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EIGENVALUE PROBLEMS ASSOCIATED WITH NONHOMOGENEOUS DIFFERENTIAL OPERATORS IN ORLICZ–SOBOLEV SPACES

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We study the boundary value problem $-\text{div}((a_1(|\nabla u|)+a_2(|\nabla u|))\nabla u) = \lambda|u|^{q(x)-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary, λ is a positive real number, q is a continuous function and a_1, a_2 are two mappings such that $a_1(|t|)t$, $a_2(|t|)t$ are increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. We establish the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of the above problem.

Keywords: Nonhomogeneous differential operator; nonlinear eigenvalue problem; Orlicz–Sobolev space.

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1. Introduction and Preliminary Results

Nonlinear eigenvalue problems associated with differential operators with variable exponent have been intensively studied in the last few years. In many cases (see, e.g., [12, 13, 19–22, 26]), the model example is the $p(x)$-Laplace operator defined by $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$, where $p(x)$ is a continuous positive function. This operator in nonhomogeneous and thus, many techniques which can be applied in the homogeneous case (when $p(x)$ is a positive constant) fail in this new setting.

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A typical example is the Lagrange multiplier theorem, which does not apply to the eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This is due to the fact that the associated Rayleigh quotient is not homogeneous, provided both p and q are not constant.

On the other hand, problems like (1) have been largely considered in the literature in the recent years. We give in what follows a concise but complete image of the actual state of research on this topic.

- In the case when $p(x) = q(x)$ on $\overline{\Omega}$, Fan, Zhang and Zhao [13] established the existence of infinitely many eigenvalues for problem (1) by using an argument based on the Ljusternik–Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang, and Zhao showed that Λ is discrete, $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).
- If $\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth, Mihăilescu and Rădulescu [22] used the Ekeland's variational principle [11] in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin.
- In the case when $\max_{x\in\overline{\Omega}} p(x) < \min_{x\in\overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth, then standard mountain-pass arguments (similar to those used by Fan and Zhang in the proof of Theorem 4.7 in [12]) can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem (1).
- If $\max_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x)$, then the energy functional associated with problem (1) has a nontrivial minimum for any positive λ large enough (see [12, Theorem 4.7]). Clearly, in this case the result in [22] can also be applied. Consequently, in this situation there exist two positive constants $\lambda^\star$ and $\lambda^{\star\star}$ such that any $\lambda \in (0, \lambda^\star) \cup (\lambda^{\star\star}, \infty)$ is an eigenvalue of problem (1).

In this paper, we are concerned with a related nonlinear eigenvalue problem in a new framework, corresponding to Orlicz–Sobolev spaces. Our main result establishes a curious phenomenon, which does not hold in the standard setting corresponding to the Laplace operator. More precisely, we prove that there exist two constants $0 < \lambda_0 \le \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of our problem.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \ge 3)$ with smooth boundary $\partial\Omega$. Consider the nonlinear eigenvalue problem

$$\begin{cases} -\mathrm{div}((a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u) = \lambda |u|^{q(x)-2}u, & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{2}$$

We assume that for any $i = 1, 2$, the functions $a_i : (0, \infty) \to \mathbb{R}$ are such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_i(t) = \begin{cases} a_i(|t|)t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. We also suppose throughout this paper that $\lambda > 0$ and $q : \overline{\Omega} \to (0, \infty)$ is a continuous function.

Since the operator in the divergence form is nonhomogeneous we introduce an Orlicz–Sobolev space setting for problems of this type. On the other hand, the term arising in the right-hand side of Eq. (2) is also nonhomogeneous and its particular form appeals to a suitable variable exponent Lebesgue space setting.

We first recall some basic facts about Orlicz spaces. For more details, we refer to the books by Adams and Hedberg [2], Adams [1] and Rao and Ren [25] and the papers by Clément *et al.* [6,7], Garciá-Huidobro *et al.* [16] and Gossez [17]. We also refer to Chipot *et al.* [3], Ciarlet [4,5], Filippakis and Papageorgiou [14], Filippucci *et al.* [15], and Rădulescu [24] for applications and related results.

Assume $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. Define

$$\Phi_i(t) = \int_0^t \varphi_i(s)\, ds, \quad (\Phi_i)^\star(t) = \int_0^t (\varphi_i)^{-1}(s)\, ds, \quad \text{for all } t \in \mathbb{R}, \quad i = 1, 2.$$

We observe that Φ_i, $i = 1, 2$, are *Young's functions*, that is, $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{x\to\infty} \Phi_i(x) = +\infty$. Furthermore, since $\Phi_i(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \Phi_i(x)/x = 0$, and $\lim_{x\to\infty} \Phi_i(x)/x = +\infty$, then Φ_i are called *N-functions*. The functions $(\Phi_i)^\star$, $i = 1, 2$, are called the *complementary* functions of Φ_i, $i = 1, 2$, and they satisfy

$$(\Phi_i)^\star(t) = \sup\{st - \Phi_i(s); s \geq 0\}, \quad \text{for all } t \geq 0.$$

We also observe that $(\Phi_i)^\star$, $i = 1, 2$, are also N-functions and Young's inequality holds true

$$st \leq \Phi_i(s) + (\Phi_i)^\star(t), \quad \text{for all } s, t \geq 0.$$

The Orlicz spaces $L_{\Phi_i}(\Omega)$, $i = 1, 2$, defined by the N-functions Φ_i (see [2,1,6]) are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup\left\{\int_\Omega uv\, dx;\ \int_\Omega (\Phi_i)^\star(|g|)\, dx \leq 1\right\} < \infty.$$

Then, $(L_{\Phi_i}(\Omega), \|\cdot\|_{L_{\Phi_i}})$, $i = 1, 2$, are Banach spaces whose norm is equivalent to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf\left\{k > 0;\ \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right) dx \leq 1\right\}.$$

For Orlicz spaces Hölder's inequality reads as follows (see [25, Inequality 4, p. 79]):

$$\int_\Omega uv\, dx \leq 2\, \|u\|_{L_{\Phi_i}} \|v\|_{L_{(\Phi_i)^\star}} \quad \text{for all } u \in L_{\Phi_i}(\Omega) \text{ and } v \in L_{(\Phi_i)^\star}(\Omega), \quad i = 1, 2.$$

We denote by $W^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, the Orlicz–Sobolev spaces defined by

$$W^1L_{\Phi_i}(\Omega) := \left\{ u \in L_{\Phi_i}(\Omega);\ \frac{\partial u}{\partial x_i} \in L_{\Phi_i}(\Omega),\ i = 1, \ldots, N \right\}.$$

These are Banach spaces with respect to the norms

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}, \quad i = 1, 2.$$

We also define the Orlicz–Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, as the closure of $C_0^\infty(\Omega)$ in $W^1L_{\Phi_i}(\Omega)$. By [17, Lemma 5.7] we obtain that on $W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, we may consider some equivalent norms

$$\|u\|_i := \||\nabla u|\|_{\Phi_i}.$$

The spaces $W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, are also reflexive Banach spaces.

In this paper, we will work with functions Φ_i and $(\Phi_i)^\star$, $i = 1, 2$, satisfying the Δ_2-condition (at infinity), namely

$$1 < \liminf_{t\to\infty} \frac{t\varphi_i(t)}{\Phi_i(t)} \leq \limsup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \infty.$$

Then, $L_{\Phi_i}(\Omega)$ and $W_0^1L_{\Phi_i}(\Omega)$, $i = 1, 2$, are reflexive Banach spaces.

Now, we introduce the Orlicz–Sobolev conjugate $(\Phi_i)_\star$ of Φ_i, $i = 1, 2$, defined as

$$(\Phi_i)_\star^{-1}(t) = \int_0^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds.$$

We assume that

$$\lim_{t\to 0} \int_t^1 \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_1^t \frac{(\Phi_i)^{-1}(s)}{s^{(N+1)/N}}\, ds = \infty, \quad i = 1, 2. \tag{3}$$

Finally, we define

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)},\ i = 1, 2.$$

Next, we recall some background facts concerning the variable exponent Lebesgue spaces. For more details we refer to the book by Musielak [23] and the papers by Edmunds *et al.* [8–10], Kovacik and Rákosník [18], Mihăilescu and Rădulescu [19], and Samko and Vakulov [26].

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $q(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ (see [18]). On $L^{q(x)}(\Omega)$, we define the Luxemburg norm by the formula

$$|u|_{q(x)} = \inf\left\{ \mu > 0;\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{q(x)} dx \leq 1 \right\}.$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and q_1, q_2 are variable exponents so that $q_1(x) \leq q_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$.

If (u_n), $u \in L^{q(x)}(\Omega)$, then the following relations hold true

$$|u|_{q(x)} > 1 \Rightarrow |u|_{q(x)}^{q^-} \leq \int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^+} \tag{4}$$

$$|u|_{q(x)} < 1 \Rightarrow |u|_{q(x)}^{q^+} \leq \int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-} \tag{5}$$

$$|u_n - u|_{q(x)} \to 0 \Leftrightarrow \int_\Omega |u_n - u|^{q(x)} dx \to 0. \tag{6}$$

In this paper, we analyze problem (2) under the following basic assumptions:

$$1 < (p_2)_0 \leq (p_2)^0 < q(x) < (p_1)_0 \leq (p_1)^0, \quad \forall\, x \in \overline{\Omega} \tag{7}$$

and

$$\lim_{t\to\infty} \frac{|t|^{q^+}}{(\Phi_2)_\star(kt)} = 0, \quad \text{for all } k > 0. \tag{8}$$

2. Auxiliary Results

In this section, we point out certain useful auxiliary results.

Lemma 1. *The following relations hold true*:

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \leq \|u\|_i^{(p_i)_0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_i < 1, \quad i = 1, 2;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \|u\|_i^{(p_i)_0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_i > 1, \quad i = 1, 2;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \|u\|_i^{(p_i)^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_i < 1, \quad i = 1, 2;$$

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \leq \|u\|_i^{(p_i)^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \text{ with } \|u\|_i > 1, \quad i = 1, 2.$$

Proof. The proof of the first two inequalities can be carried out as in [7, Lemma C.9].

Next, assume $\|u\|_i < 1$. Let $\xi \in (0, \|u\|_i)$. By the definition of $(p_i)^0$, we deduce that

$$\Phi_i(t) \geq \tau^{(p_i)^0} \Phi_i(t/\tau), \quad \forall\, t > 0, \quad \tau \in (0, 1).$$

Using the above relation, we have

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \xi^{(p_i)^0} \cdot \int_\Omega \Phi_i\left(\frac{|\nabla u(x)|}{\xi}\right) dx. \tag{9}$$

Defining $v(x) = u(x)/\xi$, for all $x \in \Omega$, we have $\|v\|_i = \|u\|_i/\xi > 1$. Using the first inequality of this lemma, we find

$$\int_\Omega \Phi_i(|\nabla v(x)|)\, dx \geq \|v\|_i^{(p_i)_0} > 1. \tag{10}$$

Relations (9) and (10) show that

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \xi^{(p_i)^0}.$$

Letting $\xi \nearrow \|u\|_i$ in the above inequality, we obtain

$$\int_\Omega \Phi_i(|\nabla u(x)|)\, dx \geq \|u\|_i^{{p_i}^0}, \quad \forall\, u \in W_0^1 L_{\Phi_i}(\Omega) \quad \text{with} \quad \|u\|_i < 1.$$

Finally, we prove the last inequality in the lemma. A straightforward computation shows that

$$\frac{\Phi_i(\sigma t)}{\Phi_i(t)} \leq \sigma^{{p_i}^0}, \quad \forall\, t > 0 \quad \text{and} \quad \sigma > 1. \tag{11}$$

Then, for all $u \in W_0^1 L_{\Phi_i}(\Omega)$ with $\|u\|_i > 1$, relation (11) implies

$$\begin{aligned}\int_\Omega \Phi_i(|\nabla u(x)|)\, dx &= \int_\Omega \Phi_i\left(\|u\|_i \frac{|\nabla u(x)|}{\|u\|_i}\right) dx \\ &\leq \|u\|_i^{(p_i)^0} \int_\Omega \Phi_i\left(\frac{|\nabla u(x)|}{\|u\|_i}\right) dx \\ &\leq \|u\|_i^{(p_i)^0}.\end{aligned}$$

The proof of Lemma 1 is complete. □

Lemma 2. *Assume relation* (7) *holds true. Then, the continuous embedding*

$$W_0^1 L_{\Phi_1}(\Omega) \subset W_0^1 L_{\Phi_2}(\Omega)$$

holds true.

Proof. By [1, Lemma 8.12(b)] it is enough to show that Φ_1 dominates Φ_2 near infinity, that is, there exist $k > 0$ and $t_0 > 0$ such that

$$\Phi_2(t) \leq \Phi_1(k \cdot t), \quad \forall\, t \geq t_0.$$

Indeed, since by (7), we have $(p_2)^0 < (p_1)_0$, it follows that

$$\frac{\varphi_2(t)}{\Phi_2(t)} < \frac{\varphi_1(t)}{\Phi_1(t)}, \quad \forall\, t > 0.$$

The above relation and some elementary computations imply

$$\left(\frac{\Phi_1(t)}{\Phi_2(t)}\right)' > 0, \quad \forall\, t > 0.$$

Thus, we deduce that $\Phi_1(t)/\Phi_2(t)$ is increasing for any $t \in (0, \infty)$. It follows that for a fixed $t_0 \in (0, \infty)$ we have

$$\frac{\Phi_1(t_0)}{\Phi_2(t_0)} < \frac{\Phi_1(t)}{\Phi_2(t)}, \quad \forall\, t > t_0.$$

Let $k \in (0, \min\{1, \Phi_1(t_0)/\Phi_2(t_0)\})$ be fixed. The above relations yield

$$\Phi_2(t) < \frac{1}{k} \cdot \Phi_1(t), \quad \forall\, t > t_0.$$

Finally, we point out that in order to end the proof of the lemma it is enough to show that

$$\frac{1}{k} \cdot \Phi_1(t) \le \Phi_1\left(\frac{1}{k} \cdot t\right), \quad \forall\, t > 0.$$

Indeed, define the function $H : [0, \infty) \to \mathbb{R}$ by

$$H(t) = \Phi_1\left(\frac{1}{k} \cdot t\right) - \frac{1}{k} \cdot \Phi_1(t).$$

Therefore,

$$H'(t) = \frac{1}{k} \cdot \left(\varphi_1\left(\frac{1}{k} \cdot t\right) - \varphi_1(t)\right).$$

Since φ_1 is an increasing function and $1/k > 1$ we deduce that H is an increasing function. That fact combined with the remark that $H(0) = 0$ implies

$$H(t) \ge H(0) = 0, \quad \forall\, t \ge 0,$$

or

$$\frac{1}{k} \cdot \Phi_1(t) \le \Phi_1\left(\frac{1}{k} \cdot t\right), \quad \forall\, t > 0.$$

The proof of Lemma 2 is complete. □

Lemma 3. *Assume relation* (7) *holds true. Then, there exists* $c > 0$ *such that the following inequality holds true*

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \ge t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \ge 0.$$

Proof. Using the definition of $(p_1)_0$, we deduce that

$$\left(\frac{\Phi_1(t)}{t^{(p_1)_0}}\right)' > 0, \quad \forall\, t > 0,$$

or, the function $\Phi_1(t)/t^{(p_1)_0}$ is increasing for $t \in (0, \infty)$. Thus, we deduce that

$$\Phi_1(t) \geq \Phi_1(1) \cdot t^{(p_1)_0}, \quad \forall\, t > 1,$$

or letting $c_1 = 1/\Phi_1(1)$

$$c_1 \cdot \Phi_1(t) \geq t^{(p_1)_0}, \quad \forall\, t > 1. \tag{12}$$

Next, by the definition of $(p_2)^0$, it is easy to prove that

$$\Phi_2(t) \geq \tau^{(p_2)^0} \Phi_2(t/\tau), \quad \forall\, t > 0, \quad \tau \in (0, 1).$$

Letting $t \in (0, 1)$ and $\tau = t$, the above inequality implies

$$\Phi_2(t) \geq t^{(p_2)^0} \cdot \Phi_2(1), \quad \forall\, t \in (0, 1),$$

or letting $c_2 = 1/\Phi_2(1)$,

$$c_2 \cdot \Phi_2(t) \geq t^{(p_2)^0}, \quad \forall\, t \in (0, 1). \tag{13}$$

Finally, let $c = 2 \cdot \max\{c_1, c_2\}$. Then, since by relation (7) we have $(p_2)^0 < (p_1)_0$ and since relations (12) and (13) hold true, we deduce that

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \geq 2 \cdot t^{(p_1)_0} \geq t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \geq 1,$$

and

$$c \cdot [\Phi_1(t) + \Phi_2(t)] \geq 2 \cdot t^{(p_2)^0} \geq t^{(p_1)_0} + t^{(p_2)^0}, \quad \forall\, t \in (0, 1).$$

The proof of Lemma 3 is complete. □

3. The Main Result

Since we study problem (2) under the hypothesis (7), it follows by Lemma 2 that $W_0^1 L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_2}(\Omega)$. Thus, a solution for a problem of type (2) will be sought in the variable exponent space $W_0^1 L_{\Phi_1}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (2) if there exists $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|)) \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx = 0,$$

for all $v \in W_0^1 L_{\Phi_1}(\Omega)$. We point out that if λ is an eigenvalue of problem (2) then the corresponding $u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}$ is a *weak solution* of (2).

Define

$$\lambda_1 := \inf_{u \in W_0^1 L_{\Phi_1}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \Phi_1(|\nabla u|) \, dx + \int_\Omega \Phi_2(|\nabla u|) \, dx}{\displaystyle\int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx}.$$

Our main result is given by the following theorem.

Theorem 1. *Assume that conditions* (3), (7) *and* (8) *are fulfilled. Then* $\lambda_1 > 0$*. Moreover, any* $\lambda \in [\lambda_1, \infty)$ *is an eigenvalue of problem* (2)*. Furthermore, there exists a positive constant* λ_0 *such that* $\lambda_0 \le \lambda_1$ *and any* $\lambda \in (0, \lambda_0)$ *is not an eigenvalue of problem* (2).

Remark 1. Relations (3) and (8) enable us to apply [16, Theorem 2.2] (see also [1, Theorem 8.33]) in order to obtain that $W_0^1 L_{\Phi_2}(\Omega)$ is compactly embedded in $L^{q^+}(\Omega)$. That fact combined with the continuous embedding of $L^{q^+}(\Omega)$ in $L^{q(x)}(\Omega)$ and with the result of Lemma 2 ensures that $W_0^1 L_{\Phi_1}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$.

4. Proof of Theorem 1

Let E denote the generalized Sobolev space $W_0^1 L_{\Phi_1}(\Omega)$. In this section, we denote by $\|\cdot\|_1$ the norm on $W_0^1 L_{\Phi_1}(\Omega)$ and by $\|\cdot\|_2$ the norm on $W_0^1 L_{\Phi_2}(\Omega)$.

In order to prove our main result, we introduce four functionals J, I, J_1, I_1 : $E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \Phi_1(|\nabla u|)\, dx + \int_\Omega \Phi_2(|\nabla u|)\, dx,$$

$$I(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx,$$

$$J_1(u) = \int_\Omega a_1(|\nabla u|)|\nabla u|^2\, dx + \int_\Omega a_2(|\nabla u|)|\nabla u|^2\, dx,$$

$$I_1(u) = \int_\Omega |u|^{q(x)}\, dx.$$

Standard arguments imply that $J, I \in C^1(E, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v\, dx,$$

$$\langle I'(u), v\rangle = \int_\Omega |u|^{q(x)-2} uv\, dx$$

for all $u, v \in E$. We split the proof of Theorem 1 into four steps.

• **Step 1.** We show that $\lambda_1 > 0$.

By Lemma 3 and relation (7), we deduce that the following relations hold true

$$\begin{aligned} 2 \cdot c \cdot (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) &\ge 2 \cdot (|\nabla u(x)|^{(p_1)_0} + |\nabla u(x)|^{(p_2)^0}) \\ &\ge |\nabla u(x)|^{q^+} + |\nabla u(x)|^{q^-} \end{aligned}$$

and

$$|u(x)|^{q^+} + |u(x)|^{q^-} \ge |u(x)|^{q(x)}.$$

Integrating the above inequalities, we find

$$2c \cdot \int_\Omega (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|))\, dx \geq \int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx, \quad \forall\, u \in E \tag{14}$$

and

$$\int_\Omega (|u|^{q^+} + |u|^{q^-})\, dx \geq \int_\Omega |u|^{q(x)}\, dx, \quad \forall\, u \in E. \tag{15}$$

On the other hand, there exist two positive constants λ_{q^+} and λ_{q^-} such that

$$\int_\Omega |\nabla u|^{q^+}\, dx \geq \lambda_{q^+} \int_\Omega |u|^{q^+} dx, \quad \forall\, u \in W_0^{1,q^+}(\Omega) \tag{16}$$

and

$$\int_\Omega |\nabla u|^{q^-}\, dx \geq \lambda_{q^-} \int_\Omega |u|^{q^-} dx, \quad \forall\, u \in W_0^{1,q^-}(\Omega). \tag{17}$$

Using again the fact that $q^- \leq q^+ < (p_1)_0$ and a similar technique as that used in the proof of Lemma 2, we deduce that E is continuously embedded both in $W_0^{1,q^+}(\Omega)$ and in $W_0^{1,q^-}(\Omega)$. Thus, inequalities (16) and (17) hold true for any $u \in E$.

Using inequalities (15)–(17), it is clear that there exists a positive constant μ such that

$$\int_\Omega (|\nabla u|^{q^+} + |\nabla u|^{q^-})\, dx \geq \mu \int_\Omega |u|^{q(x)} dx, \quad \forall\, u \in E. \tag{18}$$

Next, inequalities (18) and (14) yield

$$\int_\Omega (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|))\, dx \geq \frac{\mu}{2c} \int_\Omega |u|^{q(x)}\, dx, \quad \forall\, u \in E. \tag{19}$$

The above inequality implies

$$J(u) \geq \frac{\mu \cdot q^-}{2c} I(u), \quad \forall\, u \in E. \tag{20}$$

The last inequality ensures that $\lambda_1 > 0$ and thus, Step 1 is verified.

Remark 2. We point out that by the definitions of $(p_i)_0$, $i = 1, 2$, we have

$$a_i(t) \cdot t^2 = \varphi_i(t) \cdot t \geq (p_i)_0 \Phi_i(t), \quad \forall\, t > 0.$$

The above inequality and relation (19) imply

$$\lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0. \tag{21}$$

- **Step 2.** We show that λ_1 is an eigenvalue of problem (2).

We start with some auxiliary results.

Lemma 4. *The following relations hold true*:

$$\lim_{\|u\|\to\infty} \frac{J(u)}{I(u)} = \infty \tag{22}$$

and

$$\lim_{\|u\|\to 0} \frac{J(u)}{I(u)} = \infty. \tag{23}$$

Proof. Since E is continuously embedded in $L^{q^\pm}(\Omega)$ it follows that there exist two positive constants c_1 and c_2 such that

$$\|u\|_1 \geq c_1 \cdot |u|_{q^+}, \quad \forall\, u \in E \tag{24}$$

and

$$\|u\|_1 \geq c_2 \cdot |u|_{q^-}, \quad \forall\, u \in E. \tag{25}$$

For any $u \in E$ with $\|u\|_1 > 1$ by Lemma 1 and relations (15), (24), (25), we infer

$$\frac{J(u)}{I(u)} \geq \frac{\|u\|_1^{(p_1)_0}}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\|u\|_1^{(p_1)_0}}{\dfrac{c_1^{-q^+}\|u\|_1^{q^+} + c_2^{-q^-}\|u\|_1^{q^-}}{q^-}}.$$

Since $(p_1)_0 > q^+ \geq q^-$, passing to the limit as $\|u\|_1 \to \infty$ in the above inequality, we deduce that relation (22) holds true.

Next, by Lemma 2 the space $W_0^1 L_{\Phi_1}(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_2}(\Omega)$. Thus, if $\|u\|_1 \to 0$ then $\|u\|_2 \to 0$.

The above remarks enable us to affirm that for any $u \in E$ with $\|u\|_1 < 1$ small enough, we have $\|u\|_2 < 1$.

On the other hand, since (8) holds true, we deduce that $W_0^1 L_{\Phi_2}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. It follows that there exist two positive constants d_1 and d_2 such that

$$\|u\|_2 \geq d_1 \cdot |u|_{q^+}, \quad \forall\, u \in W_0^1 L_{\Phi_2}(\Omega) \tag{26}$$

and

$$\|u\|_2 \geq d_2 \cdot |u|_{q^-}, \quad \forall\, u \in W_0^1 L_{\Phi_2}(\Omega). \tag{27}$$

Thus, for any $u \in E$ with $\|u\|_1 < 1$ small enough, Lemma 1 and relations (15), (26), (27) imply

$$\frac{J(u)}{I(u)} \geq \frac{\displaystyle\int_\Omega \Phi_2(|\nabla u|)\, dx}{\dfrac{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}}{q^-}} \geq \frac{\|u\|_2^{(p_2)^0}}{\dfrac{d_1^{-q^+}\|u\|_2^{q^+} + d_2^{-q^-}\|u\|_2^{q^-}}{q^-}}.$$

Since $(p_2)^0 < q^- \leq q^+$, passing to the limit as $\|u\|_1 \to 0$ (and thus, $\|u\|_2 \to 0$) in the above inequality, we deduce that relation (23) holds true.

The proof of Lemma 4 is complete. $\square$

Lemma 5. *There exists $u \in E\setminus\{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.*

Proof. Let $\{u_n\} \subset E\setminus\{0\}$ be a minimizing sequence for λ_1, that is,

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{28}$$

By relation (22), it is clear that $\{u_n\}$ is bounded in E. Since E is reflexive, it follows that there exists $u \in E$ such that u_n converges weakly to u in E. On the other hand, similar arguments to those used in the proof of [21, Theorem 2] show that the functional J is weakly lower semi-continuous. Thus, we find

$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{29}$$

By Remark 1, it follows that E is compactly embedded in $L^{q(x)}(\Omega)$. Thus, u_n converges strongly in $L^{q(x)}(\Omega)$. Then, by relation (6), it follows that

$$\lim_{n\to\infty} I(u_n) = I(u). \tag{30}$$

Relations (29) and (30) imply that if $u \not\equiv 0$, then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that u cannot be trivial. Assume by contradiction the contrary. Then, u_n converges weakly to 0 in E and strongly in $L^{q(x)}(\Omega)$. In other words, we will have

$$\lim_{n\to\infty} I(u_n) = 0. \tag{31}$$

Letting $\epsilon \in (0, \lambda_1)$ be fixed by relation (28), we deduce that for n large enough, we have

$$|J(u_n) - \lambda_1 I(u_n)| < \epsilon I(u_n),$$

or

$$(\lambda_1 - \epsilon)I(u_n) < J(u_n) < (\lambda_1 + \epsilon)I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (31) holds true, we find

$$\lim_{n\to\infty} J(u_n) = 0.$$

That fact combined with the conclusion of Lemma 1 implies that actually u_n converges strongly to 0 in E, that is, $\lim_{n\to\infty} \|u_n\|_1 = 0$. From this information and relation (23), we get

$$\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \not\equiv 0$.

The proof of Lemma 5 is complete. □

By Lemma 5 we conclude that there exists $u \in E\setminus\{0\}$ such that

$$\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w\in E\setminus\{0\}} \frac{J(w)}{I(w)}. \tag{32}$$

Then, for any $v \in E$ we have

$$\frac{d}{d\epsilon} \frac{J(u+\epsilon v)}{I(u+\epsilon v)}\bigg|_{\epsilon=0} = 0.$$

A simple computation yields

$$\int_\Omega (a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u \nabla v; dx \cdot I(u) \\ - J(u) \cdot \int_\Omega |u|^{q(x)-2}uv\, dx = 0, \quad \forall\, v \in E. \tag{33}$$

Relation (33) combined with the fact that $J(u) = \lambda_1 I(u)$ and $I(u) \neq 0$ implies the fact that λ_1 is an eigenvalue of problem (2). Thus, Step 2 is verified.

- **Step 3.** We show that any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (2).

Let $\lambda \in (\lambda_1, \infty)$ be arbitrary but fixed. Define $T_\lambda : E \to \mathbb{R}$ by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly, $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T'_\lambda(u), v\rangle = \langle J'(u), v\rangle - \lambda\langle I'(u), v\rangle, \quad \forall\, u \in E.$$

Thus, λ is an eigenvalue of problem (2) if and only if there exists $u_\lambda \in E\setminus\{0\}$ a critical point of T_λ.

With similar arguments as in the proof of relation (22) we can show that T_λ is coercive, that is, $\lim_{\|u\|\to\infty} T_\lambda(u) = \infty$. On the other hand, as we have already remarked, similar arguments to those used in the proof of [21, Theorem 2] (see also [20]) show that the functional T_λ is weakly lower semi-continuous. These two facts enable us to apply [27, Theorem 1.2] in order to prove that there exists $u_\lambda \in E$, a global minimum point of T_λ, and thus, a critical point of T_λ. In order to conclude that Step 3 holds true, it is enough to show that u_λ is not trivial. Indeed, since

$\lambda_1 = \inf_{u\in E\setminus\{0\}} \frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$ it follows that there exists $v_\lambda \in E$ such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or, equivalently,

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0$$

and we conclude that u_λ is a nontrivial critical point of T_λ, that is, λ is an eigenvalue of problem (2). Thus, Step 3 is verified.

• **Step 4.** We show that any $\lambda \in (0, \lambda_0)$, where λ_0 is given by relation (21), is not an eigenvalue of problem (2).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (2) it follows that there exists $u_\lambda \in E\setminus\{0\}$ such that

$$\langle J'(u_\lambda), v\rangle = \lambda\langle I'(u_\lambda), v\rangle, \quad \forall\, v \in E.$$

Thus, for $v = u_\lambda$ we find

$$\langle J'(u_\lambda), u_\lambda\rangle = \lambda\langle I'(u_\lambda), u_\lambda\rangle,$$

or

$$J_1(u_\lambda) = \lambda I_1(u_\lambda).$$

The fact that $u_\lambda \in E\setminus\{0\}$ ensures that $I_1(u_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information implies

$$J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Thus, Step 4 is verified.

By Steps 2–4, we deduce that $\lambda_0 \leq \lambda_1$. The proof of Theorem 1 is now complete. □

Remark 3. We point out that by the proof of Theorem 1 we cannot conclude whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. Such a study remains an open problem and we expect that the answer strongly depends on a_1, a_2, and q. In the case where $\lambda_0 < \lambda_1$, another interesting open problem concerns the existence of eigenvalues of problem (2) in the interval $[\lambda_0, \lambda_1)$.

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References

[1] R. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).

[2] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Vol. 314 (Springer-Verlag, Berlin, 1996).

[3] M. Chipot, W. Gangbo and B. Kawohl, On some nonlocal variational problems, *Anal. Appl.* (*Singap.*) **4** (2006) 345–356.

[4] Ph. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, Vol. 40 (SIAM, Philadelphia, 2002).

[5] Ph. G. Ciarlet, *An Introduction to Differential Geometry with Applications to Elasticity* (Springer-Verlag, Heidelberg, 2005).

[6] Ph. Clément, M. García-Huidobro, R. Manásevich and K. Schmitt, Mountain pass type solutions for quasilinear elliptic equations, *Calc. Var. Partial Differential Equations* **11** (2000) 33–62.

[7] Ph. Clément, B. de Pagter, G. Sweers and F. de Thélin, Existence of solutions to a semilinear elliptic system through Orlicz–Sobolev spaces, *Mediterr. J. Math.* **1** (2004) 241–267.

[8] D. E. Edmunds, J. Lang and A. Nekvinda, On $L^{p(x)}$ norms, *Proc. Roy. Soc. London Ser. A* **455** (1999) 219–225.

[9] D. E. Edmunds and J. Rákosník, Density of smooth functions in $W^{k,p(x)}(\Omega)$, *Proc. Roy. Soc. London Ser. A* **437** (1992) 229–236.

[10] D. E. Edmunds and J. Rákosník, Sobolev embedding with variable exponent, *Studia Math.* **143** (2000) 267–293.

[11] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974) 324–353.

[12] X. L. Fan and Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.* **52** (2003) 1843–1852.

[13] X. Fan, Q. Zhang and D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005) 306–317.

[14] M. Filippakis and N. Papageorgiou, Multiple solutions for nonlinear elliptic problems with a discontinuous nonlinearity, *Anal. Appl.* (*Singap.*) **4** (2006) 1–18.

[15] R. Filippucci, P. Pucci and V. Rădulescu, Existence and nonexistence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, to appear in *Commun. Partial Differ. Equations.*

[16] M. Garciá-Huidobro, V. K. Le, R. Manásevich and K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: An Orlicz–Sobolev space setting, *Nonlinear Differential Equations Appl.* (*NoDEA*) **6** (1999) 207–225.

[17] J. P. Gossez, Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients, *Trans. Amer. Math. Soc.* **190** (1974) 163–205.

[18] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{1,p(x)}$, *Czechoslovak Math. J.* **41** (1991) 592–618.

[19] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, *Proc. Roy. Soc. London Ser. A* **462** (2006) 2625–2641.

[20] M. Mihăilescu and V. Rădulescu, Nonhomogeneous boundary value problems in Orlicz–Sobolev spaces, *C. R. Acad. Sci. Paris, Ser. I* **344** (2007) 15–20.

[21] M. Mihăilescu and V. Rădulescu, Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: An Orlicz–Sobolev space setting, *J. Math. Anal. Appl.* **330** (2007) 416–432.

[22] M. Mihăilescu and V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135** (2007) 2929–2937.

[23] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Vol. 1034 (Springer, Berlin, 1983).

[24] V. Rădulescu, Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities, in *Handbook of Differential Equations: Stationary Partial Differential Equations*, ed. M. Chipot, Vol. 4 (Elsevier, 2007), pp. 483–591.

[25] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces* (Marcel Dekker, Inc., New York, 1991).

[26] S. Samko and B. Vakulov, Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, *J. Math. Anal. Appl.* **310** (2005) 229–246.

[27] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems* (Springer-Verlag, Heidelberg, 1996).