in $H^1(\mathbb{R}^3)$. Therefore, together with Remark 3.1, we see that $\deg(\mathcal{Q}_\lambda, B_r(0), 0)$ can be well defined for every $r > 0$, where $\mathcal{Q}_\lambda$ is given in (3.1).

In what follows, we define the operator $\mathcal{Q}_{\lambda s} : H^1(\mathbb{R}^3) \rightarrow H^{-1}(\mathbb{R}^3)$ by

$$\mathcal{Q}_{\lambda s}(u) = \mathfrak{A}_\lambda(u) + s\mathcal{E}(u) - s\mathcal{F}(u) \text{ for all } u \in H^1(\mathbb{R}^3) \text{ and all } s \in [0, 1].$$

We assert that $\mathcal{Q}_{\lambda s}$ satisfies the fact that there is $r_1 > 0$ such that

$$\text{if } u \in B_{r_1}(0) \setminus \{0\}, \text{ then } \mathcal{Q}_{\lambda s}(u) \neq 0 \text{ for all } s \in [0, 1]. \quad (3.12)$$

By contradiction, we suppose that there are the sequences $\{v_n\}$ and $\{s_n\}$, with $\{v_n\} \subset H^1(\mathbb{R}^3) \setminus \{0\}$ and $\|v_n\| \rightarrow 0$, while $s_n \in [0, 1]$, such that $\mathcal{Q}_{\lambda s}(v_n) = 0$ for all n . Consequently, $v_n \rightarrow 0$ in $L^6(\mathbb{R}^3)$ by the continuous embedding $H^1(\mathbb{R}^3) \hookrightarrow L^\vartheta(\mathbb{R}^3)$ for all $\vartheta \in [2, 6]$. Taking $\hat{v}_n = \frac{v_n}{\|v_n\|}$ such that $\|\hat{v}_n\| = 1$. Furthermore, still denote a subsequence by itself, by the reflexivity of $H^1(\mathbb{R}^3)$, then there exists $\hat{v} \in H^1(\mathbb{R}^3)$ such that

$$\hat{v}_n \rightharpoonup \hat{v} \text{ in } L^6(\mathbb{R}^3) \text{ and } \hat{v}_n \rightarrow \hat{v} \text{ in } H^1(\mathbb{R}^3)$$

as $n \rightarrow \infty$. Thus, dividing by $\|v_n\|$ in equation $\mathcal{Q}_{\lambda s}(v_n) = 0$, we deduce

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \hat{v}_n \nabla \varphi + \hat{v}_n \varphi) dx + s_n \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^3 \hat{v}_n \varphi dx \\ &= \int_{\mathbb{R}^3} h(x) \hat{v}_n \varphi dx + \frac{s_n}{\|v_n\|} \int_{\mathbb{R}^3} |v_n|^4 v_n \varphi dx \end{aligned} \quad (3.13)$$

for all $\varphi \in H^1(\mathbb{R}^3)$. By Lemma 2.2-(ii) and Hölder inequality, we have that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^3 \hat{v}_n \varphi dx \\ &\leq \left(\int_{\mathbb{R}^3} |\phi_{v_n}^a|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} |v_n|^6 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\hat{v}_n|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} |\varphi|^6 dx \right)^{\frac{1}{6}} \rightarrow 0. \end{aligned} \quad (3.14)$$

Therefore, together with (3.14) and (2.5), we obtain that

$$\int_{\mathbb{R}^3} (\nabla \hat{v}_n \nabla \varphi + \hat{v}_n \varphi) dx = \int_{\mathbb{R}^3} h(x) \hat{v}_n \varphi dx. \quad (3.15)$$

In the following, we assert $\hat{v} \neq 0$. If $\hat{v} = 0$, then taking the test function as $\hat{v}_n$ and dividing by $\|v_n\|$ in $\mathcal{Q}_{\lambda s_n}(v_n) = 0$, we obtain

$$1 \leq \|\hat{v}_n\|^2 + s_n \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^3 \hat{v}_n^2 dx = \int_{\mathbb{R}^3} h(x) |\hat{v}_n|^2 dx + \frac{s_n}{\|v_n\|^2} \int_{\mathbb{R}^3} |v_n|^6 dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

which is impossible by Lemma 2.2-(ii), Lemma 2.4 and $\|\hat{v}_n\| = 1$. Therefore, we obtain this claim is true. Since λ is arbitrary in either $(0, \lambda_1)$ or $(\lambda_1, \lambda_1 + \delta)$ and there exists no eigenvalues in $(0, \lambda_1) \cup (\lambda_1, \lambda_1 + \delta)$, we deduce that (3.15) is not satisfied. Therefore, taking $r \in (0, r_1)$, we obtain

$$\deg(\mathcal{Q}_\lambda, B_r(0), 0) = \deg(\mathcal{A}_\lambda, B_r(0), 0)$$

by the homotopy invariance of the degree. It follows from (3.2) that (3.11) is true. Therefore, by (3.2) and (3.11), we have that the (weak) solutions of (SBP) have a bifurcation point $(\lambda_1, 0)$.

Step 3. We assert that there is a component $\mathcal{C}$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ of nontrivial (weak) solutions for (SBP), such that $\bar{\mathcal{C}}$ contains $(\lambda_1, 0)$, and that $\mathcal{C}$ either is unbounded, or contains a pair $(\lambda, 0)$ such that $\lambda > \lambda_1$ and λ is an eigenvalue of (P).

Taking the same arguments as [34, Theorem 1.3], we show that this conclusion holds. In fact, arguing by contradiction as the proof of (1.11) in [34, Theorem 1.3], we have that there are the numbers $\zeta > 0, \lambda_*, \lambda^*$ such that $\lambda_1 - \zeta < \lambda^* < \lambda_1 < \lambda_* < \lambda_1 + \zeta$ and $\text{ind}(\mathcal{Q}_{\lambda_*}, 0) = \text{ind}(\mathcal{Q}_{\lambda^*}, 0)$. Therefore, (3.11) implies

$$1 = \text{ind}(\mathcal{Q}_{\lambda^*}, 0) = \text{ind}(\mathcal{Q}_{\lambda_*}, 0) = -1$$

which is impossible. Therefore, by [34, Theorem 1.3], we deduce that there is component $\mathcal{C}$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$, such that $(\lambda_1, 0) \in \bar{\mathcal{C}}$ and $\mathcal{C}$ either is unbounded, or meets $(\lambda, 0)$ for other eigenvalue λ of (P). Therefore, we complete the proof of Theorem 1.1. $\square$

4 Existence and Asymptotic Analysis

In this section, by means of the arguments in [10] and Theorem 1.1, we prove the existence and asymptotic behavior of (weak) solutions for (SBP). In fact, we study the unbounded component $\mathcal{C}$ of nontrivial solutions for (SBP) and its properties, which play crucial roles in the proof of the existence of (weak) solutions.

Lemma 4.1 *Let $(\mathcal{H}_1)$ be satisfied, then the component $\mathcal{C}$ is unbounded in $\mathbb{R} \times H^1(\mathbb{R}^3)$ and*

$$\bar{\mathcal{C}} \subset \Upsilon := \{(\lambda_1, 0)\} \cup (\mathbb{R} \times \mathcal{M}),$$

where $\mathcal{C}$ is given in Theorem 1.1 and $\mathcal{M} = \{u \in H^1(\mathbb{R}^3), u \text{ has one-sign in } \mathbb{R}^3\}$.

Proof We verify that the following claims are true.

Claim 1. We claim that there is a neighborhood $\mathcal{W}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ such that

$$\mathcal{C} \cap (\mathcal{W} \setminus \{(\lambda_1, 0)\}) \subset \mathbb{R} \times \mathcal{M}. \quad (4.1)$$

In fact, arguing by contradiction that there is a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^1(\mathbb{R}^3)$ such that $(\lambda_n, u_n) \rightarrow (\lambda_1, 0)$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$ and every u_n is sign changing (weak) solution of (SBP). In the following, we are interested in the measure of the set

$$\Lambda^- = \{x \in \mathbb{R}^3 : u_n < 0\}.$$

We set $\hat{u}_n = \frac{u_n}{\|u_n\|}$ so that $\|\hat{u}_n\| = 1$. Therefore, up to a subsequence, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\hat{u}_n \rightharpoonup \hat{u} \text{ in } H^1(\mathbb{R}^3) \text{ and } \hat{u}_n \rightarrow \hat{u} \text{ in } L^\vartheta(\mathbb{R}^3) \text{ for all } \vartheta \in [2, 6]$$

as $n \rightarrow \infty$. Indeed, if $\hat{u}$ is trivial, since u_n solves (SBP) as $\lambda = \lambda_n$, Lemmas 2.2 and 2.4 imply that

$$1 \leq \|\hat{u}_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 \hat{u}_n^2 dx = \int_{\mathbb{R}^3} h(x) |\hat{u}_n|^2 dx + \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^3} |u_n|^6 dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

which is a contradiction. Therefore, $\hat{u}$ is nontrivial.

Fix $\hat{u}^- = \max\{-\hat{u}, 0\}$. Taking the test function $\varphi = \hat{u}^-$ and dividing by $\|u_n\|$ in (SBP), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \hat{u}_n \nabla \hat{u}^- + \hat{u}_n \hat{u}^-) dx + \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 \hat{u}_n \hat{u}^- dx \\ &= \int_{\mathbb{R}^3} h(x) \hat{u}_n \hat{u}^- dx + \frac{1}{\|u_n\|} \int_{\mathbb{R}^3} |u_n|^4 u_n \hat{u}^- dx. \end{aligned} \quad (4.2)$$

Hence, passing the limit as $n \rightarrow \infty$ in (4.2) and applying Lemma 2.2-(ii), (2.5) and Hölder inequality, we get

$$\|\hat{u}^-\|^2 \leq \|\hat{u}^-\|^2 + \int_{\mathbb{R}^3} \phi_u^a |u|^3 (\hat{u}^-)^2 dx$$

$$= \int_{\Lambda^-} h(x) |\hat{u}^-|^2 dx \leq \lambda_1 \|h\|_{L^{3/2}(\Lambda^-)} \|\hat{u}^-\|_{L^6(\Lambda^-)}^2 \leq S^2 \lambda_1 \|h\|_{L^{3/2}(\Lambda^-)} \|\hat{u}^-\|^2.$$

Therefore, we obtain

$$\|h\|_{L^{3/2}(\Lambda^-)} \geq (S^2 \Lambda_1)^{-1}$$

and

$$|\Lambda^-| \geq C_0 \quad (4.3)$$

for some $C_0 > 0$. Similarly, $\Lambda^+ = \{x \in \mathbb{R}^3 : u_n > 0\}$ and $|\Lambda^+| > 0$. Moreover, we claim that $\hat{u}$ is an eigenfunction corresponding to the first eigenvalue λ_1 . In fact, by the definitions of u_n and $\hat{u}_n$, we obtain that

$$\langle \mathcal{T}(\hat{u}_n), \varphi \rangle + \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 \hat{u}_n \varphi dx = \lambda \langle \mathcal{H}(\hat{u}_n), \varphi \rangle + \frac{1}{\|u_n\|} \langle \mathcal{F}(u_n), \varphi \rangle \quad (4.4)$$

for any $\varphi \in H^1(\mathbb{R}^3)$. Hence, using this fact, (3.14) and (2.5) and taking the limit as $n \rightarrow \infty$ in (4.4), we obtain

$$\langle \mathcal{T}(\hat{u}), \varphi \rangle = \lambda_1 \langle \mathcal{H}(\hat{u}), \varphi \rangle \quad (4.5)$$

which implies that the assertion is true.

It follows from Lemma 2.3 that $\hat{u} = t\tilde{u}_1$ where $t \in \mathbb{R}$, $\tilde{u}_1 > 0$ is the first eigenfunction. This is impossible via definition of $\hat{u}$ and (4.1) is true.

Claim 2. We claim that $\bar{\mathcal{C}} \subset \Upsilon$.

By contradiction, we assume that $\bar{\mathcal{C}}$ leaves Υ at some point $(\lambda_0, v_0) \neq (\lambda_1, 0)$. We define

$$\mathcal{C}^+ = \{(\lambda, u) \in \mathcal{C} : u(x) > 0\} \cup \{(\lambda_1, 0)\} \text{ and } \mathcal{C}^- = \{(\lambda, u) \in \mathcal{C} : u(x) < 0\} \cup \{(\lambda_1, 0)\}.$$

Now we define the sign-changing function set $\mathcal{L} = \mathcal{C} \setminus (\mathcal{C}^+ \cup \mathcal{C}^-)$ in $\mathcal{C}$. Note that $(\lambda_0, v_0) \in \mathcal{L}$ and so $\bar{\mathcal{L}} \neq \emptyset$. Since $\bar{\mathcal{C}} \subset \mathbb{R} \times H^1(\mathbb{R}^3)$ is a connected closed set, we have

$$\overline{\mathcal{C}^+ \cup \mathcal{C}^-} \cap \bar{\mathcal{L}} \neq \emptyset.$$

Hence, there is a sequence $\{(\eta_n, v_n)\} \subset \mathcal{C}^+ \cup \mathcal{C}^-$, such that $(\eta_n, v_n) \rightarrow (\lambda_0, v_0)$ in $\mathbb{R} \times H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$.

If $v_0 = 0$, then $v_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$ and similarly, arguing as (4.3) in Step 1, we deduce that each (weak) solution v_n is sign-changing. This is impossible by $v_n \in \mathcal{C}^+ \cup \mathcal{C}^-$. Therefore, $v_0 \neq 0$. Hence, without loss of generality, we assume that $(\lambda_n, v_n) \in \mathcal{C}^+$. Furthermore, as $\lambda = \lambda_0$, the fact that v_0 solves (SBP) can be deduced from $v_n \rightarrow v_0$ in $H^1(\mathbb{R}^3)$. By the maximum principle, we have $v_0 > 0$ a.e. in $\mathbb{R}^3$. This is impossible by the definition of v_0 . Therefore, we obtain $\bar{\mathcal{C}} \subset \Upsilon$ and finish the proof of this claim.

Claim 3. We claim that $\mathcal{C}$ is unbounded.

If $\mathcal{C}$ is bounded, then Theorem 1.1 yields that $\mathcal{C}$ contains $(\lambda, 0)$, where $\lambda > \lambda_1$ is an eigenvalue of (SBP). Consequently, there is a sequence $\{(\tilde{\eta}_n, \varpi_n)\} \subset \mathcal{C}$ such that

$$\tilde{\eta}_n \rightarrow \lambda, \text{ and } \|\varpi_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Furthermore, since $\|\overline{\varpi}_n\| = 1$, up to the subsequence, there exists $\overline{\varpi}$, such that $\overline{\varpi}_n \rightharpoonup \overline{\varpi}$ in $H^1(\mathbb{R}^3)$ and $\overline{\varpi}_n \rightarrow \overline{\varpi}$ in $L^\vartheta(\mathbb{R}^3)$ for all $\vartheta \in [2, 6]$. Taking the same arguments as (4.5) and [10, Lemma 2.3], we obtain that $\overline{\varpi}$ related to λ changes sign in $\mathbb{R}^3$. Consequently, we obtain a contradiction by $\overline{\mathcal{C}} \subset \Upsilon$ in Claim 2 of Lemma 4.1. Therefore, $\mathcal{C}$ is unbounded which finishes the proof of Lemma 4.1. $\square$

Lemma 4.2 *Let $(\mathcal{H}_1)$ be satisfied, the following properties are satisfied.*

- (i) *There is a constant $C_0 > 0$, such that $\lambda \leq C_0$ for all $(\lambda, u) \in \mathcal{C}$.*
- (ii) *For all $\lambda_* \in (0, \lambda_1)$, there exists a constant $C = C(\lambda_*) > 0$ such that $\|u\| \leq C$ for all $(\lambda, u) \in \mathcal{C}$ and $\lambda \geq \lambda_*$.*

Proof For (i), suppose by contradiction that there exists a sequence $n \mapsto (\eta_n, u_n) \in \mathcal{C}$, such that $\eta_n \rightarrow \infty$ as $n \rightarrow \infty$. Now we suppose $u_n > 0$ from Lemma 4.1. In what follows, we divide the proof into two cases.

Case 1. The sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

In this case, we obtain $\sup_n \|u_n\| \leq C_1$ for some $C_1 > 0$. Hence, up to a sequence,

$$u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3), \quad u_n \rightarrow u \text{ in } L^6(\mathbb{R}^3) \text{ and } u_n \rightarrow u \text{ a.e. in } (\mathbb{R}^3)$$

for some $u \in H^1(\mathbb{R}^3)$. Choosing $v_0 > 0$ as the first eigenfunction of (P), together with $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $h(x)v_0 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3} h(x)u_nv_0 dx \rightarrow \int_{\mathbb{R}^3} h(x)uv_0 dx \text{ as } n \rightarrow \infty.$$

In the following, we claim

$$\int_{\mathbb{R}^3} h(x)uv_0 dx \neq 0. \tag{4.6}$$

Arguing by contradiction that we assume $u = 0$ a.e. in $\mathbb{R}^3$. Since u_n solves (SBP), taking the test function as $v_0 > 0$ in (SBP), we have

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla v_0 + u_n v_0) dx + \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n v_0 dx = \eta_n \int_{\mathbb{R}^3} h(x)u_nv_0 dx + \int_{\mathbb{R}^3} |u_n|^4 u_n v_0 dx.$$

Therefore, one has

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n v_0 dx - \int_{\mathbb{R}^3} |u_n|^4 u_n v_0 dx &= - \int_{\mathbb{R}^3} (\nabla u_n \nabla v_0 + u_n v_0) dx + \eta_n \int_{\mathbb{R}^3} h(x)u_nv_0 dx \\ &= (\eta_n - \lambda_1) \int_{\mathbb{R}^3} g(x)u_nv_0 dx. \end{aligned} \tag{4.7}$$

By the similar arguments to Lemma 2.2-(iv), for any $\varepsilon > 0$, then there exists $R_\varepsilon > 0$ large enough such that

$$\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}^\varepsilon} \phi_{u_n}^a |u_n|^4 v_0 dx \leq \frac{\varepsilon}{2} \text{ for all } n, \quad (4.8)$$

since $\{u_n\}$ is bounded sequence in $H^1(\mathbb{R}^3)$ and $v_0 \in L^6(\mathbb{R}^3)$.

Furthermore, we see that for all $R > 0$, there exists the compact embedding $H^1(\mathbb{R}^3) \hookrightarrow L^\vartheta(B_{R_\varepsilon})$ for all $\vartheta \in [2, 6)$. Let $\varepsilon > 0$ be fixed and denote a subsequence by $\{u_n\}$, such that $u_n \rightarrow 0$ in $L^\vartheta(B_{R_\varepsilon})$ for all $\vartheta \in [2, 6)$, then we get $u_n \rightarrow 0$ a.e. in B_{R_ε} . Consequently,

$$|u_n|^4 u_n v_0 \rightarrow 0 \text{ a.e. in } B_{R_\varepsilon}.$$

Moreover, for each measurable subset $E \subset B_R$, by Hölder inequality, (2.1) and the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^3)$, we obtain

$$\int_E |u_n|^4 u_n v_0 dx \leq \left(\int_E |u_n|^6 dx \right)^{\frac{5}{6}} \left(\int_E |u_n|^6 dx \right)^{\frac{1}{6}} \leq S^{-\frac{1}{2}} \|u_n\|^5 \|v_0\| < \infty.$$

Therefore, $|u_n|^4 u_n v_0$ is equi-integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$. With the aid of Vitali convergence theorem, we obtain that

$$\lim_{n \rightarrow \infty} \int_{B_{R_\varepsilon}} |u_n|^4 u_n v_0 dx = 0. \quad (4.9)$$

Similarly, Lemma 2.2-(iv) implies that for $R_\varepsilon > 0$ large enough

$$\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} \phi_{u_n}^a |u_n|^3 u_n v_0 dx \leq \frac{\varepsilon}{2}. \quad (4.10)$$

Hence, using (4.8)–(4.10) and Lemma 2.2-(iv), we have that there exists $R_\varepsilon > 0$ so large that

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n v_0 dx - \int_{\mathbb{R}^3} |u_n|^4 u_n v_0 dx &= \int_{B_{R_\varepsilon}} \phi_{u_n}^a |u_n|^3 u_n v_0 dx + \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} \phi_{u_n}^a |u_n|^3 u_n v_0 dx \\ &\quad - \int_{B_{R_\varepsilon}} |u_n|^4 u_n v_0 dx - \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} |u_n|^4 u_n v_0 dx \\ &\leq \varepsilon + \int_{B_{R_\varepsilon}} \phi_{u_n}^a |u_n|^3 u_n v_0 dx - \int_{B_{R_\varepsilon}} |u_n|^3 u_n v_0 dx < \varepsilon \text{ as } n \rightarrow \infty. \end{aligned}$$

Since ε is arbitrary, we have

$$\int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n v_0 dx - \int_{\mathbb{R}^3} |u_n|^3 u_n v_0 dx < 0.$$

By (4.7), we get

$$(\eta_n - \lambda_1) \int_{\mathbb{R}^3} h(x) u_n v_0 dx < 0.$$

This implies that $\eta_n < \lambda_1$. This contradicts the fact that $\eta_n \rightarrow \infty$. Therefore, (4.6) holds.

Since each u_n is a (weak) solution of (SBP), using Lemma 2.2-(iv) and taking the test function as the first eigenfunction $v_0 > 0$, we obtain

$$\begin{aligned} \eta_n \int_{\mathbb{R}^3} h(x) u_n v_0 dx &\leq \int_{\mathbb{R}^3} \nabla u_n \nabla v_0 dx + \int_{\mathbb{R}^3} u_n v_0 dx + \int_{\mathbb{R}^3} \phi_{u_n}^a |u_n|^3 u_n v_0 dx \\ &\leq \lambda_1 \int_{\mathbb{R}^3} h(x) u_n v_0 dx + S^{-1} \|u_n\|^9 \|v_0\| \leq \lambda_1 \int_{\mathbb{R}^3} h(x) u_n v_0 dx + \tilde{C}, \end{aligned}$$

where $\tilde{C} = S^{-1} C^9 \|v_0\| > 0$. Therefore, we get

$$0 \leq \int_{\mathbb{R}^3} h(x) u_n v_0 dx \leq \frac{\tilde{C}}{\eta_n - \lambda_1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

which yields

$$\int_{\mathbb{R}^3} h(x) u_n v_0 dx = 0.$$

This is impossible.

Case 2. The sequence $\{u_n\}$ is unbounded in $H^1(\mathbb{R}^3)$, with $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$.

Now, we define the scaling function $\omega_n = s_n^\ell u_n(s_n^\beta x)$, with $\frac{3\beta}{2} < \ell < \frac{\beta}{4} < 0$. Choosing $s_n > 0$ so small that there exist $C_2, C_3 > 0$ such that

$$C_3 \leq \|\omega_n\|^2 = s_n^{2\ell-\beta} \|\nabla u_n\|_2^2 + s_n^{2\ell-3\beta} \|u_n\|_2^2 \leq C_2$$

for all n . In fact, by the change of variable $x' = s_n^\beta x$, we have

$$\|\nabla \omega_n\|_2^2 = s_n^{2\ell-\beta} \|\nabla u_n\|_2^2 \quad (4.11)$$

and

$$\|\omega_n\|_2^2 = s_n^{2\ell-3\beta} \|u_n\|_2^2. \quad (4.12)$$

Since $2\ell - \beta > 0, 2\ell - 3\beta > 0$ and $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$, we obtain that $s_n \rightarrow 0$ as $n \rightarrow \infty$. Applying the change of variables $x' = s_n^\beta x$ and $y' = s_n^\beta y$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{\omega_n}^a |\omega_n|^5 dx &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} |\omega_n(x)|^5 |\omega_n(y)|^5 dx dy \\ &= s_n^{10\ell-5\beta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x'-y'|}{s_n^\beta a}}}{|x'-y'|} |u_n(x')|^5 |u_n(y')|^5 dx' dy' \end{aligned}$$

and

$$\int_{\mathbb{R}^3} |\omega_n|^6 dx = s_n^{6\ell-3\beta} \|u_n\|_6^6.$$

Furthermore, since u_n is a (weak) solution of (SBP), we have that ω_n satisfies

$$\begin{aligned} s_n^{-2\beta} \|\nabla \omega_n\|_2^2 + \|\omega_n\|_2^2 + s_n^{2\beta-8\ell} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x'-y'|}{s_n^\beta a}}}{|x'-y'|} |\omega_n(x')|^5 |\omega_n(y')|^5 dx' dy' \\ = \eta_n \int_{\mathbb{R}^3} h(x) |\omega_n|^2 dx' + s_n^{-4\ell} \int_{\mathbb{R}^3} |\omega_n|^6 dx'. \end{aligned} \quad (4.13)$$

Therefore, using this fact, Lemma 2.2-(iv), Lemma 2.1 and passing the limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \eta_n \int_{\mathbb{R}^3} h(x) |\omega_n|^2 dx &\leq s_n^{-2\beta} \|\nabla \omega_n\|_2^2 + \|\omega_n\|_2^2 \\ &\quad + s_n^{2\beta-8\ell} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x'-y'|}{s_n^\beta a}}}{|x'-y'|} |\omega_n(x')|^5 |\omega_n(y')|^5 dx' dy' \\ &\leq \|\omega_n\|_2^2 + s_n^{2\beta-8\ell} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|x'-y'|} |\omega_n(x')|^5 |\omega_n(y')|^5 dx' dy' \\ &\leq (1 + S^{-5} \|\omega_n\|^8) \|\omega_n\|_2^2, \end{aligned}$$

since $-2\beta > 0$, $2\beta - 8\ell > 0$, $s_n \leq 1$ as $n \rightarrow \infty$ and $C_3 \leq \|\omega_n\| \leq C_2$. Therefore, we get

$$\eta_n \leq (1 + S^{-5} C_2^8) \frac{\|\omega_n\|_2^2}{\int_{\mathbb{R}^3} h(x) |\omega_n|^2 dx}$$

which is impossible. Consequently, we complete the proof of Lemma 4.2-(i).

(ii) Suppose by contradiction that the sequence $\{u_n\}$ is unbounded in $H^1(\mathbb{R}^3)$. Therefore, up to subsequence, there is $\{(\lambda_n, u_n)\} \subset \mathcal{C}$ such that

$$\|u_n\| > 0, \lambda_n \in [\lambda_*, \infty) \text{ and } \|u_n\| \rightarrow \infty.$$

We define the scaling function $v_n(x) = s_n^d u_n(s_n^m x)$, where

$$s_n > 0, d > 0 \text{ and } m \in (0, \frac{2d}{3}). \quad (4.14)$$

Taking $s_n > 0$ small enough such that there are $C_4, \tilde{C}'_4 > 0$ such that

$$\tilde{C}'_4 \leq \|v_n\|^2 = s_n^{2d-m} \|\nabla u_n\|_2^2 + s_n^{2d-3m} \|u_n\|_2^2 \leq C_4 \quad (4.15)$$

for all n . Invoking the fact that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$, we obtain that $s_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, the reflexivity of $H^1(\mathbb{R}^3)$ implies that, up to a subsequence, there exists $v \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^3) \text{ and } v_n \rightarrow v \text{ a.e. in } \mathbb{R}^3. \quad (4.16)$$

Furthermore, arguing as in (4.13), since u_n solves problem (SBP), we deduce that

$$\begin{aligned} & s_n^{-2m} \|\nabla v_n\|_2^2 + \|v_n\|_2^2 + s_n^{2m-8d} \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^5 dx \\ &= s_n^{-2m} \|\nabla v_n\|_2^2 + \|v_n\|_2^2 + s_n^{2m-8d} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} |v_n(x)|^5 |v_n(y)|^5 dx dy \\ &\leq s_n^{-2m} \|\nabla v_n\|_2^2 + \|v_n\|_2^2 + s_n^{2m-8d} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{s_n^{\frac{\beta}{d}}}}}{|x-y|} |v_n(x)|^5 |v_n(y)|^5 dx dy \\ &= \lambda_n \int_{\mathbb{R}^3} h(x) |v_n|^2 dx + s_n^{-4\ell} \int_{\mathbb{R}^3} |v_n|^6 dx \leq C_0 \int_{\mathbb{R}^3} h(x) |v_n|^2 dx + s_n^{-4\ell} S^{-3} \|v_n\|^6 \end{aligned}$$

by (2.1) and (i). Applying $\ell < 0$, Lemma 2.2, Hölder inequality and (4.13), we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^5 dx &\leq s_n^{8d-2m} \left(\int_{\mathbb{R}^3} h(x) |v_n|^2 dx + s_n^{-4\ell} S^{-3} \|v_n\|^6 \right) \\ &\leq s_n^{8d-2m} \left(C_4^2 C_0 \|h\|_{\frac{3}{2}} \|v_n\|^2 + s_n^{-4\ell} S^{-3} \|v_n\|^6 \right) \rightarrow 0. \end{aligned} \quad (4.17)$$

By Fatou's lemma and (4.16), we obtain

$$\int_{\mathbb{R}^3} \phi_v^a |v|^5 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \phi_{v_n}^a |v_n|^5 dx = 0$$

which yields $v = 0$ a.e. in $\mathbb{R}^3$ and contradicts (4.15). This proves the assertion and finishes the proof of Lemma 4.2. $\square$

Proof of Theorem 1.2 By Lemmas 4.1 and 4.2, we obtain that $\mathcal{C} \cap [\lambda_*, \infty) \times H^1(\mathbb{R}^3)$ is bounded. On the other hand, since $\mathcal{C}$ is unbounded and connected via Theorem 1.1 and Lemma 4.1, it must cross at some point (λ_*, u_*) . Moreover, taking the same arguments as Lemma 4.1, we have that u_* has constant-sign in $\mathbb{R}^3$. Consequently, $u_* \neq 0$ is a (weak) solution of (SBP). $\square$

Proof of Theorem 1.3 First, by Theorem 1.1, let $\{u_*^a\}_{a>0}$ be a family of weak solutions of (SBP) obtained in Theorem 1.2. On the other hand, by [9, Lemma 6.1], as $a \rightarrow 0$, we have $\phi_{u_*^a}^a \rightarrow \phi_{u_*^0}^0$ in $H^1(\mathbb{R}^3)$, which solves $-\Delta \phi_{u_*^0} = (u_*^0)^2$. Finally, taking the similar the proof process of Theorem 1.2, we obtain that u_*^0 is actually a solution of (SP). $\square$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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