A Note on Elliptic Equations Involving the Critical Sobolev Exponent

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Abstract In this work we obtain some existence results for a class of elliptic Dirichlet problems involving the critical Sobolev exponent and containing a parameter. Through a weak lower semicontinuity result and by using a critical point theorem for differentiable functionals, the existence of a precise open interval of positive eigenvalues for which the treated problems admit at least one non-trivial weak solution is established. The attained results represent a more precise version of some contributions on the treated subject.

1 Introduction

In this we study the existence of one non-trivial weak solution for the following elliptic Dirichlet problem:

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$$(P_{\mu,\lambda}) \qquad \begin{cases} -\Delta_p u = \mu \left(\int_\Omega |u(x)|^{p^*} dx \right)^{p/p^*-1} |u|^{p^*-2}u + \lambda f(x,u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $1 < p < N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ stands for the usual p-Laplace operator, $p^* := pN/(N-p)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1}, \;\; \forall\,(x,t) \in \Omega \times \mathbb{R}, \tag{h_∞}$$

where a_1, a_2 are non-negative constants and $q \in]1, pN/(N-p)[$. Finally, λ and μ are two real parameters, respectively, positive and non-negative.

Problem $(P_{0,\lambda})$ has been extensively studied during the last few years, where the nonlinearity being a continuous function provided with certain growth properties at zero and infinity, respectively. We just mention, in the large literature on the subject, the papers [14, 15, 19]; see also the recent monograph by Kristály, Rădulescu and Varga [16] as general reference for this topic.

When $\mu \neq 0$, the classical variational approach cannot be applied due to the presence of the term

$$\left(\int_\Omega |u(x)|^{p^*} dx \right)^{p/p^*-1} |u|^{p^*-2}u.$$

Indeed, the classical Sobolev inequality ensures that the embedding of the space $W_0^{1,p}(\Omega)$ into the Lebesgue space $L^{p^*}(\Omega)$ is continuous but not compact. Due to this lack of compactness the classical methods cannot be used in order to prove the weak lower semicontinuity of the energy functional associated to $(P_{\mu,\lambda})$. In our setting we overcome this difficulty through a lower semicontinuity result obtained by Montefusco in [17]. In this paper, bearing in mind the well-known inequality

$$\left(\int_\Omega |u(x)|^{p^*} dx \right)^{1/p^*} \leq \frac{1}{S^{1/p}} \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p}, \;\; \forall\, u \in W_0^{1,p}(\Omega) \tag{1}$$

where S is the best constant in the Sobolev inclusion $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ (see Sect. 2), fixing $\mu \in [0, S[$ and requiring a suitable behaviour of the nonlinearity f at zero, we determine a precise open interval of positive parameters λ, for which problem $(P_{\mu,\lambda})$ admits at least one non-trivial weak solution in $W_0^{1,p}(\Omega)$.

The proof of our main results are based on a recent abstract critical point theorem proved by Bonanno and Candito in [1, Theorem 3.1, part (a)] which is substantially a refinement of the variational principle established by Ricceri in [18]; see also Bonanno and Molica Bisci [3, Theorem 2.1, part (a)].

We explicitly observe that our results are a more precise form of the contributions obtained by Faraci and Livrea in [13]. Indeed, in Theorem 3.1 of the cited paper, fixing $\mu \in [0,S[$, the authors proved the existence of a positive parameter ν_μ^* such that, for every $\lambda \in]0,\nu_\mu^*[$, the problem $(P_{\mu,\lambda})$ admits at least one non-trivial weak solution. However, by using their approach, no concrete expression of this parameter is given.

Here, through a different strategy previously developed by Bonanno and Molica Bisci in [5], an explicit value of the parameter ν_μ^* is presented. A particular case of our results (see Theorem 3 and Remark 1 below) reads as follows.

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the following subcritical growth condition:*

$$|f(t)| \leq a_1 + a_2|t|^{p-1}, \quad \forall t \in \mathbb{R},$$

where a_1, a_2 are non-negative constants. Furthermore, assume that

$$\lim_{\xi\to 0^+} \frac{\displaystyle\int_0^\xi f(t)dt}{\xi^p} = +\infty. \tag{h_0'}$$

Then, for every $\mu \in [0,S[$ there exists a positive number ν_μ^* given by

$$\nu_\mu^* := \frac{S-\mu}{a_2 S}\left(\frac{\omega_N}{\mathrm{meas}(\Omega)}\right)^{p/N},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$, such that, for every $\lambda \in]0,\nu_\mu^*[$, the Dirichlet problem

$$(\tilde{P}_{\mu,\lambda}) \qquad \begin{cases} -\Delta_p u = \mu\left(\displaystyle\int_\Omega |u(x)|^{p^*}dx\right)^{p/p^*-1} |u|^{p^*-2}u + \lambda f(u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in W_0^{1,p}(\Omega)$. Moreover, one has that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

2 Abstract Framework and Main Results

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and denote by X the space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p}.$$

Fixing $q \in [1, p^*[$, from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \quad u \in X, \tag{2}$$

and, in particular, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. Moreover, the best constant that appears in inequality (1) is given by $S = 1/c^p$, where

$$c = \frac{1}{N\sqrt{\pi}} \left(\frac{N!\Gamma\left(\frac{N}{2}\right)}{2\Gamma\left(\frac{N}{p}\right)\Gamma\left(N+1-\frac{N}{p}\right)} \right)^{1/N} \eta^{1-1/p}, \tag{3}$$

and

$$\eta := \frac{N(p-1)}{N-p};$$

see, for instance, the quoted paper [20]. Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$, for every $(x,\xi) \in \Omega \times \mathbb{R}$, and consider the functional $E_{\mu,\lambda} : X \to \mathbb{R}$ given by

$$E_{\mu,\lambda}(u) := \Phi_\mu(u) - \lambda \Psi(u), \;\; u \in X,$$

where

$$\Phi_\mu(u) := \frac{1}{p} \int_\Omega |\nabla u(x)|^p dx - \frac{\mu}{p} \left(\int_\Omega |u(x)|^{p^*} dx \right)^{p/p^*}, \quad \Psi(u) := \int_\Omega F(x,u(x))dx.$$

Fixing $\mu \in [0, S[$, from the Sobolev inequality (1), it follows that

$$\left(\frac{S-\mu}{pS} \right) \|u\|^p \leq \Phi_\mu(u) \leq \frac{\|u\|^p}{p}, \tag{4}$$

for every $u \in X$. Furthermore, Montefusco, in [17], proved that Φ_μ is sequentially weakly lower semicontinuous for $\mu \in [0, S[$ and in this setting, since (1) holds, it is also a coercive functional. Moreover, note that $E_{\mu,\lambda} \in C^1(X, \mathbb{R})$ and a critical point $u \in X$ is a weak solution of the non-local problem

$$-\Delta_p u = \mu \left(\int_\Omega |u(x)|^{p^*} dx \right)^{p/p^*-1} |u|^{p^*-2} u + \lambda f(x,u) \quad \text{in} \quad \Omega.$$

Our main tool in order to obtain the existence of one non-trivial solution to problem $(P_{\mu,\lambda})$ is the following critical point theorem.

Theorem 2. *Let X be a reflexive real Banach space, and let $\Phi,\Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive. Further, assume that Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r-\Phi(u)}.$$

Then, for each $r > \inf_X \Phi$ and each $\lambda \in\,]0,1/\varphi(r)[$, the restriction of $J_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty,r[)$ admits a global minimum, which is a critical point (local minimum) of J_λ in X.

As pointed out in Introduction, this result is a refinement of the variational principle of Ricceri; see the quoted paper [18]. Moreover, we recall that Theorem 2 has been used in order to obtain some theoretical contributions on the existence of either three or infinitely many critical points for suitable functionals defined on reflexive Banach spaces; see [2, 3]. As consequences of the above-cited results, on the vast literature on the subject, we mention here some recent works [4, 6–12] on the existence of weak solutions for some different classes of elliptic problems.

The main result reads as follows.

Theorem 3. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $f(x,0) \neq 0$ in Ω and satisfying condition (h_∞). Then, for every $\mu \in [0,S[$ there exists a positive number ν^*_μ given by*

$$\nu^*_\mu := q\sup_{\gamma>0}\left(\frac{\gamma^{p-1}}{qa_1c_1\left(\dfrac{pS}{S-\mu}\right)^{1/p} + a_2c_q^q\left(\dfrac{pS}{S-\mu}\right)^{q/p}\gamma^{q-1}}\right),$$

such that, for every $\lambda \in]0,\nu^*_\mu[$, the following elliptic Dirichlet problem

$$(P_{\mu,\lambda}) \qquad \begin{cases} -\Delta_p u = \mu\left(\displaystyle\int_\Omega |u(x)|^{p^*}dx\right)^{p/p^*-1} |u|^{p^*-2}u + \lambda f(x,u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in X$. Moreover,

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the function $\lambda \to E_{\lambda,\mu}(u_\lambda)$ is negative and decreasing in $]0,\nu^*_\mu[$.

Proof. Fix $\mu \in [0,S[$ and $\lambda \in]0, \nu_\mu^*[$. Our aim is to apply Theorem 2 with $X = W_0^{1,p}(\Omega)$; $\Phi := \Phi_\mu$ and Ψ are the functionals introduced before. Since $\mu \in [0,S[$, $\Phi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional as well as the map $\Psi : X \to \mathbb{R}$ is continuously Gâteaux differentiable and sequentially weakly upper semicontinuous. Moreover, Φ is coercive and clearly $\inf_{u\in X} \Phi(u) = 0$. Thanks to the growth condition (h_∞), one has that

$$F(x,\xi) \le a_1|\xi| + a_2\frac{|\xi|^q}{q}, \tag{5}$$

for every $(x,\xi) \in \Omega \times \mathbb{R}$. Since $0 < \lambda < \nu_\mu^*$, there exists $\overline{\gamma} > 0$ such that

$$\lambda < \nu_\mu^\star(\overline{\gamma}) := \frac{q\overline{\gamma}^{p-1}}{qa_1c_1\left(\dfrac{pS}{S-\mu}\right)^{1/p} + a_2c_q^q\left(\dfrac{pS}{S-\mu}\right)^{q/p}\overline{\gamma}^{q-1}}. \tag{6}$$

Now, set $r \in]0,+\infty[$ and consider the function

$$\chi(r) := \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}{r}.$$

Taking into account (5) it follows that

$$\Psi(u) = \int_\Omega F(x,u(x))dx \le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{q}\|u\|^q_{L^q(\Omega)}.$$

Then, due to (4), we get

$$\|u\| < \left(\frac{pSr}{S-\mu}\right)^{1/p}, \tag{7}$$

for every $u \in X$ and $\Phi(u) < r$. Now, from (2) and by using (7), for every $u \in X$ such that $\Phi(u) < r$, one has

$$\Psi(u) < c_1a_1\left(\frac{pS}{S-\mu}\right)^{1/p} r^{1/p} + a_2\frac{c_q^q}{q}\left(\frac{pS}{S-\mu}\right)^{q/p} r^{q/p}.$$

Hence

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) \le c_1a_1\left(\frac{pS}{S-\mu}\right)^{1/p} r^{1/p} + a_2\frac{c_q^q}{q}\left(\frac{pS}{S-\mu}\right)^{q/p} r^{q/p}.$$

Then

$$\chi(r) \leq c_1 a_1 \left(\frac{pS}{S-\mu}\right)^{1/p} r^{1/p-1} + a_2 \frac{c_q^q}{q} \left(\frac{pS}{S-\mu}\right)^{q/p} r^{q/p-1}, \tag{8}$$

for every $r > 0$.

Hence, in particular

$$\chi(\overline{\gamma}^p) \leq c_1 a_1 \left(\frac{pS}{S-\mu}\right)^{1/p} \overline{\gamma}^{1-p} + a_2 \frac{c_q^q}{q} \left(\frac{pS}{S-\mu}\right)^{q/p} \overline{\gamma}^{q-p}. \tag{9}$$

Now, observe that

$$\varphi(\overline{\gamma}^p) := \inf_{u\in\Phi^{-1}(]-\infty,\overline{\gamma}^p[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,\overline{\gamma}^p[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)} \leq \chi(\overline{\gamma}^p),$$

because $u_0 \in \Phi^{-1}(]-\infty,\overline{\gamma}^p[)$ and $\Phi(u_0) = \Psi(u_0) = 0$, where $u_0 \in X$ is the identically zero function. In conclusion, bearing in mind (6), the above inequality together with (9) gives

$$\varphi(\overline{\gamma}^p) \leq \chi(\overline{\gamma}^p) \leq c_1 a_1 \left(\frac{pS}{S-\mu}\right)^{1/p} \overline{\gamma}^{1-p} + a_2 \frac{c_q^q}{q} \left(\frac{pS}{S-\mu}\right)^{q/p} \overline{\gamma}^{q-p} < \frac{1}{\lambda}.$$

In other words,

$$\lambda \in \left]0, \frac{q\overline{\gamma}^{p-1}}{qa_1c_1 \left(\frac{pS}{S-\mu}\right)^{1/p} + a_2 c_q^q \left(\frac{pS}{S-\mu}\right)^{q/p} \overline{\gamma}^{q-1}}\right[\subseteq]0, 1/\varphi(\overline{\gamma}^p)[.$$

Thanks to Theorem 2, there exists a function $u_\lambda \in \Phi^{-1}(]-\infty,\overline{\gamma}^p[)$ such that

$$E'_{\mu,\lambda}(u_\lambda) = \Phi'(u_\lambda) - \lambda\Psi'(u_\lambda) = 0,$$

and, in particular, u_λ is a global minimum of the restriction of $E_{\mu,\lambda}$ to $\Phi^{-1}(]-\infty,\overline{\gamma}^p[)$. Further, since $f(x,0) \neq 0$ in Ω, the function u_λ cannot be trivial, that is, $u_\lambda \neq 0$. Hence, for $\mu \in [0,S[$ and for every $\lambda \in]0, v_\mu^*[$ the problem $(P_{\mu,\lambda})$ admits a non-trivial solution $u_\lambda \in X$. From now, we argue in similar way of [13, Theorem 3.1] in order to prove that $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$ and that the function $\lambda \to E_{\mu,\lambda}(u_\lambda)$ is negative and decreasing in $]0, v_\mu^*[$. The proof is complete. □

Remark 1. Also when $f(x,0) = 0$ in Ω the statements of Theorem 3 are still true if, in addition to assumption (h_∞), the function f satisfies the following hypothesis:

There are a non-empty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\inf_{x\in B} F(x,\xi)}{\xi^p} = +\infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{\inf_{x\in D} F(x,\xi)}{\xi^p} > -\infty. \qquad (\mathrm{h}_0)$$

Condition (h_0) ensures that the solution, achieved by using Theorem 3, is non-trivial.

Remark 2. In conclusion, we just mention that the technical approach adopted in this manuscript has been used in different settings in order to obtain existence and multiplicity results for several kinds of differential problems, for instance, by Bonanno, Molica Bisci and Rădulescu in [8] for elliptic problems on compact Riemannian manifolds without boundary and by D'Aguì and Molica Bisci for an elliptic Neumann problem involving the p-Laplacian; see [10].

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