Computers and Mathematics with Applications 70 (2015) 691–704

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Hodge decomposition of variable exponent spaces of Clifford-valued functions and applications to Dirac and Stokes equations

Yongqiang Fu [a], Vicenţiu D. Rădulescu [b,c,*], Binlin Zhang [a,d]

[a] Department of Mathematics, Harbin Institute of Technology, 150001 Harbin, China
[b] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[c] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania
[d] Department of Mathematics, Heilongjiang Institute of Technology, Harbin, 150050, China

ARTICLE INFO

Article history:
Received 5 February 2015
Accepted 14 May 2015
Available online 12 June 2015

Keywords:
Clifford analysis
$L^{p(x)}$-decomposition
A-Dirac equation
Stokes equation
Variable exponent

ABSTRACT

In this paper, we establish a Hodge-type decomposition of variable exponent Lebesgue spaces of Clifford-valued functions, where one of the subspaces is the space of all monogenic $L^{p(x)}$-functions. Using this decomposition, we obtain the existence and uniqueness of solutions to the homogeneous A-Dirac equations with variable growth under certain appropriate conditions and to the Stokes equations in the setting of variable exponent spaces of Clifford-valued functions.

© 2015 Elsevier Ltd. All rights reserved.

1. Introduction

The central purpose of this paper is the study of the Hodge decomposition of the variable exponent Lebesgue space of Clifford-valued functions. In the compact case, the Hodge theory is a central tool for characterizing the topology of the underlying manifold. In the second part of this paper, we apply this decomposition in order to solve the homogeneous A-Dirac equations with variable growth and the Stokes equations in the setting of variable exponent spaces of Clifford-valued functions. The approach we develop is in the spirit of Helmholtz [1858]. He first formulated a result on the splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the "Hodge decomposition". We also point out the pioneering papers by Hodge [1,2], de Rham [3,4], and Weyl [5].

Variable exponent Lebesgue spaces $L^{p(x)}$ appeared in the literature for the first time already in an article by W. Orlicz in 1931, who considered the variable exponent function space $L^{p(x)}$ on the real line, see [6–8]. The next major step in the research of variable exponent spaces was the paper by O. Kováčik and J. Rákosník [9], which established many of the basic properties of Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev spaces $W^{k,p(x)}$. In the last twenty years, these spaces have attracted more and more attention. The study of these spaces has been stimulated by problems in elastic mechanics, fluid dynamics, calculus of variations and differential equations with variable growth conditions, see [6,10,11,9,12]. In particular, one of the reasons that forced the rapid expansion of the theory of variable exponent spaces have been the models of

* Corresponding author at: Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania.
E-mail addresses: fuyongqiang@hit.edu.cn (Y. Fu), vicentiu.radulescu@imar.ro (V.D. Rădulescu), zhangbinlin2012@163.com (B. Zhang).

http://dx.doi.org/10.1016/j.camwa.2015.05.020
0898-1221/© 2015 Elsevier Ltd. All rights reserved.

electrorheological fluids introduced by Rajagopal and Ružička [13], which can be described by the boundary value problem for the generalized Navier–Stokes equations, where the extra stress tensor is coercive and satisfies appropriate growth assumptions. For the detailed accounts we refer to the monograph of Ružička [12].

As a powerful tool for solving elliptic boundary value problems in the plane, the methods of complex functions theory play an important role. One way to extend these ideas to higher dimension is to begin with a generalization of algebraic and geometrical properties of the complex numbers. In this way, W.R. Hamilton studied the algebra of quaternions in 1843. Further generalizations were investigated by W.K. Clifford in 1878. He introduced the so-called *geometric algebras* or *Clifford algebras*, which are generalizations of the complex numbers, the quaternions, and the exterior algebras, see [14]. Initially developed in the 1930–1940 and very intensively since 1970, theories for functions with values in Clifford algebras were developed. Many advantages of the complex function theory were preserved in this process. Clifford analysis usually studies the solutions of the Dirac equation or of a generalized Cauchy–Riemann system for functions defined on domains in Euclidean space and taken value in Clifford algebras, see [15,16]. Of course, the motivation for this development is closely related with systems of partial differential equations in physics, see [17].

In [18,19], C.A. Nolder first introduced A-Dirac equations $DA(x, Du) = 0$ and developed tools for the study of weak solutions to nonlinear A-Dirac equations in space $W_0^{1,p}(\Omega, C\ell_n)$. Note that under appropriate identifications, the scalar part of $DA(x, Du) = 0$ is actually the *A-harmonic equation* $\operatorname{div} A(x, \nabla u) = 0$. These equations have been extensively studied with many applications, see for instance [20]. Recently, L. Diening and P. Kaplicky [17] studied the interior regularity of the local weak solutions $u \in W^{1,\varphi}(\Omega)$ and $\omega \in L^{\varphi^*}(\Omega)$ of the following stationary generalized Stokes system:

$$\begin{cases} -\operatorname{div} A(Du) + \nabla\omega = -\operatorname{div} G & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega, \end{cases}$$

where D is the symmetric part of the gradient, the extra stress tensor A determines properties of the fluid. For more detailed discussions, we refer to [17,21,22] and the references therein. Obviously, A-Dirac equations correspond to the extra stress tensor in the real-valued case.

However, the existence of weak solutions to the A-Dirac equations has not been proved. Y. Fu and B. Zhang [23–25] proved the existence of weak solutions for A-Dirac equations with variable growth. For this purpose, they also established a theory of variable exponent spaces of Clifford-valued functions. But so far they have proved the existence of weak solutions to the scalar part of the A-Dirac equations in space $W_0^{1,p(x)}(\Omega, C\ell_n)$, see [13]. With the help of the orthogonal decomposition of the space $L^2(\Omega)$, K. Gürlebeck and W. Sprößig [26,20] obtained the existence of uniqueness of solutions to the Stokes equations. Therefore, it is natural to consider an extension of the orthogonal decomposition of the spaces $L^2(\Omega, C\ell_n)$.

One of the most interesting results of complex and hyper-complex function theory is the orthogonal decomposition of the space $L^2(\Omega, C\ell_n)$, namely

$$L^2(\Omega, C\ell_n) = (\ker D \cap L^2(\Omega, C\ell_n)) \oplus DW_0^{1,2}(\Omega, C\ell_n), \tag{1.1}$$

where $\ker D$ denotes the set of all monogenic functions on Ω. This decomposition has a number of applications, especially to the theory of partial differential equations, see [27] for the complex case and [20] for the hypercomplex case. In [28], U. Kähler extended the orthogonal decomposition (1.1) to the spaces $L^p(\Omega)$ in form of a direct decomposition in the case of Clifford analysis. In [29], J. Dubinski and M. Reissig considered decompositions for the spaces $W^{m,p}(\Omega)$, which can be applied to study nonlinear variational problems.

The goal of this paper is to generalize decomposition (1.1) in the framework of the variable exponent spaces and to give applications to the A-Dirac equations and the Stokes equations. Next, using the Hodge-type decomposition that we obtain in this paper, we prove the existence and uniqueness of weak solutions to A-Dirac equations with variable growth under certain suitable assumptions. At the same time, the existence and uniqueness of solutions to Stokes equations are showed in variable exponent product spaces of Clifford-valued functions. The whole treatment applies to a much larger class of elliptic problems.

This paper is organized as follows. In Section 2, we start with a brief summary of basic knowledge of Clifford algebras and variable exponent spaces of Clifford-valued functions, then discuss properties of some operators, which will be needed in the sequel. In Section 3, we establish a Hodge-type decomposition for the space $L^{p(x)}(\Omega, C\ell_n)$. In Section 4, appealing to this decomposition in combination with the Minty–Browder theory, we prove the existence and uniqueness of a solution to the A-Dirac equations with variable growth in $W_0^{1,p(x)}(\Omega, C\ell_n)$. On the other hand, with the help of this decomposition, we prove the existence and uniqueness of a solution in $W_0^{1,p(x)}(\Omega, C\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$ of the Stokes problem, provided that the right-hand side is in $W^{-1,p(x)}(\Omega, C\ell_n)$.

We refer to the excellent book by Ciarlet [30] for necessary abstract notions and useful related examples.

2. Preliminaries

2.1. Clifford algebra

We first recall some related notions and results from Clifford algebra. For a detailed account we refer to [15,31,20] and the references therein.

Let $C\ell_n$ be the real universal Clifford algebra over $\mathbb{R}^n$. Then

$$C\ell_n = \text{span}\{e_0, e_1, e_2, \dots, e_n, e_1e_2, \dots, e_{n-1}e_n, \dots, e_1e_2\cdots e_n\}$$

where $e_0 = 1$ (the identity element in $\mathbb{R}^n$), $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j + e_je_i = -2\delta_{ij}e_0$. Thus, the dimension of $C\ell_n$ is 2^n. In particular, we denote by $C\ell_2 = \mathbb{H}$ the algebra of real quaternions. For $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ with $1 \le i_1 < i_2 < \cdots < i_n \le n$, put $e_I = e_{i_1}e_{i_2}\dots e_{i_r}$, while for $I = \emptyset$, $e_\emptyset = e_0$. For $0 \le r \le n$ fixed, the space $C\ell_n^r$ is defined by

$$C\ell_n^r = \text{span}\{e_I : |I| := \text{card}(I) = r\}.$$

The Clifford algebra $C\ell_n$ is a graded algebra as

$$C\ell_n = \bigoplus_{1\le r\le n} C\ell_n^r.$$

Any element $a \in C\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \cdots + [a]_n,$$

where $[\ ]_r : C\ell_n \to C\ell_n^r$ denotes the projection of $C\ell_n$ onto $C\ell_n^r$. It is customary to identify $\mathbb{R}$ with $C\ell_n^0$ and identify $\mathbb{R}^n$ with $C\ell_n^1$ respectively. This means that each element x of $\mathbb{R}^n$ may be represented by

$$x = \sum_{i=1}^n x_ie_i.$$

For $u \in C\ell_n$, we denotes by $[u]_0$ the scalar part of u, that is, the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{e_{i_1}e_{i_2}\dots e_{i_r}} = (-1)^{\frac{r(r+1)}{2}} e_{i_1}e_{i_2}\dots e_{i_r}.$$

For $A \in C\ell_n, B \in C\ell_n$, we have

$$\overline{AB} = \overline{B}\,\overline{A}, \qquad \overline{\overline{A}} = A.$$

We denote

$$(A, B) = [\overline{A}B]_0.$$

Then an inner product is thus obtained, which defines the norm $|\cdot|$ on $C\ell_n$ by

$$|A|^2 = [\overline{A}A]_0.$$

From [32] we know that this norm is submultiplicative, namely

$$|AB| \le C|A|\,|B|,$$

where C is a positive constant depending only on n and smaller than $2^{n/2}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. A Clifford-valued function $u : \Omega \to C\ell_n$ can be written as $u = \Sigma_I u_I e_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real-valued functions.

The Dirac operator on the Euclidean space used here is introduced by

$$D = \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$

This is a special case of the Atiyah–Singer–Dirac operator acting on sections of a spinor bundle. We also point out that the most famous Dirac operator describes the propagation of a free fermion in three dimensions.

If u is a real-valued function defined on Ω, then $Du = \nabla u$. Moreover, $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n}\frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. We refer the readers to [20,14,28,16] for basic properties of left monogenic functions.

2.2. Variable exponent spaces of Clifford-valued functions

Next, we investigate some basic properties of variable exponent spaces of Clifford-valued functions. Note that in what follows, we use the short notation $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, etc., instead of $L^{p(x)}(\Omega, \mathbb{R})$, $W^{1,p(x)}(\Omega, \mathbb{R})$, etc.

Throughout this paper we assume that

$$p \in \mathrm{P}^{\log}(\Omega) \quad \text{and} \quad 1 < p_- := \inf_{x\in\overline{\Omega}} p(x) \le p(x) \le \sup_{x\in\overline{\Omega}} p(x) =: p_+ < \infty, \tag{2.1}$$

where $\mathrm{P}^{\log}(\Omega)$ is the set of exponents p satisfying the so-called log-Hölder continuity, that is,

$$|p(x)-p(y)| \le \frac{C}{\log(\mathrm{e}+|x-y|^{-1})}$$

holds for all $x, y \in \Omega$, see [6]. Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p'(x) \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x)-1}, \quad \text{for each } x \in \Omega.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{u \in \mathcal{P}(\Omega) : \int_\Omega |u|^{p(x)} dx < \infty\right\},$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{t > 0 : \int_\Omega \left|\frac{u(x)}{t}\right|^{p(x)} dx \le 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)}. \tag{2.2}$$

Denote by $W_0^{1,p(x)}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm (2.2). The space $W^{-1,p(x)}(\Omega)$ is defined as the dual of the space $W_0^{1,p'(x)}(\Omega)$. For more details we refer to [6,33,9] and the references therein.

In what follows, we understand $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ coordinatewisely. For example, $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ means that $\{u_I\} \subset L^{p(x)}(\Omega)$ for $u = \Sigma_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega)}$. In this way, the spaces $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $C_0^\infty(\Omega, \mathrm{C}\ell_n)$, etc., can be understood similarly. In particular, the space $L^2(\Omega, \mathrm{C}\ell_n)$ can be converted into a right Hilbert $\mathrm{C}\ell_n$-module by defining the following Clifford-valued inner product (see [20, Definition 3.74])

$$(f, g)_{\mathrm{C}\ell_n} = \int_\Omega \overline{f(x)} g(x) dx. \tag{2.3}$$

Remark 2.1. A simple computation shows that $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}$ is equivalent to $\big\| |u| \big\|_{L^{p(x)}(\Omega)}$. Furthermore, we also have that for every $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}$ is an equivalent norm of $\|u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$ (see [23]).

Lemma 2.2 ([10]). *Let $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$. For $u \in L^{p(x)}(\Omega)$, we have*

(1) *If $\|u\|_{L^{p(x)}(\Omega)} \ge 1$, then $\|u\|^{p-}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p+}_{L^{p(x)}(\Omega)}$.*
(2) *If $\|u\|_{L^{p(x)}(\Omega)} \le 1$, then $\|u\|^{p+}_{L^{p(x)}(\Omega)} \le \rho(u) \le \|u\|^{p-}_{L^{p(x)}(\Omega)}$.*

Lemma 2.3 ([23]). *Assume that $p(x) \in \mathcal{P}(\Omega)$. Then the inequality*

$$\int_\Omega |uv| dx \le C(n, p)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\|v\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}$$

holds for every $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ and $v \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$.

Lemma 2.4 ([24]). *Assume that $p(x) \in \mathcal{P}(\Omega)$. Then the following properties are true.*

(1) *The dual of the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is $L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, i.e., $\left(L^{p(x)}(\Omega, \mathrm{C}\ell_n)\right)^* = L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$. Thus, the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is a reflexive Banach space.*
(2) *The space $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is a reflexive Banach space.*

Definition 2.5 ([20]). Let $u \in C(\Omega, \mathrm{C}\ell_n)$. The Teodorescu operator is defined by

$$Tu(x) = \int_\Omega G(x-y)u(y)dy,$$

where $G(x)$ is the generalized Cauchy kernel mentioned above.

Definition 2.6. Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The Hardy–Littlewood maximal operator is defined by

$$Mu(x) = \sup_{r>0} \frac{1}{\mathrm{meas}\big(B(x,r)\big)} \int_{B(x,r)} \big|u(y)\big| dy,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$.

Lemma 2.7 ([6]). *Let $x \in \Omega$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\int_\Omega \frac{1}{|x-y|^{n-1}} |u(y)| dy \le C(n)\, (\mathrm{diam}\Omega)\, Mu(x).$$

Lemma 2.8 ([6]). *If $p(x)$ satisfies (2.1), then M is bounded in $L^{p(x)}(\Omega)$. Thus, there exists a constant $C = C(n,p)$ such that*

$$\|Mu\|_{L^{p(x)}(\Omega)} \le C(n,p) \|u\|_{L^{p(x)}(\Omega)}.$$

Lemma 2.9 ([6]). *Let Φ be a Calderón–Zygmund operator with Calderón–Zygmund kernel K on $\mathbb{R}^n \times \mathbb{R}^n$. Then Φ is bounded on $L^{p(x)}(\mathbb{R}^n)$.*

Lemma 2.10 ([24]). *The operator $D : W^{1,p(x)}(\Omega, C\ell_n) \to L^{p(x)}(\Omega, C\ell_n)$ is continuous.*

Lemma 2.11 ([24]). *The operator $T : L^{p(x)}(\Omega, C\ell_n) \to W^{1,p(x)}(\Omega, C\ell_n)$ is continuous.*

Lemma 2.12. *Let $p(x) \in \mathscr{P}(\Omega)$. Then for $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$, the Borel–Pompeiu formula $TDu(x) = u(x)$ holds for all $x \in \Omega$. Moreover, for $u \in L^{p(x)}(\Omega, C\ell_n)$, the equation $DTu(x) = u(x)$ holds for all $x \in \Omega$.*

Proof. According to Remark 4.21 in [20], the conclusions are implied by $W^{1,p(x)}(\Omega, C\ell_n) \hookrightarrow W^{1,p-}(\Omega, C\ell_n)$ and $L^{p(x)}(\Omega, C\ell_n) \hookrightarrow L^{p-}(\Omega, C\ell_n)$. $\square$

Lemma 2.13. *There exists a unique linear extension $\widetilde{T}$ of the operator T such that the operator $\widetilde{T} : W^{-1,p(x)}(\Omega, C\ell_n) \to L^{p(x)}(\Omega, C\ell_n)$ is continuous.*

Proof. In view of Proposition 12.3.2 in [6], we know that for each $f \in W^{-1,p(x)}(\Omega)$, there exists $f_k \in L^{p(x)}(\Omega)$, $k = 0, 1, \dots, n$, such that

$$\langle f, \varphi\rangle = \sum_{k=0}^{n} \int_\Omega f_k \frac{\partial \varphi}{\partial x_k} dx, \tag{2.4}$$

for all $\varphi \in W_0^{1,p'(x)}(\Omega)$. Moreover, $\|f\|_{W^{-1,p(x)}(\Omega)}$ is equivalent to $\sum_{k=0}^n \|f_k\|_{L^{p(x)}(\Omega)}$. Obviously, for every $f \in W^{-1,p(x)}(\Omega, C\ell_n)$ the equality (2.4) still holds for $f_k \in L^{p(x)}(\Omega, C\ell_n)$, $k = 0, 1, \dots, n$. Moreover, $\|f\|_{W^{-1,p(x)}(\Omega, C\ell_n)}$ is equivalent to $\sum_{k=0}^n \|f_k\|_{L^{p(x)}(\Omega, C\ell_n)}$. On the other hand, by Proposition 12.3.4 in [6], the space $C_0^\infty(\Omega, C\ell_n)$ is dense in $W^{-1,p(x)}(\Omega, C\ell_n)$. Thus, we may choose

$$u^j = u_0^j + \sum_{k=1}^n \frac{\partial u_k^j}{\partial x_k},$$

where $u_0^j, u_k^j \in C_0^\infty(\Omega, C\ell_n)$, such that $\|u^j - f\|_{W^{-1,p(x)}(\Omega, C\ell_n)} \to 0$ and $\|u_k^j - f_k\|_{L^{p(x)}(\Omega, C\ell_n)} \to 0$ as $j \to \infty$, where $k = 0, 1, \dots, n$. Here, we are using the fact that $C_0^\infty(\Omega, C\ell_n)$ is dense in $L^{p(x)}(\Omega, C\ell_n)$ (see [24]). Set

$$Tu^j = \int_\Omega G(x-y) u^j(y) dy,$$

where $G(x)$ is the above-mentioned generalized Cauchy kernel. Therefore

$$\begin{aligned} Tu^j &= \int_\Omega G(x-y) \Big(u_0^j(y) + \sum_{k=1}^n \frac{\partial}{\partial y_k} u_k^j(y) \Big) dy \\ &= \int_\Omega G(x-y) u_0^j(y) dy + \sum_{k=1}^n \int_\Omega \frac{\partial}{\partial x_k} G(x-y) u_k^j(y) dy. \end{aligned}$$

Since

$$\Big| \int_\Omega G(x-y) u_0^j(y) dy \Big| \le \int_\Omega \frac{1}{|x-y|^{n-1}} \big| u_0^j(y) \big| dy,$$

Remark 2.1, Lemmas 2.7 and 2.8 imply that there exists a constant $C_0 > 0$ such that

$$\left\| \int_\Omega G(x-y)u_0^j(y)dy \right\|_{L^{p(x)}(\Omega,C\ell_n)} \le C_0 \|u_0^j\|_{L^{p(x)}(\Omega,C\ell_n)}. \tag{2.5}$$

Now, let us extend $u_k^j(x)$ by zero to $\mathbb{R}^n \setminus \Omega$. Then $\frac{\partial}{\partial x_k}G(x-y)$ satisfies the conditions of Calderón–Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ (see [24]). In view of Lemma 2.9, there exist positive constant C_k $(k = 1, \dots, n)$ such that

$$\left\| \int_\Omega \frac{\partial}{\partial x_k} G(x-y)u_k^j(y) \right\|_{L^{p(x)}(\Omega,C\ell_n)} \le C_k \|u_k^j\|_{L^{p(x)}(\Omega,C\ell_n)}. \tag{2.6}$$

Combining (2.5) with (2.6), we have

$$\begin{aligned} \|Tu^j\|_{L^{p(x)}(\Omega,C\ell_n)} &\le \left\| \int_\Omega G(x-y)u_0^j(y)dy \right\|_{L^{p(x)}(\Omega,C\ell_n)} + \sum_{k=1}^n \left\| \int_\Omega \frac{\partial}{\partial x_k} G(x-y)u_k^j(y) \right\|_{L^{p(x)}(\Omega,C\ell_n)} \\ &\le C_0 \left\| u_0^j \right\|_{L^{p(x)}(\Omega,C\ell_n)} + \sum_{k=1}^n C_k \left\| u_k^j \right\|_{L^{p(x)}(\Omega,Cl_n)}. \end{aligned}$$

Letting $j \to \infty$, by means of the continuous linear extension theorem, the operator T can be uniquely extended to a bounded linear operator $\widetilde{T}$ such that for all $f \in W^{-1,p(x)}(\Omega, C\ell_n)$, there exists a constant $\widetilde{C} > 0$ such that

$$\|\widetilde{T}f\|_{L^{p(x)}(\Omega,C\ell_n)} \le C\left(\|f_0\|_{L^{p(x)}(\Omega,Cl_n)} + \sum_{k=1}^n \|f_k\|_{L^{p(x)}(\Omega,C\ell_n)} \right) \le \widetilde{C}\|f\|_{W^{-1,p(x)}(\Omega,C\ell_n)}.$$

This completes the proof of Lemma 2.13. □

In [34], L. Diening, D. Lengeler and M. Ružička showed that the Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

possesses a unique weak solution $u \in W^{1,p(x)}(\Omega)$ for each $f \in W^{-1,p(x)}(\Omega)$. Moreover, the following estimate holds

$$\|u\|_{W^{1,p(x)}(\Omega)} \le C(n, p, \Omega)\|f\|_{W^{-1,p(x)}(\Omega)}.$$

We say that u is a *weak solution* of problem (2.7) provided that

$$\langle f, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla\varphi dx, \quad \forall\, \varphi \in W_0^{1,p'(x)}(\Omega).$$

Then it is easy to see that for all $f \in W^{-1,p(x)}(\Omega, C\ell_n)$ the problem (2.7) still has a unique weak solution $u \in W^{1,p(x)}(\Omega, C\ell_n)$. We denote by Δ_0^{-1} the solution operator. On the other hand, the operator

$$\Delta : W^{1,p(x)}(\Omega, C\ell_n) \to W^{-1,p(x)}(\Omega, C\ell_n)$$

is continuous, so we obtain that the operator $\widetilde{D} = -\Delta T : L^{p(x)}(\Omega, C\ell_n) \to W^{-1,p(x)}(\Omega, C\ell_n)$ is continuous from Lemma 2.11, where the operator $\widetilde{D}$ can be considered as a unique continuous linear extension of the Dirac operator. Hence we can derive two useful results which will be needed later.

Lemma 2.14. *Assume that $p(x)$ satisfies relation (2.1).*

(i) *If $u \in L^{p(x)}(\Omega, C\ell_n)$, then the equation $\widetilde{T}\widetilde{D}u(x) = u(x)$ holds for all $x \in \Omega$.*
(ii) *If $u \in W^{-1,p(x)}(\Omega, C\ell_n)$, then the equation $\widetilde{D}\widetilde{T}u(x) = u(x)$ holds for all $x \in \Omega$.*

Proof. (i) follows from Lemma 2.12 and the density of $W_0^{1,p(x)}(\Omega, C\ell_n)$ in $L^{p(x)}(\Omega, C\ell_n)$.

(ii) follows from Lemma 2.12 and the density of $C_0^\infty(\Omega, C\ell_n)$ in $W^{-1,p(x)}(\Omega, C\ell_n)$. □

Lemma 2.15. *Assume that $f \in W_0^{1,p(x)}(\Omega, C\ell_n)$ and $g \in L^{p'(x)}(\Omega, C\ell_n)$. Then the following equality holds*

$$(Df, g)_{C\ell_n} = (f, \widetilde{D}g)_{C\ell_n},$$

where $(\cdot,\cdot)_{C\ell_n}$ is Clifford-valued product (2.3) above mentioned.

Proof. Let $g_k \in W_0^{1,p'(x)}(\Omega, C\ell_n)$ with $g_k \to g$ in $L^{p'(x)}(\Omega, C\ell_n)$. Then we have

$$(Df, g_k)_{C\ell_n} = (f, Dg_k)_{C\ell_n}.$$

By the density of $W_0^{1,p'(x)}(\Omega, C\ell_n)$ in $L^{p'(x)}(\Omega, C\ell_n)$, together with the continuity of $\widetilde{D}$ and Lemma 2.3, the desired conclusion follows. □

3. The Hodge decomposition of variable exponent Lebesgue spaces

Now we are ready to generalize (1.1) to the case of variable exponent Lebesgue spaces.

Theorem 3.1. *The space $L^{p(x)}(\Omega, C\ell_n)$ allows the Hodge-type decomposition*

$$L^{p(x)}(\Omega, C\ell_n) = (\ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)) \oplus DW_0^{1,p(x)}(\Omega, C\ell_n) \tag{3.1}$$

with respect to the Clifford-valued product (2.3).

Proof. Similar to the proof of Theorem 6 in [28], we first prove that

$$(\ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)) \cap DW_0^{1,p(x)}(\Omega, C\ell_n) = \{0\}.$$

Suppose $f \in (\ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)) \cap DW_0^{1,p(x)}(\Omega, C\ell_n)$, then $\widetilde{D}f = 0$. Because of $f \in DW_0^{1,p(x)}(\Omega, C\ell_n)$ there exists a function $v \in W_0^{1,p(x)}(\Omega, C\ell_n)$ such that $Dv = f$. Hence, we get that $-\Delta v = 0$ and $v = 0$ on $\partial\Omega$. From the uniqueness of Δ_0^{-1} we obtain $v = 0$. Consequently, $f = 0$. Therefore, the sum of the two subspaces is a direct one.

Now let $u \in L^{p(x)}(\Omega, C\ell_n)$. Then we have $u_2 = D\Delta_0^{-1}\widetilde{D}u \in DW_0^{1,p(x)}(\Omega, C\ell_n)$. Let $u_1 = u - u_2$. Then $u_1 \in L^{p(x)}(\Omega, C\ell_n)$. Furthermore, we take $u_k \in W_0^{1,p(x)}(\Omega, C\ell_n)$ such that $u_k \to u$ in $L^{p(x)}(\Omega, C\ell_n)$, then by the density of $W_0^{1,p(x)}(\Omega, C\ell_n)$ in $L^{p(x)}(\Omega, C\ell_n)$ and Lemma 2.3, we have for any $\varphi \in W_0^{1,p'(x)}(\Omega, C\ell_n)$

$$\begin{aligned}\left(u_1, D\varphi\right)_{C\ell_n} &= \left(u - u_2, D\varphi\right)_{C\ell_n} = \lim_{k\to\infty}\left(Du_k - DD\Delta_0^{-1}Du_k, \varphi\right)_{C\ell_n} \\ &= \lim_{k\to\infty}\left(Du_k - Du_k, \varphi\right)_{C\ell_n} = 0.\end{aligned}$$

Thus, we get $u_1 \in \ker\widetilde{D}$. Since $u \in L^{p(x)}(\Omega, C\ell_n)$ is arbitrary, the desired result follows immediately. □

Beginning with this decomposition we get the following projections

$$P : L^{p(x)}(\Omega, C\ell_n) \to \ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)$$
$$Q : L^{p(x)}(\Omega, C\ell_n) \to DW_0^{1,p(x)}(\Omega, C\ell_n).$$

For $p(x) \equiv 2$, these are ortho-projections. Notice that directly from the proof of Theorem 3.1 we obtain

$$Q = D\Delta_0^{-1}\widetilde{D}, \qquad P = I - Q. \tag{3.2}$$

It follows from (3.2) that the operator Q as well as P maps the space $L^{p(x)}(\Omega, C\ell_n)$ into itself.

In the following we discuss the properties of the operator Q, which will be used later.

Theorem 3.2. *The space $L^{p(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q$ is a closed subspace of $L^{p(x)}(\Omega, C\ell_n)$, that is, the space $DW_0^{1,p(x)}(\Omega, C\ell_n)$ is closed in $L^{p(x)}(\Omega, C\ell_n)$.*

Proof. Let $u \in \overline{DW_0^{1,p(x)}(\Omega, C\ell_n)}$. Then there exists $u_k \in DW_0^{1,p(x)}(\Omega, C\ell_n)$ such that $\|Du_k - u\|_{L^{p(x)}(\Omega, C\ell_n)} \to 0$ as $k \to \infty$. Since $W_0^{1,p(x)}(\Omega, C\ell_n)$ is a reflexive Banach space, we can extract a subsequence of $\{u_k\}$ (still denote by $\{u_k\}$), such that $u_k \rightharpoonup v$ weakly in $W_0^{1,p(x)}(\Omega, C\ell_n)$. Since the norm in a Banach space is weakly lower semicontinuous and the operator $D : W_0^{1,p(x)}(\Omega, C\ell_n) \to L^{p(x)}(\Omega, C\ell_n)$ is continuous, we obtain

$$\|Dv - u\|_{L^{p(x)}(\Omega, C\ell_n)} \le \liminf_{n\to\infty} \|Du_n - u\|_{L^{p(x)}(\Omega, C\ell_n)} = 0.$$

Thus $u = Dv$. Now we complete the proof of Theorem 3.2. □

Theorem 3.3. *We have $\left(L^{p(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q\right)^* = L^{p'(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q$, that is, the linear operator*

$$\Phi : DW_0^{1,p'(x)}(\Omega, C\ell_n) \to \left(DW_0^{1,p(x)}(\Omega, C\ell_n)\right)^*$$

given by

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} := \int_\Omega [\overline{D\varphi}Du]_0 dx$$

is a Banach space isomorphism.

Proof. First, in view of Theorem 3.2, $DW_0^{1,p(x)}(\Omega, C\ell_n)$ and $DW_0^{1,p'(x)}(\Omega, C\ell_n)$ are reflexive Banach spaces since they are closed in $L^{p(x)}(\Omega, C\ell_n)$ and $L^{p'(x)}(\Omega, C\ell_n)$ respectively. The linearity of Φ is clear. For injectivity, suppose

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} = 0 \tag{3.3}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega, C\ell_n)$ and some $u \in W_0^{1,p'(x)}(\Omega, C\ell_n)$. For any $\omega \in L^{p(x)}(\Omega, C\ell_n)$, according to Theorem 3.1, we may write $\omega = \alpha + \beta$ with $\alpha \in \ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)$ and $\beta \in DW_0^{1,p(x)}(\Omega, C\ell_n)$. From Lemma 2.15 we obtain

$$(\omega, Du)_{Sc} = (\alpha + \beta, Du)_{Sc} = (\alpha, Du)_{Sc} + (\beta, Du)_{Sc} = (\beta, Du)_{Sc}.$$

This together with (3.3) gives $(\omega, Du)_{Sc} = 0$, hence $Du = 0$. It follows that Φ is injective. To get surjectivity, let $f \in \big(DW_0^{1,p(x)}(\Omega, C\ell_n)\big)^*$. By the Hahn–Banach theorem, there is $F \in \big(L^{p(x)}(\Omega, C\ell_n)\big)^*$ with $\|F\| = \|f\|$ and $F|_{DW_0^{1,p(x)}(\Omega,C\ell_n)} = f$. In terms of Lemma 2.4, there exists $\varphi \in L^{p'(x)}(\Omega, C\ell_n)$ such that $F(u) = (u, \varphi)_{Sc}$ for any $u \in L^{p(x)}(\Omega, C\ell_n)$. According to Theorem 3.1, we can write $\varphi = \eta + D\alpha$, where $\eta \in \ker\widetilde{D} \cap L^{p'(x)}(\Omega, C\ell_n)$, $D\alpha \in DW_0^{1,p'(x)}(\Omega, C\ell_n)$. For any $Du \in DW_0^{1,p(x)}(\Omega, C\ell_n)$, from Lemma 2.15 we have

$$\begin{aligned} f(Du) &= (Du, \varphi)_{Sc} = (Du, \eta + D\alpha)_{Sc} = (Du, \eta)_{Sc} + (Du, D\alpha)_{Sc} \\ &= (Du, D\alpha)_{Sc} = \Phi(D\alpha)(Du). \end{aligned}$$

Consequently, $\Phi(D\alpha) = f$. It follows that Φ is surjective. By means of Lemma 2.3 we have

$$|\Phi(Du)(D\varphi)| = |(D\varphi, Du)_{Sc}| \le C\|D\varphi\|_{L^{p(x)}(\Omega,C\ell_n)}\|Du\|_{L^{p'(x)}(\Omega,C\ell_n)}.$$

This means that Φ is continuous. Furthermore, it is immediate that Φ^{-1} is continuous from the inverse function theorem. This concludes the proof of Theorem 3.3. □

4. Applications

4.1. A-Dirac equations with variable growth

The Dirac equation arises in the study of nonlinear spinor fields in the unified theory of elementary particles, see Heisenberg [35] and Weyl [36]. The stationary states of the nonlinear Dirac field have been proposed as a model for elementary extended fermions and nucleons, see Thaller [37].

In this subsection, we are interested in the existence of solutions to the following A-Dirac equations:

$$DA(Du) = 0, \tag{4.1}$$

where $A : C\ell_n(\Omega) \to C\ell_n(\Omega)$ satisfies the following conditions:

(A1) $|A(\xi) - A(\eta)| \le C_1(|\xi| + |\eta|)^{p(x)-2}|\xi - \eta|$;
(A2) $\big[\big(\overline{A(\xi) - A(\eta)}\big)(\xi - \eta)\big]_0 \ge C_2(|\xi| + |\eta|)^{p(x)-2}|\xi - \eta|^2$;
(A3) $A(0) \in L^{p'(x)}(\Omega, C\ell_n)$,

where ξ and η are arbitrary elements in $C\ell_n$, and C_1 and C_2 are positive constants. Of course, $A(Du)(x) = A(Du(x))$ and $DA(Du) = 0$ are meant in the distributional sense.

Notice that when $A(\xi) = |\xi - \alpha|^{p(x)-2}(\xi - \alpha)$, where $\alpha \in L^{p(x)}(\Omega, C\ell_n)$ is fixed, then Eq. (4.1) generalizes the important case of the equation $D\big(|Du - \alpha|^{p(x)-2}(Du - \alpha)\big) = 0$. As $p(x) \equiv p$ and $\alpha \equiv 0$, p-Dirac equations were introduced and their conformal invariance was investigated in [38]. These equations are nonlinear generalizations of the Dirac Laplace equation as well as generalizations of elliptic equations of A-harmonic type $\operatorname{div} A(x, \nabla u) = 0$. The study of these equations is partially motivated by the fact that some arise as the Euler–Lagrange equations to variational integrals.

If u is a real-valued function, then the $p(x)$-Dirac equation becomes the so-called $p(x)$-Laplacian equation. In recent years these equations have been extensively studied, see [6,33,39] and the references therein.

In order to get the existence of a solution to the A-Dirac equations, we need a theorem of Minty–Browder as follows.

Proposition 4.1 *([40, Theorem 5.16]). Let X be a reflexive Banach space and let $G : X \to X^*$ be a continuous nonlinear mapping such that*

(i) *(strict monotonicity)* $(Gv - Gw, v - w) > 0\ \forall\, v, w \in X,\ v \ne w$.
(ii) *(coerciveness)* $\lim_{\|v\|\to\infty} \|v\|^{-1}(Gv, v) = \infty$.

Then for every $f \in X^*$ there exists a unique solution $u \in X$ of the equation $Gu = f$.

Now we are ready to prove our result as follows.

Theorem 4.2. *Under conditions* (A1), (A2) *and* (A3), *there exists a weak solution $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$ to the A-Dirac equations (4.1), hence there exists a Clifford-valued function $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$ such that*

$$\int_\Omega \overline{A(Du)} Dv\,dx = 0$$

for any $v \in W_0^{1,p(x)}(\Omega, C\ell_n)$. Furthermore, the solution is unique up to a monogenic function.

Proof. We first claim that $A(u) \in L^{p'(x)}(\Omega, C\ell_n)$ for every $u \in L^{p(x)}(\Omega, C\ell_n)$. Indeed, from (A1) and (A3) we obtain

$$\begin{aligned}\int_\Omega |A(u)|^{p'(x)}dx &= \int_\Omega (|A(u) - A(0) + A(0)|^{p'(x)})dx \\ &\le 2^{p'_+} \int_\Omega |A(u) - A(0)|^{p'(x)}dx + 2^{p'_+} \int_\Omega |A(0)|^{p'(x)}dx \\ &\le 2^{p'_+} \int_\Omega \left(|u|^{p(x)-2}|u|\right)^{p'(x)} dx + 2^{p'_+} \int_\Omega |A(0)|^{p'(x)}dx. \end{aligned} \tag{4.2}$$

This estimate together with Remark 2.1 and Lemma 2.2 yields the previous assertion.

Next we show that $QA\big(DW_0^{1,p(x)}(\Omega, C\ell_n)\big) = \big(DW_0^{1,p(x)}(\Omega, C\ell_n)\big)^* = DW_0^{1,p'(x)}(\Omega, C\ell_n)$. Evidently, it follows from (3.2) that $QA\big(DW_0^{1,p(x)}(\Omega, C\ell_n)\big) \subset DW_0^{1,p'(x)}(\Omega, C\ell_n)$. By Theorem 3.2, we get that $DW_0^{1,p(x)}(\Omega, C\ell_n)$ is a reflexive Banach space. In the following, to get surjectivity of the operator QA, we need to verify the conditions of Proposition 4.1 respectively.

(1) The operator QA is continuous. Suppose that $Du_k, Dv \in DW_0^{1,p(x)}(\Omega, C\ell_n)$ and $\|Du_k - Dv\|_{L^{p(x)}(\Omega, C\ell_n)} \to 0$ as $k \to \infty$. From (A1), we obtain

$$\big\| |A(Du_k) - A(Dv)| \big\|_{L^{p'(x)}(\Omega)} \le C_1 \Big\| \big(|Du_k| + |Dv|\big)^{p(x)-2} |Du_k - Dv| \Big\|_{L^{p'(x)}(\Omega)}.$$

Divide Ω into two parts: $\Omega_1 = \{x \in \Omega : p(x) \le 2\}$ and $\Omega_2 = \{x \in \Omega : p(x) > 2\}$. Then $\Omega = \Omega_1 \cup \Omega_2$. On Ω_1 we have

$$\int_{\Omega_1} \left(\big(|Du_k| + |Dv|\big)^{p(x)-2} |Du_k - Dv|\right)^{p'(x)} dx \le \int_\Omega \left(|Du_k - Dv|^{p(x)-1}\right)^{p'(x)} dx.$$

On Ω_2, according to Hölder inequality we have

$$\int_{\Omega_2} \left(\big(|Du_k| + |Dv|\big)^{p(x)-2} |Du_k - Dv|\right)^{p'(x)} dx \le 2 \Big\| \big(|Du_k| + |Dv|\big)^{p'(x)(p(x)-2)} \Big\|_{L^{\frac{p(x)-1}{p(x)-2}}(\Omega)} \Big\| \big|Du_k - Dv\big|^{p'(x)} \Big\|_{L^{\frac{p(x)}{p'(x)}}(\Omega)}.$$

From Remark 2.1 and Lemma 2.2 we deduce that

$$\int_\Omega \left(\big(|Du_k| + |Dv|\big)^{p(x)-2} |Du_k - Dv|\right)^{p'(x)} dx \to 0$$

as $k \to \infty$. By means of Remark 2.1 and Lemma 2.2, we obtain

$$\big\|A(Du_k) - A(Dv)\big\|_{L^{p'(x)}(\Omega, C\ell_n)} \to 0$$

as $k \to \infty$. Finally, the continuity follows immediately from the boundedness of the operator Q.

(2) The operator QA is strictly monotone. In view of Theorem 3.1, we have

$$QA(Du) = A(Du) - PA(Du)$$

for each $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$. Thus, for any $u, v \in W_0^{1,p(x)}(\Omega, C\ell_n)$, (3.2) and Lemma 2.15 give

$$\big(QA(Du), Dv\big)_{Sc} = \big(A(Du), Dv\big)_{Sc} - \big(PA(Du), Dv\big)_{Sc} = \big(A(Du), Dv\big)_{Sc}. \tag{4.3}$$

Then the condition (A2) yields

$$\begin{aligned}\big(QA(Du) - QA(Dv), Du - Dv\big)_{Sc} &= \big(A(Du) - A(Dv), Du - Dv\big)_{Sc} \\ &= \int_\Omega \Big[\overline{\big(A(Du) - A(Dv)\big)}(Du - Dv)\Big]_0 dx \\ &\ge C_2 \int_\Omega \big(|Du| + |Dv|\big)^{p(x)-2} |Du - Dv|^2 dx > 0,\end{aligned}$$

as $Du \ne Dv$.

(3) The operator QA is coercive. By means of (4.3) and (A2) we have

$$\begin{aligned}\frac{\big(QA(Du), Du\big)_{Sc}}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} - \frac{\big(QA(0), Du\big)_{Sc}}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} &= \frac{\big(A(Du) - A(0), Du\big)_{Sc}}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} \\ &= \frac{\int_\Omega \Big[\overline{\big(A(Du) - A(0)\big)}(Du - 0)\Big]_0 dx}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} \\ &\ge C_2 \frac{\int_\Omega |Du|^{p(x)} dx}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}}.\end{aligned}$$

Since

$$\frac{\int_\Omega |Du|^{p(x)}dx}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} = \int_\Omega \left(\frac{|Du|}{2^{-1}\big\||Du|\big\|_{L^{p(x)}(\Omega)}}\right)^{p(x)} \cdot \frac{\left(2^{-1}\big\||Du|\big\|_{L^{p(x)}(\Omega)}\right)^{p(x)}}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} dx.$$

When $\big\||Du|\big\|_{L^{p(x)}(\Omega)} \geq 1$, we have

$$\frac{\int_\Omega |Du|^{p(x)}dx}{\big\||Du|\big\|_{L^{p(x)}(\Omega)}} \geq 2^{-p_+}\big\||Du|\big\|^{p_- -1}_{L^{p(x)}(\Omega)}.$$

Hence we get from Remark 2.1

$$\frac{\big(QA(Du), Du\big)_{Sc}}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} - \frac{\big(QA(0), Du\big)_{Sc}}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} \to \infty$$

as $\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \to \infty$. By Lemma 2.3, we obtain

$$\frac{\big(QA(0), Du\big)_{Sc}}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} = \frac{\big(A(0), Du\big)_{Sc}}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} = \frac{\int_\Omega \big[\overline{A(0)}Du\big]_0 dx}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} \leq C\|A(0)\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Therefore,

$$\frac{\big(QA(Du), Du\big)_{Sc}}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} \to \infty$$

as $\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \to \infty$.

According to Proposition 4.1, let $X = DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ and $G = QA$. Then the operator QA is surjective. Consequently, there exists $u \in W_0^{1,p(x)}(\Omega, Cl_n)$ such that $QA(Du) = 0$. Furthermore, Theorem 3.1 gives

$$\int_\Omega \overline{A(Du)}D\varphi dx = \int_\Omega \overline{\big(QA(Du) + PA(Du)\big)}D\varphi dx = \int_\Omega \overline{\widetilde{D}PA(Du)}\varphi dx = 0,$$

for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Therefore, u is a weak solution of the A-Dirac equation (4.1).

Finally, if u_1, u_2 are solutions to (4.1), then $\big(A(Du_i), D\varphi\big)_{Sc} = 0$ $(i = 1, 2)$ for all $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Set $\varphi = u_1 - u_2$, then the condition (A2) yields

$$\begin{aligned}0 &= \big(A(Du_1) - A(Du_2), Du_1 - Du_2\big)_{Sc} \\ &= \int_\Omega \Big[\overline{\big(A(Du_1) - A(Du_2)\big)}(Du_1 - Du_2)\Big]_0 dx \\ &\geq C_2 \int_\Omega \big(|Du_1| + |Du_2|\big)^{p(x)-2}|Du_1 - Du_2|^2 dx \geq 0.\end{aligned}$$

Thus, $Du_1 = Du_2$, hence $u_1 - u_2 \in \ker D$. The proof is now complete. □

4.2. Stokes equations in variable exponent spaces

In the study of the stationary Navier–Stokes equations, the corresponding Stokes equations plays a crucial role. It can be said that any open question about Navier–Stokes equations, such as global existence of strong solutions, uniqueness and regularity of weak solutions, and asymptotic behaviour, is closely related with the qualitative and quantitative properties of the solutions of Stokes equations, see [41].

The Stokes system consists in finding a pair of function (u, ω) solution of the following equations:

$$-\Delta u + \frac{1}{\eta}\nabla\omega = \frac{\rho}{\eta}f, \quad \text{in } \Omega, \tag{4.4}$$

$$\operatorname{div} u = f_0, \quad \text{in } \Omega, \tag{4.5}$$

$$u = v_0, \quad \text{on } \partial\Omega. \tag{4.6}$$

With $\int_\Omega f_0 dx = \int_{\partial\Omega} n\cdot v_0 dx$ the necessary compatibility condition for the solvability is given. Here, u is the velocity, ω the hydrostatic pressure, ρ the density, η the viscosity, f the vector of the external forces and the scalar function f_0 a measure of the compressibility of fluid. The boundary condition (4.6) describes the adhesion on the boundary for $v_0 = 0$. This system describes the stationary flow of a homogeneous viscous incompressible fluid for small Reynold numbers. For more details we refer to [26,42,28].

In [34], L. Diening, D. Lengeler and M. Ružička proved the existence and uniqueness of a weak solution in $\left(W^{1,p(x)}(\Omega)\right)^n \times L^{p(x)}(\Omega)$ of the Stokes problem (4.3)–(4.5), provided that the right-hand side $f \in \left(W^{-1,p(x)}(\Omega)\right)^n$, $f_0 \in L^{p(x)}(\Omega)$ and $v_0 \in \mathrm{tr}((W^{1,p(x)}(\Omega))^n)$. The result is based on generalizations of the classical theories of Calderón–Zygmund and Agmon–Douglis–Nirenberg to variable exponent spaces. In this subsection, with the help of our Hodge-type decomposition, we obtain an analogous result in the context of Clifford analysis. Furthermore, we also give a representation of the solution.

We suppose that $f = \sum_{i=1}^n f_i e_i$ and $u = \sum_{i=1}^n u_i e_i$. Then the Stokes system may be written in the following hypercomplex formulation (see [26,20]):

$$-\Delta u + \frac{1}{\eta} D\omega = \frac{\rho}{\eta} f, \quad \text{in } \Omega, \tag{4.7}$$

$$[Du]_0 = f_0, \quad \text{in } \Omega, \tag{4.8}$$

$$u = v_0, \quad \text{on } \partial\Omega. \tag{4.9}$$

For the sake of simplicity, we consider the following Stokes system:

$$\widetilde{D}Du + \widetilde{D}\omega = f, \quad \text{in } \Omega, \tag{4.10}$$

$$[Du]_0 = 0, \quad \text{in } \Omega, \tag{4.11}$$

$$u = 0, \quad \text{on } \partial\Omega. \tag{4.12}$$

In the following, we require two basic results. The first one is an abstract algebraic result which is known as the "Peetre–Tartar Lemma".

Proposition 4.3 (*[43], [44, Lemma 11.1]*). *Let E_1 be a Banach space, E_2, E_3 be normed spaces, A an operator in $\mathscr{L}(E_1, E_2)$ and B a compact operator in $\mathscr{L}(E_1, E_3)$. If $\|u\|_{E_1}$ is equivalent to $\|Au\|_{E_2} + \|Bu\|_{E_3}$ for each $u \in E_1$, then the range space $R(A)$ of the operator A is a closed subspace of E_2.*

The second result we will use is a generalization of Nečas' theorem which implies an important equivalence of norms. We first give the following definition.

Definition 4.4. The operator $\widetilde{\nabla} : L^{p(x)}(\Omega) \to (W^{-1,p(x)}(\Omega))^n$ is defined by

$$\langle \widetilde{\nabla} f, \varphi \rangle = -\langle f, \operatorname{div} \varphi \rangle := -\int_\Omega f \operatorname{div} \varphi \, dx$$

for all $f \in L^{p(x)}(\Omega)$ and $\varphi \in (C_0^\infty(\Omega))^n$.

Proposition 4.5 (*[6, Theorem 14.3.18]*). *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Then there exists a positive constant $C = C(n, p, \Omega)$ such that*

$$\|f\|_{L^{p(x)}(\Omega)} \le C(\|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} + \|f\|_{W^{-1,p(x)}(\Omega)}) \tag{4.13}$$

holds for every $f \in L^{p(x)}(\Omega)$.

Remark 4.6. From Theorem 14.3.18 in [6] we know that Proposition 4.5 holds for the domain Ω satisfying the emanating chain condition. From [6, p. 239] it follows that a bounded domain satisfies the emanating chain condition if and only if it is a John domain. From [6, p. 237] it follows that any bounded Lipschitz domain is a John domain.

As a first application of these two results, the following corollary derives an important property of the above-mentioned gradient operator.

Corollary 4.7. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Then the range space of the operator $\widetilde{\nabla} \in \mathscr{L}(L^{p(x)}(\Omega), (W^{-1,p(x)}(\Omega))^n)$ is a closed subspace of $(W^{-1,p(x)}(\Omega))^n$.*

Proof. To apply Proposition 4.3, we suppose $E_1 = L^{p(x)}(\Omega)$, $E_2 = (W^{-1,p(x)}(\Omega))^n$, $E_3 = W^{-1,p(x)}(\Omega)$ $A = \widetilde{\nabla}$, $B = I$ (the identity operator). Since the domain is bounded, the canonical embedding B of E_1 into E_3 is compact (see [34]). On the other hand, by virtue of Hölder's inequality, we have for every $f \in L^{p(x)}(\Omega)$

$$\begin{aligned} \|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} &= \sup_{\|g\|_{W_0^{1,p'(x)}(\Omega)} \le 1} |\langle \widetilde{\nabla} f, g \rangle| \\ &= \sup_{\|g\|_{W_0^{1,p'(x)}(\Omega)} \le 1} \left| \int_\Omega f \operatorname{div} g dx \right| \le C\|f\|_{L^{p(x)}(\Omega)}. \end{aligned}$$

Therefore

$$\|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} + \|f\|_{W^{-1,p(x)}(\Omega)} \le C\|f\|_{L^{p(x)}(\Omega)} \quad \text{for all } f \in L^{p(x)}(\Omega). \tag{4.14}$$

Then (4.13) and (4.14) yield the equivalence of the norms of both sides above. Thus, the desired conclusion follows from Proposition 4.3. □

Now we begin our investigations with the following lemma which is a generalization of De Rham's theorem to the variable exponent spaces.

Lemma 4.8. *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Let $f \in (W^{-1,p(x)}(\Omega))^n$ satisfy:*

$$\langle f, \varphi\rangle := \int_\Omega f \cdot \varphi dx = 0,$$

for any $\varphi \in \mathscr{W}(\Omega) := \left\{v \in (W_0^{1,p'(x)}(\Omega))^n : \operatorname{div} v = 0\right\}$. Then there exists $q \in L^{p(x)}(\Omega)$ such that $f = \widetilde{\nabla} q$.

Proof. Since $\langle -\widetilde{\nabla} u, g\rangle = \langle u, \operatorname{div} g\rangle$ for every $u \in L^{p(x)}(\Omega)$ and $g \in (W_0^{1,p'(x)}(\Omega))^n$, it follows that the operator $-\widetilde{\nabla} \in \mathscr{L}(L^{p(x)}(\Omega), (W^{-1,p(x)}(\Omega))^n)$ is the adjoint of the operator

$$\operatorname{div} \in \mathscr{L}((W_0^{1,p'(x)}(\Omega))^n, L^{p'(x)}(\Omega)).$$

But according to Corollary 4.7, the range space $R(\widetilde{\nabla})$ is a closed subspace of $(W^{-1,p(x)}(\Omega))^n$. Then the closed range theorem of Banach (see [45]) implies that

$$R(\widetilde{\nabla}) = (\ker(\operatorname{div}))^\perp := \left\{y \in (W^{-1,p(x)}(\Omega))^n : \ \langle y, v\rangle = 0, \forall\, v \in \mathscr{W}(\Omega)\right\}.$$

This implies the desired conclusion. □

Remark 4.9. Note that Lemma 4.8 remains valid if the set $\mathscr{W}(\Omega)$ is replaced by

$$\tilde{\mathscr{W}}(\Omega) = \{v \in (C_0^\infty(\Omega))^n : \operatorname{div} v = 0\}.$$

The proof is analogous to that of Theorem 2.8 in [46] and thus we omit it.

Definition 4.10. We call $(u, \omega) \in W_0^{1,p(x)}(\Omega, C\ell_n) \times L^{p(x)}(\Omega)$ a solution of (4.10)–(4.12) provided that it satisfies the system (4.10)–(4.12) for every $f \in W^{-1,p(x)}(\Omega, C\ell_n)$.

Lemma 4.11. *Assume that $f \in W^{-1,p(x)}(\Omega, C\ell_n)$. Then for every solution $(u, \omega) \in W_0^{1,p(x)}(\Omega, C\ell_n) \times L^{p(x)}(\Omega)$ of system (4.10)–(4.12) we have the representation*

$$TQ\widetilde{T}f = u + TQ\omega.$$

Proof. Let $\varphi_n \in W_0^{1,p(x)}(\Omega, C\ell_n)$ with $\varphi_n \to \varphi$ in $L^{p(x)}(\Omega, C\ell_n)$. By Lemma 2.12, we have

$$TQT(D\varphi_n) = TQ\varphi_n.$$

Since $W_0^{1,p(x)}(\Omega, C\ell_n)$ is dense in $L^{p(x)}(\Omega, C\ell_n)$, it follows that $TQ\widetilde{T}\widetilde{D}\varphi = TQ\varphi$. Thus, for $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$ and $\omega \in L^{p(x)}(\Omega)$ we obtain

$$TQ\widetilde{T}f = TQ\widetilde{T}(\widetilde{D}Du + \widetilde{D}\omega) = u + TQ\omega,$$

as desired. □

We are now in a position to prove our result as follows.

Theorem 4.12. *Suppose $f \in W^{-1,p(x)}(\Omega, C\ell_n)$. Then the Stokes system (4.10)–(4.12) has a unique solution $(u, \omega) \in W_0^{1,p(x)}(\Omega, C\ell_n) \times L^{p(x)}(\Omega)$ in the form*

$$u + TQ\omega = TQ\widetilde{T}f,$$

with respect to the estimate

$$\|u\|_{W_0^{1,p(x)}(\Omega, C\ell_n)} + \|Q\omega\|_{L^{p(x)}(\Omega)} \le C\|f\|_{W^{-1,p(x)}(\Omega, C\ell_n)}.$$

Here, the hydrostatic pressure ω is unique up to a constant.

Proof. Lemma 4.11 implies that our system (4.10)–(4.12) is equivalent to the system

$$u + TQ\omega = TQ\widetilde{T}f, \tag{4.15}$$

$$[Q\omega]_0 = [Q\widetilde{T}f]_0. \tag{4.16}$$

Observe that the equality (4.15) is equivalent to the following equality

$$Du + Q\omega = Q\widetilde{T}f. \tag{4.17}$$

Indeed, due to Lemma 2.12 we know that when D is applied to (4.15), this leads to (4.17). When T is applied to (4.17), this reduces to (4.15).

Therefore we need to show that for each $f \in W^{-1,p(x)}(\Omega, C\ell_n^1)$, the function QTf can be decomposed into two functions Du and $Q\omega$. Notice that we are not decomposing the whole space $L^{p(x)}(\Omega, C\ell_n)$ but only the subspace $L^{p(x)}(\Omega, C\ell_n^1) \cap \mathrm{im}Q$. Therefore, suppose $Du + Q\omega = 0$ for $u \in W_0^{1,p(x)}(\Omega, C\ell_n^1) \cap \ker \operatorname{div}$ and $\omega \in L^{p(x)}(\Omega)$. Then (4.11) gives $[Q\omega]_0 = 0$. Notice that the operator Q maps the space $L^{p(x)}(\Omega)$ into itself. Thus, $Q\omega = 0$. Hence, $Du = Q\omega = 0$. This means that $Du + Q\omega$ is a direct sum, which is a subset of $\mathrm{im}Q$.

Next we have to ask about the existence of a functional $\mathcal{F} \in (L^{p(x)}(\Omega, C\ell_n^1) \cap \mathrm{im}Q)^*$ with $\mathcal{F}(Du) = 0$ and $\mathcal{F}(Q\omega) = 0$ but $\mathcal{F}(Q\widetilde{T}f) \neq 0$. This amounts to asking if there exists $g \in W^{-1,p'(x)}(\Omega, C\ell_n^1)$, such that for all $u \in W_0^{1,p(x)}(\Omega, C\ell_n^1) \cap \ker \operatorname{div}$ and $\omega \in L^{p(x)}(\Omega)$

$$(Du, Q\widetilde{T}g)_{Sc} = 0, \tag{4.18}$$

$$(Q\omega, Q\widetilde{T}g)_{Sc} = 0, \tag{4.19}$$

but $(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} \neq 0$. Here, we have applied Theorem 3.3 and Lemma 2.13.

Thus, let us consider the system (4.18) and (4.19) with $g \in W^{-1,p'(x)}(\Omega, C\ell_n^1)$. Observe that, with the help of Lemmas 2.14 and 2.15, (4.19) yields

$$(Du, Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}\widetilde{T}g - \widetilde{D}P\widetilde{T}g)_{Sc} = (u, g)_{Sc} = 0,$$

which implies $g = \widetilde{\nabla}h = \widetilde{D}h$ with $h \in L^{p'(x)}(\Omega)$ because of Corollary 4.7. Notice that, from Lemma 2.14, (4.19) gives

$$(Q\omega, Q\widetilde{T}g)_{Sc} = (Q\omega, Q\widetilde{T}\widetilde{D}h)_{Sc} = (Q\omega, Qh)_{Sc} = 0$$

holds for each $\omega \in L^{p(x)}(\Omega)$. Hence, $Q\omega = |Qh|^{p'(x)-2}Qh$ gives $Qh = 0$. Then we obtain

$$g = \widetilde{D}h = \widetilde{D}Qh + \widetilde{D}Ph = 0.$$

Furthermore, we get

$$(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} = 0, \quad \forall f \in W^{-1,p(x)}(\Omega, C\ell_n^1).$$

Finally, since (4.17) yields

$$\|Du\|_{L^{p(x)}(\Omega,C\ell_n)} + \|Q\omega\|_{L^{p(x)}(\Omega)} \geq \|Q\widetilde{T}f\|_{L^{p(x)}(\Omega,C\ell_n)}.$$

By the norm equivalence theorem, we obtain

$$\|Du\|_{L^{p(x)}(\Omega,C\ell_n)} + \|Q\omega\|_{L^{p(x)}(\Omega)} \leq C\|Q\widetilde{T}f\|_{L^{p(x)}(\Omega,C\ell_n)}.$$

By Remark 2.1, Lemma 2.13 and the boundedness of the operator Q, we get

$$\|u\|_{W_0^{1,p(x)}(\Omega,C\ell_n)} + \|Q\omega\|_{L^{p(x)}(\Omega)} \leq C\|f\|_{W^{-1,p(x)}(\Omega,C\ell_n)}. \tag{4.20}$$

From (4.20) the uniqueness of the solution follows. Notice that $Q\omega = 0$ implies $\omega \in \ker\widetilde{D}$. Therefore, ω is unique up to a constant. Now we complete the proof of Lemma 4.11. □

The system (4.10)–(4.12) is just an example for the general way of treating such kinds of problems with the help of our Hodge-type decomposition. We expect that this study can be extended to investigate Navier–Stokes problems.

Acknowledgements

Binlin Zhang would like to express his sincere thanks to Professor U. Kähler for his willingness to give many helps. V. Rădulescu acknowledges the support through Grant CNCS PCCA-23/2014. Y. Fu was supported by National Natural Science Foundation of China (No. 11371110). B. Zhang was supported by Natural Science Foundation of Heilongjiang Province of China (No. A201306), Research Foundation of Heilongjiang Educational Committee (No. 12541667) and Doctoral Research Foundation of Heilongjiang Institute of Technology (No. 2013BJ15).

References

[1] W.V.D. Hodge, A Dirichlet problem for harmonic functions with applications to analytic varieties, Proc. Lond. Math. Soc. 36 (1933) 257–303.
[2] W.V.D. Hodge, The Theory and Applications of Harmonic Integrals, Cambridge University Press, Cambridge, 1941.
[3] G. de Rham, Sur l'analysis situs des variétés à n dimensions, J. Math. Pures Appl. 10 (1931) 115–200.
[4] G. de Rham, Variétés Différentielles, Hermann, Paris, 1955.
[5] H. Weyl, The method of orthogonal projection in potential theory, Duke Math. J. 7 (1940) 411–444.
[6] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, Lebesgue and Sobolev Spaces with Variable Exponents, Springer-Verlag, Berlin, 2011.
[7] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, Nonlinear Anal. 121 (2015) 336–369.
[8] V. Rădulescu, D. Repovš, Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics, Taylor & Francis, Boca Raton, FL, 2015.
[9] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{m,p(x)}$, Czechoslovak Math. J. 41 (1991) 592–618.
[10] X.L. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, J. Math. Anal. Appl. 263 (2001) 424–446.
[11] Y.Q. Fu, Weak solution for obstacle problem with variable growth, Nonlinear Anal. 59 (2004) 371–383.
[12] M. Ružička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.
[13] K.R. Rajagopal, M. Ružička, On the modeling of electrorheological materials, Mech. Res. Comm. 23 (1996) 401–407.
[14] J. Gilbert, M.A.M. Murray, Clifford Algebra and Dirac Oprators in Harmonic Analysis, Oxford University Press, Oxford, 1993.
[15] R. Ablamowicz, Clifford Algebras and their Applications in Mathematical Physics, Volume 1: Algebra and Physics, Birkhäuser, Boston, 2000.
[16] J. Ryan, W. Sprößig, Clifford Algebras and their Applications in Mathematical Physics, Volume 2: Clifford Analysis, Birkhäuser, Boston, 2000.
[17] L. Diening, P. Kaplicky, Campanato estimates for the generalized Stokes system, Ann. Mat. Pura Appl. 193 (2014) 1779–1794.
[18] C.A. Nolder, A-harmonic equations and the Dirac operator, J. Inequal. Appl., 2010, Article ID 124018.
[19] C.A. Nolder, Nonlinear A-Dirac equations, Adv. Appl. Clifford Algebr. 21 (2011) 429–440.
[20] K. Gürlebeck, W. Sprößig, Quaternionic and Clifford Calculus for Physicists and Engineers, John Wiley & Sons, New York, 1997.
[21] L. Diening, P. Kaplicky, L^q theory for a generalized Stokes system, Manuscripta Math. 141 (2013) 333–361.
[22] L. Diening, P. Kaplicky, S. Schwarzacher, BMO estimates for the p-Laplacian, Nonlinear Anal. 75 (2012) 637–650.
[23] Y.Q. Fu, B.L. Zhang, Clifford valued weighted variable exponent spaces with an application to obstacle problems, Adv. Appl. Clifford Algebr. 23 (2013) 363–376.
[24] Y.Q. Fu, B.L. Zhang, Weak solutions for elliptic systems with variable growth in Clifford analysis, Czechoslovak Math. J. 63 (138) (2013) 643–670.
[25] B.L. Zhang, Y.Q. Fu, Weak solutions for A-Dirac equations with variable growth in Clifford analysis, Electron. J. Differential Equations 227 (2012) 1–10.
[26] K. Gürlebeck, W. Sprößig, Quaternionic Analysis and Elliptic Boundary Value Problems, Birkhäuser, Boston, 1990.
[27] J.A. Dubinski, On a nonlinear analytic problem, Dokl. Math. 48 (1998) 370–375.
[28] U. Kähler, On a direct decomposition in the space $L_p(\Omega)$, Z. Anal. Anwend. 4 (1999) 839–848.
[29] J. Dubinskii, M. Reissig, Variational problems in Clifford analysis, Math. Methods Appl. Sci. 25 (2002) 1161–1176.
[30] Ph.G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.
[31] C. Doran, A. Lasenby, Geometric Algebra for Physicists, Cambridge University Press, Cambridge, 2003.
[32] K. Gürlebeck, K. Habetha, W. Sprößig, Holomorphic Functions in the Plane and n-dimensional Space, Birkhäuser, Boston, 2008.
[33] Y.Q. Fu, The principle of concentration compactness in spaces $L^{p(x)}(\Omega)$ and its application, Nonlinear Anal. 71 (2009) 1876–1892.
[34] L. Diening, D. Lengeler, M. Ružička, The Stokes and Poisson problem in variable exponent spaces, Complex Var. Elliptic Equ. 56 (2011) 789–811.
[35] W. Heisenberg, Doubts and hopes in quantum-electrodynamics, Physica 19 (1953) 897–908.
[36] H. Weyl, A remark on the coupling of gravitation and electron, Phys. Rev. 77 (1950) 699–701.
[37] B. Thaller, The Dirac Equation, Springer, Berlin, 2010.
[38] C.A. Nolder, J. Ryan, p-Dirac operators, Adv. Appl. Clifford Algebr. 19 (2009) 391–402.
[39] P. Harjulehto, P. Hästö, U.V. Lê, M. Nuortio, Overview of differential equations with non-standard growth, Nonlinear Anal. 72 (2010) 4551–4574.
[40] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.
[41] P. Cerejeiras, U. Kähler, Elliptic boundary value problems of fluid dynamics over unbounded domains, Math. Methods Appl. Sci. 23 (2000) 81–101.
[42] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Springer, New York, 2011.
[43] J. Peetre, Espaces d'interpolation et théorème de Sobolev, Ann. Inst. Fourier 6 (1966) 279–317.
[44] L. Tartar, An Introduction to Sobolev Spaces and Interpolation Spaces, Springer-Verlag, Berlin, 2007.
[45] K. Yosida, Functional Analysis, Springer-Verlag, Berlin, 1980.
[46] C. Amrouche, V. Girault, Decomposition of vector space and application to the Stokes problem in arbitrary dimension, Czechoslovak Math. J. 44 (1994) 109–140.