Research Article

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Infinitely Many Nodal Solutions for Nonlinear Nonhomogeneous Robin Problems

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Abstract: We consider a nonlinear Robin problem driven by a nonhomogeneous differential operator which incorporates the \( p \)-Laplacian as special case. The reaction \( f(z, x) \) is a Carathéodory function which need not satisfy a global growth condition and is only assumed to be odd near zero. Using variational tools, we show that the problem has a whole sequence of distinct nodal (that is, sign-changing) solutions.

Keywords: Nonhomogeneous Differential Operator, Nodal Solution, Nonlinear Regularity Theory, Nonlinear Maximum Principle

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1 Introduction

Let \( \Omega \subseteq \mathbb{R}^N \) be a bounded domain with a \( C^2 \)-boundary \( \partial \Omega \). We consider the nonlinear, nonhomogeneous Robin problem

\[
\begin{cases}
- \text{div} (a(Du(z))) = f(z, u(z)) & \text{in } \Omega, \\
\frac{\partial u}{\partial n_a} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

In this problem, \( a : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a strictly monotone, continuous map which satisfies certain other regularity and growth conditions listed in the hypotheses \( H(a) \) below. These conditions are general enough to incorporate in our framework many differential operators of interest, such as the \( p \)-Laplace operator, \( 1 < p < \infty \), and the sum of a \( p \)-Laplacian with a \( q \)-Laplacian, \( 1 < q < p < \infty \). The reaction term \( f(z, x) \) is a Carathéodory function (that is, for all \( x \in \mathbb{R} \), the map \( z \mapsto f(z, x) \) is measurable, while, for almost all \( z \in \Omega \), the function \( x \mapsto f(z, x) \) is continuous).

The interesting feature of our work here is that we do not impose any global growth condition on \( f(z, \cdot) \). Instead, we assume a local symmetry condition, namely, we require that, for almost all \( z \in \Omega \), the function \( x \mapsto f(z, x) \) is odd in the bounded interval \([-\eta, \eta]\). In the boundary condition, \( \frac{\partial u}{\partial n_a} \) denotes the generalized normal derivative corresponding to the differential operator \( \text{div} a(Du) \) and is defined by

\[
\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \quad \text{for all } u \in W^{1,p}(\Omega)
\]

with \( n(\cdot) \) being the outward unit normal on \( \partial \Omega \). This kind of normal derivative is dictated by the nonlinear Green's identity (see, for example, Gasiński and Papageorgiou [7, p. 210]) and can be also found in the work...

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of Lieberman [13]. The boundary weight function $\beta \in C^{0,a}(\partial \Omega)$ with $a \in (0,1)$ satisfies $\beta(z) \geq 0$ for all $z \in \partial \Omega$. When $\beta \equiv 0$, then we have the Neumann problem.

Under these general hypotheses on the data of (1.1), we show that there exists a whole sequence $\{u_n\}_{n \geq 1} \subseteq C^1(\overline{\Omega})$ of distinct nodal (that is, sign-changing) solutions. Our approach uses variational tools together with suitable truncation-perturbation techniques. Recently, nodal solutions for nonlinear, nonhomogeneous Robin problems were produced by Papageorgiou and Rădulescu [20, 22]. However, in the aforementioned works, the authors establish the existence of only one nodal solution.

## 2 Mathematical Background and Hypotheses

Let $X$ be a Banach space and let $\varphi \in C^1(X, \mathbb{R})$. We say that $\varphi$ satisfies the Palais–Smale condition (PS-condition for short) if every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \rightharpoonup 0 \quad \text{as } n \to \infty$$

admits a strongly convergent subsequence.

Our main variational tool will be a variant due to Heinz [10] of a classical result of Clark [4]. The next result is essentially due to Heinz [10] and can be found in Wang [28]. Further extensions with applications to semilinear elliptic Dirichlet problems and to Hamiltonian systems can be found in the works of Liu and Wang [15] and Kajikiya [12].

**Theorem 2.1.** Let $X$ be a Banach space and assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the PS-condition, it is even, bounded from below, $\varphi(0) = 0$ and, for every $n \in \mathbb{N}$, there exist an $n$-dimensional subspace $Y_n$ of $X$ and $\rho_n > 0$ such that

$$\sup \{\varphi(u) : u \in Y_n \cap \partial B_{\rho_n}\} < 0,$$

where $\partial B_{\rho_n} = \{u \in X : \|u\| = \rho_n\}$. Then, there exists a sequence $\{u_n\}_{n \geq 1}$ of critical points of $\varphi$ such that $\varphi(u_n) < 0$ for all $n \in \mathbb{N}$

and

$$\varphi(u_n) \to 0 \quad \text{as } n \to \infty.$$

Let $\vartheta \in C^1(0, +\infty)$ with $\vartheta(t) > 0$ for all $t > 0$ and assume that there exists $p > 1$ such that

$$0 < \hat{c} \leq \vartheta'(t) t \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \vartheta(t) \leq c_2 (1 + t^{p-1})$$

(2.1)

for all $t > 0$ and for some $c_1, c_2 > 0$. Then, our hypotheses on the map $a(\cdot)$ involved in the definition of the differential operator are that $H(a)$ $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, +\infty)$, $t \mapsto a_0(t) t$ is strictly increasing on $(0, +\infty)$, $a_0(t) t \to 0$ as $t \to 0^+$ and

$$\lim_{t \to 0^+} \frac{a_0'(t) t}{a_0(t)} = -1;$$

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \leq c_3 \frac{a(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $\nabla a(y) \xi, \xi \in \mathbb{R}^N \geq \frac{a(|y|)}{|y|} |\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$ and all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s) s \, ds$ for $t > 0$, then there exists $q \in (1, p)$ such that

$$\lim_{t \to 0^+} \sup_{t \in \mathbb{R}} \frac{q G_0(t)}{t^q} \leq c^* \quad \text{and} \quad t \mapsto G_0(t^{1/q})$$

is convex.

**Remark 2.2.** Hypotheses $H(a)$ (i)–(iii) come from the nonlinear regularity theory of Lieberman [13] and the nonlinear maximum principle of Pucci and Serrin [26]. Hypothesis $H(a)$ (iv) serves the needs of our problem, but it is a mild condition which is satisfied in all the main cases of interest, as the examples which follow illustrate.
From the above hypotheses it is clear that the primitive \( G(y) \) is strictly convex and strictly increasing. We set 
\[
G(y) = G_0(|y|) \quad \text{for all } y \in \mathbb{R}^N. 
\]
Then, \( G(\cdot) \) is convex, \( G(0) = 0 \) and 
\[
\nabla G(0) = 0 \quad \text{and} \quad \nabla G(y) = G'_0(|y|) \frac{y}{|y|} = a_0(|y|) y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.
\]
So, \( G(\cdot) \) is the primitive of \( a(\cdot) \). The convexity of \( G(\cdot) \), since \( G(0) = 0 \), implies that 
\[
G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. 
\]
(2.2)

The next lemma summarizes the main properties of the map \( a(\cdot) \). It is a straightforward consequence of hypotheses \( H(a) \) (i)–(iii) and of (2.1).

**Lemma 2.3.** If hypotheses \( H(a) \) (i)–(iii) hold, then 
(a) \( y \mapsto a(y) \) is continuous and strictly monotone, hence, maximal monotone too;
(b) \( |a(y)| \leq c_a(1 + |y|^{p-1}) \) for all \( y \in \mathbb{R}^N \) and for some \( c_a > 0 \);
(c) \( (a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p(p-1)} |y|^p \) for all \( y \in \mathbb{R}^N \).

The last lemma and (2.2) lead to the following growth estimates for the primitive \( G(\cdot) \).

**Corollary 2.4.** If hypotheses \( H(a) \) (i)–(iii) hold, then 
\[
\frac{c_1}{p(p-1)} |y|^p \leq G(y) \leq c_5(1 + |y|^p) \quad \text{for all } y \in \mathbb{R}^N \quad \text{and for some} \quad c_5 > 0.
\]

The examples that follow illustrate that our conditions on the map \( a(\cdot) \) cover many cases of interest.

**Example 2.5.** The following maps satisfy the hypotheses \( H(a) \).

(i) The map \( a(y) = |y|^{p-2} y \) with \( 1 < p < \infty \), which corresponds to the \( p \)-Laplacian differential operator defined by 
\[
\Delta_p u = \text{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W^{1,p}(\Omega).
\]
(ii) The map \( a(y) = |y|^{p-2} y + |y|^{q-2} y \) with \( 1 < q < p < \infty \), which corresponds to the \((p, q)\)-differential operator defined by 
\[
\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).
\]

Such operators arise in problems of mathematical physics. We mention the works of Benci, D’Avenia, Fortunato and Pisani [1] (quantum physics) and Cherfils and Ilyasov [2] (plasma physics). Recently, existence and multiplicity results for such equations with Dirichlet boundary conditions were proved by Cingolani and Degiovanni [3], Gasiński and Papageorgiou [9], Mugnai and Papageorgiou [17], Papageorgiou and Rădulescu [19, 21, 23] and Sun, Zhang and Su [27].

(iii) The map \( a(y) = (1 + |y|^2)^{(p-2)/2} y \) with \( 1 < p < \infty \), which corresponds to the generalized \( p \)-mean curvature differential operator defined by 
\[
\text{div}\left((1 + |Du|^2)^{(p-2)/2} Du\right) \quad \text{for all } u \in W^{1,p}(\Omega).
\]
(iv) The map \( a(y) = |y|^{p-2} y (1 + \frac{1}{|y|^p}) \) with \( 1 < p < \infty \), which corresponds to the differential operator 
\[
\Delta_p u + \text{div}\left(\frac{|Du|^{p-2} Du}{1 + |Du|^p}\right) \quad \text{for all } u \in W^{1,p}(\Omega),
\]
which is used in problems of plasticity.

Finally, we impose the hypothesis that 
\( H(\beta) \beta \in C^{0,a}(\partial \Omega) \) with \( a \in (0, 1) \) and \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \) and our hypotheses on the reaction term \( f(z, u) \) are that 
\( H(f) f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that, for almost all \( z \in \Omega \), \( f(z, 0) = 0 \), \( f(z, \cdot) \) is odd on \( [-\eta, \eta] \) for some \( \eta > 0 \) with \( f(z, \eta) \leq 0 \leq f(z, -\eta) \) and

(i) there exists \( a_0 \in L^\infty(\Omega) \), such that \( |f(z, x)| \leq a_0(z) \) for almost all \( z \in \Omega \) and all \( |x| \leq \eta; \)
(ii) if \( q \in (1, p) \) is as is in \( H(\alpha) \) (iv), then we have

\[
\lim_{x \to 0} \frac{f(z, x)}{|x|^{q-2}x} = +\infty \quad \text{uniformly for almost all } z \in \Omega.
\]

**Remark 2.6.** We stress that the above hypotheses do not impose any global growth condition on \( f(z, \cdot) \). Instead, we assume that \( f(z, \cdot) \) has a kind of oscillatory behavior near zero and that it is symmetric in that interval. Hypothesis \( H(f) \) (ii) implies the presence of a “concave” term near zero. We mention the work of Liu and Wang [14] who produced infinitely many nodal solutions for a semilinear Schrödinger equation without assuming the existence of zeros. We should point out that the idea of using cut-off techniques to produce an infinity of solutions converging to zero goes back to the work of Wang [28] who modified the reaction term in the interval \([-\eta, \eta]\) and applied the result of Clark and Heinz to the modified functional (see Wang [28, Lemma 2.3]).

Using hypothesis \( H(f) \) (ii), we see that, given any \( \xi > 0 \) and recalling that \( q < p \), we can find \( \delta = \delta(\xi) \in (0, \bar{\eta}) \) with \( \bar{\eta} = \min\{1, \eta\} \) such that

\[
f(z, x)x \geq \xi |x|^q \geq \xi |x|^p \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \delta.
\]

Then, given \( r \in (p, +\infty) \), we can find \( c_\delta = c_\delta(r, \delta) > 0 \) such that

\[
f(z, x)x \geq \xi |x|^q - c_\delta |x|^r \quad \text{for almost all } z \in \Omega \text{ and all } x \in [-\eta, \eta].
\]

From (2.3) we have

\[
F(z, x) \geq \frac{\xi}{q} |x|^q \quad \text{for almost all } z \in \Omega \text{ and all } |x| \leq \delta.
\]

In our analysis of (1.1), in addition to the Sobolev space \( W^{1,p}(\Omega) \), we will also use the Banach space \( C^1(\overline{\Omega}) \). This is an ordered Banach space with positive cone

\[
C_+ = \{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}.
\]

This cone has a nonempty interior and, if \( u \in C_+ \) with \( u(z) > 0 \) for all \( z \in \overline{\Omega} \), then \( u \in \text{int } C_+ \). On \( \partial \Omega \), we consider the \((N-1)\)-dimensional Hausdorff (surface) measure \( \sigma(\cdot) \). Using this measure, we can define the “boundary” Lebesgue spaces \( L^q(\partial \Omega) \), \( 1 \leq q \leq \infty \). From the theory of Sobolev spaces we know that there exists a unique continuous linear map \( y_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega) \), known as the “trace map”, such that \( y_0(u) = u|_{\partial \Omega} \) for all \( u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \). We have

\[
\text{Im } y_0 = W^{1,p-\frac{1}{p}}(\partial \Omega)(\frac{1}{p} + \frac{1}{p'} = 1) \quad \text{and } \ker y_0 = W^{1,p}_0(\Omega).
\]

Moreover, the trace map \( y_0 \) is compact into \( L^q(\partial \Omega) \) for \( q \in [1, \frac{(N-1)p}{N-p}) \). Hereafter, for the sake of notational simplicity, we will drop the use of the trace map \( y_0 \). The restrictions of Sobolev functions on \( \partial \Omega \) are understood in the sense of traces. By \( \| \cdot \| \) we denote the norm of the Sobolev space \( W^{1,p}(\Omega) \) defined by

\[
\|u\| = \left( \|u\|_p^p + \|Du\|_p^{p'} \right)^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).
\]

For every \( x \in \mathbb{R} \), let \( x^\pm = \max\{\pm x, 0\} \). Then, for \( u \in W^{1,p}(\Omega) \), we set \( u^\pm(\cdot) = u(\cdot)^\pm \). We know that

\[
u = u^+ - u^- \quad |u| = u^+ + u^- \quad \text{and} \quad u^+, u^- \in W^{1,p}(\Omega).
\]

Also, by \( | \cdot |_N \) we denote the Lebesgue measure on \( \mathbb{R}^N \) and \( A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) is the nonlinear map defined by

\[
\langle A(u), h \rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N} \, dz \quad \text{for all } u, h \in W^{1,p}(\Omega).
\]

The map \( A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) is continuous, monotone and of type \((S)_+\), that is,

\[
u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0 \text{ implies that } u_n \to u \text{ in } W^{1,p}(\Omega).
\]

Here, by \( \langle \cdot, \cdot \rangle \) we denote the duality brackets for the pair \((W^{1,p}(\Omega)^*, W^{1,p}(\Omega))\) (see Gasiński and Papageorgiou [8]). Finally, for any \( \varphi \in C^1(X, \mathbb{R}) \), by \( K_\varphi \) we denote the critical set of \( \varphi \), that is,

\[
K_\varphi = \{ u \in X : \varphi'(u) = 0 \}.
\]
3 Nodal Solutions

Using (2.4), we introduce the truncation

\[ e(z, x) = \begin{cases} 
-\xi \eta^{p-1} + c_6 \eta^{r-1} & \text{if } x < -\eta, \\
\xi |x|^{q-2} x - c_6 |x|^{r-2} x & \text{if } -\eta \leq x \leq \eta, \\
\xi \eta^{r-1} - c_6 \eta^{r-1} & \text{if } \eta < x 
\end{cases} \tag{3.1} \]

of the right-hand side of (2.4) and, for all \( (z, x) \in \partial \Omega \times \mathbb{R} \), the truncation

\[ b(z, x) = \begin{cases} 
-\beta(z) \eta^{p-1} & \text{if } x < -\eta, \\
\beta(z) |x|^{p-2} x & \text{if } -\eta \leq x \leq \eta, \\
\beta(z) \eta^{p-1} & \text{if } \eta < x 
\end{cases} \tag{3.2} \]

of the boundary term \( \beta(z) |x|^{p-2} x \). Both are Carathéodory functions. We consider the auxiliary nonlinear, non-homogeneous Robin problem

\[ \begin{aligned}
-\text{div} a(Du(z)) &= e(z, u(z)) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + b(z, u) &= 0 & \text{on } \partial \Omega.
\end{aligned} \tag{3.3} \]

**Proposition 3.1.** If hypotheses \( H(a), H(\beta) \) and \( H(f) \) hold, then (3.3) admits a unique positive solution \( \bar{u} \in \text{int } C_r \) and \( \bar{v} = -\bar{u} \in -\text{int } C_r \) is its unique negative solution.

**Proof.** We introduce the Carathéodory function \( \tau : \Omega \times \mathbb{R} \to \mathbb{R} \) defined by

\[ \tau(z, x) = \begin{cases} 
-\eta^{p-1} & \text{if } x < -\eta, \\
|x|^{p-2} x & \text{if } -\eta \leq x \leq \eta, \\
\eta^{p-1} & \text{if } \eta < x 
\end{cases} \tag{3.4} \]

Let

\[ T(z, x) = \int_0^x \tau(z, s) \, ds, \quad E(z, x) = \int_0^x e(z, s) \, ds \quad \text{and} \quad B(z, x) = \int_0^x b(z, s) \, ds, \]

and consider the \( C^1 \)-functional \( \psi_+ : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[ \psi_+(u) = \int_{\Omega} G(Du) \, dz + \frac{1}{p} \|u\|_p^p + \int_{\partial \Omega} B(z, u^+) \, d\sigma - \int_{\Omega} E(z, u^+) \, dz - \int_{\Omega} T(z, u^+) \, dz \quad \text{for all } u \in W^{1,p}(\Omega). \]

From Corollary 2.4 and (3.1), (3.2) and (3.4) it is clear that \( \psi_+ \) is coercive. Also, using the Sobolev embedding theorem and the trace theorem, we see that \( \psi_+ \) is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find \( \bar{u} \in W^{1,p}(\Omega) \) such that

\[ \psi_+(\bar{u}) = \inf \{ \psi_+(u) : u \in W^{1,p}(\Omega) \}. \tag{3.5} \]

Hypothesis \( H(a) \) (iv) implies that we can find \( c^*_1 > c^* \) and \( \delta_1 \in (0, \bar{\eta}) \) such that

\[ G_0(t) \leq \frac{c^*_1}{q} t^q \quad \text{for all } t \in [0, \delta_1]. \tag{3.6} \]

Let \( u \in \text{int } C_r \) and choose \( t \in (0, 1) \) small such that

\[ tu(z) \leq \delta_1 \quad \text{and} \quad t|Du(z)| \leq \delta_1 \quad \text{for all } z \in \overline{\Omega}. \tag{3.7} \]

Using (3.6), (3.7), (3.1), (3.2) and (3.4), we have (see hypothesis \( H(\beta) \) and the trace theorem)

\[ \psi_+(tu) \leq \frac{t^p c_1^*}{p} \|Du\|_p^p + \frac{t^p}{p} \int_{\partial \Omega} \beta(z)|u|^p \, d\sigma - \frac{t^q}{q} \|u\|_q^q + \frac{t^{q-1} c_6}{r} \|u\|_r^r \]

\[ \leq \left( \frac{t^{p-q} c_1^*}{p} \|Du\|_p^p + \frac{t^{p-q}}{p} c_6 \|u\|_p^p - \frac{c^*_1}{q} t^q \|u\|_q^q + \frac{t^{q-1}}{r} c_6 \|u\|_r^r \right) t^q \]
for some $c_8 > 0$. Since $1 < q < p < r$, choosing $t \in (0, 1)$ even smaller if necessary, we have that $\psi_+(tu) < 0$ implies (see (3.5))

$$\psi_+(\bar{u}) < 0 = \psi_+(0)$$

and, hence, $\bar{u} \not= 0$. From (3.5) we have that $\psi_n(\bar{u}) = 0$ implies

$$\langle A(\bar{u}), h \rangle + \int_{\Omega} |\bar{u}|^{p-2}\bar{u}h \, dz + \int_{\partial\Omega} b(z, u^+)h \, d\sigma = \int_{\Omega} \left( (e(z, u^+) + \tau(z, u^+))h \right) \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \quad (3.8)$$

In (3.8), first we choose $h = -\bar{u}^+ \in W^{1,p}(\Omega)$ and then we have that (see Lemma 2.3 and (3.1), (3.2) and (3.4))

$$\frac{c_1}{p-1} \|D\bar{u}^+\|_p^p + \|u^-\|_p^p \leq 0$$

implies $\bar{u} \geq 0$ and $\bar{u} \not= 0$. Also, in (3.8), we choose $h = (\bar{u} - \eta)^+ \in W^{1,p}(\Omega)$ and then we have (see (3.1), (3.2) and (3.4) for the equality and (2.4) for the first inequality)

$$\langle A(\bar{u}), (\bar{u} - \eta)^+ \rangle + \int_{\Omega} \bar{u}^{p-1}(\bar{u} - \eta)^+ \, dz + \int_{\partial\Omega} \beta(z)\eta^{p-1}(\bar{u} - \eta)^+ \, d\sigma$$

since $A(\eta) = 0$ and $f(z, \eta) \leq 0$ for almost all $z \in \Omega$, which implies that

$$\langle A(\bar{u}) - A(\eta), (\bar{u} - \eta)^+ \rangle + \int_{\Omega} (\bar{u}^{p-1} - \eta^{p-1})(\bar{u} - \eta)^+ \, dz \leq 0.$$ 

Therefore,

$$||\bar{u} - \eta||_W = 0,$$

that is,

$$\bar{u} \leq \eta.$$

Thus, we have proved that

$$\bar{u} \in [0, \eta] = \left\{ u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \eta \text{ for almost all } z \in \Omega \right\}, \quad \text{and} \quad \bar{u} \not= 0. \quad (3.9)$$

Then, using (3.1), (3.2), (3.4) and (3.9), we see that (3.8) becomes

$$\langle A(\bar{u}), h \rangle + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}h \, d\sigma = \int_{\Omega} e(z, \bar{u})h \, dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

which gives (see Papageorgiou and Rădulescu [18])

$$\left\{ \begin{array}{l}
- \text{div} \, a(D\bar{u}(z)) = e(z, \bar{u}(z)) \quad \text{for almost all } z \in \Omega, \\
\frac{\partial \bar{u}}{\partial n} + \beta(z)\bar{u}^{p-1} = 0 \quad \text{on } \partial\Omega,
\end{array} \right.$$

that is, $\bar{u}$ is a positive solution of (3.3). From Papageorgiou and Rădulescu [24] we have that $\bar{u} \in L^\infty(\Omega)$ and, then, the nonlinear regularity result of Lieberman [13, p. 320]) implies that $\bar{u} \in C_+ \setminus \{0\}$. Because of (3.9) we have

$$- \text{div} \, a(D\bar{u}(z)) = \xi \bar{u}(z)^{q-1} - c_8 \bar{u}(z)^{r-1} \quad \text{for almost all } z \in \Omega,$$
which gives
\[ \text{div } a(D\tilde{u}(z)) \leq c_6 h^{-p} \tilde{u}(z)^{p-1} \quad \text{for almost all } z \in \Omega, \]
that is (see Pucci and Serrin [26, pp. 111, 120]),
\[ \tilde{u} \in \text{int } C^+. \]

Next, we show the uniqueness of this positive solution. To this end, we consider the integral functional
\[ j : L^1(\Omega) \to \mathbb{R} = \mathbb{R} \cup \{ +\infty \} \]
defined by
\[
j(u) = \begin{cases} 
\int_{\Omega} G(Du^{1/q}) \, dz + \frac{1}{p} \int_{\partial \Omega} \beta(z)u^{\theta/q} \, d\sigma & \text{if } u \geq 0, \ u^{1/q} \in W^{1,p}(\Omega), \\
+ \infty & \text{otherwise.}
\end{cases}
\]

Let \( u_1, u_2 \in \text{dom } j = \{ u \in L^1(\Omega) : j(u) < +\infty \} \) (the effective domain of \( j \)). We set
\[ u = ((1 - t)u_1 + tu_2)^{1/q} \quad \text{for } t \in [0, 1]. \]

Using Díaz and Saá [5, Lemma 1], we have
\[
|Du(z)| \leq \left( (1 - t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q \right)^{1/q} \quad \text{for almost all } z \in \Omega
\]
and because \( G_0(\cdot) \) is increasing and from hypothesis \( H(a) \) (iv), for almost all \( z \in \Omega \), we have
\[
G_0(|Du(z)|) \leq G_0\left( (1 - t)|Du_1(z)^{1/q}|^q + t|Du_2(z)^{1/q}|^q \right)^{1/q} \leq (1 - t)G_0(|Du_1(z)^{1/q}|) + tG_0(|Du_2(z)^{1/q}|),
\]
which gives
\[ G(Du(z)) \leq (1 - t)G(Du_1(z)^{1/q}) + tG(Du_2(z)^{1/q}) \quad \text{for almost all } z \in \Omega, \]
that is, \( j(\cdot) \) is convex (recall that \( q < p \) and see hypothesis \( H(\beta) \)) By Fatou’s lemma, \( j(\cdot) \) is lower semicontinuous.

Let \( \tilde{y} \in W^{1,p}(\Omega) \) be another positive solution of (3.3). As we did for \( \tilde{u} \) in the first part of the proof, we can show that
\[ \tilde{y} \in [0, \eta] \cap \text{int } C^+. \]

For any \( h \in C^1(\overline{\Omega}) \) and for \( |t| < 1 \) small, we have
\[ \tilde{u}^q + th \in \text{dom } j \quad \text{and} \quad \tilde{y}^q + th \in \text{dom } j. \]

Then, we see that the functional \( j(\cdot) \) is Gâteaux differentiable at \( \tilde{u}^q \) and \( \tilde{y}^q \) in the direction \( h \). Moreover, via the chain rule and the nonlinear Green’s identity, we have
\[
j'(\tilde{u}^q)(h) = \frac{1}{q} \int_{\Omega} - \frac{\text{div } a(D\tilde{u})}{\tilde{u}^{q-1}} h \, dz \quad \text{and} \quad j'(\tilde{y}^q)(h) = \frac{1}{q} \int_{\Omega} - \frac{\text{div } a(D\tilde{y})}{\tilde{y}^{q-1}} h \, dz.
\]

Choose \( h = \tilde{u}^q - \tilde{y}^q \). Since \( j(\cdot) \) is convex, \( j'(\cdot) \) is monotone, and so we have (see (3.1))
\[
0 \leq \int_{\Omega} \left( - \frac{\text{div } a(D\tilde{u})}{\tilde{u}^{q-1}} - \frac{\text{div } a(D\tilde{y})}{\tilde{y}^{q-1}} \right) (\tilde{u}^q - \tilde{y}^q) \, dz = \int_{\Omega} c_6 (\tilde{y}^{q-2} - \tilde{u}^{q-2}) (\tilde{u}^q - \tilde{y}^q) \, dz,
\]
which gives
\[ \tilde{u} = \tilde{y} \]
and, then, \( \tilde{u} \in [0, \eta] \cap \text{int } C^+ \) is the unique positive solution of (3.3). Evidently, since \( x \mapsto \xi |x|^{q-2} x - c_6 |x|^{r-2} x \) is odd, we have that \( \nu = -\tilde{u} \in (-\eta, 0] \cap (-\text{int } C^+) \) is the unique negative solution of (3.3). \(\)
We introduce the sets

\[ S_+ = \{ u \in W^{1,p}(\Omega) : u \text{ is a positive solution of (1.1) with } u \in [0, \eta]\}, \]
\[ S_- = \{ v \in W^{1,p}(\Omega) : v \text{ is a negative solution of (1.1) with } v \in [-\eta, 0]\}. \]

As before, the nonlinear maximum principle implies that

\[ S_+ \subseteq \text{int } C_+ \quad \text{and} \quad S_- \subseteq -\text{int } C_. \]

Moreover, as in Filippakis and Papageorgiou [6], we have that

\[ S_+ \text{ is downward directed,} \]

that is, if \( u_1, u_2 \in S_+ \), then we can find \( u \in S_+ \) such that \( u \leq \min\{u_1, u_2\} \), and

\[ S_- \text{ is upward directed,} \]

that is, if \( v_1, v_2 \in S_- \), then we can find \( v \in S_- \) such that \( v \geq \max\{v_1, v_2\} \) (see also Motreanu, Motreanu and Papageorgiou [16, p. 421]).

**Proposition 3.2.** If hypotheses \( H(a), H(\beta) \) and \( H(f) \) hold, then \( \bar{u} \leq u \) for all \( u \in S_+ \) and \( v \leq \bar{v} \) for all \( v \in S_- \).

**Proof.** Let \( u \in S_+ \). We consider the Carathéodory functions \( k_+(z, x), \hat{b}_+(z, x) \) and \( \hat{r}_+(z, x) \) defined by

\[
\begin{align*}
  k_+(z, x) &= \begin{cases} 
  0 & \text{if } x < 0, \\
  e(z, x) & \text{if } 0 \leq x \leq u(z), \\
  e(z, u(z)) & \text{if } u(z) < x,
  \end{cases} 
  
  & \quad (3.10) \\
  \hat{b}_+(z, x) &= \begin{cases} 
  0 & \text{if } x < 0, \\
  \beta(z)x^{p-1} & \text{if } 0 \leq x \leq u(z), \quad \text{for all } (z, x) \in \partial \Omega \times \mathbb{R}, \\
  \beta(z)u(z)^{p-1} & \text{if } u(z) < x,
  \end{cases} 
  
  & \quad (3.11) \\
  \hat{r}_+(z, x) &= \begin{cases} 
  0 & \text{if } x < 0, \\
  x^{p-1} & \text{if } 0 \leq x \leq u(z), \\
  u(z)^{p-1} & \text{if } u(z) < x.
  \end{cases} 
  
  & \quad (3.12)
\end{align*}
\]

We set

\[ K_+(z, x) = \int_0^x k_+(z, s) \, ds, \quad \hat{B}_+(z, x) = \int_0^x \hat{b}_+(d, s) \, ds \quad \text{and} \quad \hat{T}_+(z, x) = \int_0^x \hat{r}_+(z, s) \, ds. \]

Consider the \( C^1 \)-functional \( y_+ : W^{1,p}(\Omega) \rightarrow \mathbb{R} \) defined by

\[ y_+(u) = \int_\Omega G(Du) \, dz + \frac{1}{p}\|u\|_p^p + \int_\Omega \hat{B}_+(z, u) \, d\sigma - \int_\Omega K_+(z, u) \, dz - \int_\Omega \hat{T}_+(z, u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega). \]

From Corollary 2.4 and (3.10), (3.11) and (3.12) we see that \( y_+ \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find \( \bar{u}_0 \in W^{1,p}(\Omega) \) such that

\[ y_+(\bar{u}_0) = \inf \{ y_+(u) : u \in W^{1,p}(\Omega) \}. \]

(3.13)

As before (see the proof of Proposition 3.1), since \( 1 < q < p < r \), for \( \bar{u} \in \text{int } C_+ \) and \( t \in (0, 1) \) small, we have (see hypothesis \( H(a) \) (iv))

\[ y_+(tu) < 0 = y_+(0), \]

which implies (see (3.13))

\[ y_+(u_0) < 0 = y_+(0) \]
and, hence, \( \bar{u}_0 \neq 0 \). From (3.13) we have that \( y'(\bar{u}_0) = 0 \) implies

\[
\langle A(\bar{u}_0), h \rangle + \int_{\Omega} |\bar{u}_0|^{p-2} \bar{u}_0 h \, dz + \int_{\partial \Omega} \tilde{b}_+(z, \bar{u}_0) h \, d\sigma = \int_{\partial \Omega} (k_+(z, \bar{u}_0) + \tilde{\tau}_+(z, \bar{u}_0)) h \, d\sigma \quad \text{for all } h \in W^{1,p}(\Omega). 
\]  

(3.14)

In (3.14), we choose \( h = -\bar{u}_0 \). Using Lemma 2.3 and (3.10), (3.11) and (3.12), we obtain that

\[
\frac{c_1}{p-1} \|D\bar{u}_0\|^p_{p} + \|\bar{u}_0\|^p_{p} \leq 0
\]

implies \( \bar{u}_0 \geq 0 \) and \( \bar{u}_0 \neq 0 \). Also, in (3.14), we choose \( h = (\bar{u}_0 - u)^+ \in W^{1,p}(\Omega) \). Then, we have (see (3.10), (3.11) and (3.12) for the first equality, see (3.1) and recall that \( u \in [0, \eta] \) for the second one and see (2.4) for the first inequality)

\[
\langle A(\bar{u}_0), (\bar{u}_0 - u)^+ \rangle + \int_{\Omega} \bar{u}_0^{p-1}(\bar{u}_0 - u)^+ \, dz + \int_{\partial \Omega} \beta(z)(\bar{u}_0 - u)^+ \, d\sigma \\
= \int_{\Omega} (e(z, u) + u^{p-1})(\bar{u}_0 - u)^+ \, dz \\
= \int_{\Omega} (\xi u^{p-1} - c_\lambda u^{p-1} + u^{p-1})(\bar{u}_0 - u)^+ \, dz \\
\leq \int_{\Omega} (f(z, u) + u^{p-1})(\bar{u}_0 - u)^+ \, dz \\
= \langle A(u), (\bar{u}_0 - u)^+ \rangle + \int_{\Omega} u^{p-1}(\bar{u}_0 - u)^+ \, dz + \int_{\partial \Omega} \beta(z)u^{p-1}(u_0 - u)^+ \, d\sigma
\]

since \( u \in S_+ \), which implies that

\[
\langle A(\bar{u}_0) - A(u), (\bar{u}_0 - u)^+ \rangle + \int_{\Omega} \bar{u}_0^{p-1}(\bar{u}_0 - u)^+ \, dz \leq 0.
\]

Therefore,

\[
||\bar{u}_0 > u||_N = 0,
\]

that is,

\[
\bar{u}_0 \leq u.
\]

Thus, we have proved that

\[
\bar{u}_0 \in [0, u] = \{ y \in W^{1,p}(\Omega) : 0 \leq y(z) \leq u(z) \text{ for almost all } z \in \Omega \} \quad \text{and} \quad \bar{u}_0 \neq 0. 
\]  

(3.15)

Because of (3.10), (3.11), (3.12) and (3.15) we have that (3.14) becomes

\[
\langle A(\bar{u}_0), h \rangle + \int_{\partial \Omega} \beta(z)\bar{u}_0^{p-1} h \, d\sigma = \int_{\partial \Omega} e(z, \bar{u}_0) h \, d\sigma \quad \text{for all } h \in W^{1,p}(\Omega),
\]

which implies that \( \bar{u}_0 \) is a positive solution of (3.3) (see Papageorgiou and Rădulescu [18]). Then, from Proposition 3.1 we have \( \bar{u}_0 = \bar{u} \) and, as a result, \( \bar{u} \leq u \) for all \( u \in S_+ \).

In a similar fashion, we show that \( v \leq \bar{v} \) for all \( v \in S_+ \).

Next, we produce extremal constant-sign solutions, that is, a smallest positive solution and a biggest negative solution.

**Proposition 3.3.** If hypotheses \( H(a), H(\beta) \) and \( H(f) \) hold, then (1.1) admits a smallest positive solution

\[
u_* \in [0, \eta] \cap \text{int } C_+
\]

and a biggest negative solution

\[
u_* \in [-\eta, 0] \cap (- \text{int } C_+).
\]
Proof. Evidently, we restrict ourselves to the sets $S_+$ and $S_-$. From Hu and Papageorgiou [11, Lemma 3.10] we know that we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n \geq 1} u_n.$$ 

For all $n \in \mathbb{N}$, we have

$$\langle A(u_n), h \rangle + \int_{\partial \Omega} \beta(z) u_n^{p-1} h \, dz = \int_{\Omega} f(z, u_n) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \quad (3.16)$$

Clearly, $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \rightharpoonup^w u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial \Omega). \quad (3.17)$$

In (3.16), we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, we pass to the limit as $n \to \infty$ and we use (3.17). Then, we have that

$$\lim_{n \to \infty} \langle A(u_n), u_n - u_* \rangle = 0$$

implies

$$u_n \to u_* \text{ in } W^{1,p}(\Omega) \quad (3.18)$$

since $A(\cdot)$ is of type $(S)_+$. So, if in (3.16) we pass to the limit as $n \to \infty$ and use (3.18), then

$$\langle A(u_*), h \rangle + \int_{\partial \Omega} \beta(z) u_*^{p-1} h \, d\sigma = \int_{\Omega} f(z, u_*) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega). \quad (3.19)$$

Also (see Proposition 3.2),

$$\tilde{u} \leq u_* \quad (3.20)$$

From (3.19) and (3.20) we infer that

$$u_* \in S_+ \subseteq \text{int } C_+ \quad \text{and} \quad u_* = \inf S_+.$$ 

Similarly, we produce

$$v_* \in S_- \quad \text{and} \quad v_* = \sup S_-.$$ 

Using these two extremal constant-sign solutions, we introduce the Carathéodory functions

$$\mu(z, x) = \begin{cases} f(z, v_*(z)) + |v_*(z)|^{p-2} v_*(z) & \text{if } x < v_*(z), \\ f(z, x) + |x|^{p-2} x & \text{if } v_*(z) \leq x \leq u_*(z), \\ f(z, u_*(z)) + u_*(z)^{p-1} & \text{if } u_*(z) < x, \end{cases} \quad (3.21)$$

$$\bar{b}(z, x) = \begin{cases} \beta(z)|v_*(z)|^{p-2} v_*(z) & \text{if } x < v_*(z), \\ \beta(z)|x|^{p-2} x & \text{if } v_*(z) \leq x \leq u_*(z), \\ \beta(z)u_*(z)^{p-1} & \text{if } u_*(z) < x. \end{cases} \quad (3.22)$$

We set

$$M(z, x) = \int_0^x \mu(z, s) \, ds \quad \text{and} \quad \bar{B}(z, x) = \int_0^x \bar{b}(z, s) \, ds,$$

and we consider the $C^1$-functional $\tilde{\phi} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\phi}(u) = \int_{\Omega} G(Du) \, dz + \frac{1}{p} \|u\|_p^p + \int_{\partial \Omega} \bar{B}(z, u) \, d\sigma - \int_{\Omega} M(z, u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$
Proposition 3.4. If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then $\hat{\varphi}$ satisfies the PS-condition, it is even, bounded from below, $\hat{\varphi}(0) = 0$ and $K_\varphi \subseteq [v_*, u_*]$. 

Proof. From (3.21) and (3.22) it is clear that $\varphi$ is coercive. So, it is bounded from below and satisfies the PS-condition (see Papageorgiou and Winkert [25]). Hypotheses $H(f)$ imply that $\hat{\varphi}$ is even (recall that $u_* \in S_*$ and $v_* \in S_*$) and $\varphi(0) = 0$. Finally, let $u \in K_\varphi$. Then, $\varphi'(u) = 0$ implies that

$$
\langle A(u), h \rangle + \int_{\Omega} |u|^{p-2} uh \, dx + \int_{\partial \Omega} \beta(z) h \, d\sigma = \int_{\Omega} \mu(z, u) h \, dx
$$

for all $h \in W^{1,p}(\Omega)$. (3.23)

In (3.23), we first choose $h = (u - u_*)^+ \in W^{1,p}(\Omega)$. Then, we have (see (3.21) and (3.22) for the first equality and recall that $u_* \in S_*$ for the second one)

$$
\langle A(u), (u - u_*)^+ \rangle + \int_{\Omega} |u|^{p-1}(u - u_*)^+ \, dx + \int_{\partial \Omega} \beta(z) u_*^{p-1}(u - u_*)^+ \, d\sigma
$$

$$
= \int_{\Omega} (f(z, u_*) + u_*^{p-1})(u - u_*)^+ \, dx
$$

$$
= \langle A(u_*) , (u - u_*)^+ \rangle + \int_{\Omega} u_*^{p-1}(u - u_*)^+ \, dx + \int_{\partial \Omega} \beta(z) u_*^{p-1}(u - u_*)^+ \, d\sigma,
$$

which implies that

$$
\langle A(u) - A(u_*) , (u - u_*)^+ \rangle + \int_{\Omega} (u^{p-1}_* - u^{p-1})(u - u_*)^+ \, dx = 0.
$$

Therefore,

$$
|\{u > u_*\}|_N = 0,
$$

that is,

$$
u_* \leq u.*
$$

Similarly, if in (3.23) we choose $h = (v_* - v_*)^+ \in W^{1,p}(\Omega)$, then we obtain that $v_* \leq u$ implies

$$
K_\varphi \subseteq [v_*, u_*].
$$

The extremality of $v_* \in \text{int} C_*$ and of $u_* \in \text{int} C_*$ implies the following property.

Corollary 3.5. If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then the elements of $K_\varphi \setminus \{0, v_*, u_*\}$ are nodal solutions of (1.1).

Now, we are ready to produce a whole sequence of distinct nodal solutions for (1.1).

Theorem 3.6. Assume that hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold. Then, (1.1) has a whole sequence $\{u_n\}_{n \geq 1} \subseteq C^1(\bar{\Omega})$ of distinct nodal solutions.

Proof. Let $\bar{\eta} = \min\{\min\Omega u_* - \max\Omega v_*\}$ (recall that $u_* \in \text{int} C_*$ and $v_* \in - \text{int} C_*$). Hypothesis $H(a)$ (iv) implies that we can find $\delta_0 \in (0, \bar{\eta})$ such that

$$
G(y) \leq c_0 |y|^q \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \leq \delta_0 \text{ and for some } c_0 > 0. \quad (3.24)
$$

Also, from (2.5) we have

$$
F(z, x) \geq \frac{\xi}{q} |x|^q \quad \text{for almost all } z \in \Omega \text{ and for all } |x| \leq \delta \text{ with } \xi > 0. \quad (3.25)
$$

Let $n \in \mathbb{N}$ and let $Y_n \subseteq W^{1,p}(\Omega)$ be an $n$-dimensional subspace. Then, all norms are equivalent on $Y_n$. So, we can find $\rho_n > 0$ such that $u \in Y_n$ and $\|u\| \leq \rho_n$ imply

$$
|u(z)| \leq \delta \quad \text{for almost all } z \in \Omega. \quad (3.26)
$$
Using (3.24), (3.25) and (3.26) together with (3.21) and (3.22), for all \( u \in Y_n \) with \( \| u \| \leq \rho_n \), we have

\[
\varphi(u) \leq c_0 \| Du \|_q^q + \frac{1}{p} \int_{\Omega} \beta(z) |u|^p \, d\sigma - \frac{\xi}{q} \| u \|_q^q \leq (c_{10} - \xi c_{11}) \| u \|^q
\]

(3.27)

with \( c_{10}, c_{11} > 0 \) independent of \( \xi > 0 \) (use the trace theorem and recall that all norms are equivalent on \( Y_n \)). Recall that \( \xi > 0 \) is arbitrary (see (2.3)). So, we choose \( \xi > \frac{c_{10}}{c_{11}} \) and we have that

\[
\varphi(u) < 0 \quad \text{for all } u \in Y_n \text{ with } \| u \| = \rho_n.
\]

Because of Proposition 3.4 we can apply Theorem 2.1 to find \( \{ u_n \}_{n \geq 1} \subseteq W^{1,p}(\Omega) \setminus \{ 0 \} \) such that

\[
u_n \in K_\varphi \setminus \{ 0 \} \quad \text{for all } n \in \mathbb{N}
\]

and

\[
\varphi(u_n) \to 0 \quad \text{as } n \to \infty.
\]

Since \( K_\varphi \subseteq C^1(\overline{\Omega}) \) (nonlinear regularity theory), we have

\[
u_n \in C^1(\overline{\Omega}) \setminus \{ 0 \} \quad \text{for all } n \in \mathbb{N}.
\]

Finally, Corollary 3.5 implies that \( \{ u_n \}_{n \geq 1} \subseteq C^1(\overline{\Omega}) \) is a sequence of distinct nodal solutions for (1.1). \( \square \)

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