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Partial differential equations

Bifurcation near infinity for the Neumann problem with concave–convex nonlinearities

Bifurcation à l'infini pour le problème de Neumann avec terme concave–convexe

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ABSTRACT

In this Note, we study a class of Neumann parametric elliptic equations driven by a nonhomogeneous differential operator and with a reaction that exhibits competing terms (concave–convex nonlinearities). Using the Ambrosetti–Rabinowitz condition and related topological and variational arguments, we prove a bifurcation result for large values of the parameter.

RÉSUMÉ

Dans cette Note, nous étudions le problème elliptique paramétrique de Neumann pour un opérateur différentiel non homogène et avec une réaction qui présente des termes du type concave–convexe. En utilisant la condition d'Ambrosetti–Rabinowitz en combinaison avec des outils topologiques et variationnels, nous prouvons un théorème de bifurcation pour de grandes valeurs du paramètre réel.

Version française abrégée

Soit $p > 1$; supposons que $\Omega \subset \mathbb{R}^N$ soit un domaine borné et régulier. Soit $\eta \in C^1(0,\infty)$ tel que

$$0 < \hat{c} \le \frac{t\eta'(t)}{\eta(t)} \le \hat{c_0} \quad \text{et} \quad \hat{c_1} t^{p-1} \le \eta(t) \le \hat{c_2}\left(1 + t^{p-1}\right) \quad \text{pour tout } t > 0 \text{ avec } \hat{c_1}, \hat{c_2} > 0.$$

Supposons que $a : \mathbb{R}^N \to \mathbb{R}^N$ soit une fonction continue et strictement monotone qui satisfait les hypothèses suivantes : $H(a)$: $a(x) = a_0(\|x\|)x$ pour tout $x \in \mathbb{R}^N$, avec $a_0 > 0$ sur $(0,\infty)$ et

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(i) $a_0 \in C^1(0,\infty)$, $t \longmapsto a_0(t)t$ est strictement croissante sur $(0,\infty)$, $a_0(t)t \to 0$ si $t \to 0^+$ et

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(x)| \le c_1 \frac{\eta(|x|)}{|x|}$ avec $c_1 > 0$, pour tout $x \in \mathbb{R}^N \backslash \{0\}$;

(iii) $\frac{\eta(|x|)}{|x|}|\xi|^2 \le (\nabla a(x)\xi, \xi)_{\mathbb{R}^N}$ pour tout $x \in \mathbb{R}^N \backslash \{0\}$ et $\xi \in \mathbb{R}^N$;

(iv) si $G_0(t) = \int_0^t a_0(s)s\mathrm{d}s$ pour $t \ge 0$, alors $pG_0(t) - a_0(t)t^2 \ge -\hat{\xi}$ pour tout $t \ge 0$, avec $\hat{\xi} > 0$;

(v) il existe $\tau \in (1,p)$ tel que l'application $t \longmapsto G_0(t^{1/\tau})$ est convexe sur $(0,\infty)$, $\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0$ et

$$a_0(t)t^2 - \tau G_0(t) \ge \tilde{c}t^p \quad \text{avec } \tilde{c} > 0, \text{ pour tout } t > 0.$$

Supposons que f_0 satisfait les conditions suivantes :

H : $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ est une fonction de Carathéodory telle que $f_0(x,0) = 0$ p.p. dans Ω et

(i) $|f_0(x,u)| \le a(x)(1 + u^{r-1})$ p.p. $x \in \Omega$ et pour tout $u \ge 0$, avec $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) si $F_0(x,u) = \int_0^u f_0(x,s)\mathrm{d}s$, alors il existe $c_1 > 0$ et $\eta > p$ tels que

$$c_1 u^\eta \le \eta F_0(x,u) \le u f_0(x,u) \quad \text{p.p. } x \in \Omega, \text{ et pour tout } u \ge 0;$$

(iii) il existe $q \in (1,\tau)$ (voir $H(a)$(v)) tel que

$$0 < c_2 \le \liminf_{u\to 0^+} \frac{f_0(x,u)}{u^{q-1}} \le \limsup_{u\to 0^+} \frac{f_0(x,u)}{u^{q-1}} \le c_3 < \infty \quad \text{uniformément p.p. } x \in \Omega;$$

(iv) pour tout $\rho > 0$, il existe $\xi_\rho > 0$ tel que p.p. $x \in \Omega$, l'application $u \longmapsto f_0(x,u) + \xi_\rho u^{p-1}$ est croissante sur $[0,\rho]$.

Dans cette Note, nous étudions le problème de Neumann non linéaire suivant :

$$\begin{cases} -\mathrm{div}\, a(Du(x)) = f_0(x,u(x)) - \lambda u(x)^{p-1} & \text{dans } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{sur } \partial\Omega, \ u > 0. \end{cases} \qquad (P)$$

Soit $C_+ = \{u \in C^1(\overline{\Omega}) : u(x) \ge 0 \text{ pour tout } x \in \overline{\Omega}\}$ et $\mathrm{int}\, C_+ = \{u \in C_+ : u(x) > 0 \text{ pour tout } x \in \overline{\Omega}\}$.

Le résultat principal de cette Note est le suivant.

Théorème 0.1. *Supposons que les hypothèses $H(a)$ et H soient satisfaites. Alors il existe $\lambda_* > 0$ tel que*

(a) *pour tout $\lambda > \lambda_*$, le problème (P) admet au moins deux solutions :*

$$u_\lambda, \hat{u}_\lambda \in \mathrm{int}\, C_+, \quad u_\lambda \ne \hat{u}_\lambda;$$

(b) *si $\lambda = \lambda_*$, le problème (P) admet au moins une solution $u_* \in \mathrm{int}\, C_+$;*

(c) *pour tout $\lambda \in (0,\lambda_*)$, le problème (P) n'a pas de solutions.*

English version

Let $p > 1$ be a real number and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$. In this Note, we study the following nonlinear, nonhomogeneous parametric Neumann problem:

$$\begin{cases} -\mathrm{div}\, a(Du(x)) = f_0(x,u(x)) - \lambda u(x)^{p-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \partial\Omega, \ u > 0. \end{cases} \qquad (P)$$

Let $\eta \in C^1(0,\infty)$ and assume that

$$0 < \hat{c} \le \frac{t\eta'(t)}{\eta(t)} \le \hat{c_0} \quad \text{and} \quad \hat{c_1}t^{p-1} \le \eta(t) \le \hat{c_2}\big(1 + t^{p-1}\big) \quad \text{for all } t > 0 \text{ with } \hat{c_1}, \ \hat{c_2} > 0.$$

We assume that $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone mapping, which satisfies the following hypotheses:

$H(a)$: $a(x) = a_0(\|x\|)x$ for all $x \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \longmapsto a_0(t)t$ is strictly increasing on $(0,\infty)$, $a_0(t)t \to 0$ as $t \to 0^+$ and

$$\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(x)| \le c_1 \frac{\eta(|x|)}{|x|}$ for some $c_1 > 0$, all $x \in \mathbb{R}^N\setminus\{0\}$;

(iii) $\frac{\eta(|x|)}{|x|}|\xi|^2 \le (\nabla a(x)\xi,\xi)_{\mathbb{R}^N}$ for all $x \in \mathbb{R}^N\setminus\{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)s\,\mathrm{d}s$ for all $t \ge 0$, then $pG_0(t) - a_0(t)t^2 \ge -\hat{\xi}$ for all $t \ge 0$, some $\hat{\xi} > 0$;

(v) there exists $\tau \in (1,p)$ such that the function $t \longmapsto G_0(t^{1/\tau})$ is convex on $(0,\infty)$, $\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0$ and

$$a_0(t)t^2 - \tau G_0(t) \ge \tilde{c}t^p \quad \text{for some } \tilde{c} > 0, \text{ all } t > 0.$$

These conditions on $a(\cdot)$ are motivated by the regularity results of Lieberman [4] and the nonlinear maximum principle of Pucci and Serrin [7]. According to the above conditions, the potential function $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(x) = G_0(|x|)$ for all $x \in \mathbb{R}^N$. Then G is convex and differentiable, $\nabla G(0) = 0$, and

$$\nabla G(x) = G_0'(|x|)\frac{x}{|x|} = a_0(|x|)x = a(x) \quad \text{for all } x \in \mathbb{R}^N\setminus\{0\}.$$

So, $G(\cdot)$ is the primitive of the mapping $a(\cdot)$. Because $G(0) = 0$ and $x \longmapsto G(x)$ is convex, from the properties of convex functions, we have $G(x) \le (a(x),x)_{\mathbb{R}^N}$ for all $x \in \mathbb{R}^N$. By direct computation we deduce that hypotheses $H(a)$(i), (ii), (iii) imply the following properties of $a(\cdot)$:

(a) the mapping $x \longmapsto a(x)$ is continuous and strictly monotone, hence maximal monotone too;
(b) $|a(x)| \le c_2(1 + |x|^{p-1})$ for some $c_2 > 0$, all $x \in \mathbb{R}^N$;
(c) $(a(x),x)_{\mathbb{R}^N} \ge \frac{\hat{c}_1}{p-1}|x|^p$ for all $x \in \mathbb{R}^N$.

The following functions satisfy hypotheses $H(a)$:
(a) $a(x) = |x|^{p-2}x$ with $1 < p < \infty$. This mapping corresponds to the p-Laplacian operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right) \quad \text{for all } u \in W^{1,p}(\Omega);$$

(b) $a(x) = |x|^{p-2}x + \mu|x|^{q-2}x$ with $1 < q < p < \infty$ and $\mu > 0$. This mapping corresponds to the (p,q)-differential operator defined by

$$\Delta_p u + \mu\Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega);$$

(c) $a(x) = (1 + |x|^2)^{\frac{p-2}{2}}x$ with $1 < p < \infty$. This mapping corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}\left[\left(1 + |Du|^2\right)^{\frac{p-2}{2}}Du\right] \quad \text{for all } u \in W^{1,p}(\Omega);$$

(d) $a(x) = |x|^{p-2}x + \frac{|x|^{p-2}x}{1+|x|^p}$ with $1 < p < \infty$.

We assume that f_0 is a Carathéodory function that exhibits competing nonlinearities, namely it is $(p-1)$-superlinear near $+\infty$ and admits a concave term near zero. The superlinearity of $f_0(x,\cdot)$ is expressed via the Ambrosetti–Rabinowitz condition, see [1]. The presence of the term $-\lambda u^{p-1}$ changes the geometry of the problem and we will show that the bifurcation occurs for large values of the parameter $\lambda > 0$ (bifurcation near infinity).

We assume that f_0 satisfies the following hypotheses:

H: $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f_0(x,0) = 0$ for a.a. $x \in \Omega$ and

(i) $|f_0(x,u)| \le a(x)(1 + u^{r-1})$ for a.a. $x \in \Omega$, all $u \ge 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;

(ii) if $F_0(x,u) = \int_0^u f_0(x,s)\mathrm{d}s$, then there exist $c_1 > 0$ and $\eta > p$ such that

$$c_1 u^\eta \le \eta F_0(x,u) \le u f_0(x,u) \quad \text{for a.a. } x \in \Omega, \text{ all } u \ge 0;$$

(iii) there exists $q \in (1,\tau)$ (see hypothesis $H(a)$(v)) such that

$$0 < c_2 \le \liminf_{u\to 0^+} \frac{f_0(x,u)}{u^{q-1}} \le \limsup_{u\to 0^+} \frac{f_0(x,u)}{u^{q-1}} \le c_3 < \infty \quad \text{uniformly for a.a. } x \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $x \in \Omega$, the mapping $u \longmapsto f_0(x,u) + \xi_\rho u^{p-1}$ is nondecreasing on $[0,\rho]$.

Hypotheses H(ii), (iii) reveal the competing nonlinearities (concave–convex nonlinearities). Observe that in this case the superlinearity of $f_0(x,\cdot)$ is expressed using a global version of the unilateral Ambrosetti–Rabinowitz condition. Since we are interested in positive solutions, we can assume without any loss of generality that $f_0(x,u)=0$ for all $(x,u)\in\Omega\times(-\infty,0]$. We also point out that the model example for a nonlinearity that fulfills these assumptions is

$$f_0(x,u)=f_0(u)=u^{q-1}+u^{r-1}\quad\text{for all } x\geq 0$$

with $1<q<\tau<p<r<p^*$.

In the analysis of problem (P), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space, with positive cone

$$C_+=\left\{u\in C^1(\overline{\Omega}): u(x)\geq 0 \text{ for all } x\in\overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+=\left\{u\in C_+: u(x)>0 \text{ for all } x\in\overline{\Omega}\right\}.$$

In the Sobolev space $W^{1,p}(\Omega)$, we use the norm $\|u\|=[\|u\|_p^p+\|Du\|_p^p]^{1/p}$ for all $u\in W^{1,p}(\Omega)$. To distinguish, we use $|\cdot|$ to denote the norm of $\mathbb{R}^N$.

We point out that the first work concerning positive solutions to problems with concave and convex nonlinearities is due to Ambrosetti, Brezis and Cerami [2]. A related problem concerning the combined effects of a concave–convex nonlinearity and a parameter has been studied by Marano and Papageorgiou [5]. We also refer to Ciarlet [3] for related results and complements.

The main result in this Note is the following bifurcation property.

Theorem 0.1. *Assume that hypotheses $H(a)$ and $H(f)$ are fulfilled. Then there exists $\lambda_*>0$ such that*

(a) *for every $\lambda>\lambda_*$, problem (P) has at least two positive solutions:*

$$u_\lambda,\ \hat{u}_\lambda\in\operatorname{int} C_+,\quad u_\lambda\neq\hat{u}_\lambda;$$

(b) *for $\lambda=\lambda_*$, problem (P) has at least one positive solution $u_*\in\operatorname{int} C_+$;*
(c) *for every $\lambda\in(0,\lambda_*)$, problem (P) has no positive solution.*

Sketch of the proof. We introduce the following sets

$$\mathcal{S}_0=\left\{\lambda>0: \text{problem (P) admits a positive solution}\right\}$$
$$S_0(\lambda)=\text{the set of positive solutions to problem (P)}.$$

Step 1. We have $\mathcal{S}_0\neq\varnothing$ and for all $\lambda>0$, $S_0(\lambda)\subseteq\operatorname{int} C_+$ and for $\lambda\in\mathcal{S}_0$, we have $[\lambda,+\infty)\subseteq\mathcal{S}_0$. The main idea is to prove that if $\lambda\in\mathcal{S}_0$, then $u_\lambda\in S_0(\lambda)\subseteq\operatorname{int} C_+$ and if $\mu>\lambda$, then $\mu\in\mathcal{S}_0$, and we can find $u_\mu\in S_0(\mu)\subseteq\operatorname{int} C_+$ such that $u_\mu\leq u_\lambda$. In the following, we improve this conclusion. More exactly, we establish the following property.

Step 2. If $\lambda\in\mathcal{S}_0$, then $u_\lambda\in S(\lambda)\subseteq\operatorname{int} C_+$ and if $\mu>\lambda$, then we can find $u_\mu\in S_0(\mu)\subseteq\operatorname{int} C_+$ such that $u_\lambda-u_\mu\in\operatorname{int} C_+$. Indeed, by Step 1, we can find $u_\mu\in S_0(\mu)\subseteq\operatorname{int} C_+$ such that $u_\mu\leq u_\lambda$.

Let $m_\mu=\min_{\overline{\Omega}} u_\mu>0$ and let $\delta\in(0,\frac{m_\mu}{2})$. We set $u_\lambda^\delta=u_\lambda-\delta\in\operatorname{int} C_+$. Also, for $\rho=\|u_\lambda\|_\infty$, we let $\xi_\rho>0$ be as postulated by hypothesis $H(iv)$. Then

$$\begin{aligned}
&-\operatorname{div} a\big(Du_\lambda^\delta\big)+(\mu+\xi_\rho)\big(u_\lambda^\delta\big)^{p-1}\\
&\quad\geq-\operatorname{div} a(Du_\lambda)+(\mu+\xi_\rho)u_\lambda^{p-1}-\chi(\delta)\quad\text{with } \chi(\delta)\to 0^+ \text{ as } \delta\to 0^+\\
&\quad=f_0(x,u_\lambda)+(\mu-\lambda)u_\lambda^{p-1}+\xi_\rho u_\lambda^{p-1}-\chi(\delta)\quad\big(\text{since } u_\lambda\in S_0(\lambda)\big)\\
&\quad\geq f_0(x,u_\mu)+\xi_\rho u_\mu^{p-1}+(\mu-\lambda)m_\mu-\chi(\delta)\quad\big(\text{since } m_\mu\leq u_\mu\leq u_\lambda \text{ and use hypothesis } H(\text{iv})\big)\\
&\quad\geq f_0(x,u_\mu)+\xi_\rho u_\mu^{p-1}\quad\text{for } \delta>0 \text{ small}\\
&\quad=-\operatorname{div} a(Du_\mu)+(\mu+\xi_\rho)u_\mu^{p-1}\quad\big(\text{since } u_\mu\in S_0(\mu)\big)\\
&\quad\Rightarrow\quad u_\mu\leq u_\lambda^\delta,\quad\text{for all } \delta>0 \text{ small}\Rightarrow u_\lambda-u_\mu\in\operatorname{int} C_+.
\end{aligned}$$

Let $\lambda_*=\inf\mathcal{S}_0$.

Step 3. We have $\lambda_*>0$. For this purpose, consider a sequence $\{\lambda_n\}_{n\geq 1}\subseteq\mathcal{S}_0$ such that $\lambda_n\downarrow\lambda_*$. We can find a corresponding sequence $\{u_n\}_{n\geq 1}$ such that $u_n\in S_0(\lambda_n)\subseteq\operatorname{int} C_+$ for all $n\geq 1$. We claim that $\{u_n\}_{n\geq 1}$ can be chosen to be increasing. To see this, note that since $\lambda_2<\lambda_1$, the function $u_{\lambda_2}\in S_0(\lambda_2)\subseteq\operatorname{int} C_+$ satisfies for all $h\in W^{1,p}(\Omega)$ with $h\geq 0$:

$$\langle A(u_2), h\rangle + \lambda_1 \int_\Omega u_2^{p-1} h \, dx \geq \langle A(u_2), h\rangle + \lambda_2 \int_\Omega u_2^{p-1} h \, dx. \tag{1}$$

Considering problem (P) for $\lambda = \lambda_1$ and truncating $f_0(x, \cdot)$ at $\{0, u_2(x)\}$, we obtain $u_1 \in [0, u_2] \cap S_0(\lambda_1)$. Therefore

$$\begin{aligned}
&\langle A(u_1), h\rangle + \lambda_2 \int_\Omega u_1^{p-1} h \, dx \\
&= \int_\Omega f_0(x, u_1) h \, dx + (\lambda_2 - \lambda_1) \int_\Omega u_1^{p-1} h \, dx \quad \big(\text{since } u_1 \in S_0(\lambda_1)\big) \\
&\leq \int_\Omega f_0(x, u_1) h \, dx \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } h \geq 0 \quad (\text{recall } \lambda_1 > \lambda_2).
\end{aligned} \tag{2}$$

Also, we have for all $h \in W^{1,p}(\Omega)$ with $h \geq 0$:

$$\langle A(u_3), h\rangle + \lambda_2 \int_\Omega u_3^{p-1} h \, dx \geq \langle A(u_3), h\rangle + \lambda_3 \int_\Omega u_3^{p-1} h \, dx. \tag{3}$$

Truncating $f_0(x, \cdot)$ at $\{u_1(x), u_3(x)\}$, we produce $u_2 \in [u_1, u_3] \cap S_0(\lambda_2)$. Continuing this way, we see that we can choose $\{u_n\}_{n\geq 1}$ to be increasing. We have:

$$\langle A(u_n), h\rangle + \lambda_n \int_\lambda u_n^{p-1} h \, dx = \int_\Omega f_0(x, u_n) h \, dx \quad \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \geq 1.$$

Choose $h \equiv 1 \in W^{1,p}(\Omega)$. Then for all $n \geq 1$ we have:

$$\begin{aligned}
&\lambda_n \int_\Omega u_n^{p-1} dx = \int_\Omega f_0(x, u_n) \, dx \\
&\Rightarrow \quad \lambda_n \|u_n\|_{p-1}^{p-1} \geq c \|u_n\|_{p-1}^{\eta-1} \quad \text{for some } c > 0 \quad (\text{see hypothesis } H(\text{ii}) \text{ and recall } p < \eta) \\
&\Rightarrow \quad \lambda_n \geq c \, \|u_1\|_{p-1}^{\eta-p} \quad (\text{recall } u_n \geq u_1, \text{ for all } n \geq 1) \\
&\Rightarrow \quad \lambda_* \geq c \, \|u_1\|_{p-1}^{\eta-p} > 0.
\end{aligned}$$

Step 4. For all $\lambda > \lambda_*$, problem (P) admits at least two positive solutions.
Note that hypotheses H(i), (iii) imply that we can find $A, B > 0$ such that

$$f_0(x, u) \geq A u^{q-1} - B u^{r-1} \quad \text{for a.a. } x \in \Omega, \text{ all } u \geq 0. \tag{4}$$

This leads to the following auxiliary Neumann problem

$$-\operatorname{div} a\big(Du(x)\big) + \lambda u(x)^{p-1} = A u(x)^{q-1} - B u(x)^{r-1} \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \ u > 0. \tag{Q}$$

Step 5. For all $\lambda > 0$, problem (Q) admits a unique positive solution $u_\lambda^* \in \operatorname{int} C_+$.
In fact, the mapping $\lambda \longmapsto u_\lambda^*$ has useful monotonicity and continuity properties.

Step 6. The function $\lambda \longmapsto u_\lambda^*$ is nonincreasing and continuous from $(0, \infty)$ into $C^1(\overline{\Omega})$.

Step 7. If $\lambda > 0$, then $u_\lambda^* \leq u$ for all $u \in S_0(\lambda)$.

We now consider the critical case that corresponds to $\lambda = \lambda_*$.

Step 8. We have $\lambda_* \in \mathcal{S}_0$, that is, $\mathcal{S} = [\lambda_*, \infty)$. The proof relies on the qualitative analysis of the associated energy functional for problem (P), which is defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du) \, dx + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega F_0(x, u) \, dx \quad \text{for all } u \in W^{1,p}(\Omega).$$

Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathcal{S}_0$ such that $\lambda_n \downarrow \lambda_*$. Then there exists a sequence $\{u_n\}_{n\geq 1}$ such that $u_n \in S_0(\lambda_n)$ for all $n \geq 1$. We first prove that this sequence of solutions can be chosen such that

$$\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \geq 1. \tag{5}$$

Thus, by relation (5),

$$\int_\Omega \eta G(Du_n)\,dx + \frac{\lambda_n\eta}{p}\|u_n\|_p^p \le \int_\Omega \eta F_0(x,u_n)\,dx \quad \text{for all } n\ge 1. \tag{6}$$

Also, since $u_n \in S_0(\lambda_n)$, we have:

$$-\int_\Omega \big(a(Du_n),Du_n\big)_{\mathbb{R}^N}dx - \lambda_n\|u_n\|_p^p = -\int_\Omega f_0(x,u_n)u_n\,dx \quad \text{for all } n\ge 1. \tag{7}$$

Adding relations (6) and (7) and using hypothesis $H(a)$(iv), we obtain for all $n\ge 1$:

$$(\eta-p)\int_\Omega G(Du_n)\,dx + \int_\Omega \big[f_0(x,u_n)u_n - pF_0(x,u_n)\big]dx + \lambda_n\left(\frac{\eta}{p}-1\right)\|u_n\|_p^p \le 0.$$

Therefore

$$\frac{(\eta-p)c_1}{p(p-1)}\|Du_n\|_p^p + \lambda_*\left(\frac{\eta}{p}-1\right)\|u_n\|_p^p \le 0,$$

hence $\{u_n\}_{n\ge1} \subseteq W^{1,p}(\Omega)$ is bounded (recall that $\eta > p$). So, we can assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \quad\text{and}\quad u_n \to u_* \text{ in } L^r(\Omega). \tag{8}$$

But

$$\langle A(u_n),h\rangle + \lambda_n\int_\Omega u_n^{p-1}h\,dx = \int_\Omega f_0(x,u_n)h\,dx \quad \text{for all } h\in W^{1,p}(\Omega),\ \text{all } n\ge 1. \tag{9}$$

If in (9) we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (8), then

$$\lim_{n\to\infty}\langle A(u_n),u_n-u_*\rangle = 0 \quad\Rightarrow\quad u_n\to u_* \quad \text{in } W^{1,p}(\Omega) \text{ as } n\to\infty. \tag{10}$$

Passing to the limit as $n\to\infty$ in (9) and using (10), we obtain

$$\langle A(u_*),h\rangle + \lambda_*\int_\Omega u_*^{p-1}h\,dx = \int_\Omega f_0(x,u_*)h\,dx \quad \text{for all } h\in W^{1,p}(\Omega). \tag{11}$$

From Step 7, we deduce that $u^*_{\lambda_n} \le u_n$ for all $n\ge1$, hence

$$u^*_{\lambda_*} \le u_*. \tag{12}$$

From (11) and (12) it follows that $u_* \in S_0(\lambda_*)$, hence $\lambda_* \in \mathcal{S}_0$. □

We refer to Papageorgiou and Rădulescu [6] for more details, complete proofs, and related problems.

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