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Multiplicity of solutions for Robin problems with double resonance

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Abstract. We consider Robin boundary value problems with a reaction exhibiting double resonance at $\pm\infty$ with respect to any nonprincipal spectral interval. We prove several multiplicity theorems, producing nontrivial smooth solutions with sign information. We also prove an exact multiplicity theorem. We employ variational tools from critical point theory, together with truncation-perturbation techniques, flow invariance arguments and Morse theory (critical groups). We produce up to seven nontrivial smooth solutions all with sign information.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem:

$$\begin{cases} -\Delta u(z) = f(z, u(z)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

When $\beta \equiv 0$ we recover the Neumann problem. Our aim is to prove multiplicity theorems for problem (1.1), under conditions of double resonance at $\pm\infty$. Most of the earlier works on resonant equations deal with Dirichlet problems, impose more restrictive conditions on the reaction term, do not cover the case of double resonance and prove weaker multiplicity results. We mention the works of Furtado and Silva [13], Landesman, Robinson and Rumbos [20], Liang and Su [21], de Paiva [28,29], Su [36], Zou and Liu [39] (Dirichlet problems) and Gasinski and Papageorgiou [15,16], Li [22], Li and Li [23], Motreanu, Motreanu and Papageorgiou [26], Papageorgiou and Rădulescu [31,33] (Neumann problems).

Suppose that $\{\hat{\lambda}_k\}_{k\geqslant 1}$ denote the distinct eigenvalues of the differential operator of the problem (Dirichlet, Neumann or Robin). In the double resonance situation there is a spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$ $(m \geqslant 1)$ such that asymptotically as

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$x \to \pm\infty$, the quotient $\dfrac{f(z,x)}{x}$ reaches that spectral interval with possible interaction (resonance) with the two end points $\hat{\lambda}_m$ and $\hat{\lambda}_{m+1}$. More precisely, we have

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_{m+1}$$

uniformly for almost all $z \in \Omega$.

Such equations were first investigated in the context of Dirichlet problems by Berestycki and de Figueiredo [5], who also coined the term "double resonance". Since then all the studies on doubly resonant equations , concern Dirichlet problems. We refer to the works of Furtado and Silva [13], Liang and Su [21], de Paiva [28], Su [36]. The results proved in these works impose more restrictive conditions on the reaction term $f(z,x)$ and are not as strong and sharp as our multiplicity results here, producing fewer solutions with less information about them.

Our approach combines variational tools coming from the critical point theory with truncation-perturbation techniques, flow invariance arguments and Morse theory (critical groups). In the next section, for easy reference, we recall the main mathematical notions and results which we will use in the sequel.

2. Mathematical background

Let X be a Banach space. By X^* we denote the topological dual of X and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$ we say that φ satisfies the *Cerami condition* (the *C-condition* for short), if the following is true:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This compactness-type condition on the functional φ is a basic ingredient in the minimax theory for the critical values of φ. A basic result in that theory, is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [3], stated here in a slightly more general from (see Gasinski and Papageorgiou [14, page 648]).

Theorem 2.1. *Assume that $\varphi \in C^1(X,\mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, and $||u_1 - u_0||_X > r > 0$, it holds*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0||_X = r] = m_r$$

and $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geqslant m_r$ and c is a critical value of φ.

The analysis of problem (1.1) involves the following three spaces: the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesque space $L^2(\partial\Omega)$. In what follows, by $||\cdot||$ we denote the norm of $H^1(\Omega)$ defined by

$$||u|| = \left[||u||_2^2 + ||Du||_2^2\right]^{1/2} \text{ for all } u \in H^1(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

We let $D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}$. This is an open subset of C_+. On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leqslant q \leqslant \infty)$ in the usual way. From the theory of Sobolev spaces, there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the *trace map*, such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map gives meaning to the boundary values of a Sobolev function. The trace map is compact into $L^q(\Omega)$ for $q \in \left[1, \dfrac{2N-2}{N-2}\right)$ if $N \geqslant 3$ and into $L^q(\Omega)$ for all $q \in [1, +\infty)$ if $N = 2$. Also, we know that

$$\ker \gamma_0 = H_0^1(\Omega) \text{ and } \operatorname{im} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega).$$

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. The restriction of a Sobolev function u on $\partial\Omega$ is understood in the sense of traces.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. If $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function, that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z, x)$ is continuous), we define

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \text{ for all } u \in H^1(\Omega)$$

(the Nemytskii map corresponding to g). Also, for $x \in \mathbb{R}$ we set $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in H^1(\Omega)$, we set $u^{\pm}(\cdot) = u(\cdot)^{\pm}$ and we have $u^{\pm} \in H^1(\Omega)$, and

$$|u| = u^+ + u^-, \text{ and } u = u^+ - u^-.$$

As we already indicated in the introduction, we will make extensive use of the spectrum of $-\Delta$ with Robin boundary condition. So, we consider the following eigenvalue problem:

$$-\Delta u(z) = \hat{\lambda} u(z) \text{ in } \Omega,\ \frac{\partial u}{\partial n} + \beta(z) u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

We say that $\hat{\lambda}$ is an "eigenvalue" of (2.1), if there is a nontrivial solution $\hat{u} \in H^1(\Omega)$ of (2.1), known as an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. Using the spectral theorem for compact self-adjoint operators, we show that the spectrum of (2.1) consists of a sequence $\{\hat{\lambda}_k\}_{k \geqslant 1} \subseteq \mathbb{R}_+ = [0, +\infty)$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$ as $k \to +\infty$. Also, there is a corresponding sequence $\{\hat{u}_k\}_{k \geqslant 1}$ of eigenfunctions which form an orthonormal basis of $H^1(\Omega)$ and an orthogonal basis of $L^2(\Omega)$. These eigenvalues admit convenient variational characterizations. So, suppose that

$$\beta \in W^{1,\infty}(\partial\Omega) \text{ and } \beta(z) \geqslant 0 \text{ for all } z \in \partial\Omega$$

and consider the C^1-functional $\gamma : H^1(\Omega) \to \mathbb{R}$ defined by

$$\gamma(u) = ||Du||_2^2 + \int_{\partial\Omega} \beta(z) u^2 d\sigma \text{ for all } u \in H^1(\Omega).$$

Also, let $E(\hat{\lambda}_k)$ be the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. We know that $E(\hat{\lambda}_k)$ is finite dimensional and standard regularity theory implies that

$$E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega}) \text{ for all } k \in \mathbb{N}$$

(see Wang [37]).

We set $\bar{H}_m = \oplus_{k=1}^m E(\hat{\lambda}_k)$ and $\hat{H}_{m+1} = \overline{\oplus_{k \geqslant m+1} E(\hat{\lambda}_k)} = \bar{H}_m^\perp$. We have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1}$$

and so every $u \in H^1(\Omega)$ can be expressed in a unique way as

$$u = \bar{u} + \hat{u} \text{ with } \bar{u} \in \bar{H}_m, \hat{u} \in \hat{H}_{m+1}.$$

We have

$$\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{||u||_2^2} : u \in H^1(\Omega), u \neq 0\right] \tag{2.2}$$

$$\begin{aligned} \hat{\lambda} &= \inf\left[\frac{\gamma(u)}{||u||_2^2} : u \in \hat{H}_m, u \neq 0\right] \\ &= \sup\left[\frac{\gamma(u)}{||u||_2^2} : u \in \bar{H}_m, u \neq 0\right], m \geqslant 2. \end{aligned} \tag{2.3}$$

We know that $\hat{\lambda}_1 \geqslant 0$ and it is simple (that is, $\dim E(\hat{\lambda}_1) = 1$). The infimum in (2.2) is realized on $E(\hat{\lambda}_1)$, while both the infimum and supremum in (2.3) are realized on $E(\hat{\lambda}_m)$. Evidently the elements of $E(\hat{\lambda}_1)$ have constant sign. Let

$\hat{u}_1 \in H^1(\Omega)$ be the L^2-normalized, positive eigenfunction corresponding to $\hat{\lambda}_1$. Regularity theory and the strong maximum principle imply that $\hat{u}_1 \in D_+$. The eigenfunctions corresponding to higher eigenvalues are all nodal (sign changing). The eigenspaces $E(\hat{\lambda}_k)$ have the so-called *unique continuation property* (*UCP* for short), namely if $u \in E(\hat{\lambda}_k)$ and u vanishes on a set of positive measure, then $u \equiv 0$.

Using the UCP we have the following useful inequalities (see Papageorgiou and Rădulescu [31,34]).

Proposition 2.2. *The following facts hold:*

(a) *If* $\eta \in L^\infty(\Omega)$, *and* $\eta(z) \geqslant \hat{\lambda}_n$ $(n \in \mathbb{N})$ *for almost all* $z \in \Omega$, *and* $\eta \not\equiv \hat{\lambda}_n$, *then there exists* $c_1 > 0$ *such that*

$$\gamma(u) - \int_\Omega \eta(z) u^2 dz \leqslant -c_1 ||u||^2 \text{ for all } u \in \bar{H}_n;$$

(b) *If* $\eta \in L^\infty(\Omega)$, *and* $\eta(z) \leqslant \hat{\lambda}_n$ $(n \in \mathbb{N})$ *for almost all* $z \in \Omega$, *and* $\eta \not\equiv \hat{\lambda}_n$, *then there exists* $c_2 > 0$ *such that*

$$\gamma(u) - \int_\Omega \eta(z) u^2 dz \geqslant c_2 ||u||^2 \text{ for all } u \in \hat{H}_n.$$

In addition to the eigenvalue problem (2.1), we will also consider a weighted version of it. So, let $m \in L^\infty(\Omega)$, $m(z) \geqslant 0$ for almost all $z \in \Omega$, $m \not\equiv 0$. We consider the following eigenvalue problem:

$$-\Delta u(z) = \tilde{\lambda} m(z) u(z) \text{ in } \Omega, \ \frac{\partial u}{\partial n} + \beta(z) u = 0 \text{ on } \partial\Omega. \tag{2.4}$$

The analysis of this eigenvalue problem follows the analysis of (2.1) and so we have that the spectrum consists of a sequence $\{\tilde{\lambda}_k(m)\}_{k \geqslant 1} \subseteq \mathbb{R}_+ = [0, +\infty)$ of distinct eigenvalues such that $\tilde{\lambda}_k(m) \to +\infty$ and $\tilde{\lambda}_1(m)$ is simple. In this case in the variational expressions for the eigenvalues, the Rayleigh quotient has the form

$$\frac{\gamma(u)}{\int_\Omega m(z) u^2 dz} \ u \in H^1(\Omega), \text{ with } u \neq 0.$$

As a consequence of the UCP we have the following monotonicity property of the map $m \mapsto \hat{\lambda}_k(m)$.

Proposition 2.3. *If* $m, \hat{m} \in L^\infty(\Omega) \setminus \{0\}$, $0 \leqslant m(z) \leqslant \hat{m}(z)$ *for almost all* $z \in \Omega$, $m \not\equiv \hat{m}$, *then* $\tilde{\lambda}_n(\hat{m}) < \tilde{\lambda}_n(m)$ *for all* $n \in \mathbb{N}$.

For more details we refer to Papageorgiou and Rădulescu [31,34].

Suppose that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f_0(z, x)| \leqslant a_0(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \ \text{all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and

$$1 < r < 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 3 \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and consider the C^1-functional $\varphi_0 : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2}\gamma(u) - \int_\Omega F_0(z, u(z))dz \text{ for all } u \in H^1(\Omega).$$

The following property is a special case of a more general result of Papageorgiou and Rădulescu [32].

Proposition 2.4. *Assume that $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \textit{ with } ||h||_{C^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1)$ and u_0 is also a local $H^1(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \textit{ for all } h \in H^1(\Omega) \textit{ with } ||h|| \leqslant \rho_1.$$

Next we recall some basic definitions and facts from Morse theory, which will be needed in what follows.

So, let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leqslant c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$, let $H_k(Y_1, Y_2)$ be the k-th singular homology group for the pair (Y_1, Y_2) with integer coefficients (recall that for $k \in -\mathbb{N}$ we have $H_k(Y_1, Y_2) = 0$). Suppose that $u_0 \in K_\varphi^c$ is isolated. The critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{0\}) \text{ for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see Gasinski and Papageorgiou [14, page 628]), implies that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that K_φ is finite and consider the following quantities:

$$M(t,u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \text{ for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t), \tag{2.5}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

Suppose $X = H =$ a Hilbert space, $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$. We give the following definitions:

- The *Morse index* of u, denoted by $\mu(u)$, is defined as the supremum of dimensions of the subspaces of H on which $\varphi''(u)$ is negative definite;
- The *nullity* of u, denoted by $\nu(u)$, is the dimension of $\ker \varphi''(u)$;
- We say that $u \in K_\varphi$ is *nondegenerate*, if $\nu(u) = 0$, that is, $\varphi''(u)$ is invertible.

Evidently by the inverse function theorem a nondegenerate critical point is automatically isolated. Sometimes when it is clear which critical point we are using, then we simply write μ and ν instead of $\mu(u)$ and $\nu(u)$ for the Morse index and nullity respectively. The shifting theorem, is a useful tool in identifying new critical points or distinguishing among critical points.

Theorem 2.5. *If $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$ is isolated with finite Morse index μ and nullity ν, then either one of the following holds:*

(a) $C_k(\varphi, u) = 0$ *for all $k \leqslant \mu$ and all $k \geqslant \mu + \nu$;*
(b) $C_k(\varphi, u) = \delta_{k,\mu}\mathbb{Z}$ *for all $k \in \mathbb{N}_0$;*
(c) $C_k(\varphi, u) = \delta_{k,\mu+\nu}\mathbb{Z}$ *for all $k \in \mathbb{N}_0$.*

In fact this result has been extended by Li, Li and Liu [24] to functionals $\varphi \in C^{2-0}(H, \mathbb{R})$. Recall that $\varphi \in C^{2-0}(H, \mathbb{R})$, if $\varphi \in C^1(H, \mathbb{R})$ and its derivative is locally Lipschitz.

In the sequel $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is the operator defined by

$$\langle A(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega).$$

Recall that a Banach space X has the *Kadec-Klee property*, if the following is true

$$\text{“}u_n \xrightarrow{w} u \text{ in } X,\ ||u_n||_X \to ||u||_X \text{ implies } u_n \to u \text{ in } X\text{”}.$$

We know that a uniformly convex Banach space (in particular, a Hilbert space) has the Kadec-Klee property (see Gasinski and Papageorgiou [17, pages 853 and 901]).

3. Three and four nontrivial solutions

In this section under general conditions on the reaction term $f(z, x)$ which permit double resonance, we prove two multiplicity theorems producing three and four nontrivial smooth solutions respectively.

Recall that on the boundary coefficient $\beta(z)$ we impose the following conditions:

$H(\beta)$: it holds $\beta \in W^{1,\infty}(\partial\Omega)$, $\beta(z) \geqslant 0$ for all $z \in \Omega$.

Remark 3.1. We can have $\beta \equiv 0$ and so our analysis incorporates the Neumann problem.

The first multiplicity theorem does not require any differentiability properties on the reaction term $f(z, \cdot)$. Specifically the condition on $f(z, x)$ are the following.

H_1 : the map $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) For every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z, x)| \leqslant a_\rho(z) \text{ for almost all } z \in \Omega, \text{ and all } |x| \leqslant \rho;$$

(ii) There exists an integer $m \geqslant 2$ such that

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z, x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z, x)}{x} \leqslant \hat{\lambda}_{m+1}$$

uniformly for almost all $z \in \Omega$;

(iii) If $F(z, x) = \displaystyle\int_0^x f(z, s)ds$, then

$$\lim_{x\to\pm\infty} [f(z, x)x - 2F(z, x)] = +\infty$$

uniformly for almost all $z \in \Omega$;

(iv) There exists a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z) \leqslant \hat{\lambda}_1 \text{ for almost all } z \in \Omega,\ \vartheta \not\equiv \hat{\lambda}_1,$$

$$-c_0 \leqslant \liminf_{x\to 0} \frac{f(z, x)}{x} \leqslant \limsup_{x\to 0} \frac{f(z, x)}{x} \leqslant \vartheta(z)$$

uniformly for almost all $z \in \Omega$.

Remark 3.2. Hypothesis H_1(ii) implies that we have double resonance at any nonprincipal spectral interval. Hypothesis H_1(iv) says that at zero we have nonuniform nonresonance with respect to the principal eigenvalue.

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)dz \text{ for all } u \in H^1(\Omega)$$

(recall that $\gamma(u) = ||Du||_2^2 + \int_{\partial\Omega} \beta(z)u^2 d\sigma$ for all $u \in H^1(\Omega)$). Evidently, $\varphi \in C^1(H^1(\Omega))$.

Also, consider the following Carathéodory functions

$$\hat{f}_+(z,x)=\begin{cases} 0 & \text{if } x\leqslant 0 \\ f(z,x)+x & \text{if } 0<x \end{cases} \text{ and } \hat{f}_-(z,x)=\begin{cases} f(z,x)+x & \text{if } x<0 \\ 0 & \text{if } 0\leqslant x. \end{cases} \tag{3.1}$$

We set $\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)ds$ and consider the C^1-functionals $\hat{\varphi}_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega \hat{F}_\pm(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Proposition 3.3. *If hypotheses* $H(\beta)$, H_1(i), (ii), (iii) *hold, then the functional* φ *satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3.2}$$
$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{3.3}$$

From (3.3) we have

$$\left| \langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z)u_n h d\sigma - \int_\Omega f(z,u_n)hdz \right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \tag{3.4}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$. In (3.4) we choose $h = u_n \in H^1(\Omega)$ and obtain

$$-\gamma(u_n) + \int_\Omega f(z,u_n)u_n dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

Also, from (3.2) we have

$$\gamma(u_n) - \int_\Omega 2F(z,u_n)dz \leqslant 2M_1 \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

We add (3.5) to (3.6) and obtain

$$\int_\Omega [f(z,u_n)u_n - 2F(z,u_n)]dz \leqslant M_2 \text{ for some } M_2 > 0, \text{ all } n \in \mathbb{N}. \tag{3.7}$$

Claim 3.4. $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded.

We argue indirectly. So, suppose that the claim is not true. Then by passing to a subsequence if necessary, we may assume that $||u_n|| \to \infty$. Let $y_n = \dfrac{u_n}{||u_n||}$ for all $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.8}$$

From (3.4) we have

$$\begin{aligned} &\left|\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma - \int_\Omega \frac{N_f(u_n)}{||u_n||} h dz\right| \\ &\leqslant \frac{\epsilon_n ||h||}{(1+||u_n||)||u_n||} \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Hypotheses H_1(i), (ii) imply that

$|f(z,x)| \leqslant c_3(1+|x|)$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, some $c_3 > 0$,

hence,

$$\left\{\frac{N_f(u_n)}{||u_n||}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, from (3.6) and hypothesis H_1(ii) we have (at least for a subsequence), that

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \eta y \text{ in } L^2(\Omega) \text{ with } \hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega \tag{3.10}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 16]).

In (3.9) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.8), (3.10) and hypothesis $H(\beta)$. Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \tag{3.11}$$

hence $||Dy_n||_2 \to ||Dy||_2$, from which it follows that $y_n \to y$ in $H^1(\Omega)$ (by the Kadec-Klee property, (see (3.8))), and we obtain $||y|| = 1$.

In (3.9) we pass to the limit as $n \to \infty$ and using (3.8) and (3.10) we obtain

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega \eta(z) y h dz \text{ for all } h \in H^1(\Omega),$$

from which it follows that

$$\Rightarrow -\Delta y(z) = \eta(z) y(z) \text{ for almost all } z \in \Omega,\ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega \tag{3.12}$$

(see Papageorgiou and Rădulescu [32]).

Recall that $\hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1}$ for almost all $z \in \Omega$ (see (3.10)). First we assume that

$$\eta \not\equiv \hat{\lambda}_m \text{ and } \eta \not\equiv \hat{\lambda}_{m+1}.$$

Then using Proposition 2.3 we can say that

$$\tilde{\lambda}_m(\eta) < \tilde{\lambda}_m(\hat{\lambda}_m) = 1 \text{ and } 1 = \tilde{\lambda}_{m+1}(\hat{\lambda}_{m+1}) < \hat{\lambda}_m(\eta), \Rightarrow y = 0$$

(see (3.12)), which contradicts (3.11).

Next we assume that

$$\eta(z) = \hat{\lambda}_m \text{ or } \eta(z) = \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega\,.$$

We assume that $\eta(z) = \hat{\lambda}_m$ for almost all $z \in \Omega$. Then from (3.11) and (3.12) we have

$$y \in E(\hat{\lambda}_n)\backslash\{0\}.$$

Then the UCP implies that $y(z) \neq 0$ for almost all $z \in \Omega$ and so

$$\begin{aligned} & |u_n(z)| \to +\infty \text{ for almost all } z \in \Omega, \\ \Rightarrow\ & f(z, u_n(z))u_n(z) - 2F(z, u_n(z)) \to +\infty \text{ for almost all } z \in \Omega \\ & \text{(see hypothesis } H_1\text{(iii))}, \\ \Rightarrow\ & \int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]dz \to +\infty \end{aligned} \tag{3.13}$$

(see hypothesis H_1(iii) and use Fatou's lemma).

Comparing (3.7) and (3.13), we have a contradiction. This proves the claim.

Because of the claim, we may assume that

$$\begin{aligned} & u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \\ & u_n \to u \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \end{aligned} \tag{3.14}$$

In (3.4) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.10) and (3.14). Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ \Rightarrow\ & u_n \to u \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property, see (3.14))}, \\ \Rightarrow\ & \varphi \text{ satisfies the C-condition.} \end{aligned}$$

□

Proposition 3.5. *If hypotheses* $H(\beta)$, H_1(i), (ii) *hold, then the functionals* $\hat{\varphi}_\pm$ *both satisfy the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\hat{\varphi}_+(u_n)| \leqslant M_3 \text{ for some } M_3 > 0, \text{ all } n \in \mathbb{N}, \tag{3.15}$$
$$(1 + ||u_n||)\hat{\varphi}_+'(u_n) \to 0 \text{ in } H^1(\Omega)^*. \tag{3.16}$$

From (3.16) we have

$$\left| \langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n h d\sigma + \int_\Omega u_n h dz - \int_\Omega \hat{f}_+(z, u_n) h dz \right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \tag{3.17}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$. In (3.17) we choose $h = -u_n^- \in H^1(\Omega)$ and obtain

$$\gamma(u_n^-) + ||u_n^-||_2^2 \leqslant \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (3.1))},$$
$$\Rightarrow u_n^- \to 0 \text{ in } H^1(\Omega) \text{ (see hypothesis } H(\beta)). \tag{3.18}$$

Claim 3.6. $\{u_n^+\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded.

Again we argue by contradiction. So, suppose that the claim is not true and we have $||u_n^+|| \to \infty$. Let $y_n = \dfrac{u_n^+}{||u_n^+||}$, $n \in \mathbb{N}$. Then $||y_n|| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$\begin{aligned} & y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \\ & y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \end{aligned} \tag{3.19}$$

From (3.17), (3.18) and (3.1), we have

$$\left| \left\langle A(u_n^+), h\right\rangle + \int_{\partial\Omega} \beta(z) u_n^+ h d\sigma - \int_\Omega f(z, u_n^+) h dz \right| \leqslant \epsilon_n' ||h||$$

for all $h \in H^1(\Omega)$, with $\epsilon_n' \to 0^+$, implying that

$$\left| \langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma - \int_\Omega \frac{N_f(u_n^+)}{||u_n^+||} h dz \right| \leqslant \frac{\epsilon_n' ||h||}{||u_n||} \text{ for all } n \in \mathbb{N}. \tag{3.20}$$

From the proof of Proposition 3.3, we know that at least for a subsequence, we have

$$\frac{N_f(u_n^+)}{||u_n^+||} \xrightarrow{w} \eta y \text{ in } L^2(\Omega) \text{ with } \hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega. \tag{3.21}$$

In (3.20) we choose $h = y_n - y \in H^1(\Omega)$ and pass to the limit as $n \to \infty$. Then

$$\begin{aligned} & \lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ & \Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (be the Kadec-Klee property)}, \\ & \Rightarrow ||y|| = 1, y \geqslant 0. \end{aligned} \tag{3.22}$$

In (3.16) we pass to the limit as $n \to \infty$ and use (3.19) and (3.21). Then

$$\begin{aligned} &\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega f(z,u) h dz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta y(z) = \eta(z) y(z) \text{ for almost all } z \in \Omega, \ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega \end{aligned} \tag{3.23}$$

(see Papageorgiou and Rădulescu [32]).

From (3.21) we have

$$\tilde{\lambda}_m(\eta) \leqslant \tilde{\lambda}_m(\lambda_m) = 1.$$

Since $m \geqslant 2$ (see hypothesis H_1(ii)) from (3.23) it follows that y is nodal or zero, both contradicting (3.22). This proves the claim.

From (3.18) and the claim, it follows that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. From this, via the Kadec-Klee property, we conclude that $\hat{\varphi}_+$ satisfies the C-condition as before.

We argue similarly for the functional $\hat{\varphi}_-$. □

Proposition 3.7. *If hypotheses $H(\beta)$, H_1(iv) hold, then $u = 0$ is a local minimizer for the functionals $\hat{\varphi}_\pm$ and φ.*

Proof. Hypothesis H_1(iv) implies that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\vartheta(z) + \epsilon) x^2 \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \delta. \tag{3.24}$$

Let $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta$. Then

$$\begin{aligned} \hat{\varphi}_+(u) &= \frac{1}{2}\gamma(u) + \frac{1}{2}||u^-||_2^2 - \int_\Omega F(z,u^+) dz \text{ (see (3.1))} \\ &\geqslant \frac{1}{2}\left[\gamma(u^+) - \int_\Omega \vartheta(z)(u^+)^2 dz\right] + \frac{1}{2}\left[\gamma(u^-) + ||u^-||_2^2\right] - \frac{\epsilon}{2}||u||^2 \text{ (see (3.24))} \\ &\geqslant \frac{c_4 - \epsilon}{2}||u||^2 \text{ for some } c_4 > 0 \end{aligned} \tag{3.25}$$

(see Proposition 2.2(b) and hypothesis $H(\beta)$).

Choosing $\epsilon \in (0, c_4)$, from (3.25) we see that

$$\begin{aligned} &u = 0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \hat{\varphi}_+, \\ \Rightarrow\ &u = 0 \text{ is a local } H^1(\Omega) - \text{minimizer of } \hat{\varphi}_+ \end{aligned}$$

(see Proposition 2.4).

Similarly for the functionals $\hat{\varphi}_-$ and φ. □

Now we are ready to produce two nontrivial constant sign smooth solutions for problem (1.1).

Proposition 3.8. *If hypotheses* $H(\beta)$, H_1 *hold, then problem* (1.1) *admits two constant sign solutions*

$$u_0 \in D_+ \text{ and } v_0 \in -D_+.$$

Proof. First we show that

$$K_{\hat{\varphi}_+} \setminus \{0\} \subseteq D_+. \tag{3.26}$$

So, let $u \in K_{\hat{\varphi}_+}$, $u \neq 0$. Then

$$\begin{aligned} &\hat{\varphi}'_+(u) = 0, \\ \Rightarrow\ &\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z) u h d\sigma + \int_\Omega u h dz \\ &= \int_\Omega \hat{f}_+(z, u) h dz \text{ for all } h \in H^1(\Omega). \end{aligned} \tag{3.27}$$

In (3.27) we choose $h = -u^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u^-) + ||u^-||_2^2 = 0 \text{ (see (3.1))}, \\ \Rightarrow\ &||u^-||^2 \leqslant 0 \text{ (see hypothesis } H(\beta)), \\ \Rightarrow\ &u \geqslant 0,\ u \neq 0. \end{aligned}$$

From (3.27) and (3.1), we have

$$\begin{aligned} &\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z) u h d\sigma = \int_\Omega f(z, u) h dz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta u(z) = f(z, u(z)) \text{ for almost all } z \in \Omega,\ \frac{\partial u}{\partial n} + \beta(z) u = 0 \text{ on } \partial\Omega \end{aligned} \tag{3.28}$$

(see Papageorgiou and Rădulescu [32]).

Standard regularity theory (see Wang [37]), implies that

$$u \in C_+ \setminus \{0\}.$$

Let $\rho = ||u||_\infty$. Hypotheses $H_1(i)$, (iv) imply that we can find $\hat{\xi}_\rho > 0$ such that

$$f(z, x)x + \hat{\xi}_\rho x^2 \geqslant 0 \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho. \tag{3.29}$$

From (3.28) and (3.29) we obtain that

$$\begin{aligned} &\Delta u(z) \leqslant \hat{\xi}_\rho u(z) \text{ for almost all } z \in \Omega, \\ \Rightarrow\ &u \in D_+ \end{aligned}$$

(by the maximum principle, see Gasinski and Papageorgiou [14, page 738]).

So, we have proved (3.26).

Then (3.26) and (3.1) suggest that we may assume that $K_{\hat{\varphi}_+}$ is finite or otherwise we already have an infinity of positive solutions. On account of Proposition 3.7, we can find $r \in (0, 1)$ small such that

$$\hat{\varphi}_+(0) = 0 < \inf[\hat{\varphi}_+(u) : ||u|| = r] = \hat{m}_+ \tag{3.30}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29]).

Since $m \geqslant 2$, hypothesis $H_1(ii)$ implies that

$$\hat{\varphi}_+(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty\,. \tag{3.31}$$

Moreover, from Proposition 3.5 we have

$$\hat{\varphi}_+ \text{ satisfies the C-condition.} \tag{3.32}$$

Then (3.30), (3.31), (3.32) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in H^1(\Omega)$ such that

$$\begin{aligned} &u_0 \in K_{\hat{\varphi}_+} \text{ and } \hat{m}_+ \leqslant \hat{\varphi}_+(u_0), \\ \Rightarrow\ &u_0 \in D_+ \text{ (see (3.26) and (3.30)) and solves (1.1) (see (3.1)).} \end{aligned} \tag{3.33}$$

Similarly, working now with the functional $\hat{\varphi}_-$, we produce a negative smooth solution $v_0 \in -D_+$. □

Remark 3.9. The above proof leads to the observation that

$$C_1(\varphi, u_0) \neq 0 \text{ and } C_1(\varphi, v_0) \neq 0.$$

Indeed from the proof we have that $u_0 \in D_+$ is a critical point of mountain pass type for the functional $\hat{\varphi}_+$. So, Corollary 6.81 of Motreanu, Motreanu and Papageorgiou [27, page 168] implies that

$$C_1(\hat{\varphi}_+, u_0) \neq 0. \tag{3.34}$$

From (3.1) we see that $\hat{\varphi}_+|_{C_+} = \varphi|_{C_+}$. Also $u_0 \in D_+$ (see Proposition 3.8). Therefore

$$C_k(\hat{\varphi}_+|_{C^1(\overline{\Omega})}, u_0) = C_k(\varphi|_{C^1(\overline{\Omega})}, u_0) \text{ for all } k \in \mathbb{N}_0. \tag{3.35}$$

Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, from Palais [30, Theorem 16] (see also Chang [8, page 14]), we have

$$\begin{aligned} &C_k(\hat{\varphi}_+|_{C^1(\overline{\Omega})}, u_0) = C_k(\hat{\varphi}_+, u_0) \text{ and} \\ &C_k(\varphi|_{C^1(\overline{\Omega})}, u_0) = C_k(\varphi, u_0) \text{ for all } k \in \mathbb{N}_0. \end{aligned} \tag{3.36}$$

So, from (3.35) and (3.36) it follows that

$$\begin{aligned} &C_k(\hat{\varphi}_+, u_0) = C_k(\varphi, u_0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ &C_1(\varphi, u_0) \neq 0 \text{ (see (3.34)).} \end{aligned}$$

Similarly, working this time with the pair $\{\hat{\varphi}_-, \varphi\}$, we show that $C_1(\varphi, v_0) \neq 0$.

In fact we can show that problem (1.1) has extremal constant sign solutions. More precisely, we show that there exists a smallest positive solution $u_* \in D_+$ and a biggest negative solution $v_* \in -D_+$. These extremal constant sign solutions, will lead to a nodal (that is, sign changing) solution (see Theorem 3.19).

In what follows S_+ (respectively S_-) denotes the set of positive (respectively negative) solutions of problem (1.1). From Proposition 3.8 and its proof, we have

$$\emptyset \neq S_+ \subseteq D_+ \text{ and } \emptyset \neq S_- \subseteq -D_+.$$

Moreover, as in Filippakis and Papageorgiou [11] (see Lemmata 4.1 and 4.2) (see also Mariconda and Treu [25, Lemma 3.1]), we have that

- S_+ is downward directed (that is, if $u, u' \in S_+$, then there exists $y \in S_+$ such that $y \leqslant u$, and $y \leqslant u'$).
- S_- is upward directed (that is, if $v, v' \in S_-$, then there exists $y \in S_-$ such that $v \leqslant y$, and $v' \leqslant y$).

Proposition 3.10. *If hypotheses $H(\beta)$, H_1 hold, then problem* (1.1) *admits a smallest positive solution $u_* \in D_+$ and a biggest negative solution $v_* \in -D_+$.*

Proof. Invoking Lemma 3.10 of Hu and Papageorgiou [19, page 178], we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+ \subseteq D_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

We have

$$\begin{aligned} &-\Delta u_n(z) = f(z, u_n(z)) \text{ for almost all } z \in \Omega,\\ &\frac{\partial u_n}{\partial n} + \beta(z)u_n = 0 \text{ on } \partial\Omega, \ \text{ with } n \in \mathbb{N}. \end{aligned} \tag{3.37}$$

From (3.37) and since $0 \leqslant u_n \leqslant u_1 \in D_+$ for all $n \in \mathbb{N}$, it follows that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$\begin{aligned} &u_n \xrightarrow{w} u_* \text{ in } H^1(\Omega)\\ &u_n \to u_* \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \end{aligned}$$

As before, using (3.37) and the Kadec-Klee property, we have

$$u_n \to u_* \text{ in } H^1(\Omega). \tag{3.38}$$

Suppose that $u_* = 0$. Then $||u_n|| \to 0$ (see (3.38)). Let $y_n = \dfrac{u_n}{||u_n||}$. Then $||y_n|| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$\begin{aligned} &y_n \xrightarrow{w} y \text{ in } H^1(\Omega)\\ &y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \end{aligned} \tag{3.39}$$

From (3.37), we have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma = \int_\Omega \frac{N_f(u_n)}{||u_n||} h dz \text{ for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N}. \quad (3.40)$$

Hypotheses H_1(i), (ii), (iv) imply that

$$\begin{aligned} &|f(z,x)| \leqslant c_6|x| \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_6 > 0, \\ &\Rightarrow \left\{\frac{N_f(u_n)}{||u_n||}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.} \end{aligned} \quad (3.41)$$

From (3.41) and hypothesis H_1(iv) we have, at least for a subsequence, that

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \vartheta_0 y \text{ in } L^2(\Omega), \text{ with } -c_0 \leqslant \vartheta_0(z) \leqslant \vartheta(z) \text{ for almost all } z \in \Omega \quad (3.42)$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 14]). In (3.40) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.39), (3.41). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ &\Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property)}, \\ &\Rightarrow ||y|| = 1, \ y \geqslant 0. \end{aligned} \quad (3.43)$$

In (3.40) we pass to the limit as $n \to \infty$ and use (3.39), (3.42). Then

$$\begin{aligned} &\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega \vartheta_0 y h dz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta y(z) = \vartheta_0(z) y(z) \text{ for almost all } z \in \Omega, \text{ and } \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega. \end{aligned} \quad (3.44)$$

Recall that

$$-c_0 \leqslant \vartheta_0(z) \leqslant \vartheta(z) \leqslant \hat{\lambda}_1 \text{ for almost all } z \in \Omega, \ \vartheta \not\equiv \hat{\lambda}_1$$

(see (3.42) and hypothesis H_1(iv)). Using Proposition 2.3, we have

$$1 = \tilde{\lambda}_1(\hat{\lambda}_1) < \tilde{\lambda}_1(\vartheta) \leqslant \tilde{\lambda}_1(\vartheta_0). \quad (3.45)$$

Then from (3.44) and (3.45) it follows that

$$y = 0,$$

which contradicts (3.43). This proves that $u_* \not\equiv 0$. We have

$$\begin{aligned} &-\Delta u_*(z) = f(z, u_*(z)) \text{ for almost all } z \in \Omega, \ \frac{\partial u_*}{\partial n} + \beta(z) u_* = 0 \text{ on } \partial\Omega, \\ &\Rightarrow u_* \in S_+ \subseteq D_+ \text{ and } u_* = \inf S_+. \end{aligned}$$

Similarly, we produce $v_* \in S_- \subseteq -D_+$ the biggest negative solution of problem (1.1). □

Next we produce a third nontrivial smooth solution, distinct from u_0 and v_0. To do this we will use tools from Morse theory. So, we compute the critical groups of φ at infinity.

Proposition 3.11. *If hypotheses $H(\beta)$, H_1 hold, then $C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$, with $d_m = \dim \bar{H}_m = \dim \bigoplus_{k=1}^{m} E(\hat{\lambda}_k)$.*

Proof. Recall that $\bar{H}_m = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k)$, and $\hat{H}_{m+1} = \overline{\bigoplus_{k\geqslant m+1} E(\hat{\lambda}_k)} = \bar{H}_m^{\perp}$ and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1}.$$

Let $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and consider the C^2-functional $\tau : H^1(\Omega) \to \mathbb{R}$ defined by

$$\tau(u) = \frac{1}{2}\gamma(u) - \frac{\lambda}{2}||u||_2^2 \text{ for all } u \in H^1(\Omega).$$

From the choice of λ it follows that

$$K_\tau = \{0\} \text{ and } u = 0 \text{ is a nondegenerate critical point of } \tau\,.$$

Then from Theorem 6.51 of Motreanu, Motreanu and Papageorgiou [27, page 155], we have

$$C_k(\tau, \infty) = C_k(\tau, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{3.46}$$

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\varphi(u) + t\tau(u) \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Claim 3.12. There exist $\mu \in \mathbb{R}$ and $\delta > 0$ such that

$$h(t, u) \leqslant \mu \Rightarrow (1 + ||u||)||h'_u(t, u)||_* \geqslant \delta \text{ for all } t \in [0, 1].$$

We argue by contradiction. Evidently the homotopy h maps bounded sets to bounded sets. So, if the claim is not true, then we can find $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$\begin{aligned} &t_n \to t,\ ||u_n|| \to \infty,\ h(t_n, u_n) \to -\infty \ \text{ and} \\ &(1 + ||u_n||)h'_u(t_n, u_n) \to 0 \ \text{ in } \ H^1(\Omega)^*. \end{aligned} \tag{3.47}$$

From the last convergence in (3.47), we have

$$\begin{aligned} &\left|\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z)u_n h d\sigma - (1 - t_n)\int_\Omega f(z, u_n)hdz - t_n\int_\Omega \lambda u_n hdz\right| \\ &\leqslant \frac{\epsilon_n||h||}{1 + ||u_n||} \end{aligned} \tag{3.48}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$.

We set $y_n = \dfrac{u_n}{||u_n||}$, and $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.49}$$

From (3.48) we have

$$\begin{aligned}&\left|\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma - (1-t_n)\int_\Omega \frac{N_f(u_n)}{||u_n||} h dz - t_n \int_\Omega \lambda y_n h dz\right| \\ &\leqslant \frac{\epsilon_n ||h||}{(1+||u_n||)||u_n||} \text{ for all } n \in \mathbb{N}.\end{aligned} \tag{3.50}$$

Recall that (see (3.21))

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \eta y \text{ in } L^2(\Omega),\ \text{ with } \hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega\,. \tag{3.51}$$

As before, if in (3.50) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.49), (3.51) and the Kadec-Klee property, then

$$y_n \to y \text{ in } H^1(\Omega) \text{ and so } ||y|| = 1. \tag{3.52}$$

From (3.50) in the limit as $n \to \infty$, we obtain

$$\begin{aligned}&\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega [(1-t)\eta(z) + t\lambda] y h dz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta y(z) = \eta_t(z) y(z) \text{ for almost all } z \in \Omega,\ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega\,.\end{aligned} \tag{3.53}$$

Note that

$$\hat{\lambda}_m \leqslant \eta_t(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega,\ \text{ with } \eta_1 = \lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1}).$$

So, if $t = 1$ or if $t \in (0,1)$, then $\eta_t \not\equiv \hat{\lambda}_m$, and $\eta_t \not\equiv \hat{\lambda}_{m+1}$, then from (3.53) it follows that $y = 0$, a contradiction to (3.52).

Therefore, $t = 0$ and $\eta_0(z) = \hat{\lambda}_m$ or $\eta_0(z) = \hat{\lambda}_{m+1}$ for almost all $z \in \Omega$. From (3.52) and (3.53) we have

$$y \in E(\hat{\lambda}_m)\backslash\{0\} \text{ or } y \in E(\hat{\lambda}_{m+1})\backslash\{0\}.$$

We assume that $y \in E(\hat{\lambda}_m)\backslash\{0\}$. From the UCP we have that $y(z) \neq 0$ for almost all $z \in \Omega$. Hence

$$\begin{aligned}&|u_n(z)| \to +\infty \text{ for almost all } z \in \Omega, \\ \Rightarrow\ & f(z, u_n(z))u_n(z) - 2F(z, u_n(z)) \to +\infty \text{ for almost all } z \in \Omega \\ & \text{(see hypothesis } H_1\text{(iii))}, \\ \Rightarrow\ & \int_\Omega [f(z,u_n)u_n - 2F(z,u_n)]dz \to +\infty\end{aligned} \tag{3.54}$$

(by Fatou's lemma).

On the other hand, from the third convergence in (3.47), we see that we can find $n_0 \in \mathbb{N}$ such that

$$\gamma(u_n) - (1 - t_n) \int_\Omega 2F(z, u_n)dz - t_n \int_\Omega \lambda u_n^2 dz \leqslant -1 \text{ for all } n \geqslant n_0. \quad (3.55)$$

Also, in (3.48) we choose $h = u_n \in H^1(\Omega)$ and have

$$-\gamma(u_n) + (1 - t_n) \int_\Omega f(z, u_n)u_n dz + t_n \int_\Omega \lambda u_n^2 dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}. \quad (3.56)$$

By choosing $n_0 \in \mathbb{N}$ even bigger if necessary, we can have $\epsilon_n \in (0, 1)$ for all $n \geqslant n_0$. Adding (3.55), (3.56) we obtain

$$(1 - t_n) \int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]dz \leqslant 0 \text{ for all } n \geqslant n_0.$$

Since $h(1, \cdot) = \tau(\cdot)$, it follows that we can have $t_n \in [0, 1)$ for all $n \geqslant n_0$. Hence

$$\int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]dz \leqslant 0 \text{ for all } n \geqslant n_0. \quad (3.57)$$

Comparing (3.54) and (3.57), we have a contradiction. This proves the claim.

As in the proof of Proposition 3.3, we can show that for every $t \in [0, 1]$, $h(t, \cdot)$ satisfies the C-condition. Therefore using Proposition 3.2 of Liang and Su [21] (see also Chang [9, Theorem 5.1.2, page 334]), we have

$$\begin{aligned} & C_k(h(0, \cdot), \infty) = C_k(h(1, \cdot), \infty) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi, \infty) = C_k(\tau, \infty) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (3.46))}. \end{aligned}$$

The proof is complete. □

An analogous argument leads to the computation of the critical groups of $\hat{\varphi}_\pm$ at infinity.

Proposition 3.13. *If hypotheses $H(\beta), H_1$ hold, then $C_k(\hat{\varphi}_\pm, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Again let $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and consider the C^1-functional $\hat{\psi}_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_+(u) = \frac{1}{2}\gamma(u) + \frac{1}{2}||u^-||_2^2 - \frac{\lambda}{2}||u^+||_2^2 \text{ for all } u \in H^1(\Omega).$$

We consider the homotopy $\hat{h}_+(t, u)$ defined by

$$\hat{h}_+(t, u) = (1 - t)\hat{\varphi}_+(u) + t\hat{\psi}_+(u) \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Claim 3.14. There exist $\mu \in \mathbb{R}$ and $\delta > 0$ such that

$$\hat{h}_+(t,u) \leqslant \mu \Rightarrow (1+||u||)||(\hat{h}_+)'(t,u)||_* \geqslant \delta \text{ for all } t \in [0,1].$$

As before we proceed by contradiction. So, suppose that Claim 3.14 is not true. Then because $\hat{h}_+(\cdot,\cdot)$ maps bounded sets to bounded sets, there exist two sequence $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ such that

$$\begin{aligned} &t_n \to t, ||u_n|| \to \infty, \hat{h}_+(t_n,u_n) \to -\infty \text{ and} \\ &(1+||u_n||)(\hat{h}_+)'_u(t_n,u_n) \to 0 \text{ in } H^1(\Omega)^*. \end{aligned} \tag{3.58}$$

From the last convergence in (3.58) we have

$$\begin{aligned} &\Big| \langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n h d\sigma \\ &- \int_\Omega (u_n^-) h dz - (1-t_n)\int_\Omega f(z,u_n^+) h dz - t_n \int_\Omega \lambda(u_n^+) h dz \Big| \\ &\leqslant \frac{\epsilon_n ||h||}{1+||u_n||} \text{ for all } h \in H^1(\Omega), \text{ with } \epsilon_n \to 0^+. \end{aligned} \tag{3.59}$$

In (3.59) we choose $h = -u_n^- \in H^1(\Omega)$. Then

$$\begin{aligned} &\gamma(u_n^-) + ||u_n^-||_2^2 \leqslant \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see hypothesis } H(\beta)), \\ &\Rightarrow u_n^- \to 0 \text{ in } H^1(\Omega) \text{ as } n \to \infty. \end{aligned}$$

From (3.58) we have $||u_n|| \to \infty$. Hence (3.60) implies that $||u_n^+|| \to \infty$. We set $y_n = \dfrac{u_n^+}{||u_n^+||}$ for all $n \in \mathbb{N}$. Then $||y_n|| = 1$, and $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and } L^2(\partial\Omega). \tag{3.60}$$

From (3.59) and (3.60) it follows that

$$\begin{aligned} &\left| \left\langle A(u_n^+), h \right\rangle + \int_{\partial\Omega} \beta(z) u_n^+ h d\sigma - (1-t_n)\int_\Omega N_f(u_n^+) h dz - t_n \int_\Omega \lambda(u_n^+) h dz \right| \\ &\leqslant \epsilon_n' ||h|| \end{aligned} \tag{3.61}$$

for all $h \in H^1(\Omega)$, with $\epsilon_n' \to 0^+$, hence it holds

$$\left| \langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma - (1-t_n) \int_\Omega \frac{N_f(u_n^+)}{||u_n^+||} h dz - t_n \int_\Omega \lambda y_n h dz \right| \leqslant \frac{\epsilon_n' ||h||}{||u_n^+||}$$

for all $n \in \mathbb{N}$.

Hypotheses H_1(i), (ii) imply that

$$\left\{\frac{N_f(u_n^+)}{||u_n^+||}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.} \tag{3.62}$$

So, passing to a subsequence if necessary and using hypothesis H_1(ii), we have

$$\frac{N_f(u_n^+)}{||u_n^+||} \xrightarrow{w} \eta y \text{ in } L^2(\Omega), \text{ with } \hat{\lambda}_m \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega\,. \tag{3.63}$$

Choosing $h = y_n - y \in H^1(\Omega)$, passing to the limit as $n \to \infty$ and using (3.60), (3.62), we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,\\
\Rightarrow\ & y_n \to y \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property)},\\
\Rightarrow\ & ||y|| = 1,\ y \geqslant 0.
\end{aligned} \tag{3.64}$$

From (3.61), using (3.60) and (3.63), in the limit as $n \to \infty$, we have

$$\begin{aligned}
&\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega \eta_t(z) y h dz\\
&\text{for all } h \in H^1(\Omega), \text{ with } \eta_+(z) = (1-t)\eta(z) + t\lambda\\
\Rightarrow\ & -\Delta y(z) = \eta_t(z) y(z) \text{ for almost all } z \in \Omega,\ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega\,.
\end{aligned} \tag{3.65}$$

From (3.65) as in the proof of Proposition 3.5, we infer that $y \equiv 0$ or y is nodal, both contradicting (3.64). This proves Claim 3.14.

In fact this argument with minor changes shows that for all $t \in [0, 1]$, the functional $\hat{h}_+(t, \cdot)$ satisfies the C-condition. So, Proposition 3.2 of Liang and Su [21] (see also Chang [9, Theorem 5.1.21, page 334]) implies that

$$\begin{aligned}
&C_k(\hat{h}_+(0, \cdot), \infty) = C_k(\hat{h}_+(1, \cdot), \infty) \text{ for all } k \in \mathbb{N}_0,\\
\Rightarrow\ & C_k(\hat{\varphi}_+, \infty) = C_k(\hat{\psi}_+, \infty) \text{ for all } k \in \mathbb{N}_0.
\end{aligned} \tag{3.66}$$

Next consider the homotopy $\tilde{h}_+(t, u)$ defined by

$$\tilde{h}_+(t, u) = \hat{\psi}_+(u) - t \int_\Omega u dz \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Claim 3.15. $(\tilde{h}_+)'_u(t, u) \neq 0$ for all $t \in [0, 1]$ and all $u \in H^1(\Omega)\backslash\{0\}$.

We argue indirectly. So, suppose we can find $t \in [0, 1]$ and $u \in H^1(\Omega)\backslash\{0\}$ such that

$$\begin{aligned}
&(\tilde{h}_+)'_u(t, u) = 0 \\
\Rightarrow\ &\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z)uh d\sigma - \int_\Omega (u^-)h dz \\
&= \lambda \int_\Omega (u^+)h dz + t \int_\Omega h dz \ \text{ for all } h \in H^1(\Omega).
\end{aligned} \tag{3.67}$$

In (3.67) we choose $h = -u^- \in H^1(\Omega)$. Then

$$\begin{aligned}
&\gamma(u^-) + ||u^-||_2^2 \leqslant 0 \text{ (see hypothesis } H(\beta)), \\
\Rightarrow\ & u \geqslant 0,\ u \neq 0.
\end{aligned}$$

Then (3.67) becomes

$$\begin{aligned}
&\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z)uh d\sigma = \int_\Omega (\lambda u + t)h dz \text{ for all } h \in H^1(\Omega), \\
&\Rightarrow -\Delta u(z) = \lambda u(z) + t \text{ for almost all } z \in \Omega, \text{ and } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega\,.
\end{aligned} \tag{3.68}$$

From (3.68), regularity theory (see Wang [37]) and the strong maximum principle (see Gasinski and Papageorgiou [14, page 738]), we infer that

$$u \in D_+.$$

Let $v \in D_+$ and consider the function

$$R(v, u)(z) = |Dv(z)|^2 - \left(Du(z), D\left(\frac{v^2}{u}\right)(z)\right)_{\mathbb{R}^N}.$$

Using Picone's identity (see, for example, Motreanu, Motreanu and Papageorgiou [27, page 255]), we have

$$\begin{aligned}
0 &\leqslant \int_\Omega R(v,u) dz \\
&= ||Dv||_2^2 - \int_\Omega (-\Delta u)\frac{v^2}{u} dz + \int_{\partial\Omega} \beta(z)u\frac{v^2}{u} d\sigma \\
&\quad \text{(using Green's identity, see Gasinski and Papageorgiou [14, page 211])} \\
&\leqslant ||Dv||_2^2 + \int_{\partial\Omega} \beta(z)v^2 d\sigma - \int_\Omega \lambda v^2 dz \text{ (see (3.68))} \\
&= \gamma(v) - \lambda \int_\Omega v^2 dz.
\end{aligned} \tag{3.69}$$

In (3.69), let $v = \hat{u}_1 \in D_+$. Then

$$0 \leqslant (\hat{\lambda}_1 - \lambda) < 0$$

(since $||\hat{u}_1||_2 = 1$ and $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ with $m \geqslant 2$), a contradiction. This proves Claim 3.15.

The homotopy invariance of the singular homology groups implies that for $r > 0$ small, we have

$$H_k(\tilde{h}_+(0,\cdot)^\circ \cap B_r, \tilde{h}_+(0,\cdot)^\circ \cap B_r \backslash \{0\}) = H_k(\tilde{h}_+(1,\cdot)^\circ \cap B_r, \tilde{h}_+(1,\cdot)^\circ \cap B_r \backslash \{0\}) \quad (3.70)$$

for all $k \in \mathbb{N}_0$ with $B_r = \{y \in H^1(\Omega) : ||y|| < r\}$.

From Claim 3.15, we infer that

$$H_k(\tilde{h}_+(1,\cdot)^\circ \cap B_r, \tilde{h}_+(1,\cdot)^\circ \cap B_r \backslash \{0\}) = 0 \text{ for all } k \in \mathbb{N}_0 \quad (3.71)$$

(by the second deformation theorem, see Gasinski and Papageorgiou [14, page 628]).

Also, since $\tilde{h}_+(0,\cdot) = \hat{\psi}_+(\cdot)$, from the definition of critical groups, we have

$$H_k(\tilde{h}_+(0,\cdot)^\circ \cap B_r, \tilde{h}_+(0,\cdot)^\circ \cap B_r \backslash \{0\}) = C_k(\hat{\psi}_+, 0) \text{ for all } k \in \mathbb{N}_0. \quad (3.72)$$

From (3.70), (3.71), (3.72) we infer that

$$C_k(\hat{\psi}_+, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \quad (3.73)$$

Since $\lambda \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ we have $K_{\hat{\psi}_+} = \{0\}$ and so

$$\begin{aligned} & C_k(\hat{\psi}_+, \infty) = C_k(\psi_+, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\hat{\psi}_+, \infty) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see (3.73))}, \\ \Rightarrow\ & C_k(\hat{\varphi}_+, \infty) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see (3.66))}. \end{aligned}$$

Similarly we show that $C_k(\hat{\varphi}_-, \infty) = 0$ for all $k \in \mathbb{N}_0$. □

Thanks to this proposition, we can compute precisely the critical groups of the energy functional φ at the two constant sign solutions $u_0 \in D_+$ and $v_0 \in -D_+$ produced in Proposition 3.8.

Proposition 3.16. *If hypotheses $H(\beta)$, and H_1 hold, the solutions $u_0 \in D_+$ and $v_0 \in -D_+$ produced in Proposition* 3.8 *are the only nontrivial constant sign solutions of* (1.1) *and K_φ is finite, then $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof. The hypotheses and (3.26) imply that

$$K_{\hat{\varphi}_+} = \{0, u_0\}. \quad (3.74)$$

Let $\xi < 0 < \tau < \hat{m}_+$ (see (3.30)) and consider the following triple of sets

$$\hat{\varphi}_+^{\xi} \subseteq \hat{\varphi}_+^{\tau} \subseteq H^1(\Omega).$$

We consider the corresponding long exact sequence of singular homology groups. We have

$$\begin{aligned} \ldots \to H_k(H^1(\Omega), \hat{\varphi}_+^{\xi}) \xrightarrow{i_*} H_k(H^1(\Omega), \hat{\varphi}_+^{\tau}) \xrightarrow{\hat{\partial}_*} H_{k-1}(\hat{\varphi}_+^{\tau}, \hat{\varphi}_+^{\xi}) \to \ldots \\ \text{for all } k \in \mathbb{N} \end{aligned} \tag{3.75}$$

with i_* being the group homomorphism corresponding to the inclusion

$$(H^1(\Omega), \hat{\varphi}_+^{\xi}) \overset{i}{\hookrightarrow} (H^1(\Omega), \hat{\varphi}_+^{\tau})$$

and $\hat{\partial}_*$ is the composed boundary homomorphism (see Motreanu, Motreanu and Papageorgiou [27, Proposition 6.14, page 143]). From the rank theorem, we have

$$\begin{aligned} \operatorname{rank} H_k(H^1(\Omega), \hat{\varphi}_+^{\tau}) &= \operatorname{rank}\ker \hat{\partial}_* + \operatorname{rank}\operatorname{im} \hat{\partial}_* \\ &= \operatorname{rank}\operatorname{im} i_* + \operatorname{rank}\operatorname{im} \hat{\partial}_* \end{aligned} \tag{3.76}$$

(since (3.75) is exact). From the choice of the levels ξ, τ and (3.74) we have

$$\begin{aligned} &H_k(H^1(\Omega), \hat{\varphi}_+^{\xi}) = C_k(\hat{\varphi}_+, \infty) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see Proposition 3.13)}, \\ \Rightarrow\ &\operatorname{im} i_* = \{0\}. \end{aligned} \tag{3.77}$$

Also, we have

$$\begin{aligned} &H_k(H^1(\Omega), \hat{\varphi}_+^{\tau}) = C_k(\hat{\varphi}_+, u_0) \text{ for all } k \in \mathbb{N}_0, \\ &H_{k-1}(\hat{\varphi}_+^{\tau}, \hat{\varphi}_+^{\xi}) = C_{k-1}(\hat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \end{aligned} \tag{3.78}$$

(see Proposition 3.7). So returning to (3.76), we see that

$$\begin{aligned} &\operatorname{rank} C_1(\hat{\varphi}_+, u_0) \leqslant 1, \\ \Rightarrow\ &\operatorname{rank} C_1(\hat{\varphi}_+, u_0) = 1 \end{aligned}$$

(see the remark after Proposition 3.8). But note that (3.75), due to (3.77), (3.78), implies that only the tail $k = 1$ is nontrivial. Hence

$$C_k(\hat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{3.79}$$

Now consider the homotopy

$$h_+^*(t, u) = (1 - t)\varphi(u) + t\hat{\varphi}_+(u) \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Suppose we can find $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ such that

$$t_n \to t,\, u_n \to u_0 \text{ in } H^1(\Omega) \text{ and } (h_+^*)'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{3.80}$$

We have

$$-\Delta u_n(z) = (1-t_n) f(z, u_n(z)) + t_n f(z, u_n^+(z)) + t_n(u_n^-)(z) \text{ for almost all } z \in \Omega,$$
$$\frac{\partial u_n}{\partial n} + \beta(z) u_n = 0 \text{ on } \partial\Omega, n \in \mathbb{N}.$$

Then from regularity theory (see Wang [37]), we know that there exist $\alpha \in (0, 1)$ and $c_7 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_7 \text{ for all } n \in \mathbb{N}. \tag{3.81}$$

From the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (3.80), we have

$$\begin{aligned} & u_n \to u_0 \text{ in } C^1(\overline{\Omega}), \\ \Rightarrow\ & u_n \in D_+ \text{ for all } n \geqslant n_0, \\ \Rightarrow\ & \{u_n\}_{n \geqslant n_0} \subseteq K_\varphi \end{aligned}$$

(see (3.1)), a contradiction to the hypothesis that u_0 is the only nontrivial positive solution of (1.1). Therefore (3.80) cannot happen and so from the homotopy invariance of critical groups (see, for example, Gasinski and Papageorgiou [18, page 838]), we have

$$\begin{aligned} & C_k(h_+^*(0, \cdot), u_0) = C_k(h_+^*(1, \cdot), u_0) \text{ for all } k \in \mathbb{N}, \\ \Rightarrow\ & C_k(\varphi, u_0) = C_k(\hat{\varphi}_+, u_0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \end{aligned}$$

Similarly, using this time the functional $\hat{\varphi}_-$, we show that $C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$. □

Now we are ready for our first multiplicity theorem for problem (1.1) (three solutions theorem). We stress that the result is proved without imposing any differentiability condition on the reaction terms $f(z, \cdot)$. This is in contrast to the corresponding three solutions theorems of Liang and Su [21, Theorem 1.1] and de Paiva [28, Theorem 1.2] for Dirichlet doubly resonant equations, where $f \in C^1(\overline{\Omega} \times \mathbb{R})$.

Theorem 3.17. *If hypotheses $H(\beta)$, and H_1 hold, then problem* (1.1) *admits at least three distinct nontrivial smooth solutions*

$$u_0 \in D_+,\ v_0 \in -D_+ \text{ and } y_0 \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.8 we already have two nontrivial constant sign smooth solutions

$$u_0 \in D_+ \text{ and } v_0 \in -D_+.$$

From Proposition 3.16 we have that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{3.82}$$

Also, Proposition 3.7 implies that

$$C_k(\varphi, 0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{3.83}$$

From Proposition 3.11 we know that

$$C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{3.84}$$

Therefore, we can find $y_0 \in K_\varphi$ such that

$$C_{d_m}(\varphi, y_0) \neq 0 \text{ and } d_m \geqslant 2 \tag{3.85}$$

(since $m \geqslant 2$). Comparing (3.85) with (3.82), (3.83), we see that $y_0 \notin \{0, u_0, v_0\}$. So, y_0 is a nontrivial solution of (1.1) and by regularity theory, $y_0 \in C^1(\overline{\Omega})$. □

If we strengthen the conditions on the reaction $f(z, \cdot)$, we can generate a fourth nontrivial smooth solution, which is nodal. To produce this new solution, the reasoning is based on flow invariance arguments and uses the two extremal constant sign solutions $u_* \in D_+$ and $v_* \in -D_+$ produced in Proposition 3.10.

The new conditions on the reaction term $f(z, x)$ are the following:

H_2 : The map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, while $f(z, \cdot) \in C^1(\mathbb{R})$ and:

(i) It holds $|f'_x(z, x)| \leqslant a(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, and $2 \leqslant r < 2^*$;
(ii) There exists an integer $m \geqslant 3$ such that

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z, x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z, x)}{x} \leqslant \hat{\lambda}_{m+1}$$

uniformly for almost all $z \in \Omega$;
(iii) It holds $\lim_{x\to\pm\infty} [f(z, x)x - 2F(z, x)] = +\infty$ uniformly for almost all $z \in \Omega$ (recall $F(z, x) = \int_0^x f(z, s)ds$);
(iv) It holds $f'_x(z, 0) = \lim_{x\to 0} \frac{f(z, x)}{x}$ uniformly for almost all $z \in \Omega$ and

$$f'_x(z, 0) \leqslant \hat{\lambda}_1 \text{ for almost all } z \in \Omega,\ f'_x(\cdot, 0) \not\equiv \hat{\lambda}_1;$$

(v) There exists $\xi^* > 0$ such that for almost all $z \in \Omega$ the function

$$x \mapsto f(z, x) + \xi^* x$$

is strictly increasing on $\mathbb{R}$ and for every $M > 0$ there exists $\Omega_M \subseteq \Omega$ with $|\Omega_M|_N > 0$ such that

$$\frac{f(z, x)}{x} < f'_x(z, x) \text{ for almost all } z \in \Omega_M, \text{ all } |x| \leqslant M.$$

Remark 3.18. The second part of hypothesis $H_2(v)$ is satisfied if, for example, for almost all $z \in \Omega$

$$x \mapsto \frac{f(z, x)}{x} \text{ is strictly increasing on } (0, +\infty),$$
$$x \mapsto \frac{f(z, x)}{x} \text{ is strictly decreasing on } (-\infty, 0).$$

Similarly, if for almost all $z \in \Omega, \ x \mapsto f(z, x)$ is strictly convex on $[0, +\infty)$ and strictly concave on $(-\infty, 0]$.

Theorem 3.19. *If hypotheses $H(\beta)$, and H_2 hold, then problem* (1.1) *admits at least four distinct nontrivial smooth solutions*

$$u_0 \in D_+, \ v_0 \in -D_+, \ y_0 \in C^1(\overline{\Omega})$$
$$\hat{y} \in C^1(\overline{\Omega}) \textit{ nodal.}$$

Proof. From Theorem 3.17 we already have three distinct nontrivial smooth solutions

$$u_0 \in D_+, v_0 \in -D_+ \text{ and } y_0 \in C^1(\overline{\Omega}).$$

Let $\xi^* > 0$ be as postulated by hypothesis $H_2(v)$. On $H^1(\Omega)$ we introduce the following inner product

$$(u, h)_0 = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz + \int_{\partial\Omega} \beta(z) uh d\sigma + \int_\Omega \xi^* uh dz \text{ for all } u, h \in H^1(\Omega).$$

By $|| \cdot ||_0$ we denote the corresponding norm. Hypothesis $H(\beta)$ and the trace theorem imply that $|| \cdot ||_0$ and $|| \cdot ||$ are equivalent norms on $H^1(\Omega)$. Let $K \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ be defined by

$$\langle K(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz + \int_{\partial\Omega} \beta(z) uh d\sigma + \int_\Omega \xi^* uh dz = (u, h)_0$$

for all $u, h \in H^1(\Omega)$. Evidently K is (strictly) monotone, coercive, thus surjective. Invoking the Banach theorem (see Gasinski and Papageorgiou [17, Theorem 5.48, page 845]), we have that $K^{-1} \in \mathcal{L}(H^1(\Omega)^*, H^1(\Omega))$. Let

$$L = K^{-1} \circ (N_f + \xi^* I). \tag{3.86}$$

Then $L \in \mathcal{L}(H^1(\Omega), H^1(\Omega))$ and the regularity theory (see Wang [37]) implies that $L(C^1(\overline{\Omega})) \subseteq C^1(\overline{\Omega})$.

Claim 3.20. L is compact and $u - v \in C_+\backslash\{0\} \Rightarrow L(u) - L(v) \in D_+$.

According to Proposition 3.1.7 of Gasinski and Papageorgiou [14, page 268], to show the compactness of L, it suffices to show that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \Rightarrow L(u_n) \to L(u) \text{ in } H^1(\Omega). \tag{3.87}$$

From the Sobolev embedding theorem and the compactness of the trace map, we have

$$u_n \to u \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{3.88}$$

Let $v_n = L(u_n)$ for $n \in \mathbb{N}$ and $v = L(u)$. We have

$$\begin{aligned}
\langle K(v_n), h\rangle &= \int_\Omega [f(z, u_n) + \xi^* u_n] h dz \text{ for all } h \in H^1(\Omega), \text{ all } n \in \mathbb{N}, \\
\langle K(v), h\rangle &= \int_\Omega [f(z, u) + \xi^* u] h dz \text{ for all } h \in H^1(\Omega)
\end{aligned}$$

(see (3.86)). Then

$$\begin{aligned}
&\langle K(u_n) - K(u), h\rangle = \int_\Omega [f(z, u_n) - f(z, u)] h dz + \int_\Omega \xi^*(u_n - v_n) h dz, \\
&\Rightarrow (u_n - u, h)_0 \leqslant (||N_f(u_n) - N_f(u)||_2 + \xi^* ||u_n - u||_2) ||h||_2, \\
&\Rightarrow ||u_n - u||_0 \leqslant c_7 [||N_f(u_n) - N_f(u)||_2 + \xi^* ||u_n - u||_2] \text{ for some } c_7 > 0, \\
&\quad \text{all } n \in \mathbb{N}, \\
&\Rightarrow u_n \xrightarrow{||\cdot||_0} u \text{ (see (3.87))}, \\
&\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ and thus } L(u_n) \to L(u) \text{ in } H^1(\Omega).
\end{aligned}$$

So, we have established (3.87), which means that L is compact.

Next suppose that $u, v \in H^1(\Omega)$ and $u - v \in C_+\backslash\{0\}$. Let $x = L(u)$, $y = L(v)$. We have

$$\begin{aligned}
&\int_\Omega (Dx, Dh)_{\mathbb{R}^N} dz + \int_{\partial\Omega} \beta(z) x h d\sigma + \int_\Omega \xi^* x h dz = \int_\Omega [f(z, u) + \xi^* u] h dz \\
&\int_\Omega (Dy, Dh)_{\mathbb{R}^N} dz + \int_{\partial\Omega} \beta(z) y h d\sigma + \int_\Omega \xi^* y h dz = \int_\Omega [f(z, v) + \xi^* v] h dz \\
&\qquad \text{for all } h \in H^1(\Omega),
\end{aligned}$$

$$\Rightarrow -\Delta x(z) + \xi^* x(z) = f(z, u(z)) + \xi^* u(z) \tag{3.89}$$

$$\text{for almost all } z \in \Omega, \ \frac{\partial x}{\partial n} + \beta(z) x = 0 \text{ on } \partial\Omega,$$

$$-\Delta y(z) + \xi^* y(z) = f(z, v(z)) + \xi^* v(z) \tag{3.90}$$

$$\text{for almost all } z \in \Omega, \ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega$$

(see Papageorgiou and Rădulescu [32]).

Regularity theory implies that $x, y \in C^1(\overline{\Omega})$. Also hypothesis $H_2(v)$ and the fact that $u - v \in C_+\backslash\{0\}$ imply

$$f(z, v(z)) + \xi^* v(z) \leqslant f(z, u(z)) + \xi^* u(z) \text{ for almost all } z \in \Omega,$$
$$N_f(v) + \xi^* v \not\equiv N_f(u) + \xi^* u.$$

Hence $x - y \in C_+\backslash\{0\}$ and from (3.90), (3.91) we have

$$\Delta(x - y)(z) \leqslant \xi^*(x - y)(z) \text{ for almost all } z \in \Omega,$$
$$\Rightarrow x - y \in D_+$$

(by the strong maximum principle). This proves Claim 3.20.

Note that hypotheses H_2 imply $\varphi \in C^2(H^1(\Omega))$. Using the inner product $(\cdot, \cdot)_0$ we define the gradient $\nabla\varphi \in \mathcal{L}(H^1(\Omega), H^1(\Omega))$ by setting

$$\left\langle \varphi'(u), h \right\rangle = (\nabla\varphi(u), h)_0 \text{ for all } u, h \in H^1(\Omega). \tag{3.91}$$

Clearly, $u \mapsto \nabla\varphi(u)$ is continuous on $H^1(\Omega)$.

Claim 3.21. $\nabla\varphi = I - L$.

Let $u, h \in H^1(\Omega)$. Then

$$\begin{aligned}\left\langle \varphi'(u), h \right\rangle &= \langle A(u), h\rangle + \int_{\partial\Omega} \beta(z) u h d\sigma - \int_\Omega f(z, u) h dz \\ &= \langle A(u), h\rangle + \int_{\partial\Omega} \beta(z) u h d\sigma + \int_\Omega \xi^* u h dz \\ &\quad - \int_\Omega [f(z, u) + \xi^* u] h dz \\ &= (u, h)_0 - \int_\Omega [f(z, u) + \xi^* u] h dz.\end{aligned}$$

Note that

$$\begin{aligned}\int_\Omega [f(z, u) + \xi^* u] h dz &= \left\langle K \circ K^{-1}(N_f + \xi^* I)(u), h \right\rangle \\ &= \langle K \circ L(u), h\rangle \\ &= \langle A(L(u)), h\rangle + \int_{\partial\Omega} \beta(z) L(u) h d\sigma \\ &\quad + \int_\Omega \xi^* L(u) h dz \\ &= (L(u), h)_0.\end{aligned} \tag{3.92}$$

Returning to (3.92) and using (3.92), we obtain

$$\left\langle \varphi'(u), h \right\rangle = (u - L(u), h)_0 \text{ for all } u, h \in H^1(\Omega)$$
$$\Rightarrow \nabla\varphi = I - L$$

(see (3.91)). This proves Claim 3.21.

Consider the negative gradient flow $\tau(t,u)$ defined by

$$\frac{d\tau(t,u)}{dt} = -\nabla\varphi(\tau(t,u)) \text{ on } \mathbb{R}_+, \text{ with } \tau(0,u) = u.$$

On account of Claim 3.21, we can rewrite this Cauchy problem as

$$\frac{d\tau(t,u)}{dt} + \tau(t,u) = L(\tau(t,u)) \text{ on } \mathbb{R}_+, \text{ with } \tau(0,u) = u.$$

This is a linear Cauchy problem and so its flow is global and it is given by the following variation of constants formula

$$\tau(t,u) = e^{-t}u + \int_0^t e^{-(t-s)} L(\tau(s,u))ds \text{ for all } t \geqslant 0. \tag{3.93}$$

From (3.93) and the properties of the operator L, we have

$$\tau(t, C^1(\overline{\Omega})) \subseteq C^1(\overline{\Omega}) \text{ for all } t \geqslant 0$$

(that is, the space $C^1(\overline{\Omega})$ is positively τ-invariant). Also, if $u \in C_+\backslash\{0\}$, then

$$\tau(t,u) \in D_+ \text{ for all } t > 0.$$

We introduce the following set

$$E_1 = \big\{u \in C^1(\overline{\Omega}) : \text{there exists } t_0 > 0 \text{ such that for all } t \geqslant t_0,\\ \tau(t,u) \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]\big\}.$$

Here $[v_*, u_*]$ is the order interval defined by

$$[v_*, u_*] = \big\{u \in H^1(\Omega) : v_*(z) \leqslant u(z) \leqslant u_*(z) \text{ for almost all } z \in \Omega\big\}$$

and $\operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]$ denotes its interior in $C^1(\overline{\Omega})$, that is

$$\operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*] = \big\{u \in C^1(\overline{\Omega}) : u - v_* \in \operatorname{int} C_+, u_* - u \in \operatorname{int} C_+\big\}.$$

Recall that $v_* \in -D_+$ and $u_* \in D_+$ (see Proposition 3.10). Hence $0 \in E_1$. The continuous dependence of the flow on the initial condition, implies that $E_1 \subseteq C^1(\overline{\Omega})$ is open. Also, from the semigroup property of the flow, we have

$$\tau(t+s,u) = \tau(t, \tau(s,u)) \text{ for all } t,s \geqslant 0\\ \Rightarrow E_1 \text{ is positively } \tau - \text{invariant}. \tag{3.94}$$

Claim 3.22. ∂E_1 is positively τ-invariant.

Arguing by contradiction, suppose that Claim 3.22 is not true. Then we can find $\tilde{u} \in \partial E_1$ and $\tilde{t} > 0$ such that $\tau(\tilde{t}, \tilde{u}) \notin \partial E_1$. First we assume that $\tau(\tilde{t}, \tilde{u}) \in E_1$ (recall that E_1 is open). From the semigroup property of the flow, we have

$$\tau(t + \tilde{t}, \tilde{u}) = \tau(t, \tau(\tilde{t}, \tilde{u})) \text{ for all } t \geqslant 0,$$
$$\Rightarrow \tilde{u} \in E_1 \text{ (see (3.94)), a contradiction.}$$

So, suppose that $\tau(\tilde{t}, \tilde{u}) \notin \bar{E}_1$. Let $\{u_n\}_{n\geqslant 1} \subseteq E_1$ be such that $u_n \to \tilde{u}$ (recall that $\tilde{u} \in \partial E_1$). Then

$$\tau(\tilde{t}, u_n) \to \tau(\tilde{t}, \tilde{u})$$

(continuous dependence of the flow on the initial condition).

From (3.94) we have

$$\tau(\tilde{t}, u_n) \in E_1 \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \tau(\tilde{t}, \tilde{u}) \in \bar{E}_1,$$

a contradiction. This proves Claim 3.22.

Next we introduce the following set

$$E_2 = \big\{u \in C^1(\overline{\Omega}) : \text{there exists } t^* > 0 \text{ such that for all } t \geqslant t^*,$$
$$\tau(t, u) \in D_+ \cup (-D_+)\big\}.$$

Again $E_2 \subseteq C^1(\overline{\Omega})$ is open and the positive τ-invariance of $\pm D_+$, implies the positive τ-invariance of E_2. Also, we have $C_+ \backslash \{0\} \subseteq E_2$, and $0 \in \partial E_2$ and arguing as in the proof of Claim 3.22, we have that ∂E_2 is positively τ-invariant. Finally we show that

$$\partial E_1 \cap \partial E_2 \neq \emptyset. \tag{3.95}$$

Indeed, we know that $C_+ \subseteq \overline{E_2}$ and $\partial E_1 \cap C_+ \neq \emptyset$. Therefore $\partial E_1 \cap \overline{E_2} \neq \emptyset$. Because $\partial E_1 \cap (-D_+) \neq \emptyset$, we conclude that (3.95) holds. Using (3.95) we define

$$m_0 = \inf\left[\varphi(u) : u \in \partial E_1 \cap \partial E_2\right]. \tag{3.96}$$

Claim 3.23. $m_0 > -\infty$.

From (3.67) we see that it suffices to show that $\varphi|_{\partial E_1}$ is bounded from below. Note that $\varphi|_{[v_*, u_*]}$ is bounded from below (see hypotheses $H(\beta)$, and H_3(i)). If $u \in E$, then by definition there exists $t_0 > 0$ such that $\tau(t, u) \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]$ for all $t \geqslant t_0$. The flow $\tau(\cdot, u)$ being the negative gradient flow, is φ-decreasing. Hence

$$-\infty < m_* = \inf[\varphi(u) : u \in [v_*, u_*]] \leqslant \varphi(\tau(t, u)) \text{ for all } t \geqslant t_0$$
$$\leqslant \varphi(u) \text{ (since } \tau(0, u) = u).$$

Since $u \in E_1$ is arbitrary, it follows that

$$-\infty \leqslant m_* \leqslant \varphi|_{\partial E_1},$$
$$\Rightarrow -\infty < m_0.$$

This proves Claim 3.23.

Claim 3.24. m_0 is a critical value of φ (see (3.96) and Claim 3.23).

We argue indirectly. So, suppose that Claim 3.24 is not true and m_0 is a regular value of φ. Proposition 3.3 implies that $\varphi(K_\varphi) \subseteq \mathbb{R}$ is closed. So, for $\epsilon > 0$ small we have

$$(m_0 - \epsilon, m_0 + \epsilon) \cap \varphi(K_\varphi) = \emptyset.$$

Let $\hat{u} \in \partial E_1 \cap \partial E_2$ be such that

$$\varphi(\hat{u}) \leqslant m_0 + \frac{\epsilon}{2}$$

(see (3.96)).

Then the argument in the proof of the deformation theorem (see Gasinski and Papageorgiou [14, page 636]), implies that by choosing $\epsilon > 0$ even smaller if necessary, we have

$$\varphi(\tau(1, \hat{u})) \leqslant m_0 - \frac{\epsilon}{2}. \tag{3.97}$$

Let us recall that both ∂E_1 and ∂E_2 are positively τ-invariant. Hence

$$\begin{aligned} &\tau(1, \hat{u}) \in \partial E_1 \cap \partial E_2, \\ \Rightarrow\ & m_0 \leqslant \varphi(\tau(1, \hat{u})), \end{aligned}$$

a contradiction (see (3.97)). This proves Claim 3.24.

Claim 3.24 says that we can find $\hat{y} \in \partial E_1 \cap \partial E_2$ such that

$$\hat{y} \in K_\varphi \text{ and } m_0 = \varphi(\hat{y}).$$

Since $0 \in E_1$, we infer that $\hat{y} \neq 0$. Also $\hat{y} \in \partial E_2$ and so it follows that

$$\begin{aligned} &\hat{y} \notin D_+ \cup (-D_+) \\ \Rightarrow\ &\hat{y} \text{ is nodal} \end{aligned}$$

(recall the constant sign solutions of (1.1) belong to $D_+ \cup (-D_+)$).

Therefore we have proved that the set of nodal solutions of (1.1) is nonempty. We assume that is finite or otherwise we are done. According to Theorem 3.6 of Bartsch, Chang and Wang [4], we can choose the nodal solution $\hat{y}$ such that

$$\mu(\hat{y}) \in \{1, 2\} \text{ and } C_2(\varphi, \hat{y}) \neq 0. \tag{3.98}$$

Note that

$$\begin{aligned} \left\langle \varphi''(\hat{y})\hat{y}^+, \hat{y}^+ \right\rangle &= \gamma(\hat{y}^+) - \int_\Omega f'_x(z, \hat{y})(\hat{y}^+)^2 dz \\ &= \int_\Omega \left[f(z, \hat{y})\hat{y}^+ - f'_x(z, \hat{y})(\hat{y}^+)^2 \right] dz < 0 \\ \left\langle \varphi''(\hat{y})\hat{y}^-, \hat{y}^- \right\rangle &= \gamma(\hat{y}^-) - \int_\Omega f'_x(z, \hat{y})(\hat{y}^-)^2 dz \\ &= \int_\Omega \left[f(z, \hat{y})\hat{y}^- - f'_x(z, \hat{y})(\hat{y}^-)^2 \right] dz < 0 \end{aligned}$$

(see hypothesis H_2(v)).

Exploiting the fact that $\hat{y}^+ \perp \hat{y}^-$, we have

$$\left\langle \varphi''(\hat{y})h, h \right\rangle < 0 \text{ for all } h \in \operatorname{span}\{\hat{y}^+, \hat{y}^-\}.$$
$$\Rightarrow \mu(\hat{y}) = 2 \text{ and } C_k(\varphi, \hat{y}) = \delta_{k,2}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0$$

(see (3.98) and Theorem 2.5).

Recall (see (3.85)) that

$$C_{d_m}(\varphi, y_0) \neq 0 \text{ with } d_m \geqslant 3 \text{ (since } m \geqslant 3)$$
$$\Rightarrow \hat{y} \neq y_0.$$

Therefore $\hat{y} \in C^1(\overline{\Omega})$ is the fourth nontrivial smooth solution of (1.1) which is nodal. □

Remark 3.25. Bartsch, Chang and Wang [4], considered autonomous Dirichlet problems and did not allow for resonance to occur at $\pm\infty$. Under stronger conditions on the autonomous reaction term $f(x)$, they produced four nontrivial solutions, two of constant sign and two nodal. Later, de Paiva [28, Theorem 1.3] considered resonant Dirichlet problems and obtained four nontrivial solutions, two of constant sign and one nodal. His hypotheses do not allow double resonance to occur (see [28, hypothesis (g4)]) and the conditions on the reaction term are more restrictive. In particular, de Paiva [28] assumes that $f \in C^1(\Omega \times \mathbb{R})$. More recently, Liang and Su [21] also dealt with the Dirichlet problem and assumed that $f \in C^1(\Omega \times \mathbb{R})$. They allowed double resonance and they proved a four solution theorem (Liang and Su [21, Theorem 1.2]) but without producing a nodal solution and under more restrictive conditions on $f(z,x)$ – see in Liang and Su [21, hypothesis (f_0)].

4. Seven nontrivial solutions

In this section, by modifying the geometry of the problem near zero, we prove new multiplicity theorems producing up to seven nontrivial smooth solutions all with sign information.

First we impose the following conditions on the reaction term $f(z,x)$.

H_3 : The fuction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that:

(i) For every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leqslant a_\rho(z) \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(ii) There exist an integer $m \geqslant 2$ such that

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_{m+1}$$

uniformly for almost all $z \in \Omega$;

(iii) It holds $\lim_{x\to\pm\infty}[f(z,x)x - pF(z,x)] = \pm\infty$ uniformly for almost all $z \in \Omega$;

(iv) There exist functions $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} w_-(z) &\leqslant c_- < 0 < c_+ \leqslant w_+(z) \text{ for all } z \in \overline{\Omega}, \\ A(w_-) &\leqslant 0 \leqslant A(w_+) \text{ in } H^1(\Omega)^*, \\ f(z, w_+(z)) &\leqslant 0 \leqslant f(z, w_-(z)) \text{ for almost all } z \in \Omega; \end{aligned}$$

(v) There exist $l \in \mathbb{N}, l \neq m$ and $\delta_0 > 0$ such that

$$\begin{aligned} &\hat{\lambda}_l x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}x^2 \text{ for almost all } z \in \Omega,\ \text{all } |x| \leqslant \delta_0 \text{ when } l \geqslant 2 \\ &\mu(z)x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_2 x^2 \text{ for almost all } z \in \Omega,\ \text{all } |x| \leqslant \delta_0 \text{ when } l = 1 \end{aligned}$$

with $\eta \in L^\infty(\Omega), \eta(z) \geqslant \hat{\lambda}_1$ for almost all $z \in \Omega, \eta \not\equiv \hat{\lambda}_1$;

(vi) For every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 4.1. Suppose that we can find $\tau_- < 0 < \tau_+$ such that

$$f(z, \tau_-) \leqslant 0 \leqslant f(z, \tau_+) \text{ for almost all } z \in \Omega.$$

Then hypothesis $H_3(iv)$ is satisfied. Evidently, this hypothesis together with $H_3(v)$ implies a kind of oscillatory behavior near zero for $f(z,\cdot)$. Hypothesis $H_3(v)$ allows for double resonance to occur at zero with respect to any nonprincipal spectral interval. Since we have double resonance at $\pm\infty$ (see hypothesis $H_3(ii)$), we see that we have a kind of "double double resonance".

We introduce the following truncations-perturbations of the reaction $f(z,\cdot)$:

$$\begin{aligned} g_+(z,x) &= \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) + x & \text{if } 0 \leqslant x \leqslant w_+(z) \\ f(z, w_+(z)) + w_+(z) & \text{if } w_+(z) < x \end{cases} \\ g_-(z,x) &= \begin{cases} f(z, w_-(z)) + w_-(z) & \text{if } x < w_-(z) \\ f(z,x) + x & \text{if } w_-(z) \leqslant x \leqslant 0 \\ 0 & \text{if } 0 < x. \end{cases} \end{aligned} \tag{4.1}$$

Both are Carathéodory functions. We set $G_\pm(z,x) = \int_0^x g_\pm(z,s)ds$ and consider the C^1-functionals $\psi_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega G_\pm(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Proposition 4.2. *Assume that hypotheses $H(\beta)$, H_3(iv), (v) hold,*

$$|f(z,x)| \leqslant a(z) \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho = \max\{||w_+||_\infty, ||w_-||_\infty\}$$

with $a \in L^\infty(\Omega)_+$ and H_3(vi) is true for this $\rho > 0$. Then problem (1.1) admits two nontrivial constant sign solutions

$$u_0 \in D_+ \text{ with } u_0 \in [0, w_+],$$
$$v_0 \in -D_+ \text{ with } v_0 \in [w_-, 0].$$

Proof. Evidently ψ_+ is coercive (see (4.1)). Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that ψ_+ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in H^1(\Omega)$ such that

$$\psi_+(u_0) = \inf\left[\psi_+(u) : u \in H^1(\Omega)\right]. \tag{4.2}$$

Recall that $\hat{u}_1 \in D_+$. So, we can find $t \in (0,1)$ small such that

$$0 < t\hat{u}_1(z) \leqslant c_+ \text{ for all } z \in \overline{\Omega}.$$

Then when $l \geqslant 2$, clearly we have

$$\psi_+(t\hat{u}_1) < 0.$$

When $l = 1$, we have

$$\psi_+(t\hat{u}_1) = \frac{t^2}{2}\int_\Omega [\hat{\lambda}_1 - \eta(z)]\hat{u}_1^2 dz < 0$$

see hypothesis H_3(v). Therefore

$$\psi_+(u_0) < 0 = \psi_+(0)$$

(see (4.2)), hence $u_0 \neq 0$. From (4.2) we have

$$\begin{aligned} &\psi_+'(u_0) = 0, \\ &\Rightarrow \langle A(u_0), h\rangle + \int_\Omega u_0 h dz + \int_{\partial\Omega} \beta(z) u_0 h d\sigma \\ &= \int_\Omega g_+(z, u_0) h dz \text{ for all } h \in H^1(\Omega). \end{aligned} \tag{4.3}$$

In (4.3) we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\gamma(u_0^-) + ||u_0^-||_2^2 \leqslant 0,$$
$$\Rightarrow u_0 \geqslant 0, \ u_0 \neq 0.$$

Also, in (4.3) we choose $h = (u_0 - w_+)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
& \left\langle A(u_0), (u_0 - w_+)^+ \right\rangle + \int_\Omega u_0(u_0 - w_+)^+ dz + \int_{\partial\Omega} \beta(z) u_0 (u_0 - w_+)^+ d\sigma \\
= & \int_\Omega [f(z, w_+) + w_+](u_0 - w_+)^+ dz \text{ (see (4.1))} \\
\leqslant & \left\langle A(w_+), (u_0 - w_+)^+ \right\rangle + \int_\Omega w_+(u_0 - w_+)^+ dz + \int_{\partial\Omega} \beta(z) u_0 (u_0 - w_+)^+ d\sigma \\
& \text{(see hypotheses } H_3\text{(iv), } H(\beta)\text{)} \\
\Rightarrow & \ ||D(u_0 - w_+)^+||_2^2 + ||(u_0 - w_+)^+||_2^2 \leqslant 0 \text{ (see hypothesis } H(\beta)\text{)} \\
\Rightarrow & \ u_0 \leqslant w_+.
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, w_+].$$

Then equation (4.3) becomes

$$\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z) u_0 h d\sigma = \int_\Omega f(z, u_0) h dz \text{ for all } h \in H^1(\Omega) \text{ (see (4.1))}$$

$$\Rightarrow -\Delta u_0(z) = f(z, u_0(z)) \text{ for almost all } z \in \Omega, \text{ and } \frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \text{ on } \partial\Omega, \quad (4.4)$$

$$\Rightarrow u_0 \in C_+ \backslash \{0\} \quad (4.5)$$

(regularity theory). By hypothesis from (4.4) we have

$$\begin{aligned}
& \Delta u_0(z) \leqslant \hat{\xi}_\rho u_0(z) \text{ for almost all } z \in \Omega, \\
\Rightarrow & \ u_0 \in D_+
\end{aligned}$$

(by the strong maximum principle). Similarly, working with the functional ψ_-, we produce $v_0 \in -D_+$ a negative solution of problem (1.1). □

Remark 4.3. If $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ are not solutions for (1.1) and $\Delta w_\pm \in L^2(\Omega)$ or alternatively $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ and

$$f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \text{ for almost all } z \in \Omega, \ f(\cdot, w_\pm(\cdot)) \not\equiv 0$$

then

$$u_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[0, w_+] \text{ and } v_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[w_-, 0].$$

Indeed, if the first option holds, then with $\hat{\xi}_\rho > 0$ as in the hypothesis, we have

$$\begin{aligned}
- & \ \Delta u_0(z) + \hat{\xi}_\rho u_0(z) \\
= & \ f(z, u_0(z)) + \hat{\xi}_\rho u_0(z) \\
\leqslant & \ f(z, w_+(z)) + \hat{\xi}_\rho w_+(z) \text{ (since } u_0 \leqslant w_+\text{)} \\
\leqslant & \ -\Delta w_+(z) + \hat{\xi}_\rho w_+(z) \text{ for almost all } z \in \Omega \qquad (4.6) \\
& \text{(see hypothesis } H_3(iv) \text{ and recall that } \Delta w_+ \in L^2(\Omega)\text{)} \\
\Rightarrow & \ \Delta(w_+ - u_0)(z) \leqslant \hat{\xi}_\rho (w_+ - u_0)(z) \text{ for almost all } z \in \Omega. \qquad (4.7)
\end{aligned}$$

Note that $w_+ \neq u_0$ (recall that w_+ is not a solution of (1.1)). So, from (4.7) and the strong maximum principle, we have

$$\begin{aligned} & w_+ - u_0 \in \operatorname{int} C_+ \\ \Rightarrow\ & u_0 \in \operatorname{int}_{C_1(\overline{\Omega})}[0, w_+]. \end{aligned}$$

If the second option holds, then from (4.6) and the strong comparison principle of Fragnelli, Mugnai and Papageorgiou [12, Proposition 4], we have

$$\begin{aligned} & w_+ - u_0 \in \operatorname{int} C_+ \\ \Rightarrow\ & u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[0, w_+]. \end{aligned}$$

In this case, Proposition 2.4 implies that $u_0 \in D_+$ and $v_0 \in -D_+$ are both local minimizers of the energy functional φ.

As before, we can produce extremal constant sign solutions.

Proposition 4.4. *If the hypotheses of Proposition* 4.2 *hold, then problem* (1.1) *admits extremal constant sign solutions*

$$u_* \in D_+ \text{ and } v_* \in -D_+.$$

Proof. As in the proof of Proposition 3.10, we can find a decreasing sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

Evidently $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } H^1(\Omega) \text{ and } u_n \to u_* \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{4.8}$$

Suppose that $u_* = 0$ and let $y_n = \dfrac{u_n}{||u_n||}$, $n \in \mathbb{N}$. Then $||y_n|| = 1,\ y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega),\ y \geqslant 0. \tag{4.9}$$

We have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma = \int_\Omega \frac{N_f(u_n)}{||u_n||} h dz \text{ for all } n \in \mathbb{N}. \tag{4.10}$$

The hypotheses on $f(z, x)$ imply that

$$\begin{aligned} & |f(z, x)| \leqslant c_7|x| \text{ for almost all } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{some } c_7 > 0 \\ \Rightarrow\ & \left\{\frac{N_f(u_n)}{||u_n||}\right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.} \end{aligned} \tag{4.11}$$

Then from (4.11) and hypothesis H_3(v) and by passing to a subsequence if necessary, we have

$$\frac{N_f(u_n)}{||u_n||} \xrightarrow{w} \eta_0 y \text{ in } L^2(\Omega) \text{ with } \begin{cases} \hat{\lambda}_l \leqslant \eta_0(z) \leqslant \hat{\lambda}_{l+1} \text{ if } l \geqslant 2 \\ \eta(z) \leqslant \eta_0(z) \leqslant \hat{\lambda}_2 \text{ if } l = 1 \end{cases} \tag{4.12}$$

for almost all $z \in \Omega$. In (4.10) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (4.9), (4.11). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ &\Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property)}, \\ &\Rightarrow ||y|| = 1,\ y \geqslant 0. \end{aligned} \tag{4.13}$$

Passing to the limit as $n \to \infty$ in (4.10) and using (4.9), (4.12) we obtain

$$\begin{aligned} &\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega \eta_0(z) y h dz \text{ for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta y(z) = \eta_0(z) y(z) \text{ for almost all } z \in \Omega,\ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.14}$$

From (4.12), we have

$$\begin{aligned} &\widetilde{\lambda}_l(\eta_0) \leqslant \widetilde{\lambda}_l(\hat{\lambda}_l) = 1 \quad \text{if } l \geqslant 2 \\ &\widetilde{\lambda}_1(\eta_0) < \widetilde{\lambda}_1(\hat{\lambda}_1) = 1 \quad \text{if } l = 1 \end{aligned}$$

(see Proposition 2.3). So, in both cases we have

$$\begin{aligned} &\widetilde{\lambda}_1(\eta_0) < 1 \\ \Rightarrow\ & y = 0 \text{ or } y \text{ is nodal}, \end{aligned}$$

(see (4.14)) a contradiction, see (4.13). Therefore $u_* \neq 0$ and so $u_* \in S_+, u_* = \inf S_+$.

Similarly we produce $v_* \in S_-,\ v_* = \sup S_-$. ☐

Using these extremal constant sign solutions, we can produce nodal solutions. First we compute the critical groups of the energy functional φ at the origin. To do this, we need only the behavior of $f(z, \cdot)$ near zero.

Proposition 4.5. *Assume that hypothesis $H(\beta)$ holds, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that*

$$|f(z, x)| \leqslant a(z)(1 + |x|^{r-1}) \textit{ for almost all } z \in \Omega,\ \textit{all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+,\ 2 \leqslant r < 2^*$ for almost all $z \in \Omega$

$$f(z, 0) = 0 \textit{ and } f(z, \cdot) \textit{ satisfies hypothesis } H_3(\text{v}).$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z, u)dz \textit{ for all } u \in H^1(\Omega)$$

(recall $F(z, x) = \int_0^x f(z, s)ds$) and assume that φ satisfies the C-condition and $0 \in K_\varphi$ is isolated. Then $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$, where $d_l = \dim \overline{H}_l$.

Proof. Let $\vartheta \in (\hat{\lambda}_l, \hat{\lambda}_{l+1})$ and consider the C^2-functional $\xi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\xi(u) = \frac{1}{2}\gamma(u) - \frac{\vartheta}{2}||u||_2^2 \text{ for all } u \in H^1(\Omega).$$

Then $u = 0$ is a nondegenerate critical point of ξ with Morse index d_l. Hence

$$C_k(\xi, 0) = \delta_{k,d_l}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \tag{4.15}$$

(see Motreanu, Motreanu and Papageorgiou [27, Theorem 6.51, page 155]).

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1 - t)\varphi(u) + t\xi(u) \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

We have

$$\left\langle h_u^{'}(t, u), y\right\rangle = (1 - t)\left\langle \varphi^{'}(u), y\right\rangle + t\left\langle \xi^{'}(u), y\right\rangle \text{ for all } u, y \in H^1(\Omega). \tag{4.16}$$

Let $\delta_0 > 0$ be as postulated by hypothesis H_3(v) and let $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta_0$.

We write

$$u = \overline{u} + \hat{u} \text{ with } \overline{u} \in \overline{H}_l,\ \hat{u} \in \hat{H}_{l+1}$$

(this decomposition is unique). Exploiting the orthogonality of the component spaces, we have

$$\left\langle \varphi^{'}(u), \hat{u} - \overline{u}\right\rangle = \gamma(\hat{u}) - \gamma(\overline{u}) - \int_\Omega f(z, u)(\hat{u} - \overline{u})dz. \tag{4.17}$$

The choice of $u \in C^1(\overline{\Omega})$ and hypothesis H_3(v) imply that

$$f(z, u(z))(\hat{u} - \overline{u})(z) \leqslant \hat{\lambda}_{l+1}\hat{u}(z)^2 - \hat{\lambda}_l\bar{u}(z)^2 \text{ for almost all } z \in \Omega. \tag{4.18}$$

Using (4.18) in (4.17), we obtain

$$\left\langle \varphi^{'}(u), \hat{u} - \overline{u}\right\rangle \geqslant \gamma(\hat{u}) - \hat{\lambda}_{l+1}||\hat{u}||_2^2 - \left[\gamma(\overline{u}) - \hat{\lambda}_l||\overline{u}||_2^2\right] \geqslant 0. \tag{4.19}$$

Also we have

$$\left\langle \xi'(u), \hat{u} - \overline{u} \right\rangle = \gamma(\hat{u}) - \vartheta ||\hat{u}||_2^2 - \left[\gamma(\overline{u}) - \vartheta ||\overline{u}||_2^2 \right] \geqslant c_8 ||u||^2 \tag{4.20}$$

for some $c_8 > 0$ (see Proposition 2.2).

Returning to (4.16) and using (4.19), (4.20), we have

$$\left\langle h_u'(t, u), \hat{u} - \overline{u} \right\rangle \geqslant t c_8 ||u||^2 > 0 \text{ for } t \in (0, 1].$$

For $t = 0$, we have $h(0, \cdot) = \varphi(\cdot)$ and by hypothesis $0 \in K_\varphi$ is isolated. Since $K_{h(t,\cdot)} \subseteq C^1(\overline{\Omega})$ for all $t \in [0, 1]$ (regularity theory, see Wang [37]), using the homotopy invariance property of critical groups (see Corvellec and Hantoute [10, Theorem 5.2] and Gasinski and Papageorgiou [18, Theorem 5.126, page 838]), we have

$$\begin{aligned} & C_k(h(0, \cdot), 0) = C_k(h(1, \cdot), 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi, 0) = C_k(\xi, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ & C_k(\varphi, 0) = \delta_{k,d_l} \text{ for all } k \in \mathbb{N}_0 \text{ (see (4.15)).} \end{aligned}$$

□

Proposition 4.6. *If hypotheses $H(\beta)$, and H_3(iv), (v) (with $l \geqslant 2$) hold,*

$$|f(z, x)| \leqslant a(z) \textit{ for almost all } z \in \Omega, \textit{ all } |x| \leqslant \rho = \max\{||w_+||_\infty, ||w_-||_\infty\}$$

with $a \in L^\infty(\Omega)_+$ and H_3(vi) is true for this $\rho > 0$, then problem (1.1) *admits a nodal solution $y_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[v_*, u_*]$ (here $u_* \in D_+$ and $v_* \in -D_+$ are two extremal constant sign solutions produced in Theorem* 3.19*).*

Proof. Using the extremal constant sign solutions $u_* \in D_+$ and $v_* \in -D_+$ (see Proposition 4.4), we introduce the following Carathéodory function

$$k(z, x) = \begin{cases} f(z, v_*(z)) + v_*(z) & \text{if } x < v_*(z) \\ f(z, x) + x & \text{if } v_*(z) \leqslant x \leqslant u_*(z) \\ f(z, u_*(z)) + u_*(z) & \text{if } u_*(z) < x. \end{cases} \tag{4.21}$$

Also we consider the positive and negative truncations of $k(z, \cdot)$, that is, the Carathéodory functions

$$k_\pm(z, x) = k(z, \pm x^\pm) \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

We set $K(z, x) = \int_0^x k(z, s)ds$, $K_\pm(z, x) = \int_0^x k_\pm(z, s)ds$ and consider the C^1-functionals $\mu, \mu_\pm : H^1(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \mu(u) &= \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega K(z, u)dz \\ \mu_\pm(u) &= \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega K_\pm(z, u)dz \text{ for all } u \in H^1(\Omega). \end{aligned}$$

Claim 4.7. It holds $K_\mu \subseteq [v_*, u_*]$, while $K_{\mu_+} = \{0, u_*\}$, and $K_{\mu_-} = \{0, v_*\}$.
Let $u \in K_\mu$. Then

$$\langle A(u), h\rangle + \int_\Omega uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma = \int_\Omega k(z, u)hdz \text{ for all } h \in H^1(\Omega). \quad (4.22)$$

Choose $h = u - u_* \in H^1(\Omega)$. Then

$$\begin{aligned}
&\left\langle A(u), (u-u_*)^+\right\rangle + \int_\Omega u(u-u_*)^+dz + \int_{\partial\Omega} \beta(z)u(u-u_*)^+d\sigma \\
= &\int_\Omega [f(z, u_*) + u_*](u-u_*)^+dz \text{ (see (4.21))} \\
= &\left\langle A(u_*), (u-u_*)^+\right\rangle + \int_\Omega u_*(u-u_*)^+dz + \int_{\partial\Omega} \beta(z)u_*(u-u_*)^+d\sigma, \\
\Rightarrow &\ ||D(u-u_*)^+||_2^2 + ||(u-u_*)^+||_2^2 \leqslant 0 \text{ (see hypothesis H}(\beta)) \\
\Rightarrow &\ u \leqslant u_*.
\end{aligned}$$

Similarly, choosing $h = (v_* - u)^+ \in H^1(\Omega)$ in (4.22), we show that $v_* \leqslant u$. Hence

$$K_\mu \subseteq [v_*, u_*].$$

In a similar fashion, we show that

$$K_{\mu_+} \subseteq [0, u_*] \text{ and } K_{\mu_-} \subseteq [v_*, 0].$$

The extremality of $u_* \in D_+$ and $v_* \in -D_+$ implies that

$$K_{\mu_+} = \{0, u_*\} \text{ and } K_{\mu_-} = \{0, v_*\}.$$

This proves Claim 4.7.

Claim 4.8. $u_* \in D_+$ and $v_* \in -D_+$ are local minimizers of μ.

Evidently μ_+ is coercive (see (4.21)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_* \in H^1(\Omega)$ such that

$$\mu_+(\tilde{u}_*) = \inf\left[\mu_+(u) : u \in H^1(\Omega)\right]. \quad (4.23)$$

Because of hypothesis H_3(v), for $t \in (0, 1)$ small so that $t\hat{u}_1(z) \leqslant \min_{\overline{\Omega}} u_*$ (recall that $\hat{u}_1, u_* \in D_+$), we have

$$\begin{aligned}
&\mu_+(t\hat{u}_1) < 0 \text{ (see the proof of Proposition 4.2)}, \\
\Rightarrow &\ \mu_+(\tilde{u}_*) < 0 = \mu_+(0), \\
\Rightarrow &\ \tilde{u}_* \neq 0.
\end{aligned}$$

From (4.23) and Claim 4.7, we infer that $\tilde{u}_* = u_* \in D_+$. Note that

$$\begin{aligned}
&\mu_+ \big|_{C_+} = \mu\big|_{C_+} \text{ (see (4.21))},\\
\Rightarrow\ & u_* \in D_+ \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \mu,\\
\Rightarrow\ & u_* \in D_+ \text{ is a local } H^1(\Omega) - \text{minimizer of } \mu
\end{aligned}$$

(see Proposition 2.4). Similarly for $v_* \in -D_+$ using this time the functional μ_-.

This proves Claim 4.8.

Without any loss of generality, we may assume that $\mu(v_*) \leqslant \mu(u_*)$ (the reasoning is similar if the opposite inequality holds). Also, we may assume that K_μ is finite. Otherwise, thanks to Claim 4.7, we see that we have an infinity of nodal solutions belonging in $C^1(\overline{\Omega})$ (regularity theory). Using Claim 4.8, we see that we can find $\rho \in (0, 1)$ small such that

$$\mu(v_*) \leqslant \mu(u_*) < \inf[\mu(u) : \ ||u - u_*|| = \rho] = m_*, \ ||v_* - u_*|| > \rho \tag{4.24}$$

(see Aizicovici, Papageorgiou and Staicu [1, proof of Proposition 29]).

The functional μ is coercive (see (4.21)). So,

$$\mu \text{ satisfies the C-condition} \tag{4.25}$$

(see Papageorgiou and Winkert [35]). Then (4.24) and (4.25) permit the use of Theorem 2.1 (the mountain pass theorem). Therefore, we can find $y_0 \in H^1(\Omega)$ such that

$$y_0 \in K_\mu \text{ and } m_* \leqslant \mu(y_0). \tag{4.26}$$

From (4.26) we have

$$y_0 \in C^1(\overline{\Omega})$$

(regularity theory, see Wang [37]) and

$$y_0 \in [v_*, u_*] \setminus \{v_*, u_*\}$$

(see (4.24)). So, $y_0 \in C^1(\overline{\Omega})$ is a smooth solution of problem and

$$C_1(\mu, y_0) \neq 0. \tag{4.27}$$

From Proposition 4.5, we have

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.28}$$

Since $u_* \in D_+$ and $v_* \in -D_+$, we have

$$C_k\left(\mu|_{C^1(\overline{\Omega})}, 0\right) = C_k\left(\varphi|_{C^1(\overline{\Omega})}, 0\right) \text{ for all } k \in \mathbb{N}_0. \tag{4.29}$$

From Palais [30, Theorem 16] (see also Chang [8, page 14]), we know that

$$C_k\left(\mu|_{C^1(\overline{\Omega})}, 0\right) = C_k(\mu, 0) \text{ and } C_k\left(\varphi|_{C^1(\overline{\Omega})}, 0\right) = C_k(\varphi, 0) \text{ for all } k \in \mathbb{N}_0. \tag{4.30}$$

So, from (4.29), (4.30) we infer that

$$\begin{aligned} &C_k(\mu, 0) = C_k(\varphi, 0) \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow\ &C_k(\mu, 0) = \delta_{k,d_l}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \end{aligned} \tag{4.31}$$

Since $l \geqslant 2$, we have $d_l \geqslant 2$ and so from (4.27) and (4.31) it follows that

$$\begin{aligned} &y_0 \in K_\mu \setminus \{0, u_*, v_*\} \subseteq [v_*, u_*] \cap C^1(\overline{\Omega}) \setminus \{0, u_*, v_*\} \text{ (see Claim 4.7)}, \\ \Rightarrow\ &y_0 \in C^1(\overline{\Omega}) \text{ is a nodal solution of (1.1).} \end{aligned}$$

Let $\hat{\xi}_\rho > 0$ be as in the hypothesis. We have

$$\begin{aligned} &-\Delta y_0(z) + \hat{\xi}_\rho y_0(z) \leqslant -\Delta u_*(z) + \hat{\xi}_\rho u_*(z) \text{ for almost all } z \in \Omega, \\ \Rightarrow\ &\Delta(u_* - y_0)(z) \leqslant \hat{\xi}_\rho (u_* - y_0)(z) \text{ for almost all } z \in \Omega, \\ \Rightarrow\ &u_* - y_0 \in D_+ \end{aligned}$$

(by the strong maximum principle).

Similarly we show that $y_0 - v_* \in D_+$. So, finally we have

$$y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_*, u_*]. \qquad \square$$

So, we can state the following multiplicity result. We stress that in this theorem there is no growth or differentiability requirement on $f(z, \cdot)$ and we have sign information for all solutions. Moreover, at zero we allow resonance with respect to any nonprincipal eigenvalue. So, our result is a considerable improvement of the multiplicity results of Liang and Su [21, Theorem 1.1], de Paiva [28, Theorem 1.4], Su [36, Theorem 2] for Dirichlet problems.

Theorem 4.9. *If hypotheses $H(\beta)$, and H_3(iv),(v) (with $l \geqslant 2$) hold,*

$$|f(z, x)| \leqslant a(z) \textit{ for almost all } z \in \Omega \textit{ all } |x| \leqslant \rho = \max\{||w_+||_\infty, ||w_-||_\infty\}$$

with $a \in L^\infty(\Omega)_+$ and H_3(vi) is true for this $\rho > 0$, then problem (1.1) admits at least three nontrivial smooth solutions

$$u_0 \in D_+,\ v_0 \in -D_+ \textit{ and } y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal.}$$

We can improve this multiplicity theorem and produce a second nodal solution, provided that we strengthen the regularity of $f(z, \cdot)$.

So, the new conditions on the reaction term $f(z, x)$, are the following:

H_0' : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R} \setminus \{0\})$, is continuous at zero, satisfies hypotheses H_3(iv), (v),

$$|f(z, x)| \leqslant a(z) \text{ for almost all } z \in \Omega \text{ all } |x| \leqslant \rho = \max\{||w_+||_\infty, ||w_-||_\infty\},$$

with $a \in L^\infty(\Omega)_+$ and H_3(vi) is true for this $\rho > 0$.

Theorem 4.10. *If hypotheses $H(\beta)$, and $H_0^{'}$ hold, then problem* (1.1) *admits at least four nontrivial smooth solutions*

$$u_0 \in D_+, \; v_0 \in -D_+ \text{ and } y_0, \hat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0] \text{ nodal.}$$

Proof. From Theorem 4.9, we already have three nontrivial smooth solutions

$$u_0 \in D_+, v_0 \in -D_+ \text{ and } y_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0] \text{ nodal.} \tag{4.32}$$

Returning to the notation introduced in the proof of Proposition 4.6 and recalling that y_0 is a critical point of the functional μ of the mountain pass type, we have

$$\begin{aligned} &C_1(\mu, y_0) \neq 0 \text{ and } \mu \in C^{2-0}(H^1(\Omega)),\\ \Rightarrow\ &C_k(\mu, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \end{aligned} \tag{4.33}$$

(see Li, Li and Liu [24, Theorem 2.7]).

Without any loss of generality, we may assume that u_0 and v_0 are extremal (that is, $u_0 = u_* \in D_+$, $v_0 = v_* \in -D_+$). From the proof of Proposition 4.6 (see Claim 4.8), we know that u_0 and v_0 are local minimizers of μ. Hence

$$C_k(\mu, u_0) = C_k(\mu, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.34}$$

Also, we know that

$$C_k(\mu, 0) = \delta_{k,d_l}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see (4.31)).} \tag{4.35}$$

Finally recall that μ is coercive (see (4.21)). Hence

$$C_k(\mu, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.36}$$

Suppose that $K_\mu = \{y_0, u_0, v_0, 0\}$. Then from (4.33), (4.34), (4.35), (4.36) and the Morse relation with $t = -1$ (see (3.54)), we obtain

$$\begin{aligned} &(-1)^1 + 2(-1)^0 + (-1)^{d_l} = (-1)^0,\\ \Rightarrow\ &(-1)^{d_l} = 0 \text{ a contradiction.} \end{aligned}$$

So, there exists $\hat{y} \in H^1(\Omega)$ such that

$$\hat{y} \in K_\mu \text{ and } \hat{y} \notin \{y_0, u_0, v_0, 0\}. \tag{4.37}$$

Then from (4.37) and Claim 4.7 in the proof of Proposition 4.6, we have

$$\begin{aligned} &\hat{y} \in [v_0, u_0] \cap C^1(\overline{\Omega}),\\ \Rightarrow\ &\hat{y} \in C^1(\overline{\Omega}) \text{ is a second nodal solution of (1.1)} \end{aligned}$$

(see (4.21) and (4.37)). Moreover, as in the proof of Proposition 4.6, using the strong maximum principle, we have $\hat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0]$. □

Remark 4.11. Theorem 4.10 above is considerably more general than Theorem 1.1 of Zhang, Li and Xue [38]. In that work, the differential operator (left-hand side of the equation), is $-\Delta u + au$ with $a \in (0, +\infty)$, the reaction term f is autonomous and has positive and negative zeros and no resonance is allowed at zero. In addition, the boundary coefficient $\beta \not\equiv 0$ and so their framework excludes Neumann problems. They produce four nontrivial solutions, but no nodal solutions.

Up to this point we have not used the asymptotic conditions at $\pm\infty$ (see hypotheses H_3(ii), (iii)). If we activate these conditions, then we can produce additional constant sign smooth solutions for problem (1.1). In what follows, $u_0 \in D_+$ and $v_0 \in -D_+$ are the two nontrivial constant sign smooth solutions from Theorem 4.9. Also, we strengthen hypothesis H_3(iv) according the Remark 4.3. So, we introduce the following conditions on $f(z, x)$:

H_3' : The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that hypotheses H_3'(i), (ii), (iii), (v), (vi) are the same as the corresponding hypotheses H_3(i), (ii), (iii), (v), (vi) and (iv) one of the following holds:

[a] there exist functions $w_\pm \in C^1(\overline{\Omega})$ which are not solutions of (1.1), $\Delta w_\pm \in L^2(\Omega)$ and

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \text{ for all } z \in \overline{\Omega},\\
&A(w_-) \leqslant 0 \leqslant A(w_+) \text{ in } H^1(\Omega)^*\\
&f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \text{ for almost all } z \in \Omega;
\end{aligned}$$

or

[b] there exist functions $w_\pm \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \text{ for all } z \in \overline{\Omega},\\
&A(w_+) \leqslant 0 \leqslant A(w_-) \text{ in } H^1(\Omega)^*,\\
&f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \text{ for almost all } z \in \Omega, \text{ and } f(\cdot, w_\pm(\cdot)) \not\equiv 0.
\end{aligned}$$

Remark 4.12. If there exist $c_- < 0 < c_+$ such that

$$f(z, c_+) < 0 < f(z, c_-) \text{ for almost all } z \in \Omega_0 \subseteq \Omega \text{ with } |\Omega_0|_N > 0,$$

then both cases [a] and [b] in hypothesis H_3'(iv) are satisfied.

Proposition 4.13. *If hypotheses $H(\beta)$, and H_3' hold, then problem* (1.1) *admits two more nontrivial smooth solutions of constant sign*

$$\begin{aligned}
&\hat{u} \in D_+ \textit{ with } \hat{u} - u_0 \in D_+,\\
&\hat{v} \in D_+ \textit{ with } v_0 - \hat{v} \in D_+.
\end{aligned}$$

Proof. Using the positive solution $u_0 \in D_+$ from Theorem 4.9, we introduce the following Carathéodory function

$$e_+(z, x) = \begin{cases} f(z, u_0(z)) + u_0(z) & \text{if } x < u_0(z) \\ f(z, x) + x & \text{if } u_0(z) \leqslant x. \end{cases} \tag{4.38}$$

Let $E_+(z,x) = \int_0^x e_+(z,s)ds$ and consider the C^1-functional $\chi_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\chi_+(u) = \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega E_+(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Claim 4.14. $K_{\chi_+} \subseteq [u_0) = \left\{u \in H^1(\Omega) : \ u_0(z) \leqslant u(z) \text{ for almost all } z \in \Omega\right\}$.
Let $u \in K_{\chi_+}$. Then

$$\langle A(u), h\rangle + \int_\Omega uhdz + \int_{\partial\Omega} \beta(z)uhd\sigma = \int_\Omega e_+(z,u)hdz \text{ for all } h \in H^1(\Omega).$$

Choose $h = (u_0 - u)^+ \in H^1(\Omega)$. Then

$$\begin{aligned}
&\left\langle A(u), (u_0-u)^+\right\rangle + \int_\Omega u(u_0-u)^+dz + \int_{\partial\Omega} \beta(z)u(u_0-u)^+d\sigma \\
=\ & \int_\Omega [f(z,u_0)+u_0](u_0-u)^+dz \text{ (see (4.38))} \\
=\ & \left\langle A(u_0), (u_0-u)^+\right\rangle + \int_\Omega u_0(u_0-u)^+dz + \int_{\partial\Omega} \beta(z)u_0(u_0-u)^+d\sigma, \\
\Rightarrow\ & ||D(u_0-u)^+||_2^2 + ||(u_0-u)^+||_2^2 \leqslant 0 \text{ (see hypothesis } H(\beta)), \\
\Rightarrow\ & u_0 \leqslant u.
\end{aligned}$$

This proves the claim.

Recall that $u_0 \in [0, w_+]$ (see Proposition 4.2). We may assume that

$$K_{\chi_+} \cap [0, w_+] = \{u_0\}. \tag{4.39}$$

Otherwise on account of the claim, we already have a second positive solution $\hat{u} \in C^1(\overline{\Omega})$ with $u_0 \leqslant \hat{u}$ and as before, via hypothesis H_3'(vi) and the strong maximum principle, we can say that $\hat{u} - u_0 \in D_+$ and so we are done.

We consider the following truncation of $e_+(z,\cdot)$:

$$\hat{e}_+(z,x) = \begin{cases} e_+(z,x) & \text{if } x \leqslant w_+(z) \\ e_+(z,w_+(z)) & \text{if } w_+(z) < x. \end{cases} \tag{4.40}$$

This is a Carathéodory function. We set $\hat{E}_+(z,x) = \int_0^x \hat{e}_+(z,s)ds$ and consider the C^1-functional $\hat{\chi}_+ : H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\chi}_+(u) = \frac{1}{2}\gamma(u) + \frac{1}{2}||u||_2^2 - \int_\Omega \hat{E}_+(z,u)dz \text{ for all } u \in H^1(\Omega).$$

From (4.40) it is clear that $\hat{\chi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in H^1(\Omega)$ such that

$$\hat{\chi}_+(\tilde{u}_0) = \inf\left[\hat{\chi}_+(u) : \ u \in H^1(\Omega)\right]. \tag{4.41}$$

As before (see the proof of Proposition 4.6), we can show that

$$K_{\hat{\chi}_+} \subseteq \left[u_0, w_+\right]. \tag{4.42}$$

Then from (4.39), (4.41), (4.42) it follows that

$$\tilde{u}_0 = u_0.$$

Hypothesis $H_3^{'}$(iv) implies that

$$w_+ - u_0 \in \operatorname{int} C_+$$

(see Remark 4.3). We have $\hat{\chi}_+ \big|_{[0,w_+]} = \chi_+\big|_{[0,w_+]}$ (see (4.40)). So, it follows that

$$\begin{aligned} &u_0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \chi_+, \\ \Rightarrow\ &u_0 \text{ is a local } H^1(\Omega) - \text{minimizer of } \chi_+ \text{ (see Proposition 2.4).} \end{aligned}$$

We assume that K_{χ_+} is finite or otherwise on account of the claim, we already have an infinity of positive solutions in $C^1(\overline{\Omega})$ strictly bigger than u_0 and so we are done. Hence, we can find $\rho \in (0, 1)$ small such that

$$\chi_+(u_0) < \inf\left[\chi_+(u) : \ ||u - u_0|| = \rho\right] = m_+. \tag{4.43}$$

Hypothesis $H_3^{'}$(ii) implies that

$$\chi_+(t\hat{u}_1) \to -\infty \text{ as } t \to +\infty. \tag{4.44}$$

Moreover, Proposition 3.5 implies that

$$\chi_+ \text{ satisfies the C-condition.} \tag{4.45}$$

Then (4.43), (4.44), (4.45) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_{\chi_+} \subseteq [u_0) \text{ (see the claim) and } \chi_+(u_0) < m_+ \leqslant \chi_+(\hat{u}). \tag{4.46}$$

From (4.46) and (4.38) it follows that

$$\hat{u} \in C^1(\overline{\Omega}) \text{ solves problem (1.1) and } \hat{u} \neq u_0.$$

In fact, hypothesis $H_3^{'}(vi)$ and the strong maximum principle, imply that

$$\hat{u} - u_0 \in D_+.$$

Similarly, starting with the Carathéodory function

$$e_-(z,x) = \begin{cases} f(z,x) + x & \text{if } x \leqslant v_0(z) \\ f(z, v_0(z)) + v_0(z) & \text{if } v_0(z) < x \end{cases}$$

and reasoning as above, we produce a second negative solution

$$\hat{v} \in -D_+ \text{ with } v_0 - \hat{v} \in D_+. \qquad \square$$

So, we can state the following multiplicity theorem

Theorem 4.15. *If hypotheses $H(\beta)$, $H_3^{'}$ hold, then problem* (1.1) *admits at least five nontrivial smooth solutions*

$$\begin{aligned} &u_0, \hat{u} \in D_+ \textit{ with } \hat{u} - u_0 \in D_+ \\ &v_0, \hat{v} \in -D_+ \textit{ with } v_0 - \hat{v} \in D_+, \\ &y_0 \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal}. \end{aligned}$$

With stronger regularity on $f(z,\cdot)$ (namely, we require that $f \in C^1(\mathbb{R}\backslash\{0\})$), we can improve this multiplicity theorem.

The new hypotheses on $f(z,x)$ are the following:

$H_3^{''}$: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, and $f(z,\cdot) \in C^1(\mathbb{R}\backslash\{0\})$, and $f(z,\cdot)$ is continuous at zero;

(i) It holds $|f_x^{'}(z,x)| \leqslant a(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}\backslash\{0\}$ with $a \in L^\infty(\Omega)_+, 2 \leqslant r < 2^*$;
and hypotheses $H_3^{''}$(ii), (iii), (iv) and (v) are the same as the corresponding hypotheses $H_3^{'}$(ii)$\to$(v).

Remark 4.16. Note that in this case the differentiability of $f(z,\cdot)$ on $\mathbb{R}\backslash\{0\}$ and hypothesis $H_3^{''}$(i) imply that hypothesis H_3(vi) is automatically true.

Then using Theorems 4.10 and 4.15, we have the following multiplicity theorem.

Theorem 4.17. *If hypotheses $H(\beta)$, and $H_3^{''}$ hold, then problem (1.1) admits at least six nontrivial smooth solutions*

$$\begin{aligned} &u_0, \hat{u} \in D_+ \textit{ with } \hat{u} - u_0 \in D_+ \\ &v_0, \hat{v} \in -D_+ \textit{ with } v_0 - \hat{v} \in D_+, \\ &y_0, \hat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal}. \end{aligned}$$

Finally, a further strengthening of the conditions on $f(z,\cdot)$ (in the direction of hypothesis H_2(v)), gives a third nodal solution, for a total of seven nontrivial smooth solutions all with sign information.

H_3''': The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, and $f(z,0) = 0$, and $f(z,\cdot) \in C^1(\mathbb{R})$ and:

(i) It holds $|f_x'(z,x)| \leqslant a(z)(1+|x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, and $2 \leqslant r < 2^*$;

(ii) There exists an integer $m \geqslant 3$ such that

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_{m+1} \text{ uniformly for almost all } z\in\Omega;$$

(iii) It holds $\lim_{x\to\pm\infty}[f(z,x)x - 2F(z,x)] = +\infty$ uniformly for almost all $z \in \Omega$ (recall that $F(z,x) = \int_0^x f(z,s)ds$);

(iv) One of the following holds:
[a] there exist functions $w_\pm \in C^1(\overline{\Omega})$ which are not solutions of (1.1), $\Delta w_\pm \in L^2(\Omega)$ and

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \text{ for all } z \in \overline{\Omega},\\
&A(w_-) \leqslant 0 \leqslant A(w_+), \text{ in } H^1(\Omega)^*\\
&f(z,w_+(z)) \leqslant 0 \leqslant f(z,w_-(z)) \text{ for almost all } z \in \Omega;
\end{aligned}$$

[b] there exist functions $w_\pm \in H^1(\Omega)\cap C(\overline{\Omega})$ such that

$$\begin{aligned}
&w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \text{ for all } z \in \overline{\Omega},\\
&A(w_-) \leqslant 0 \leqslant A(w_+) \text{ in } H^1(\Omega)^*\\
&f(z,w_+(z)) \leqslant 0 \leqslant f(z,w_-(z)) \text{ for almost all } z \in \Omega,\ f(\cdot,w_\pm(\cdot)) \not\equiv 0;
\end{aligned}$$

(v) There exist an integer $l \geqslant 2, l \neq m$ and $\delta_0 > 0$ such that

$$\hat{\lambda}_l x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1}x^2 \text{ for almost all } z \in \Omega,\ \text{all } |x| \leqslant \delta_0;$$

(vi) There exists $\xi^* > 0$ such that for almost all $z \in \Omega$ the function

$$x \mapsto f(z,x) + \xi^* x$$

is strictly increasing on $\mathbb{R}$ and for every $M > 0$ there exists $\Omega_M \subseteq \Omega$ with $|\Omega_M|_N > 0$ such that

$$\frac{f(z,x)}{x} < f_x'(z,x) \text{ for almost all } z \in \Omega,\ \text{all } |x| \leqslant M.$$

Under these conditions on $f(z,x)$, the reasoning in the proof of Theorem 3.19 remains valid and gives a nodal solution $\tilde{y} \in C^1(\overline{\Omega})$. The fact that the hypothesis near zero is different (compare hypothesis H_3'''(v) with hypothesis H_2(iv)), does not change anything, since this condition is only used to generate extremal constant sign solutions. In the present setting this was done in Proposition 4.6 (for Theorem 3.19, we did it in Proposition 3.10). From Theorem 4.10 we know that for the two other nodal solutions that we have already produced, we have

$$y_0, \hat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0]$$

(as before we assume that $u_0 = u_* \in D_+$, and $v_0 = v_* \in -D_+$). Hence from the proof of Theorem 3.19 we infer that

$$\tilde{y} \neq y_0 \text{ and } \tilde{y} \neq \hat{y}.$$

Therefore we can state the following multiplicity result.

Theorem 4.18. *If hypotheses $H(\beta)$, H_3''' hold, then problem* (1.1) *admits seven nontrivial smooth solutions*

$$\begin{aligned}
&u_0, \hat{u} \in D_+ \text{ with } \hat{u} - u_0 \in D_+\\
&v_0, \hat{v} \in D_+ \text{ with } v_0 - \hat{v} \in D_+\\
&y_0,\ \hat{y},\ \tilde{y} \in C^1(\overline{\Omega}) \text{ nodal with } y_0, \hat{y} \in \mathrm{int}_{C^1(\overline{\Omega})}[v_0, u_0].
\end{aligned}$$

Remark 4.19. To the best of our knowledge, this is the first theorem which produces seven nontrivial smooth solutions for problems exhibiting double resonance. Even for Dirichlet problems, no such multiplicity theorem exists. We stress that in Theorem 4.18 we provide sign information for all solutions. We mention Theorem 1.3 of Liang and Su [21] who deal with Dirichlet equations and produce six nontrivial solutions. However, they do not provide sign information for all of them and their hypotheses on $f(z,x)$ are more restrictive and do not allow resonance at zero.

5. Exact multiplicity result

In the previous section, by introducing an oscillatory behavior near zero (see hypothesis H_3(iv)), we were able to prove various multiplicity theorems, reaching up to seven nontrivial smooth solutions all with sign information. Now we remove this condition, always keeping the double resonance at $\pm\infty$ and the resonance with respect to a nonprincipal eigenvalue at zero ("double double resonance"). We show that we can still have two nontrivial smooth solutions. In fact eventually we prove a sharp multiplicity theorem (see Theorem 5.3).

The hypotheses on the reaction term $f(z,x)$ are the following:

H_4: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, and $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) It holds $|f'_x(z,x)| \leqslant a(z)(1+|x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, and $2 \leqslant r < 2^*$;

(ii) There exists an integer $m \geqslant 3$ such that

$$\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_{m+1} \text{ uniformly for almost all } z \in \Omega;$$

(iii) It holds $\lim_{x\to\pm\infty} [f(z,x)x - 2F(z,x)] = +\infty$ uniformly for almost all $z \in \Omega$;

(iv) There exist $l \in \mathbb{N}$ with $l+1 < m$ and $\delta_0 > 0$ such that

$$\hat{\lambda}_l x^2 \leqslant f(z,x)x \leqslant \hat{\lambda}_{l+1} x^2 \text{ for almost all } z \in \Omega,\ \text{ all } |x| \leqslant \delta_0;$$

(v) It holds $\hat{\lambda}_l \leqslant \dfrac{f(z,x)}{x} \leqslant f'_x(z,x)$ for almost all $z \in \Omega$, all $x \neq 0$, and for every $M > 0$, there exists $\Omega_M \subseteq \Omega$ measurable with $|\Omega_M|_N > 0$ such that

$$\hat{\lambda}_l < \frac{f(z,x)}{x} < f'_x(z,x) \text{ for almost all } z \in \Omega_M,\ \text{ all } 0 < |x| \leqslant M$$

and $f'_x(z,x) \leqslant \hat{\lambda}_{m+2}$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ and for every $M > 0$, there exists $\hat{\Omega}_M \subseteq \Omega$ measurable with $|\hat{\Omega}_M|_N > 0$ such that $f'_x(z,x) < \hat{\lambda}_{m+2}$ for almost all $z \in \hat{\Omega}_M$, all $|x| \leqslant M$.

Recall that $\varphi : H^1(\Omega) \to \mathbb{R}$ is the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z,u)dz \text{ for all } u \in H^1(\Omega).$$

Under hypotheses H_4 we have $\varphi \in C^2(H^1(\Omega))$. We start with an observation concerning the critical points of φ motivated by de Paiva [28, Lemma 3.2].

Proposition 5.1. *If hypotheses $H(\beta)$, and H_5 hold and $\tilde{u} \in K_\varphi \backslash \{0\}$, then $\nu(\tilde{u}) \leqslant d_m - d_{l+1}$ (recall $d_m = \dim \bar{H}_m$, and $d_{l+1} = \dim \bar{H}_{l+1}$).*

Proof. Let

$$\xi(z) = \begin{cases} \dfrac{f(z,\tilde{u}(z))}{\tilde{u}(z)} & \text{if } \tilde{u}(z) \neq 0 \\ f'(z,0) & \text{if } \tilde{u}(z) = 0. \end{cases}$$

Then $\xi \in L^\infty(\Omega)$ and we can equivalently rewrite problem (1.1) as follows

$$-\Delta u(z) = \xi(z)u(z) \text{ in } \Omega,\ \text{ with } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \tag{5.1}$$

Consider the following weighted eigenvalue Robin problem

$$-\Delta u(z) = \tilde{\eta}\xi(z)u(z) \text{ in } \Omega,\ \text{ with } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \tag{5.2}$$

From (5.1), we see that $\tilde{u} \in C^1(\overline{\Omega})$ (by the regularity theory, see Wang [37]) is an eigenfunction for (5.2) (with eigenvalue $\tilde{\eta} = 1$) and so by the UCP, $\tilde{u}(z) \neq 0$ for almost all $z \in \Omega$ (see also Caffarelli and Friedman [6]). Let $\{\tilde{\eta}_k\}_{k\in\mathbb{N}}$ be the eigenvalue of (5.2) (evidently, by hypothesis $H_4(v)$, they exist and $\tilde{\eta} = 1$ is one of them).

Also, let $\xi_0(z) = f'_x(z, \tilde{u}(z))$. Evidently $\xi_0 \in L^\infty(\Omega)$ and we consider the following weighted eigenvalue Robin problem

$$-\Delta u(z) = \tilde{\vartheta}\xi_0(z)u(z) \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \tag{5.3}$$

Let $\{\tilde{\vartheta}_n\}_{n\in\mathbb{N}}$ be the eigenvalues of (5.3) (see hypothesis $H_4(v)$).

From hypothesis $H_4(v)$ we have

$$\xi(z) \leqslant \xi_0(z) \text{ for almost all } z \in \Omega \text{ and } \xi \not\equiv \xi_0. \tag{5.4}$$

From (5.4) and Proposition 2.3, we have

$$\tilde{\vartheta}_n = \tilde{\lambda}_n(\xi_0) < \tilde{\lambda}_n(\xi) = \tilde{\eta}_n \text{ for all } n \in \mathbb{N}. \tag{5.5}$$

Consider the following two eigenvalue Robin problems

$$-\Delta u(z) = \tilde{\sigma}\hat{\lambda}_{m+2}u(z) \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega \tag{5.6}$$

$$-\Delta u(z) = \tilde{\tau}\hat{\lambda}_{l+1}u(z) \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega\,. \tag{5.7}$$

Let $\{\tilde{\sigma}_n\}_{n\geqslant 1}$ be the eigenvalues of (5.6) and $\{\tilde{\tau}_n\}_{n\geqslant 1}$ be the eigenvalues of (5.7). We have

$$\tilde{\sigma}_n = \frac{\hat{\lambda}_n}{\hat{\lambda}_{m+2}} \text{ and } \tilde{\tau}_n = \frac{\hat{\lambda}_n}{\hat{\lambda}_{l+1}} \text{ for all } n \in \mathbb{N}\,.$$

Using hypothesis $H_4(v)$ we see that

$$\begin{aligned} &\hat{\lambda}_l \leqslant \xi(z) \leqslant \xi_0(z) \leqslant \hat{\lambda}_{m+2} \text{ for almost all } z \in \Omega \\ &\hat{\lambda}_l \not\equiv \xi \text{ and } \hat{\lambda}_{m+2} \not\equiv \xi_0. \end{aligned}$$

So, using Proposition 2.3, we obtain

$$\tilde{\eta}_l < \tilde{\tau}_l < 1 \text{ and } 1 = \tilde{\sigma}_{m+2} < \tilde{\vartheta}_{m+2} < \tilde{\eta}_{m+2}. \tag{5.8}$$

Recall that $\tilde{\eta} = 1$ is an eigenvalue of (5.2). So, from (5.8) if follows that

$$\tilde{\eta}_j = 1 \text{ for some } j \in \{l+1, \dots, m+1\}.$$

If $\tilde{\eta}_{m+1} = 1$, then $\tilde{\vartheta}_{m+1} < 1$ and so (5.8) implies that $\tilde{\vartheta} = 1$ is not an eigenvalue of (5.3). Note that $u \in \ker\varphi''(\tilde{u})$ implies that

$$-\Delta u(z) = \xi_0(z)u(z) \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega.$$

Therefore it follows that in this case $\nu(\tilde{u}) = 0$.

Next suppose that $\tilde{\eta}_m = 1$. Then from (5.5) we have $\tilde{\vartheta}_m < 1$ and so from (5.8) we have $1 < \tilde{\vartheta}_{m+2}$. It follows that $\nu(\tilde{u}) \leqslant 1$. Continuing this way, after a finite number of steps we conclude that $\nu(\tilde{u}) \leqslant d_m - d_{l+1}$. □

Now we can have a new multiplicity theorem for doubly resonant Robin problems.

Theorem 5.2. *If hypotheses $H(\beta)$, and H_4 hold, then problem* (1.1) *admits at least two nontrivial smooth solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.11, we have that $C_k(\varphi, \infty) = \delta_{k,d_m}\mathbb{Z}$ for all $k \in \mathbb{N}_0$. So, we can find $u_0 \in K_\varphi$ such that

$$C_{d_m}(\varphi, u_0) \neq 0. \tag{5.9}$$

Also, from Proposition 4.5, we know that $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}$. Since $l \neq m$ we have $d_l \neq d_m$ and so from (5.9) and Corollary 6.92 of Motreanu, Motreanu and Papageorgiou [27, page 173], we see that there exists $\hat{u} \in K_\varphi\backslash\{0\}$ such that

$$C_{d_l-1}(\varphi, \hat{u}) \neq 0 \text{ or } C_{d_l+1}(\varphi, \hat{u}) \neq 0. \tag{5.10}$$

According to Theorem 2.5 (the shifting theorem), one of the following is true:

(a) It holds $C_k(\varphi, \hat{u}) = \delta_{k,\mu(\hat{u})}\mathbb{Z}$ for all $k \in \mathbb{N}_0$;
(b) It holds $C_k(\varphi, \hat{u}) = \delta_{k,\mu(\hat{u})+\nu(\hat{u})}\mathbb{Z}$ for all $k \in \mathbb{N}_0$;
(c) It holds $C_k(\varphi, \hat{u}) = 0$ if $k \leqslant \mu(\hat{u})$ and if $k \geqslant \mu(\hat{u}) + \nu(\hat{u})$.

If (a) or (b) hold, since $d_{l-1} < d_{l+1} < d_m$ (recall that $l + 1 < m$, see hypothesis $H_4(iv)$) from (5.9), (5.10) it follows that $\hat{u} \neq u_0$.

So, suppose that (c) holds. Then from (5.10) we have

$$d_l - 1 > \mu(\hat{u}) \text{ and } d_{l+1} < \mu(\hat{u}) + \nu(\hat{u}).$$

Using Proposition 5.1, we obtain

$$\begin{aligned} &\mu(\hat{u}) + \nu(\hat{u}) < d_l - 1 + d_m - d_{l+1} < d_m, \\ \Rightarrow\ &C_{d_m}(\varphi, \hat{u}) = 0 \end{aligned} \tag{5.11}$$

(see (c)). Comparing (5.9) and (5.11), we conclude that $u_0 \neq \hat{u}$.

Since $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ (see Proposition 4.5) and $d_l \neq d_m$, we see that $u_0 \neq 0$. Similarly (5.10) implies $\hat{u} \neq 0$. Therefore $u_0, \hat{u}$ are two nontrivial solutions of (1.1) and by the regularity theory, $u_0, \hat{u} \in C^1(\overline{\Omega})$. □

In fact we can have a sharp multiplicity theorem, provided that we assume the following more restrictive version of H_4.

H_4': The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying hypotheses H_4 with two additional requirements:

- $l = m - 1$;
- $\hat{\lambda}_l$ is simple.

Theorem 5.3. *If hypotheses* $H(\beta)$, H_4' *hold, then problem* (1.1) *admits exactly two nontrivial smooth solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}).$$

Proof. As before (see the proof of Theorem 5.2) we know that there exists $u_0 \in K_\varphi$ such that

$$C_{d_m}(\varphi, u_0) \neq 0. \tag{5.12}$$

Since $C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z}$ for all $k \in \mathbb{N}_0$ (see Proposition 4.5) and $d_l \neq d_m$, from (5.12) it follows that $u_0 \neq 0$. Also, using Proposition 5.1 and the hypotheses that $l = m - 1$ and that $\hat{\lambda}_l$ is simple, we have

$$\begin{aligned} &\nu(u_0) \leqslant d_m - d_{l+1} = d_m - d_m = 0, \\ \Rightarrow\ &u_0 \text{ is nondegenerate with Morse index } d_m \end{aligned}$$

(see Theorem 2.5). Evidently the same is true for any nontrivial critical point u of φ. Hence

$$C_k(\varphi, u) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \text{ all } u \in K_\varphi \backslash \{0\} \tag{5.13}$$

(see Motreanu, Motreanu and Papageorgiou [27, Theorem 6.51, page 155]).

Let μ be the number of nontrivial critical points of φ. Using the Morse relation with $t = -1$ (see (3.54)) and (5.13) we have

$$\begin{aligned} &(-1)^{d_l} + \mu(-1)^{d_m} = (-1)^{d_m} \\ \Rightarrow\ &(-1)^{d_l} + (\mu - 1)(-1)^{d_l+1} = 0 \\ \Rightarrow\ &\mu - 1 = 1, \\ \Rightarrow\ &\mu = 2. \end{aligned}$$

Therefore there exist exactly two nontrivial smooth solutions $u_0, \hat{u} \in C^1(\overline{\Omega})$. $\square$

Remark 5.4. It appears that Theorem 5.3 is the first exact multiplicity theorem for doubly resonant equations (in fact for resonant equations). All previous such results (only for Dirichlet problems), excluded the possibility of resonance. We refer to the works of Ambrosetti and Mancini [2], Castro and Lazer [7] and de Paiva [28, Theorem 1.4].

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