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On a Class of Parametric $(p, 2)$-equations

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Abstract We consider parametric equations driven by the sum of a p-Laplacian and a Laplace operator (the so-called $(p, 2)$-equations). We study the existence and multiplicity of solutions when the parameter $\lambda > 0$ is near the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. We prove multiplicity results with precise sign information when the near resonance occurs from above and from below of $\hat{\lambda}_1(p) > 0$.

Keywords Near resonance · Local minimizer · Critical group · Constant sign and nodal solutions · Nonlinear maximum principle

Mathematics Subject Classification 35J20 · 35J60 · 58E05

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric nonlinear nonhomogeneous Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = \lambda |u(z)|^{p-2}u(z) + f(z, u(z)) \text{ in } \Omega, \ u|_{\partial\Omega} = 0, \ 2 < p < \infty. \tag{P_λ}$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, $\lambda > 0$ is a parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z, x)$ is continuous).

Our aim in this paper is to study the existence and multiplicity of nontrivial solutions when the parameter $\lambda > 0$ is near the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ either from the left or from the right. Such equations, which are near resonance, were first investigated by Mawhin and Schmitt [21,22] (for semilinear Dirichlet and periodic problems, respectively). Subsequently, their work was extended by Badiale and Lupo [4], Chiappinelli et al. [11] and Ramos and Sanchez [33]. All these papers consider semilinear elliptic equations driven by the Laplacian. Extensions to equations driven by the p-Laplacian were obtained by Ma et al. [20] and Papageorgiou and Papalini [25].

In this work we extend the analysis to $(p, 2)$-equations (that is, equations driven by the sum of a p-Laplacian ($p > 2$) and a Laplacian). We stress that the differential operator in (P_λ) is nonhomogeneous and this is a source of difficulties in the analysis of the problem (P_λ). We note that $(p, 2)$-equations arise in many physical applications (see Cherfils and Ilyasov [10]) and recently such equations were studied by Barile and Figueiredo [5], Carvalho et al. [7], Chaves et al. [9], Mugnai and Papageorgiou [23], Papageorgiou and Rădulescu [26–28] and Papageorgiou and Winkert [30,31].

Our approach is variational, based on the critical point theory, together with suitable truncation and comparison techniques, and Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the paper.

2 Mathematical Background

The topological notion of linking sets is central in the critical point theory.

Definition 1 Let Y be a Hausdorff topological space and E_0, E, D be closed subspaces of Y such that $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ is linking with D in Y, if

(a) $E_0 \cap D = \emptyset$; and
(b) for every $\gamma \in C(E, Y)$ such that $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

Now, let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). Given $\varphi \in C^1(X)$, we say that φ satisfies the Cerami condition (the C-condition for short), if the following is true:
"If $\{u_n\}_{n\geqslant 1} \subseteq X$ is a sequence such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1+||u_n||)\varphi'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p}+\frac{1}{p'}=1\right) \text{ as } n\to\infty,$$

then it admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ, which compensates for the fact that the ambient space X need not be locally compact (since X is in general, infinite dimensional). The C-condition is important in developing a minimax theory for the critical values of φ. A basic result in that theory is the following theorem which involves the notion of linking sets (see, for example, Gasinski and Papageorgiou [16, p. 644]).

Theorem 2 *If X is a Banach space, E_0, E and D are nonempty closed subsets of X such that the pair $\{E_0, E\}$ is linking with D in X (see Definition 1), $\varphi \in C^1(X)$ and satisfies the C-condition,* $\sup_{E_0}\varphi < \inf_D \varphi$ *and*

$$c = \inf_{\gamma\in\Gamma}\sup_{u\in E} \varphi(\gamma(u)) \textit{ with } \Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\},$$

then $c \geqslant \inf_D \varphi$ and c is a critical value of φ.

With suitable choices of the linking sets, we obtain the well-known mountain pass theorem, saddle point theorem and the generalized mountain pass theorem (see [16]). For future use, we state the mountain pass theorem.

Theorem 3 *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the C-condition, $u_0, u_1 \in X$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\,[\varphi(u) : ||u-u_0|| = \rho] = m_\rho,\ \ ||u_1-u_0|| > \rho > 0$$

and $c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$ *with* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$, *then $c \geqslant m_\rho$ and c is a critical value of φ.*

Remark 1 It is easy to see that Theorem 3 can be deduced from Theorem 2, if we consider $E_0 = \{u_0, u_1\}$, $E = \{u \in X : u = tu_1 + (1-t)u_0, t \in [0,1]\}$, $D = \partial B_\rho(u_0) = \{u \in X : ||u-u_0|| = \rho\}$.

In this analysis of problem (P_λ), we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}$. The latter is an ordered Banach space with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}); u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\right\}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

In what follows, by $||\cdot||$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$||u|| = ||Du||_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Next, we present some basic facts about the spectrum of $(-\Delta_q, W_0^{1,q}(\Omega))$ with $1 < q < \infty$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_q u(z) = \hat{\lambda}|u(z)|^{q-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0.$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$, if the above equation admits a nontrivial solution $\hat{u} \in W_0^{1,q}(\Omega)$. We say that $\hat{u}$ is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. We know that there exists a smallest eigenvalue $\hat{\lambda}_1(q)$ with the following properties:

(i) $\hat{\lambda}_1(q) > 0$;
(ii) $\hat{\lambda}_1(q)$ is isolated, that is, there exists $\epsilon > 0$ such that $(\hat{\lambda}_1(q), \hat{\lambda}_1(q) + \epsilon)$ contains no eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$; and
(iii) $\hat{\lambda}_1(q)$ is simple, that is, if $\hat{u}, \hat{v}$ are eigenfunctions corresponding to $\hat{\lambda}_1(q)$, then $\hat{u} = \xi\hat{v}$ for some $\xi \in \mathbb{R}\backslash\{0\}$.

Moreover, $\hat{\lambda}_1(q)$ admits the following variational characterization

$$\hat{\lambda}_1(q) = \inf\left[\frac{||Du||_q^q}{||u||_q^q} : u \in W_0^{1,q}(\Omega),\ u \neq 0\right]. \tag{1}$$

In (1) the infimum is realized on the corresponding one-dimensional eigenspace. By (1) it is clear that the elements of this eigenspace do not change the sign. By $\hat{u}_1(q)$ we denote the positive, L^q-normalized (that is, $||\hat{u}_1(q)||_q = 1$) eigenfunction corresponding to $\hat{\lambda}_1(q) > 0$. From the nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [16, pp. 737–738]), it follows that $\hat{u}_1(q) \in \operatorname{int} C_+$.

Let $\sigma(q)$ denote the set of eigenvalues of $(-\Delta_q, W_0^{1,q}(\Omega))$. It is easy to check that this set is closed. Since $\hat{\lambda}_1(q) > 0$ is isolated, the second eigenvalue $\hat{\lambda}_2^*(q)$ is well-defined by

$$\hat{\lambda}_2^*(q) = \inf[\hat{\lambda} \in \sigma(q) : \hat{\lambda} > \hat{\lambda}_1(q)].$$

If $N = 1$ (ordinary differential equations), then $\sigma(q) = \{\hat{\lambda}_k(q)\}_{k\geqslant 1}$ with each $\hat{\lambda}_k(q)$ being a simple eigenvalue and $\hat{\lambda}_k(q) \uparrow +\infty$ as $k \to \infty$ and the corresponding eigenfunctions $\{\hat{u}_k(q)\}_{k\geqslant 1}$ have exactly $k-1$ zeros. If $N \geqslant 2$ (partial differential equations), then using the Ljusternik–Schnirelmann minimax scheme, we can produce a strictly increasing sequence $\{\hat{\lambda}_k(q)\}_{k\geqslant 1} \subseteq \sigma(q)$ such that $\hat{\lambda}_k(q) \to +\infty$ as $k \to \infty$. However, we do not know if this is the complete list of all eigenvalues. We know that

$\hat{\lambda}_2^*(q) = \hat{\lambda}_2(q)$, that is, the second eigenvalue and the second Ljusternik–Schnirelmann eigenvalue coincide. The Ljusternik–Schnirelmann theory gives a minimax characterization of $\hat{\lambda}_2(q)$. For our purposes, this characterization is not convenient. Instead, we will us an alternative one due to Cuesta, de Figueiredo and Gossez [13].

Proposition 4 *If* $\partial B_1^{L^q} = \{u \in L^q(\Omega) : ||u||_q = 1\}$, $M = W_0^{1,q}(\Omega) \cap \partial B^{L^q}$, *and*

$$\Gamma_0 = \{\gamma_0 \in C([-1, 1], M) : \gamma_0(-1) = -\hat{u}_1(q),\ \gamma_0(1) = \hat{u}_1(q)\}$$

then $\hat{\lambda}_2(q) = \inf_{\gamma_0 \in \Gamma_0} \max_{-1 \leqslant t \leqslant 1} ||D\gamma_0(t)||_q^q$.

We mention that $\hat{\lambda}_1(q) > 0$ is the only eigenvalue with eigenfunctions of constant sign. Every other eigenvalue has nodal (that is, sign-changing) eigenfunctions.

When $q = 2$ (linear eigenvalue problem), then $\sigma(2) = \{\hat{\lambda}_k(2)\}_{k \geqslant 1}$. In this case, the eigenspaces are linear spaces. By $E(\hat{\lambda}_k(2))$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2)$. The regularity theory implies that $E(\hat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$. Moreover, $E(\hat{\lambda}_k(2))$ has the so-called unique continuation property, that is, if $u \in E(\hat{\lambda}_k(2))$ and vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. In this case all eigenvalues admit variational characterization, namely

$$\hat{\lambda}_1(2) = \inf \left[\frac{||Du||_2^2}{||u||_2^2} : u \in H_0^1(\Omega), u \neq 0 \right] \tag{2}$$

and for $k \geqslant 2$, we have

$$\begin{aligned} \hat{\lambda}_k(2) &= \sup \left[\frac{||Du||_2^2}{||u||_2^2} : u \in \overset{k}{\underset{\text{i=1}}{\oplus}} E(\hat{\lambda}_i(2)), u \neq 0 \right] \\ &= \inf \left[\frac{||Du||_2^2}{||u||_2^2} : u \in \overline{\underset{\text{i}\geqslant\text{k}}{\oplus} E(\hat{\lambda}_i(2))}, u \neq 0 \right]. \end{aligned} \tag{3}$$

In (2) the infimum is realized on $E(\hat{\lambda}_1(2))$, while in (3) both the supremum and the infimum are realized on $E(\hat{\lambda}_k(2))$.

From the variational characterizations in (2) and (3) and the unique continuation property, we have the following result (see Papageorgiou and Kyritsi [24]).

Proposition 5 (a) *If* $k \geqslant 1$, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \hat{\lambda}_k(2)$ *for a.a.* $z \in \Omega$ *and* $\vartheta \not\equiv \hat{\lambda}_k(2)$, *then there exists* $\hat{\xi}_0 > 0$ *such that*

$$||Du||_2^2 - \int_\Omega \vartheta(z)u(z)^2 dz \geqslant \hat{\xi}_0 ||u||^2 \textit{ for all } u \in \overline{\underset{\text{i}\geqslant\text{k}}{\oplus} E(\hat{\lambda}_k(2))}.$$

(b) *If* $k \geqslant 1$, $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \geqslant \hat{\lambda}_k(2)$ *for a.a.* $z \in \Omega$ *and* $\vartheta \not\equiv \hat{\lambda}_k(2)$, *then there exists* $\hat{\xi}_1 > 0$ *such that*

$$||Du|||_2^2 - \int_\Omega \vartheta(z)u(z)^2 dz \leqslant -\hat{\xi}_1 ||u||^2 \textit{ for all } u \in \overset{k}{\underset{\text{i=1}}{\oplus}} E(\hat{\lambda}_i(2)).$$

For $1 < q < \infty$, let $A_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ be the nonlinear map defined by

$$\langle A_q(u), h\rangle = \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,q}(\Omega).$$

If $q = 2$, then $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

By Papageorgiou and Kyritsi [24, p. 314], we have the following result summarizing the basic properties of the map A_q.

Proposition 6 *The map $A_q : W_0^{1,q}(\Omega) \to W^{-1.q'}(\Omega)$ is bounded (that is, it maps bounded sets to bounded sets), demicontinuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,q}(\Omega)$ and $\limsup_{n\to\infty} \langle A_q(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ in $W_0^{1,q}(\Omega)$ as $n \to \infty$.*

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth in the $x \in \mathbb{R}$ variable, that is,

$$|f_0(z, x)| \leqslant a_0(z)(1 + |x|^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p. \end{cases}$

We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega F_0(z, u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next result is a special case of a more general result of Aizicovici et al. [2].

Proposition 7 *Let $u_0 \in W_0^{1,p}(\Omega)$ be a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C_0^1(\overline{\Omega}),\ ||h||_{C_0^1(\overline{\Omega})} < \rho_0.$$

Then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in W_0^{1,p}(\Omega),\ ||h|| \leqslant \rho_1.$$

We also recall some basic definitions and facts from Morse theory. So, let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets.

$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\},\ K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ and } K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients. The critical groups of φ at $u \in K_\varphi^c$ which is isolated among the critical points, are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U,\ \varphi^c \cap U \backslash \{u\}) \quad \text{for all} \quad k \geqslant 0.$$

Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of the singular homology implies that this definition is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf\ \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geqslant 0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [16, p. 628]), implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

We introduce

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \text{ for all } t \in \mathbb{R},\ \text{all } u \in K_\varphi \text{ and}$$
$$P(t, \infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \tag{4}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative coefficients.

Finally, let us fix our notation in this paper. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Given $x \in \mathbb{R}$, we let $x^{\pm} = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given a measurable function $g(z, x)$ (for example, a Carathéodory function), we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \text{ for all } u \in W_0^{1,p}(\Omega)$$

(the Nemytski map corresponding to g). Evidently, $z \mapsto N_g(u)(z)$ is measurable.

3 Near Resonance from the Left of $\hat{\lambda}_1(p) > 0$

In this section we deal with problem (P_λ) in which the parameter is close to $\hat{\lambda}_1(p) > 0$ from the left (near resonance from the left). We introduce the following conditions on the perturbation $f(z, x)$:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(ii) $\lim\limits_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$ and if $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega; \text{ and}$$

(iii) there exist an integer $m \geqslant 2$ and a function $\eta \in L^\infty(\Omega)$ such that

$$\eta(z) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega, \ \eta \not\equiv \hat{\lambda}_m(2), \ \eta \not\equiv \hat{\lambda}_{m+1}(2)$$
$$\lim_{x\to 0} \frac{f(z,x)}{x} = \eta(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2 Evidently, $f(z, \cdot)$ is differentiable at $x = 0$ and $f'_x(z, 0) = \eta(z)$. Hypotheses H_1 imply that there exists $c_1 > 0$ such that $F(z, x) \geqslant -c_1x^2$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$.

For $\lambda > 0$, let $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (P_λ), defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\lambda}{p}||u||_p^p - \int_\Omega F(z, u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Proposition 8 *If hypotheses $H_1(i), (ii)$ hold and $\lambda \in (0, \hat{\lambda}_1(p))$, then the functional φ_λ is coercive.*

Proof By virtue of hypotheses $H_1(i), (ii)$, given $\epsilon > 0$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}|x|^p + c_2 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{5}$$

Then for all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\lambda}{p}||u||_p^p - \int_\Omega F(z, u(z))dz \\ &\geqslant \frac{1}{p}\left[1 - \frac{\lambda+\epsilon}{\hat{\lambda}_1(p)}\right] ||u||^p - c_2|\Omega|_N \text{ (see (1) and (4)).}\end{aligned}$$

Choosing $\epsilon \in (0, \hat{\lambda}_1(p) - \lambda)$ (recall that $\lambda < \hat{\lambda}_1(p)$), we can conclude from the last inequality that φ_λ is coercive. □

Let $V = \{u \in W_0^{1,p}(\Omega) : \int_\Omega u\, \hat{u}_1(p)^{p-1} dz = 0\}$ (recall $\hat{u}_1(p) \in \operatorname{int} C_+$). We have

$$W_0^{1,p}(\Omega) = \mathbb{R}\hat{u}_1(p) \oplus V.$$

We introduce the following quantity

$$\hat{\lambda}_V(p) = \inf\left[\frac{||Du||_p^p}{||u||_p^p} : u \in V,\ u \neq 0\right].$$

Lemma 9 $\hat{\lambda}_1(p) < \hat{\lambda}_V(p) \leqslant \hat{\lambda}_2(p)$.

Proof Clearly, $\hat{\lambda}_1(p) \leqslant \hat{\lambda}_V(p)$ (see (1)). Suppose that $\hat{\lambda}_1(p) = \hat{\lambda}_V(p)$. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq V$ such that

$$||u_n||_p = 1 \text{ and } ||Du||_p^p \to \hat{\lambda}_V(p) = \hat{\lambda}_1(p).$$

By passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty.$$

We have $u \in V$ and $||u||_p = 1$. Also,

$$\hat{\lambda}_1(p) \leqslant ||Du||_p^p \leqslant \liminf_{n\to\infty} ||Du_n||_p^p = \hat{\lambda}_V(p) = \hat{\lambda}_1(p),$$
$$\Rightarrow \hat{\lambda}_1(p) = ||Du||_p^p, \text{ hence } u = \pm\hat{u}_1(p) \text{ (recall } ||u||_p = 1).$$

But then $u \notin V$, a contradiction. So, we have proved that

$$\hat{\lambda}_1(p) < \hat{\lambda}_V(p).$$

Next, suppose that $\hat{\lambda}_2(p) < \hat{\lambda}_V(p)$. By virtue of Proposition 4, we can find $\hat{\gamma}_0 = \Gamma_0$ such that

$$||D\hat{\gamma}_0(t)||_p^p < \hat{\lambda}_V(p) \text{ for all } t \in [0, 1]. \tag{6}$$

We have $\hat{\gamma}_0(-1) = -\hat{u}_1(p)$, $\hat{\gamma}_0(1) = \hat{u}_1(p)$. Consider the function $[-1, 1] \ni t \to \sigma(t) = \int_\Omega \hat{\gamma}_0(t)\hat{u}_1(p)^{p-1} dz$. Evidently, this function is continuous and $\sigma(-1) = -||\hat{u}_1(p)||_p^p < 0 < ||\hat{u}_1(p)||_p^p = \sigma(1)$. So, by Bolzano's theorem, we can find $t_0 \in (0, 1)$ such that

$$\sigma(t_0) = \int_\Omega \hat{\gamma}_0(t_0)\hat{u}_1(p)^{p-1} dz = 0$$
$$\Rightarrow \hat{\gamma}_0(t_0) \in V, \text{ which contradicts (6).}$$

Therefore we infer that $\hat{\lambda}_V(p) \leqslant \hat{\lambda}_2(p)$. □

Proposition 10 *If hypotheses $H_1(i), (ii)$ hold and $\lambda = \hat{\lambda}_1(p)$, then $\varphi_\lambda|_V$ is bounded from below.*

Proof Let $v \in V$. We have

$$\begin{aligned}\varphi_\lambda(v) &= \frac{1}{p}||Dv||_p^p + \frac{1}{2}||Dv||_2^2 - \frac{\hat{\lambda}_1(p)}{p}||v||_p^p - \int_\Omega F(z,v)dz \\ &\geqslant \frac{\hat{\lambda}_V(p) - \hat{\lambda}_1(p) - \epsilon}{2}\,||v||_p^p - c_2|\Omega|_N \text{ (see (4))}. \end{aligned} \tag{7}$$

From Lemma 9 we know that $\hat{\lambda}_1(p) < \hat{\lambda}_V(p)$. So, we choose $\epsilon \in (0, \hat{\lambda}_V(p) - \hat{\lambda}_1(p))$. Then from (7) we infer that $\varphi_\lambda|_V$ with $\lambda = \hat{\lambda}_1(p)$, is bounded from below. □

Let $m_1 = \inf\limits_V \varphi_{\hat{\lambda}_1(p)} > -\infty$ (see Proposition 10). Note that, if $\lambda \in (0, \hat{\lambda}_1(p))$ then

$$\begin{aligned} &\varphi_{\hat{\lambda}_1(p)} \leqslant \varphi_\lambda, \\ \Rightarrow\ & m_1 \leqslant \inf_V \varphi_\lambda \text{ for all } \lambda \in (0, \hat{\lambda}_1(p)). \end{aligned} \tag{8}$$

Proposition 11 *If hypothesis H_1 holds, then we can find small $\epsilon > 0$ such that every $\lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p))$ we can find large $t_0 > 0$ such that*

$$\varphi_\lambda(\pm t_0 \hat{u}_1(p)) < m_1.$$

Proof By virtue of hypothesis $H_1(ii)$, given $\xi > 0$, we can find $M_1 = M_1(\xi) > 0$ such that

$$F(z,x) \geqslant \xi x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \geqslant M_1. \tag{9}$$

Let $t > 0$. We have

$$\begin{aligned} &\int_\Omega F(z, t\hat{u}_1(p))dz \\ &= \int_{\{t\hat{u}_1(p) \geqslant M_1\}} F(z, t\hat{u}_1(p))dz + \int_{\{0 \leqslant t\hat{u}_1(p) < M_1\}} F(z, t\hat{u}_1(p))dz \\ &\geqslant \xi t^2 \int_{\{t\hat{u}_1(p) \geqslant M_1\}} \hat{u}_1(p)^2 dz + \int_{\{0 \leqslant t\hat{u}_1(p) < M_1\}} F(z, t\hat{u}_1(p))dz \quad \text{(see (9))} \\ &\geqslant \xi t^2 ||\hat{u}_1(p)||_2^2 - (\xi + c_1)t^2 |\{0 \leqslant t\hat{u}_1(p) < M_1\}|_N. \end{aligned} \tag{10}$$

Note that $|\{0 \leqslant t\hat{u}_1(p) < M_1\}|_N \to 0$ as $t \to \infty$ (recall that $\hat{u}_1(p) \in \operatorname{int} C_+$). Also, $\xi > 0$ is arbitrary. So, we see that for all large $t > 0$, we have

$$\xi t^2 ||\hat{u}_1(p)||_2^2 - (\xi + c_1)t^2 \left|\{0 \leqslant t\hat{u}_1(p) < M_1\}\right|_N \geqslant -(m_1 - 1) + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2. \tag{11}$$

From (10) and (11) and for $t_0^1 > 0$ big, we have

$$\int_\Omega F(z, t\hat{u}_1(p))dz \geqslant -(m_1 - 1) + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2 \text{ for all } t \geqslant t_0^1. \tag{12}$$

So, we have

$$\begin{aligned}
\varphi_\lambda(t_0^1\hat{u}_1(p)) &= \frac{(t_0^1)^p}{p}||D\hat{u}_1(p)||_p^p + \frac{(t_0^1)^2}{2}||D\hat{u}_1(p)||_2^2 - \frac{\lambda(t_0^1)^p}{p}||\hat{u}_1(p)||_p^p \\
&\quad - \int_\Omega F(z, t_0^1\hat{u}_1(p))dz \\
&\leqslant \frac{(t_0^1)^p[\hat{\lambda}_1(p) - \lambda]}{p} + \frac{(t_0^1)^2}{2}||D\hat{u}_1(p)||_2^2 + m_1 - 1 - \frac{(t_0^1)^2}{2}||D\hat{u}_1(p)||_2^2 \\
&\quad \text{(see (12) and recall that } ||\hat{u}_1(p)||_p = 1) \\
&\leqslant \frac{(t_0^1)^p\epsilon}{p} + m_1 - 1 \text{ with } \epsilon > 0 \text{ (recall } \lambda < \hat{\lambda}_1(p)) \\
&< m_1 \left(\text{by choosing } \epsilon > 0 \text{ small such that } t_0^1 < \left(\frac{p}{\epsilon}\right)^{1/p}\right).
\end{aligned}$$

In a similar fashion, we can find large $t_0^2 > 0$ such that

$$\varphi_\lambda(-t_0\hat{u}_1(p)) < 0 \text{ for all } \lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p)), \text{ all } t \geqslant t_0^2.$$

Let $t_0 = \max\{t_0^1, t_0^2\}$. Then

$$\varphi_\lambda(\pm t_0\hat{u}_1(p)) < m_1 \text{ for all } \lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p)) \text{ with small } \epsilon > 0\,.$$

□

We introduce the following sets

$$\begin{aligned}
U_+ &= \{u \in W_0^{1,p}(\Omega) : u = t\hat{u}_1(p) + v,\ t > 0,\ v \in V\}, \\
U_- &= \{u \in W_0^{1,p}(\Omega) : u = -t\hat{u}_1(p) + v,\ t > 0,\ v \in V\}.
\end{aligned}$$

Proposition 12 *If hypothesis H_1 holds and $\lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p))$ with $\epsilon > 0$ as in Proposition 11, then problem (P_λ) has at least two nontrivial solutions*

$$\hat{u}_+ \in U_+ \text{ and } \hat{u}_- \in U_-$$

and both are local minimizers of the energy functional φ_λ.

Proof We introduce the functional

$$\hat{\varphi}_\lambda^+(u) = \begin{cases} \varphi_\lambda(u) & \text{if } u \in \overline{U}_+ \\ +\infty & \text{if } u \notin \overline{U}_+. \end{cases}$$

Evidently, $\hat{\varphi}_\lambda^+$ is lower semicontinuous and bounded from below (see Proposition 8). So, we can apply the Ekeland variational principle (see, for example, Gasinski and Papageorgiou [16, p. 582]) and $\{u_n\}_{n\geqslant 1} \subseteq U_+$ such that

$$\varphi_\lambda(u_n) = \hat{\varphi}_\lambda^+(u_n) \downarrow \inf \hat{\varphi}_\lambda^+ \text{ as } n \to \infty \tag{13}$$

$$\varphi_\lambda(u_n) = \hat{\varphi}_\lambda^+(u_n) \leqslant \hat{\varphi}_\lambda^+(y) + \frac{1}{n(1+||u_n||)}||y-u_n|| \tag{14}$$
$$\text{for all } y \in W_0^{1,p}(\Omega), \text{ all } n \geqslant 1.$$

Fix $n \geqslant 1$ and let $h \in W_0^{1,p}(\Omega)$. Then for small $t > 0$ we have $u_n + th \in U_+$. Using this as a test function in (14), we have

$$-\frac{||h||}{n(1+||u_n||)} \leqslant \frac{\varphi_\lambda(u_n+th)-\varphi_\lambda(u_n)}{t} \text{ (note that } \varphi_\lambda|_{\overline{U}_1} = \hat{\varphi}_\lambda^+|_{\overline{U}_+})$$
$$\Rightarrow -\frac{||h||}{n(1+||u_n||)} \leqslant \langle \varphi_\lambda'(u_n), h\rangle \text{ (recall } \varphi_\lambda \in C^1(W_0^{1,p}(\Omega))). \tag{15}$$

Since $h \in W_0^{1,p}(\Omega)$ is arbitrary, from (15) it follows that

$$(1+||u_n||)\varphi_\lambda'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty.$$

But φ_λ being coercive, satisfies the C-condition (see [30]). So, it follows that

$$u_n \to \hat{u}_+ \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

We have $\hat{u}_+ \in \overline{U}_+$ and so from (13) we infer that

$$\varphi_\lambda(\hat{u}_+) = \inf_{\overline{U}_+} \varphi_\lambda$$

Suppose that $\hat{u}_+ \in \partial U_+ = V$. Then

$$m_1 \leqslant \inf_{\overline{U}_+} \varphi_\lambda = \varphi_\lambda(\hat{u}_+) \text{ (see (8))},$$

which contradicts Proposition 11. Therefore $\hat{u}_+ \in U_+$ and it is a local minimizer of φ_λ, hence a nontrivial solution of (P_λ). By Ladyzhenskaya and Uraltseva [18, p. 286] we have $\hat{u}_+ \in L^\infty(\Omega)$. Then we can apply Theorem 1 of Lieberman [19] and obtain that $\hat{u}_+ \in C_0^1(\overline{\Omega})$.

Similarly, working with the functional

$$\hat{\varphi}_\lambda(u) = \begin{cases} \varphi_\lambda(u) & \text{if } u \in \overline{U}_- \\ +\infty & \text{if } u \notin \overline{U}_-, \end{cases}$$

we obtain a second nontrivial solution $\hat{u}_- \in U_- \cap C_0^1(\overline{\Omega})$, which is a local minimizer of φ_λ and is distinct from $\hat{u}_+$. $\square$

Next, using Morse theory, we will produce the third nontrivial solution. To this end, we need to compute the critical groups of φ_λ at the origin.

Proposition 13 *If hypotheses H_1 hold and $\lambda > 0$, then $C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geqslant 0$ with $d_m = \dim \overset{m}{\underset{\mathrm{i=1}}{\oplus}} E(\hat{\lambda}_i(2)) \geqslant 2$.*

Proof Let $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\psi(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{1}{2}\int_\Omega \eta(z)u(z)^2 dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We consider the homotopy

$$h_\lambda(t, u) = (1-t)\varphi_\lambda(u) + t\psi(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \text{ in } [0, 1], \ u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \text{ and } (h_\lambda)'_u(t_n, u_n) = 0 \quad \text{for all } n \geqslant 1. \tag{16}$$

We have

$$A_p(u_n)+A(u_n) = (1-t_n)\lambda|u_n|^{p-2}u_n+(1-t_n)N_f(u_n)+t_n\eta u_n \text{ for all } n \geqslant 1. \tag{17}$$

Let $y_n = \frac{u_n}{||u_n||}$ $n \geqslant 1$. Then $||y_n|| = 1$ for all $n \geqslant 1$ and so may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{18}$$

From (17), we have

$$||u_n||^{p-2}A_p(y_n) + A(y_n) = (1-t_n)\lambda|u_n|^{p-2}y_n + (1-t_n)\frac{N_f(u_n)}{||u_n||} + t_n\eta y_n \quad \text{for all } n \geqslant 1. \tag{19}$$

Note that hypothesis $H_1(i)$ and (16), imply that $\left\{\frac{N_f(u_n)}{||u_n||}\right\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded. This fact, in conjunction with hypothesis $H_1(iii)$, implies (at least for a subsequence) that (see [1])

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \eta y \text{ in } L^2(\Omega) \text{ as } n \to \infty. \tag{20}$$

Also, we have that $\{A_p(y_n)\}_{n\geqslant 1} \subseteq W^{-1,p'}(\Omega)$ is bounded (see (18) and Proposition 6). Therefore

$$||u_n||^{p-2}A_p(y_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty \text{ (see (16))}. \tag{21}$$

So, if in (19) we pass to the limit as $n \to \infty$ and use (18), (20), (21), then

$$\begin{aligned} & A(y) = \eta y, \\ \Rightarrow & -\Delta y(z) = \eta(z) y(z) \text{ for a.a. } z \in \Omega, \ y\,|_{\partial\Omega} = 0. \end{aligned} \tag{22}$$

From hypothesis $H_1(iii)$ and (22) it follows that $y \equiv 0$. On the other hand, from (19) we have

$$\left\{ \begin{array}{l} ||u_n||^{p-2}(-\Delta_p y_n(z)) - \Delta y_n(z) = (1-t_n)\lambda |y_n(z)|^{p-2} y_n(z) + (1-t_n)\frac{f(z,u_n(z))}{||u_n||} \\ \qquad\qquad\qquad\qquad\qquad\qquad + t_n \eta(z) y_n(z) \text{ for a.a. } z \in \Omega, \\ u_n|_{\partial\Omega} = 0 \end{array} \right\} \tag{23}$$

Then by (23) and Ladyzhenskaya and Uraltseva [18, p. 286], we know that we can find $M_2 > 0$ such that

$$||u_n||_\infty \leqslant M_2 \text{ for all } n \geqslant 1. \tag{24}$$

Since $||u_n||^{p-2} \to 0$ as $n \to \infty$ (see (16)), from (23), (24) and Theorem 1 of Lieberman [19], we know that there exist $\alpha \in (0, 1)$ and $M_3 > 0$ such that

$$y_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } ||y_n||_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant M_3 \text{ for all } n \geqslant 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and using (18), we have

$$\begin{aligned} & y_n \to y = 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty, \\ \Rightarrow & \ y_n \to y = 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty, \end{aligned}$$

which contradicts the fact that $||y_n|| = 1$ for all $n \geqslant 1$. Hence (16) cannot occur and so by the homotopy invariance of critical groups we have

$$C_k(\varphi_\lambda, 0) = C_k(\psi, 0) \text{ for all } k \geqslant 0. \tag{25}$$

From Cingolani and Vannella [12, Theorem 1.1] we know that

$$\begin{aligned} & C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geqslant 0, \\ \Rightarrow & \ C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geqslant 0 \text{ (see (25))}. \end{aligned}$$

□

Now we can generate the third nontrivial solution.

Proposition 14 *If hypotheses H_1 hold and $\lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p))$ with $\epsilon > 0$ as in Proposition 11, then problem (P_λ) admits a third nontrivial solution $\hat{y} \in C_0^1(\overline{\Omega})$.*

Proof Without any loss of generality, we may assume that $\varphi_\lambda(\hat{u}_-) \leqslant \varphi_\lambda(\hat{u}_+)$ (the analysis is similar if the opposite inequality holds). Also, we assume that K_{φ_λ} is finite (otherwise we already have infinitely many solutions for problem (P_λ)). From

Proposition 12, we know that $\hat{u}_+ \in C_0^1(\overline{\Omega})$ is a local minimizer of φ_λ. So, we can find small $\rho \in (0, 1)$ such that

$$\varphi_\lambda(\hat{u}_-) \leqslant \varphi_\lambda(\hat{u}_+) < \inf[\varphi_\lambda(u) : ||u - \hat{u}_+|| = \rho] = m_\rho^\lambda, \ ||\hat{u}_- - \hat{u}_+|| > \rho \quad (26)$$

(see Aizicovici et al. [1], proof of Proposition 29). Recall that φ_λ satisfies the C-condition. This fact and (26) permit the use of Theorem 2 (the mountain pass theorem). So, we can find $\hat{y} \in W_0^{1,p}(\Omega)$ such that

$$\hat{y} \in K_{\varphi_\lambda} \text{ and } m_\rho^\lambda \leqslant \varphi_\lambda(\hat{y}). \quad (27)$$

From (27) it follows that $\hat{y}$ is a solution of (P_λ) and $\hat{y} \notin \{\hat{u}_-, \hat{u}_+\}$. Since $\hat{y}$ is a critical point of φ_λ of mountain pass type, we have

$$C_1(\varphi_\lambda, \hat{y}) \neq 0. \quad (28)$$

On the other hand, from Proposition 13, we have

$$C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0 \text{ with } d_m \geqslant 2. \quad (29)$$

Comparing (28) and (29), we see that $\hat{y} \neq 0$. Nonlinear regularity theory (see [19]) implies $\hat{y} \in C_0^1(\overline{\Omega})$. This is the third nontrivial solution of (P_λ). □

So, we can state our first multiplicity theorem for problem (P_λ).

Theorem 15 *If hypotheses H_1 hold, then there exists $\epsilon > 0$ such that for all $\lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p))$ problem (P_λ) admits at least three nontrivial solutions*

$$\hat{u}_+, \hat{u}_-, \hat{y} \in C_0^1(\overline{\Omega}),$$

with $\hat{u}_+$ and $\hat{u}_-$ being local minimizers of the energy functional φ_λ.

By strengthening the regularity conditions on $f(z, \cdot)$, we can improve Theorem 15 and produce the fourth nontrivial solution. The new hypotheses on $f(z, x)$ are the following:

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \text{ for a.a. } z \in \Omega, \ \text{all } |x| \leqslant \rho;$$

(ii) $\lim_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$ and if $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x \to \pm\infty} \frac{F(z, x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega; \text{ and}$$

(iii) there exists an integer $m \geqslant 2$ such that

$$f'_x(z,0) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega,\ f'_x(\cdot,0) \not\equiv \hat{\lambda}_m(2),$$
$$f'_x(\cdot,0) \not\equiv \hat{\lambda}_{m+1}(2)$$
$$f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

Theorem 16 *If hypotheses H_2 hold, then there exists $\epsilon > 0$ such that for every $\lambda \in (\hat{\lambda}_1(p) - \epsilon, \hat{\lambda}_1(p))$ problem (P_λ) has at least four nontrivial solutions*

$$\hat{u}_+, \hat{u}_-, \hat{y}, \tilde{y} \in C_0^1(\overline{\Omega})$$

with $\hat{u}_+$ and $\hat{u}_-$ being local minimizers of the energy functional φ_λ.

Proof From Theorem 15, we already have three nontrivial solutions

$$\hat{u}_+, \hat{u}_-, \hat{y} \in C_0^1(\overline{\Omega}),$$

with $\hat{u}_+$ and $\hat{u}_-$ being local minimizers of φ_λ. Hence

$$C_k(\varphi_\lambda, \hat{u}_+) = C_k(\varphi_\lambda, \hat{u}_-) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{30}$$

Recall that

$$C_1(\varphi_\lambda, \hat{y}) \neq 0 \text{ (see (28))}. \tag{31}$$

Since $\varphi_\lambda \in C^2(W_0^{1,p}(\Omega))$, from (31) and Papageorgiou and Smyrlis [29] (see also Papageorgiou and Rădulescu [26]) it follows that

$$C_k(\varphi_\lambda, \hat{y}) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{32}$$

From Theorem 15, we know that

$$C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{33}$$

From Proposition 8, we know that φ_λ is coercive. Therefore

$$C_k(\varphi_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0. \tag{34}$$

Suppose that $K_{\varphi_\lambda} = \{0, \hat{u}_+, \hat{u}_-, \hat{y}\}$. Then from (30), (32), (33), (34) and the Morse relation (see (4)) with $t = -1$, we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0,$$
$$\Rightarrow (-1)^{d_m} = 0, \text{ a contradiction.}$$

So, we can find $\tilde{y} \in K_{\varphi_\lambda}$, $\tilde{y} \notin \{0, \hat{u}_+, \hat{u}_-, \hat{y}\}$. It follows that $\tilde{y}$ is the fourth nontrivial solution of (P_λ) and the nonlinear regularity theory implies $\tilde{y} \in C_0^1(\overline{\Omega})$. □

4 Near Resonance from the Right of $\hat{\lambda}_1(p) > 0$

In this section we examine problem (P_λ) as the parameter λ approaches $\hat{\lambda}_1(p) > 0$ from the above (from the right). In contrast to the previous case (Sect. 3), now the energy functional is indefinite.

We start with an existence result which is valid for all λ in the open spectral interval $(\hat{\lambda}_1(p), \hat{\lambda}_2(p))$. The hypotheses on the perturbation $f(z, x)$ are the following:

$H_3 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(ii) $\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exists $\tau \in (2, p)$ and $\beta_0 > 0$ such that

$$\beta_0 \leqslant \liminf_{x\to\pm\infty} \frac{pF(z, x) - f(z, x)x}{|x|^\tau} \text{ uniformly for a.a. } z \in \Omega; \text{ and}$$

(iv) there exists a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \leqslant \hat{\lambda}_1(2) \text{ for a.a. } z \in \Omega, \ \vartheta \not\equiv \hat{\lambda}_1(2)$$
$$\limsup_{x\to 0} \frac{2F(z, x)}{x^2} \leqslant \vartheta(z) \text{ uniformly for a.a. } z \in \Omega.$$

As before, for every $\lambda > 0$, $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional of problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\lambda}{p}||u||_p^p - \int_\Omega F(z, u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$.

Proposition 17 *If hypotheses H_3 hold and $\lambda > 0$, then φ_λ satisfies the C-condition.*

Proof Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_\lambda(u_n)| \leqslant M_3 \text{ for some } M_3 > 0, \text{ all } n \geqslant 1 \quad (35)$$
$$(1 + ||u_n||)\varphi'_\lambda(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \quad (36)$$

From (36) we have

$$
\begin{aligned}
& |\langle \varphi'_\lambda(u_n), h\rangle| \leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in W_0^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+, \\
\Rightarrow & \left| \langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \lambda \int_\Omega |u_n|^{p-2} u_n h dz - \int_\Omega f(z, u_n) h dz \right| \leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \\
& \text{for all } n \geqslant 1.
\end{aligned} \tag{37}
$$

In (37) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and obtain

$$
||Du_n||_p^p + ||Du_n||_2^2 - \lambda ||u_n||_p^p - \int_\Omega f(z, u_n) u_n dz \leqslant \epsilon_n \text{ for all } n \geqslant 1. \tag{38}
$$

On the other hand from (35), we have

$$
-||Du_n||_p^p - \frac{p}{2} ||Du_n||_2^2 + \lambda ||u_n||_p^p + \int_\Omega p F(z, u_n) dz \leqslant p M_3 \text{ for all } n \geqslant 1. \tag{39}
$$

We add (38) and (39). Then

$$
\begin{aligned}
& \int_\Omega [p F(z, u_n) - f(z, u_n) u_n] dz \leqslant M_4 + \left(\frac{p}{2} - 1\right) ||Du_n||_2^2 \\
& \text{for some } M_4 > 0, \text{ all } n \geqslant 1.
\end{aligned} \tag{40}
$$

By virtue of hypotheses $H_3(i), (iii)$, we can find $\beta_1 \in (0, \beta_0)$ and $c_3 > 0$ such that

$$
\beta_1 |x|^\tau - c_3 \leqslant p F(z, x) - f(z, x) x \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{41}
$$

We use (41) in (40) and obtain

$$
\beta_1 ||u_n||_\tau^\tau \leqslant M_5 + \left(\frac{p}{2} - 1\right) ||Du_n||_2^2 \text{ for some } M_5 > 0 \text{ and all } n \geqslant 1. \tag{42}
$$

Suppose that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. Then $||u_n|| \to \infty$ as $n \to \infty$. Set $y_n = \frac{u_n}{||u_n||}$, $n \geqslant 1$. By passing to a suitable subsequence if necessary, we may assume that

$$
y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{43}
$$

From (42) we have

$$
\begin{aligned}
& \beta_1 ||y_n||_\tau^\tau \leqslant \frac{M_5}{||u_n||^\tau} + \left(\frac{p}{2} - 1\right) \frac{1}{||u_n||^{\tau-2}} ||Dy_n||_2^2 \quad \text{for all } n \geqslant 1, \\
\Rightarrow & \ y_n \to 0 \text{ in } L^\tau(\Omega) \text{ as } n \to \infty \text{ (recall } 2 < \tau < p), \text{ hence } y = 0 \text{ (see (43))}.
\end{aligned} \tag{44}
$$

On the other hand, from (37) we have

$$\left|\left\langle A_p(y_n), h\right\rangle + \frac{1}{||u_n||^{p-2}} \langle A(y_n), h\rangle - \lambda \int_\Omega |y_n|^{p-2} y_n h dz - \int_\Omega \frac{f(z, u_n)}{||u_n||^{p-1}} h dz\right| \leqslant \epsilon_n \quad \text{for all} \quad n \geqslant 1. \tag{45}$$

Hypotheses $H_3(i)$, (ii), imply that

$$|f(z, x)| \leqslant c_4(1 + |x|^{p-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_4 > 0,$$
$$\Rightarrow \left\{\frac{N_f(u_n)}{||u_n||^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

If in (45) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (44), we obtain

$$\lim_{n\to\infty} \left\langle A_p(y_n), y_n - y\right\rangle = 0 \text{ (recall } p > 2),$$
$$\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 6)}, \text{ hence } ||y|| = 1. \tag{46}$$

Comparing (44) and (46), we reach a contradiction. This proves that $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{47}$$

In (37) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (47). Then

$$\lim_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \langle A(u_n), u_n - u\rangle\right] = 0,$$
$$\Rightarrow \limsup_{n\to\infty} \left[\left\langle A_p(u_n), u_n - u\right\rangle + \langle A(u), u_n - u\rangle\right] \leqslant 0 \text{ (since } A \text{ is monotone)},$$
$$\Rightarrow \limsup_{n\to\infty} \left\langle A_p(u_n), u_n - u\right\rangle \leqslant 0 \text{ (see (47))},$$
$$\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

This proves that the functional φ_λ satisfies the C-condition for all $\lambda > 0$. □

Proposition 18 *If hypotheses H_3 hold and $\lambda > \hat{\lambda}_1(p)$, then $\varphi_\lambda(t\hat{u}_1(p)) \to -\infty$ as $t \to \pm\infty$ (that is, $\varphi_\lambda|_{\mathbb{R}\hat{u}_1(p)}$ is anticoercive).*

Proof Hypothesis $H_3(ii)$ implies that

$$\lim_{x\to\pm\infty} \frac{F(z, x)}{|x|^p} = 0 \text{ uniformly for a.a. } z \in \Omega. \tag{48}$$

From (48) and hypothesis $H_3(i)$, we see that given $\epsilon > 0$, we can find $c_5 = c_5(\epsilon) > 0$ such that

$$F(z, x) \geqslant -\epsilon|x|^p - c_5 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{49}$$

Then for $t \neq 0$, we have

$$\begin{aligned}
\varphi_\lambda(t\hat{u}_1(p)) &= \frac{|t|^p}{p}\hat{\lambda}_1(p) + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2 - \frac{\lambda|t|^p}{p} - \int_\Omega F(z, t\hat{u}_1(p))dz \\
&\quad (\text{recall } ||\hat{u}_1(p)||_p = 1) \\
&\leqslant \frac{|t|^p}{p}[\hat{\lambda}_1(p) - \lambda] + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2 + \frac{\epsilon|t|^p}{p} + c_5|\Omega|_N \text{ (see (49))} \\
&= \frac{|t|^p}{p}[\hat{\lambda}_1(p) + \epsilon - \lambda] + \frac{t^2}{2}||D\hat{u}_1(p)||_2^2 + c_3|\Omega|_N.
\end{aligned} \tag{50}$$

Choose $\epsilon \in (0, \lambda - \hat{\lambda}_1(p))$ (recall $\lambda > \hat{\lambda}_1(p)$). Then from (50) and since $p > 2$, we have

$$\varphi_\lambda(t\hat{u}_1(p)) \to -\infty \text{ as } t \to \pm\infty.$$

This completes the proof. □

Let $D = \left\{u \in W_0^{1,p}(\Omega) : ||Du||_p^p = \hat{\lambda}_2(p)||u||_p^p\right\}$.

Proposition 19 *If hypotheses H_3 hold and $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$, then $\varphi_\lambda|_D$ is coercive.*

Proof From (48) and hypothesis $H_3(i)$, we see that given $\epsilon > 0$, we can find $c_6 = c_6(\epsilon) > 0$ such that

$$F(z, x) \leqslant \frac{\epsilon}{p}|x|^p + c_6 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{51}$$

Let $u \in D$. We have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\lambda}{p}||u||_p^p - \int_\Omega F(z, u)dz \\
&\geqslant \frac{1}{p}||Du||_p^p - \frac{\lambda}{p\hat{\lambda}_2(p)}||Du||_p^p - \frac{\epsilon}{p\hat{\lambda}_2(p)}||Du||_p^p - c_6|\Omega|_N \text{ (see (51))} \\
&= \frac{1}{p}\left[1 - \frac{\lambda + \epsilon}{\hat{\lambda}_2(p)}\right]||u||^p - c_6|\Omega|_N.
\end{aligned} \tag{52}$$

Choosing $\epsilon \in (0, \hat{\lambda}_2(p) - \lambda)$ (recall $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$), from (52) we infer that $\varphi_\lambda|_D$ is coercive. □

By virtue of Proposition 19, we have

$$m_D = \inf_D \varphi_\lambda > -\infty.$$

Then, invoking Proposition 18, we can find $t^* > 0$ such that

$$\varphi_\lambda(\pm t^* \hat{u}_1(p)) < m_D. \tag{53}$$

We introduce the following sets

$$E_0 = \{\pm t^* \hat{u}_1(p)\},\ \ E = \text{conv}\,\{\pm t^* \hat{u}_1(p)\} = \{-st^* \hat{u}_1(p) + (1-s)t^* \hat{u}_1(p) : s \in [0, 1]\}.$$

For this pair $\{E_0, E\}$ and the set D introduced above, we have the following property.

Proposition 20 *The pair $\{E_0, E\}$ is linking with D in $W_0^{1,p}(\Omega)$.*

Proof Let $\hat{G} = \{u \in W_0^{1,p}(\Omega) : ||Du||_p^p < \hat{\lambda}_2(p)||u||_p^p\}$. We claim that $-t^* \hat{u}_1(p)$ and $t^* \hat{u}_1(p)$ belong to different path components of the set $\hat{G}$. To this end, let $\gamma \in C([0, 1], W_0^{1,p}(\Omega))$ be a path such that

$$\gamma(0) = -t^* \hat{u}_1(p) \text{ and } \gamma(1) = t^* \hat{u}_1(p).$$

By virtue of Proposition 4, we have

$$\hat{\lambda}_2(p) \leqslant \max \left[\frac{||D\gamma(t)||_p^p}{||\gamma(t)||_p^p} : t \in [0, 1] \right]$$

and so we can find $t_0 \in (0, 1)$ such that $\gamma(t_0) \notin \hat{G}$, which shows that $-t^* \hat{u}_1(p)$ and $t^* \hat{u}_1(p)$ cannot be in the same path component of the set $\hat{G}$. This means that, given any $\gamma \in C([0, 1], W_0^{1,p}(\Omega))$ with

$$\gamma(0) = -t^* \hat{u}_1(p) \text{ and } \gamma(1) = t^* \hat{u}_1(p),$$

we have

$$\gamma([0, 1]) \cap \partial \hat{G} \neq \emptyset.$$

Note that $\partial \hat{G} \subseteq D$. Therefore

$$\begin{aligned} &\gamma([0, 1]) \cap D \neq \emptyset \\ \Rightarrow\ &\{E_0, E\} \text{ links with } D \text{ in } W_0^{1,p}(\Omega) \text{ (see Definition 1).} \end{aligned}$$

□

Proposition 21 *If hypothesis H_3 holds and $\lambda > 0$, then $u = 0$ is a local minimizer of the functional φ_λ.*

Proof By virtue of hypotheses $H_3(i), (iv)$ we see that given $\epsilon > 0$, we can find $c_7 = c_7(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\vartheta(z)+\epsilon)x^2 + c_7|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{54}$$

Then for every $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\varphi_\lambda(u) &\geqslant \frac{1}{2}\left[||Du||_2^2 - \int_\Omega \vartheta(z)u^2 dz\right] - \frac{\epsilon}{2\hat{\lambda}_1(2)}||u||^2 - c_8||u||^p - \frac{\lambda}{p\hat{\lambda}_1(p)}||u||^p \\ &\quad \text{for some } c_8 > 0 \text{ (see (1), (2) and (54))} \\ &\geqslant \frac{1}{2}\left[\hat{\xi}_0 - \frac{\epsilon}{\hat{\lambda}_1(2)}\right]||u||^2 - c_9||u||^p \text{ for some } c_9 > 0 \text{ (see Proposition 5).}\end{aligned}$$

We choose $\epsilon \in (0, \hat{\lambda}_1(2)\hat{\xi}_0)$ and have

$$\varphi_\lambda(u) \geqslant c_{10}||u||^2 - c_9||u||^p \text{ for some } c_{10} > 0, \text{ all } u \in W_0^{1,p}(\Omega). \tag{55}$$

Since $2 < p$, from (55) it follows that we can find small $\rho \in (0,1)$ such that

$$\begin{aligned}&\varphi_\lambda(u) > 0 = \varphi_\lambda(0) \text{ for all } u \in W_0^{1,p}(\Omega) \text{ with } 0 < ||u|| \leqslant \rho, \\ \Rightarrow\ & u = 0 \text{ is a (strict) local minimizer of } \varphi_\lambda.\end{aligned}$$

We can state the following existence result.

Theorem 22 *If hypothesis H_3 holds and $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_2(p))$, then problem (P_λ) admits a nontrivial solution $\hat{u} \in C_0^1(\overline{\Omega})$.*

Proof Propositions 17, 20, and (53), permit the use of Theorem 1 (the linking theorem). So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that (see Chang [8])

$$\hat{u} \in K_{\varphi_\lambda} \text{ and } C_1(\varphi_\lambda, \hat{u}) \neq 0. \tag{56}$$

By Proposition 21, we know that $u = 0$ is a local minimizer of φ_λ. Hence

$$C_k(\varphi_\lambda, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geqslant 0. \tag{57}$$

From (56) and (57) it follows that $\hat{u} \neq 0$ and $\hat{u}$ is a solution of (P_λ). Moreover, the nonlinear regularity theory implies that $\hat{u} \in C_0^1(\overline{\Omega})$. □

We can have multiple solutions when we restrict λ to be near $\hat{\lambda}_1(p)$ from above (near resonance from the right). To do this, we introduce the following hypotheses on the perturbation $f(z,x)$.

H_4 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1+|x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+$;

(ii) there exists a function $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant 0$ for a.a. $z \in \Omega$, $\vartheta \not\equiv 0$ such that

$$\limsup_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \leqslant \vartheta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist an integer $m \geqslant 2$ and a function $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega,\ \ \eta \not\equiv \hat{\lambda}_m(2),\ \ \eta \not\equiv \hat{\lambda}_{m+1}(2)$$
$$\lim_{x\to 0} \frac{f(z,x)}{x} = \eta(z) \text{ uniformly for a.a. } z \in \Omega;\ \text{ and}$$

(iv) for every $\rho > 0$ there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \longmapsto f(z,x) + \xi_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 3 Evidently, for a.a. $z \in \Omega$, $f(z,\cdot)$ is differentiable at $x = 0$ and $\eta(\cdot) = f'_x(\cdot, 0)$.

We will produce solutions of constant sign. For this purpose, we introduce the positive and negative truncations of $f(z,\cdot)$, namely the Carathéodory functions

$$f_\pm(z,x) = f(z, \pm x^\pm).$$

Let $F_\pm(z,x) = \int_0^x f_\pm(z,s)ds$ and consider the C^1-functionals $\varphi_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\lambda}{p}||u^\pm||_p^p - \int_\Omega F_\pm(z,u(z))dz$$
$$\text{for all } u \in W_0^{1,p}(\Omega).$$

Next, we produce a pair of nontrivial constant sign solutions.

Proposition 23 *If hypothesis H_4 holds, then we can find $\epsilon > 0$ such that for all $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ problem (P_λ) has at least two nontrivial solutions of constant sign*

$$u_n \in int\, C_+ \text{ and } v_0 \in -int\, C_+,$$

both being local minimizers of the energy functional φ_λ.

Proof By virtue of hypotheses $H_4(i), (ii)$, given $\delta > 0$, we can find $c_{11} = c_{11}(\delta) > 0$ such that

$$F(z,x) \leqslant \frac{1}{p}(\vartheta(z) + \delta)|x|^p + c_{11} \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{58}$$

Since $\lambda > \hat{\lambda}_1(p)$, we have $\lambda = \hat{\lambda}_1(p) + \mu$ with $\mu > 0$. Then for every $u \in W_0^{1,p}(\Omega)$ we have (see Papageorgiou and Kyritsi [24, p. 356])

$$\begin{aligned}
\varphi_\lambda^+(u) &= \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{\hat{\lambda}_1(p)}{p}||u^+||_p^p - \frac{\mu}{p}||u^+||_p^p - \int_\Omega F_+(z,u)dz \\
&\geqslant \frac{1}{p}||Du||_p^p - \int_\Omega (\hat{\lambda}_1(p) + \vartheta(z))(u^+)^p dz - \frac{\mu+\delta}{p\hat{\lambda}_1(p)}||u||^p - c_{11}|\Omega|_N \text{ (see (58))} \\
&\geqslant \frac{1}{p}\left[\xi^* - \frac{\mu+\delta}{\hat{\lambda}_1(p)}\right]||u||^p - c_{11}|\Omega|_N \text{ for some } \xi^* > 0.
\end{aligned}$$

Since $\delta > 0$, is arbitrary, for $\mu \in (0, \xi^* \, \hat{\lambda}_1(p))$, we have that φ_λ^+ is coercive. Also, using the Sobolev embedding theorem, we see that φ_λ^+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda^+(u_0) = \inf \left[\varphi_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{59}$$

Hypothesis $H_4(iii)$ implies that for small $t \in (0, 1)$

$$\begin{aligned}
&\varphi_\lambda^+(t\hat{u}_1(2)) < 0 \text{ (recall that } p > 2), \\
\Rightarrow\ &\varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0) \text{ (see (59)), hence } u_0 \neq 0.
\end{aligned}$$

From (59) we have

$$\begin{aligned}
&(\varphi_\lambda^+)'(u_0) = 0, \\
\Rightarrow\ &A_p(u_0) + A(u_0) = \lambda(u_0^+)^{p-1} + N_{f_+}(u_0).
\end{aligned} \tag{60}$$

On (60) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain $u_0 \geqslant 0,\ u_0 \neq 0$. So, (60) becomes

$$\begin{aligned}
&A_p(u_0) + A(u_0) = \lambda u_0^{p-1} + N_f(u_0), \\
\Rightarrow\ &u_0 \text{ is a solution of} (P_\lambda),\ u_0 \in C_+ \backslash \{0\} \\
&\qquad\qquad \text{(by the nonlinear regularity theory).}
\end{aligned}$$

Let $\rho = ||u_n||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H_2(iv)$. Then

$$\begin{aligned}
&-\Delta_p u_0(z) - \Delta u_0(z) + \xi_\rho u_0(z)^{p-1} \\
&= (\lambda + \xi_\rho)u_0(z)^{p-1} + f(z, u_0(z)) \geqslant 0 \text{ for a.a. } z \in \Omega, \\
\Rightarrow\ &\Delta_p u_0(z) + \Delta u_0(z) \leqslant \xi_\rho u_p(z) \text{ for a.a. } z \in \Omega.
\end{aligned}$$

From the nonlinear maximum principle of Pucci and Serrin [32, p. 111 and 120], we obtain that $u_0 \in \operatorname{int} C_+$. Since $\varphi_\lambda|_{C_+} = \varphi_\lambda^+|_{C_+}$, we infer that $u_0 \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$ minimizer of φ_λ. Invoking Proposition 7, we infer that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ_λ.

Similarly, working with φ_φ^- we produce $v_0 \in -\text{int}\, C_+$ a second nontrivial constant sign solution of (P_λ), which is a local minimizers of φ_λ. □

Let $\epsilon > 0$ be as in the above proposition. Hypotheses $H_4(i)$, (iii) imply that given $\delta > 0$, we can find $c_{12} = c_{12}(\delta) > \hat{\lambda}_1(p) + \epsilon$ such that

$$f(z,x)x \geqslant (\eta(z) - \delta)x^2 - c_{12}|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{61}$$

This estimate leads to the following auxiliary Dirichlet problem

$$-\Delta_p u(z) - \Delta u(z) = (\eta(z) - \delta)u(z) - c_{13}|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0 \tag{62}$$

where $c_{13} = c_{13}(\delta, \lambda) = c_{12} - \lambda$, with $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$.

Proposition 24 *For small $\delta > 0$, problem (62) has a unique nontrivial positive solution $u_* \in \text{int}\, C_+$ and because (62) is odd $v_* = -u_* \in -int\, C_+$ is the unique nontrivial negative solution of (62).*

Proof First we establish the existence of a nontrivial positive solution. To this end, let $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi_+(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \frac{1}{2}\int_\Omega (\eta(z) + \delta)(u^+)^2 dz + \frac{c_{13}}{p}||u^+||_p^p$$
$$\text{for all } u \in W_0^{1,p}(\Omega).$$

Since $p > 2$, it is clear that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(u_*) = \inf\left[\psi_+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{63}$$

Let $t > 0$. We have

$$\begin{aligned}\psi_+(t\hat{u}_1(2)) &= \frac{t^p}{p}||D\hat{u}_1(2)||_p^p + \frac{t^2}{2}\hat{\lambda}_1(2) - \frac{t^2}{2}\int_\Omega (\eta(z) - \delta)\hat{u}_1(2)^2 dz + \frac{c_{13}}{p}t^p||\hat{u}_1(2)||_p^p \\ &\qquad \text{(recall } ||\hat{u}_1(2)||_2 = 1) \\ &\leqslant \frac{t^p}{p}\left[1 + \frac{c_{13}}{\hat{\lambda}_1(p)}\right]||\hat{u}_1(2)||^p - \frac{t^2}{2}\left[\int_\Omega (\eta(z) - \hat{\lambda}_1(2))\hat{u}_1(2)^2 dz - \delta\right].\end{aligned}$$

Evidently, $\xi_0 = \int_\Omega (\eta(z) - \hat{\lambda}_1(2))\hat{u}_1(2)^2 dz > 0$. So, if $\delta \in (0, \xi_0)$, then

$$\psi_+(t\hat{u}_1(2)) \leqslant \frac{t^p}{p}c_{14} - \frac{t^2}{2}c_{15} \text{ some } c_{14}, c_{15} > 0.$$

Since $p > 2$, by choosing small $t \in (0, 1)$, we have

$$\begin{aligned} &\psi_+(t\hat{u}_1(2)) < 0, \\ \Rightarrow\ &\psi_+(u_*) < 0 = \psi_+(0) \text{ (see (63)), hence } u_* \neq 0. \end{aligned}$$

From (63) we have

$$\begin{aligned} &\psi_+'(u_*) = 0, \\ \Rightarrow\ &A_p(u_*) + A(u_*) = (\eta - \delta)u_*^+ - c_{13}(u_*^+)^{p-1}. \end{aligned} \tag{64}$$

On (64) we act with $-u_*^- \in W_0^{1,p}(\Omega)$ and obtain $u^* \geqslant 0,\ u_* \neq 0$. Then

$$\begin{aligned} &A_p(u_*) + A(u_*) = (\eta - \delta)u_* - c_{13}u_*^{p-1}, \\ \Rightarrow\ &u_* \in C_+\backslash\{0\} \text{ (nonlinear regularity solves (62)).} \end{aligned}$$

In fact, we have (see Pucci and Serrin [32, p. 111 and 120])

$$\begin{aligned} &\Delta_p u_*(z) + \Delta u_*(z) \leqslant c_{13}u_*(z)^{p-1} \text{ for a.a. } z \in \Omega \\ \Rightarrow\ &u_* \in \operatorname{int} C_+. \end{aligned}$$

Next, we show the uniqueness of this positive solutions. To this end, let

$$G_0(t) = \frac{t^p}{p} + \frac{t^2}{2} \text{ for all } t \geqslant 0.$$

Then $G_0(\cdot)$ is increasing and $t \to G_0(t^{1/2})$ is convex. We set

$$G(y) = G_0(|y|) \text{ for all } y \in \mathbb{R}^N.$$

Evidently, $G \in C^1(\mathbb{R}^N)$ (recall $p > 2$) and we have

$$\begin{aligned} &\nabla G(y) = a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N \\ &\operatorname{div} a(Du) = \Delta_p u + \Delta u \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

Let $\mu_+ : L^1(\Omega) \to \mathbb{R}$ be the integral functional defined by

$$\mu_+(u) = \begin{cases} \int_\Omega G(Du^{1/2})dz & \text{if } u \geqslant 0,\ u^{1/2} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} \mu_+ = \{u \in L^1(\Omega) : \mu_+(u) < \infty\}$ (the effective domain of μ_+) and let $y = (tu_1 + (1-t)u_2)^{1/2} \in W_0^{1,p}(\Omega)$ with $t \in [0, 1]$. From Benguria et al. [6, Lemma 4], we have

$$
\begin{aligned}
&|Dy(z)| \leqslant \left(t|Du_1(z)^{1/2}|^2 + (1-t)|Du_2(z)^{1/2}|^2\right)^{1/2} \text{ for a.a. } z \in \Omega,\\
\Rightarrow\ & G_0(|Dy(z)|) \leqslant G_0\left(t|Du_1(z)^{1/2}|^2 + (1-t)|Du_2(z)^{1/2}|^2\right) \text{ (since } G_0 \text{ is increasing)}\\
&\leqslant tG_0(|Du_1(z)^{1/2}|) + (1-t)G_0(|Du_2(z)^{1/2}|) \text{ (since } t \to G_0(t^{1/2})\\
&\text{is convex)},\\
\Rightarrow\ & G(Dy(z)) \leqslant tG(Du_1(z)^{1/2}) + (1-t)G(Du_2(z)^{1/2}) \text{ for a.a. } z \in \Omega,\\
\Rightarrow\ & \mu_+ \text{is convex.}
\end{aligned}
$$

Also, by the Fatou lemma we see that μ_+ is lower semicontinuous.

Let $y_* \in W_0^{1,p}(\Omega)$ be another positive solution of (62). From the first part of the proof, we have $y_* \in \operatorname{int} C_+$. Let $h \in C_0^1(\overline{\Omega})$ and $t \in (-1, 1)$ with $|t|$ small. Then we will have

$$
\begin{aligned}
&u_*^2 + th \in \operatorname{int} C_+ \text{ and } y_*^2 + th \in \operatorname{int} C_+\\
\Rightarrow\ & u_*^2,\ y_*^2 \in \operatorname{dom} \mu_+.
\end{aligned}
$$

So, μ_+ is Gâteaux differentiable at u_* and at y_* in the direction h. Using the chain rule, we obtain

$$
\begin{aligned}
\mu_+'(u_*^2)(h) &= \frac{1}{2}\int_\Omega \frac{-\Delta_p u_* - \Delta u_*}{u_*} h dz\\
\mu_+'(y_*^2)(h) &= \frac{1}{2}\int_\Omega \frac{-\Delta_p y_* - \Delta y_*}{y_*} h dz \text{ for all } h \in C_0^1(\overline{\Omega}).
\end{aligned}
$$

The convexity of μ_+ implies that μ_+' is monotone. Hence

$$
\begin{aligned}
0 &\leqslant \frac{1}{2}\int_\Omega \left(\frac{-\Delta_p u_* - \Delta u_*}{u_*} - \frac{-\Delta_p y_* - \Delta y_*}{y_*}\right)(u_*^2 - y_*^2) dz\\
&= \frac{1}{2}\int_\Omega c_{13}(y_*^{p-2} - u_*^{p-2})(u_*^2 - y_*^2) dz \leqslant 0 \text{ (recall } p > 2),\\
\Rightarrow\quad & u_* = y_*.
\end{aligned}
$$

This proves the uniqueness of the positive solution $u_* \in \operatorname{int} C_+$.

Since (62) is odd, $v_* = -u_* \in -\operatorname{int} C_+$ is the unique nontrivial negative solution of (62). ◻

Using the proposition, we can establish the existence of extremal nontrivial constant sign solutions, that is, a smallest positive solution and a biggest nontrivial negative solution.

Proposition 25 *If hypothesis H_4 holds and $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ with $\epsilon > 0$ as in Proposition 23, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$.*

Proof Let $S_+(\lambda)$ be the set of positive of problem (P_λ). From Proposition 23 and its proof, we have

$$S_+(\lambda) \neq \emptyset \text{ and } S_+(\lambda) \subseteq \operatorname{int} C_+.$$

As in Gasinski and Papageorgiou [17], exploiting the monotonicity of $u \to A_p(u)+A(u)$ we have that the solution set $S_+(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S_+(\lambda)$, then we can find $u \in S_+(\lambda)$ such that $u \leqslant u_1,\ u \leqslant u_2$. Since we are looking for the smallest positive solution, without any loss of generality we may assume that there exists $M_6 > 0$ such that

$$0 \leqslant u(z) \leqslant M_6 \qquad \text{for all } z \in \overline{\Omega},\ \text{all } u \in S_+(\lambda). \tag{65}$$

From Dunford and Schwartz [14, p. 336], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S_+(\lambda)$ such that $\inf S_+(\lambda) = \inf\limits_{n\geqslant 1} u_n$.

We have

$$A_p(u_n) + A(u_n) = \lambda u_n^{p-1} + N_f(u_n) \text{ for all } n \geqslant 1, \tag{66}$$
$$\Rightarrow \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (65)).}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{67}$$

On (66) we act with $u_n - u_\lambda^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (67). Then

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A(u_n), u_n - u_\lambda^*\rangle\right] = 0,$$
$$\Rightarrow u_n \to u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see the proof of Proposition 17).} \tag{68}$$

Claim 1 $u_* \leqslant u$ for all $u \in S_+(\lambda)$.

Let $u \in S_+(\lambda)$ and consider the following function

$$\beta_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ (\eta(z)-\delta)x - c_{13}x^{p-1} & \text{if } 0 \leqslant x \leqslant u(z) \quad \text{(see (61))} \\ (\eta(z)-\delta)u(z) - c_{13}u(z)^{p-1} & \text{if } u(z) < x \end{cases} \tag{69}$$

(see (61)). This is a Carathéodory function. We set $B_+(z,x) = \int_0^x \beta_+(z,s)ds$ and consider the C^1-functional $\xi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\xi_+(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega B_+(z,u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (69) it is clear that ξ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\xi_+(\hat{u}_*) = \inf\left[\xi_+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{70}$$

As before (see the proof of Proposition 24), for $y \in \operatorname{int} C_+$, and for small $t > 0$ (at least such that $ty \leqslant u$, recall that $u \in \operatorname{int} C_+$, and see Filippakis et al. [15, Lemma 3.3]), we have

$$\begin{aligned}&\xi_+(ty) < 0 = \xi_+(0),\\ \Rightarrow\ &\xi_+(\hat{u}_*) < 0 = \xi_+(0) \text{ (see (70)), hence } \hat{u}_* \neq 0.\end{aligned}$$

From (70) we have

$$\begin{aligned}&\xi_+'(\hat{u}_*) = 0,\\ \Rightarrow\ &A_p(\hat{u}_*) + A(\hat{u}_*) = N_{\beta_+}(\hat{u}_*).\end{aligned} \tag{71}$$

On (71) we act with $-\hat{u}_*^- \in W_0^{1,p}(\Omega)$ and obtain $\hat{u}_* \geqslant 0,\ \hat{u}_* \neq 0$ (see (69)). Also, on (71) we act with $(\hat{u}_* - u)^+ \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}&\left\langle A_p(\hat{u}_*), (\hat{u}_* - u)^+\right\rangle + \left\langle A(\hat{u}_*), (\hat{u}_* - u)^+\right\rangle\\ &\quad = \int_\Omega \beta_+(z, \hat{u}_*)(\hat{u}_* - u)^+ dz\\ &\quad = \int_\Omega [(\eta(z) - \delta)u - c_{13}u^{p-1}](\hat{u}_* - u)^+ dz \text{ (see (71))}\\ &\quad \leqslant \int_\Omega [\lambda u^{p-1} + f(z, u)](\hat{u}_* - u)^+ dz \text{ (see (61) and recall } c_{13} = c_{12} - \lambda > 0)\\ &\quad = \left\langle A_p(u) + A(u), (\hat{u}_* - u)^+\right\rangle \text{ (since } u \in S_+(\lambda))\\ &\quad \Rightarrow \left\langle A_p(\hat{u}_*) - A_p(u), (\hat{u}_* - u)^+\right\rangle + ||D(\hat{u}_* - u)^+||_2^2 \leqslant 0,\\ &\quad \Rightarrow \hat{u}_* \leqslant u.\end{aligned}$$

Therefore we have proved that

$$\hat{u}_* \in [0, u] = \left\{y \in W_0^{1,p}(\Omega) : 0 \leqslant y(z) \leqslant u(z) \text{ for a.a. } z \in \Omega\right\},\ \hat{u}_* \neq 0.$$

So, (71) becomes

$$\begin{aligned}&A_p(\hat{u}_*) + A(\hat{u}_*) = (\eta(z) - \delta)\hat{u}_* - c_{13}\hat{u}_*^{p-1},\\ \Rightarrow\ &\hat{u}_* \text{ is a positive solution of problem (62)},\\ \Rightarrow\ &\hat{u}_* = u_* \in \operatorname{int} C_+ \text{ (see Proposition 24)}.\end{aligned}$$

Thus we have proved the claim.

Passing to the limit as $n \to \infty$ in (66) and using (68), we obtain

$$\begin{aligned}&A_p(u_\lambda^*) + A(u_\lambda^*) = \lambda(u_\lambda^*)^{p-1} + N_f(u_\lambda^*),\ u_* \leqslant u_\lambda^*,\\ \Rightarrow\ &u_\lambda^* \in S_+(\lambda) \text{ and } u_\lambda^* = \inf S_+(\lambda).\end{aligned}$$

For the biggest negative solution we use the set $S_-(\lambda)$ which is upward directed (that is, if $v_1, v_2 \in S_-(\lambda)$, then we can find $v \in S_-(\lambda)$ such that $v_1 \leqslant v,\ v_2 \leqslant v$). Reasoning as above, we produce $v_\lambda^* \in S_-(\lambda) \subseteq -\mathrm{int}\, C_+$ a biggest negative solution of (P_λ). ☐

Using these extremal constant sign solutions, we can produce a nodal solution of problem (P_λ).

Proposition 26 *If hypothesis H_4 holds and $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p)+\epsilon)$ with $\epsilon > 0$ as in Proposition 23, then problem (P_λ) admits a nodal solution $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C_0^1(\overline{\Omega})$.*

Proof Let $u_\lambda^* \in \mathrm{int}\, C_+$ and $v_\lambda^* \in -\mathrm{int}\, C_+$ be the extremal constant sign solutions of (P_λ) produced in Proposition 26. We introduce the following truncation of the reaction in problem (P_λ)

$$g_\lambda(z,x) = \begin{cases} \lambda |v_\lambda^*(z)|^{p-2} v_\lambda^*(z) + f(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z) \\ \lambda |x|^{p-2}x + f(z,x) & \text{if } v_\lambda^*(z) \leqslant x \leqslant u_\lambda^*(z) \\ \lambda u_\lambda^*(z)^{p-1} + f(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{72}$$

This is a Carathéodory function. We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega G_\lambda(z,u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, we introduce $g_\lambda^\pm(z,x) = g_\lambda(z, \pm x^\pm)$, $G_\lambda^\pm(z,x) = \int_0^x g_\lambda^\pm(z,s)ds$ and the C^1-functional $\hat{\varphi}_\lambda^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda^\pm(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega G_\lambda^\pm(z,u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as in the proof of Proposition 25 and using (72), we obtain

$$K_{\hat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*],\ K_{\hat{\varphi}_\lambda^+} \subseteq [0, u_\lambda^*];\ K_{\hat{\varphi}_\lambda^-} \subseteq [v_\lambda^*, 0]$$

The extremality of $u_\lambda^* \in \mathrm{int}\, C_+$ and of $v_\lambda^* \in -\mathrm{int}\, C_+$, implies that

$$K_{\hat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*],\ K_{\hat{\varphi}_\lambda^+} = \{0, u_\lambda^*\},\ K_{\hat{\varphi}_\lambda^-} = \{v_\lambda^*, 0\}. \tag{73}$$

Claim 2 u_λ^* and v_λ^* are local minimizers of the functional $\hat{\varphi}_\lambda$.

Clearly $\hat{\varphi}_\lambda^+$ is coercive (see (72)). Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda^+(\hat{u}) = \inf\left[\hat{\varphi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{74}$$

As before hypothesis $H_4(iii)$ and the fact that $u_\lambda^* \in \operatorname{int} C_+$ and $2 < p$, imply that

$$\begin{aligned}
&\hat{\varphi}_\lambda^+(\pm \hat{u}_1(2)) < 0,\\
\Rightarrow\ &\hat{\varphi}_\lambda^+(\hat{u}) < 0 = \hat{\varphi}_\lambda^+(0) \text{ (see (73)), hence } \hat{u} \neq 0.
\end{aligned}$$

From (74) we have

$$\begin{aligned}
&\hat{u} \in K_{\hat{\varphi}_\lambda^+},\\
\Rightarrow\ &\hat{u} \in \{0, u_\lambda^*\},\ \hat{u} \neq 0,\\
\Rightarrow\ &\hat{u} = u_\lambda^* \in \operatorname{int} C_+.
\end{aligned}$$

Since $\hat{\varphi}_\lambda^+\big|_{C_+} = \hat{\varphi}_\lambda\big|_{C_+}$, it follows that u_λ^* is a local $C_0^1(\overline{\Omega})$-minimizer of $\hat{\varphi}_\lambda$, hence it is a local $W_0^{1,p}(\Omega)$-minimizer of $\hat{\varphi}_\lambda$ (see Proposition 7).

Similarly for v_λ^*, using this time the functional $\hat{\varphi}_\lambda$. This proves the claim.

Without any loss of generality, we may assume that

$$\hat{\varphi}_\lambda(v_\lambda^*) \leqslant \hat{\varphi}_\lambda(u_\lambda^*).$$

The analysis is similar if the opposite inequality holds. We may assume that $K_{\hat{\varphi}_\lambda}$ is finite (otherwise we already have infinity many nodal solutions, see (73)). From the claim we know that u_λ^* is a local minimizer of $\hat{\varphi}_\lambda$. So, we can find small $\rho \in (0,1)$ such that

$$\hat{\varphi}_\lambda(v_*) \leqslant \hat{\varphi}_\lambda(u_*) < \inf[\hat{\varphi}_\lambda(u) : ||u - u_\lambda^*|| = \rho] = m_\rho^\lambda,\ ||v_\lambda^* - u_\lambda^*|| > \rho \tag{75}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 22]).

The functional $\hat{\varphi}_\lambda$ is coercive, hence it satisfies the C-condition (see [30]). This fact and (75) permit the use of Theorem 2 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\hat{\varphi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \text{ (see (73)) and } m_\rho^\lambda \leqslant \hat{\varphi}_\lambda(y_0). \tag{76}$$

From (75) and (76) we have that $y_0 \notin \{v_\lambda^*, u_\lambda^*\}$ and y_0 is a solution of (P_λ) (see (72)) with $y_0 \in C_0^1(\overline{\Omega})$ (nonlinear regularity). We need to show that $y_0 \neq 0$ in order to conclude that y_0 is nodal.

Let $\rho = \max\{||u_\lambda^*||_\infty, ||v_\lambda^*||_\infty\}$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H_4(iv)$. Then

$$\begin{aligned}
&-\Delta_p y_0(z) - \Delta y_0(z) + \xi_\rho (y_0(z))^{p-2} y_0(z)\\
&\quad = (\lambda + \xi_\rho)|y_0(z)|^{p-2} y_0(z) + f(z, y_0(z))\\
&\quad \leqslant (\lambda + \xi_\rho) u_\lambda^*(z)^{p-1} + f(z, u_\lambda^*(z)) \text{ (since } y_0 \leqslant u_\lambda^*, \text{ see hypothesis } H_4(iv))\\
&\quad = -\Delta_p u_\lambda^*(z) - \Delta u_\lambda^*(z) + \xi_p u_\lambda^*(z)^{p-1} \text{ a.e. in } \Omega.
\end{aligned} \tag{77}$$

As before (see the proof of Proposition 24), we consider the map $a : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\begin{aligned} &a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N, \\ \Rightarrow\ &\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y \otimes y}{|y|^2}\right) + I, \\ \Rightarrow\ &(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \text{ for all } y, \xi \in \mathbb{R}^N. \end{aligned}$$

So, we can apply the tangency principle of Pucci and Serrin [32, p. 35], and obtain

$$y_0(z) < u_\lambda^*(z) \text{ for all } z \in \Omega.$$

Then from (77) and Arcoya and Ruiz [3, Proposition 2.6], we have

$$u_\lambda^* - y_0 \in \operatorname{int} C_+.$$

In a similar fashion, we show that

$$y_0 - v_\lambda^* \in \operatorname{int} C_+.$$

So, we have proved that

$$y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]. \tag{78}$$

We consider the deformation

$$h(t, u) = h_t(u) = (1-t)\hat{\varphi}_\lambda(u) + t\varphi_\lambda(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega)$$

Suppose we can find $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t \text{ in } [0, 1],\ u_n \to y_0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \text{ and } (h_{t_n})'_u(t_n, u_n) = 0 \text{ for all } n \geqslant 1. \tag{79}$$

We have

$$\begin{aligned} &A_p(u_n) + A(u_n) = (1-t_n)N_{g_\lambda}(u_n) + t_n\lambda|u_n|^{p-2}u_n + t_nN_f(u_n)\ n \geqslant 1 \\ \Rightarrow\ &-\Delta_p u_n(z) - \Delta u_n(z) = (1-t_n)g_\lambda(z, u_n(z)) + t_n\lambda|u_n(z)|^{p-2}u_n(z) + t_n f(z, u_n(z)) \\ &\text{for a.a. } z \in \Omega. \end{aligned}$$

From Ladyzhenskaya and Uraltseva [18, p. 286], we know that there exists $M_7 > 0$ such that

$$||u_n||_\infty \leqslant M_7 \text{ for all } n \geqslant 1.$$

Hence by virtue of Lieberman [19, Theorem 1], there exists $\alpha \in (0, 1)$ and $M_8 > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } ||u_n||_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant M_8 \text{ for all } n \geqslant 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and using (79), we have

$$\begin{aligned} & u_n \to y_0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty, \\ \Rightarrow\ & u_n \in [v_\lambda^*, u_\lambda^*] \text{ for all } n \geqslant n_0 \text{ (see (78))}. \end{aligned}$$

But from (72) we see that $\{u_n\}_{n\geqslant 1} \subseteq K_{\varphi_\lambda}$, a contradiction to our hypotheses that K_{φ_λ} is finite. So, (78) cannot happen and hence the homotopy invariance of singular homology implies that

$$C_k(\varphi_\lambda, y_0) = C_k(\hat{\varphi}_\lambda, y_0) \text{ for all } k \geqslant 0. \tag{80}$$

Recall that y_0 is a critical point of mountain pass type the functional $\hat{\varphi}_\lambda$. Therefore

$$\begin{aligned} & C_1(\hat{\varphi}_\lambda, y_0) \neq 0, \\ \Rightarrow\ & C_1(\varphi_\lambda, y_0) \neq 0 \text{ (see (80))}. \end{aligned} \tag{81}$$

From Proposition 13, we know that

$$\begin{aligned} & C_k(\varphi_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geqslant 0, \text{ with } d_m \geqslant 2, \\ \Rightarrow\ & y_0 \neq 0 \text{ (see (81))}, \\ \Rightarrow\ & y_0 \in C_0^1(\overline{\Omega}) \text{ is nodal.} \end{aligned}$$

□

So, we can state the following multiplicity theorem for problem (P_λ).

Theorem 27 *If hypothesis H_4 holds, then there exists $\epsilon > 0$ such that for all $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ problem (P_λ) has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+,\ \ v_0 \in -\operatorname{int} C_+ \textit{ and } y_0 \in \mathit{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \textit{ is nodal.}$$

Remark 4 We stress that the above theorem provides sign information for all solutions and localizes them. None of the other papers mentioned in the introduction, contains such a multiplicity result for equations near resonance from above.

In fact we can improve Theorem 27 and generate a second nodal solution provided we strengthen the regularity of $f(z,\cdot)$. The new hypotheses on $f(z,x)$ are the following:

$H_5 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f_x'(z,x)| \leqslant a(z)(1 + |x|^{p-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+$;

(ii) there exists $\vartheta \in L^\infty(\Omega)$ such that $\vartheta(z) \leqslant 0$ for a.a. $z \in \Omega$, $\vartheta \neq 0$ and

$$\limsup_{x\to\pm\infty} \frac{pF(z,x)}{|x|^p} \leqslant \vartheta(z) \text{ uniformly for a.a. } z \in \Omega; \text{ and}$$

(iii) there exists integer $m \geqslant 2$ such that

$$f'_x(z,0) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega,\ f'_x(\cdot,0) \not\equiv \hat{\lambda}_m(2),\ f'_x(\cdot,0) \not\equiv \hat{\lambda}_{m+1}(2)$$
$$f'_x(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

Remark 5 The differentiability of $f(z,\cdot)$ and hypothesis $H_5(i)$ imply that for every $\rho > 0$, there exists $\xi_\rho > 0$ for a.a. $z \in \Omega$, $x \to f(z,x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

We can now state the following multiplicity theorem.

Theorem 28 *If hypothesis H_5 holds, then there exists $\epsilon > 0$ such that for all $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ problem (P_λ) admits at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+,\ v_0 \in -\operatorname{int} C_+$$
$$\text{and}\ \ y_0, \hat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \text{ are nodal.}$$

Proof From Theorem 27 we already know that there exists $\epsilon > 0$ such that for all $\lambda \in (\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ has at least three nontrivial solutions

$$u_0 \in \operatorname{int} C_+,\ v_0 \in -\operatorname{int} C_+ \text{ and } y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \text{ is nodal.}$$

By virtue of Proposition 25, we may assume that u_0 and v_0 are extremal (that is, $u_0 = u^*_\lambda \in \operatorname{int} C_+$ and $v_0 = v^*_\lambda \in -\operatorname{int} C_+$). From the proof of Proposition 26 (see the claim), we know that u_0 and v_0 are local minimizers of the functional $\hat{\varphi}_\lambda$. Therefore

$$C_k(\hat{\varphi}_\lambda, u_0) = C_k(\hat{\varphi}_\lambda, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geqslant 0. \tag{82}$$

Since $\hat{\varphi}_\lambda\big|_{[v_0,u_0]} = \varphi_\lambda|_{[v_0,u_0]}$ (see (72)) and since $v_0 \in -\operatorname{int} C_+$, $u_0 \in \operatorname{int} C_+$ from Proposition 13, we have

$$C_k(\hat{\varphi}_\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geqslant 0,\ \text{ with } d_m \geqslant 2. \tag{83}$$

From the proof of Proposition 26, we have (see Papageorgiou and Smyrlis [29] and Papageorgiou and Rădulescu [26])

$$C_k(\hat{\varphi}_\lambda, y_0) \neq 0,$$
$$\Rightarrow C_k(\hat{\varphi}_\lambda, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geqslant 0 \tag{84}$$

Finally, since $\hat{\varphi}_\lambda$ is coercive (see (72)), we have

$$C_k(\hat{\varphi}_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geqslant 0. \tag{85}$$

Suppose $K_{\hat{\varphi}_\lambda} = \{u_0, v_0, 0, y_0\}$. From (82), (83), (84), (85) and the Morse relation with $t = -1$ (see (4)), we have

$$\begin{aligned} &2(-1)^0 + (-1)^{d_m} + (-1)^1 = (-1)^0, \\ \Rightarrow\ &(-1)^{d_m} = 0, \ \text{ a contradiction.} \end{aligned}$$

So, we have $\hat{y} \in K_{\hat{\varphi}_\lambda} \subseteq [v_0, u_0]$ (see (73)), $\hat{y} \notin \{u_0, v_0, 0\}$, thus $\hat{y}$ is nodal. Moreover, from the nonlinear regularity theory and reasoning as before (see the proof of Proposition 26), we have

$$\hat{y} \in \text{int}_{C_0^1(\overline{\Omega})}[v_0, u_0].$$

□

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