



This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier's archiving and manuscript policies are encouraged to visit:

<http://www.elsevier.com/authorsrights>

Contents lists available at [ScienceDirect](http://www.sciencedirect.com)

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

Bifurcation near the origin for the Robin problem with concave–convex nonlinearities



Bifurcation autour de l'origine pour le problème de Robin avec terme concave–convexe

Nikolaos S. Papageorgiou^a, Vicențiu D. Rădulescu^b

^a National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

^b Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

ARTICLE INFO

Article history:

Received 4 April 2014

Accepted 3 May 2014

Available online 11 June 2014

Presented by Philippe G. Ciarlet

ABSTRACT

In this Note, we deal with the Robin parametric elliptic equation driven by a nonhomogeneous differential operator and with a reaction that exhibits competing terms (concave–convex nonlinearities). Without employing the Ambrosetti–Rabinowitz condition, we prove a bifurcation theorem for small positive values of the real parameter.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Dans cette Note, nous étudions le problème elliptique paramétrique de Robin pour un opérateur différentiel non homogène et avec une réaction qui présente des termes concurrents (du type concave–convexe). Sans utiliser la condition d'Ambrosetti–Rabinowitz, nous prouvons un théorème de bifurcation pour de petites valeurs positives du paramètre réel.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

Soit $p > 1$ et supposons que $\Omega \subset \mathbb{R}^N$ soit un domaine borné et régulier. Soit $\beta \in C^{1,\alpha}(\partial\Omega)$ tel que $\beta \geq 0$ sur $\partial\Omega$ et supposons que $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ soit continue, strictement monotone et satisfasse les hypothèses suivantes :

$H(a) : a(y) = a_0(\|y\|)y$ pour tout $y \in \mathbb{R}^N$, avec $a_0 > 0$ sur $(0, \infty)$ et

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto a_0(t)t$ est strictement croissante sur $(0, \infty)$, $a_0(t)t \rightarrow 0$ si $t \rightarrow 0^+$ et

$$\lim_{t \rightarrow 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) il existe $c_1 > 0$ tel que $|\nabla a(y)| \leq c_1 \frac{\eta(|y|)}{|y|}$ pour tout $y \in \mathbb{R}^N \setminus \{0\}$;

E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@math.cnrs.fr (V.D. Rădulescu).

<http://dx.doi.org/10.1016/j.crma.2014.05.007>

1631-073X/© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

- (iii) $\frac{\eta(|y|)}{|y|} |\xi|^2 \leq (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ pour tout $y \in \mathbb{R}^N \setminus \{0\}$ et $\xi \in \mathbb{R}^N$;
- (iv) si $G_0(t) = \int_0^t a_0(s) ds$ pour $t \geq 0$, alors $pG_0(t) - a_0(t)t^2 \geq -\hat{\xi}$ pour tout $t \geq 0$, où $\hat{\xi} > 0$;
- (v) il existe $\tau \in (1, p)$ tel que $t \mapsto G_0(t^{1/\tau})$ soit convexe sur $(0, \infty)$, $\lim_{t \rightarrow 0^+} \frac{G_0(t)}{t^\tau} = 0$ et il existe $\tilde{c} > 0$ tel que $a_0(t)t^2 - \tau G_0(t) \geq \tilde{c}t^p$ pour tout $t > 0$.

Dans $H(a)$ (ii)–(iii) nous supposons que $\eta \in C^1(0, \infty)$ tel que pour $c_1, c_2 > 0$ et $1 < p < \infty$

$$0 < \hat{c} \leq \frac{t\eta'(t)}{\eta(t)} \leq c_0 \quad \text{et} \quad c_1 t^{p-1} \leq \eta(t) \leq c_2(1 + t^{p-1}) \quad \text{pour tout } t > 0.$$

Supposons que $f : \Omega \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ soit une fonction qui satisfasse les conditions suivantes :

$H(f)$:

- (i) pour tout $(x, \lambda) \in \mathbb{R} \times (0, \infty)$, la fonction $z \mapsto f(z, x, \lambda)$ est mesurable et pour presque tout $z \in \Omega$, l'application $(x, \lambda) \mapsto f(z, x, \lambda)$ est continue ;
- (ii) $|f(z, x, \lambda)| \leq a_\lambda(z)(1 + x^{r-1})$ p.p. $z \in \Omega$, pour tout $x \geq 0$ et tout $\lambda > 0$, où $a_\lambda \in L^\infty(\Omega)$, $\lambda \mapsto \|a_\lambda\|_\infty$ est borné sur les ensembles bornés de $(0, \infty)$ et $p < r < p^*$;
- (iii) si $F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds$, alors $\lim_{x \rightarrow +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty$ uniformément p.p. pour $z \in \Omega$;
- (iv) il existe $\vartheta = \vartheta(\lambda) \in ((r-p) \max\{\frac{N}{p}, 1\}, p^*)$ tel que

$$0 < \gamma_0 \leq \liminf_{x \rightarrow +\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\vartheta} \quad \text{uniformément p.p. pour } z \in \Omega;$$

- (v) il existe $1 < \mu = \mu(\lambda) < q = q(\lambda) < \tau$ et $\gamma = \gamma(\lambda) > \mu$, $\delta_0 = \delta_0(\lambda) \in (0, 1)$ tels que

$$c_3 x^q \leq f(z, x, \lambda)x \leq qF(z, x, \lambda) \leq \xi_\lambda(z)x^\mu + \tau x^\gamma \quad \text{p.p. } z \in \Omega, \quad \text{pour tout } 0 \leq x \leq \delta_0,$$

avec $c_3 = c_3(\lambda) > 0$, $c_3(\lambda) \rightarrow +\infty$ si $\lambda \rightarrow +\infty$, $\bar{c} = \bar{c}(\lambda) > 0$, $\xi_\lambda \in L^\infty(\Omega)_+$ où $\|\xi_\lambda\|_\infty \rightarrow 0$ si $\lambda \rightarrow 0^+$;

- (vi) pour tout $\rho > 0$, il existe $\xi_\rho = \xi_\rho(\lambda) > 0$ tel que pour p.p. $z \in \Omega$, l'application $x \mapsto f(z, x, \lambda) + \xi_\rho x^{p-1}$ soit croissante sur $[0, \rho]$;
- (vii) pour chaque intervalle $K = [x_0, \hat{x}]$ avec $x_0 > 0$ et pour tout $\lambda > \lambda' > 0$, il existe $d_K(x_0, \lambda)$ croissante par rapport à λ et telle que $d_K(x_0, \lambda) \rightarrow +\infty$ si $\lambda \rightarrow +\infty$, et il existe $\hat{d}_K(x_0, \lambda, \lambda')$ tel que

$$\begin{aligned} f(z, x, \lambda) &\geq d_K(x_0, \lambda) \quad \text{p.p. } z \in \Omega, \quad \text{pour tout } x \in K \\ f(z, x, \lambda) - f(z, x, \lambda') &\geq \hat{d}_K(x_0, \lambda, \lambda') \quad \text{p.p. } z \in \Omega, \quad \text{pour tout } x \in K. \end{aligned}$$

Dans cette Note, nous étudions le problème de Robin non linéaire suivant :

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z), \lambda) & \text{dans } \Omega, \\ \frac{\partial u}{\partial n_a}(z) + \beta(z)u(z)^{p-1} = 0 & \text{sur } \partial\Omega, \\ u > 0. \end{cases} \tag{P}$$

Soit $C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ pour tout } z \in \bar{\Omega}\}$ et $\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ pour tout } z \in \bar{\Omega}\}$.

Le résultat principal de cette Note est le suivant.

Théorème 0.1. *Supposons que les hypothèses $H(a)$ et $H(f)$ soient satisfaites. Alors il existe $\lambda^* > 0$ tel que*

- (a) pour tout $\lambda \in (0, \lambda^*)$, le problème (P) a au moins deux solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \leq \hat{u}$, $u_0 \neq \hat{u}$;
- (b) si $\lambda = \lambda^*$, le problème (P) admet au moins une solution $u_* \in \operatorname{int} C_+$;
- (c) pour tout $\lambda > \lambda^*$, le problème (P) n'a pas de solutions.

Full English version

Let $p > 1$ be a real number and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 -boundary $\partial\Omega$. In this Note, we study the following nonlinear, nonhomogeneous parametric Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = f(z, u(z), \lambda) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_a}(z) + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0. \end{cases} \tag{P}$$

Let $\eta \in C^1(0, \infty)$ and assume that for some $c_1, c_2 > 0$ and $1 < p < \infty$

$$0 < \hat{c} \leq \frac{t\eta'(t)}{\eta(t)} \leq c_0 \quad \text{and} \quad c_1 t^{p-1} \leq \eta(t) \leq c_2(1 + t^{p-1}) \quad \text{for all } t > 0.$$

We assume that $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuous and strictly monotone map, which satisfies the following hypotheses:

$H(a)$: $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0, \infty)$, $a_0(t)t \rightarrow 0$ as $t \rightarrow 0^+$ and

$$\lim_{t \rightarrow 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) $|\nabla a(y)| \leq c_1 \frac{\eta(\|y\|)}{\|y\|}$ for some $c_1 > 0$, all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $\frac{\eta(\|y\|)}{\|y\|} |\xi|^2 \leq (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)ds$ for all $t \geq 0$, then $pG_0(t) - a_0(t)t^2 \geq -\hat{\xi}$ for all $t \geq 0$, some $\hat{\xi} > 0$;

(v) there exists $\tau \in (1, p)$ such that $t \mapsto G_0(t^{1/\tau})$ is convex on $(0, \infty)$, $\lim_{t \rightarrow 0^+} \frac{G_0(t)}{t^\tau} = 0$ and

$$a_0(t)t^2 - \tau G_0(t) \geq \tilde{c}t^p \quad \text{for some } \tilde{c} > 0, \text{ all } t > 0.$$

These conditions on $a(\cdot)$ are motivated by the regularity results of Lieberman [5] and the nonlinear maximum principle of Pucci and Serrin [7]. According to the above conditions, the potential function $G_0(\cdot)$ is strictly convex and strictly increasing. We set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. Then G is convex and differentiable, $\nabla G(0) = 0$, and

$$\nabla G(y) = G_0'(\|y\|) \frac{y}{\|y\|} = a_0(\|y\|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}.$$

So, $G(\cdot)$ is the primitive of the map $a(\cdot)$. Because $G(0) = 0$ and $y \mapsto G(y)$ is convex, from the properties of convex functions, we have $G(y) \leq (a(y), y)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N$. By direct computation, we deduce that hypotheses $H(a)$ (i), (ii), (iii) imply the following properties of $a(\cdot)$:

(a) $y \mapsto a(y)$ is continuous and strictly monotone, hence maximal monotone too;

(b) $|a(y)| \leq c_2(1 + |y|^{p-1})$ for some $c_2 > 0$, all $y \in \mathbb{R}^N$;

(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{c}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.

The following maps satisfy hypotheses $H(a)$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p -Laplacian operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for all } u \in W^{1,p}(\Omega);$$

(b) $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ with $1 < q < p < \infty$ and $\mu > 0$. This map corresponds to the (p, q) -differential operator defined by

$$\Delta_p u + \mu \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega);$$

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$. This map corresponds to the generalized p -mean curvature differential operator defined by

$$\operatorname{div}\left[(1 + |Du|^2)^{\frac{p-2}{2}} Du\right] \quad \text{for all } u \in W^{1,p}(\Omega);$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $1 < p < \infty$.

We denote by $\frac{\partial u}{\partial n_a}$ the conormal derivative, that is, $\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$.

Throughout this paper, we assume that the boundary weight map $\beta \in C^{1,\alpha}(\partial\Omega)$ (for some $\alpha \in (0, 1)$) satisfies $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

The reaction term $f(z, x, \lambda)$ is a parametric function with $\lambda > 0$ being the parameter and $(z, x) \mapsto f(z, x, \lambda)$ is Carathéodory (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x, \lambda)$ is measurable and for a.a. $z \in \Omega$ the map $x \mapsto f(z, x, \lambda)$ is continuous). We assume that $f(z, \cdot, \lambda)$ exhibits competing nonlinearities, namely near the origin, it has a “concave” term (that is, a strictly $(p - 1)$ -sublinear term), while near $+\infty$, the reaction is “convex” (that is, $x \mapsto f(z, x, \lambda)$ is $(p - 1)$ -superlinear). More precisely, the hypotheses on f are the following:

$H(f)$: $f : \Omega \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ is a function such that for a.a. $z \in \Omega$ and all $\lambda > 0$, $f(z, 0, \lambda) = 0$ and

- (i) for all $(x, \lambda) \in \mathbb{R} \times (0, \infty)$, $z \mapsto f(z, x, \lambda)$ is measurable, while for a.a. $z \in \Omega$, $(x, \lambda) \mapsto f(z, x, \lambda)$ is continuous;
- (ii) $|f(z, x, \lambda)| \leq a_\lambda(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, all $\lambda > 0$, with $a_\lambda \in L^\infty(\Omega)$, $\lambda \mapsto \|a_\lambda\|_\infty$ bounded on bounded sets in $(0, \infty)$ and $p < r < p^*$;
- (iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds$, then $\lim_{x \rightarrow +\infty} \frac{F(z, x, \lambda)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iv) there exists $\vartheta = \vartheta(\lambda) \in ((r - p) \max\{\frac{N}{p}, 1\}, p^*)$ such that

$$0 < \gamma_0 \leq \liminf_{x \rightarrow +\infty} \frac{f(z, x, \lambda)x - pF(z, x, \lambda)}{x^\vartheta} \quad \text{uniformly for a.a. } z \in \Omega;$$

- (v) there exists $1 < \mu = \mu(\lambda) < q = q(\lambda) < \tau$ (see hypothesis $H(a)(v)$) and $\gamma = \gamma(\lambda) > \mu$, $\delta_0 = \delta_0(\lambda) \in (0, 1)$ such that

$$c_3 x^q \leq f(z, x, \lambda)x \leq qF(z, x, \lambda) \leq \xi_\lambda(z)x^\mu + \tau x^\gamma \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta_0$$

with $c_3 = c_3(\lambda) > 0$, $c_3(\lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$, $\bar{c} = \bar{c}(\lambda) > 0$, $\xi_\lambda \in L^\infty(\Omega)_+$ with $\|\xi_\lambda\|_\infty \rightarrow 0$ as $\lambda \rightarrow 0^+$;

- (vi) for every $\rho > 0$, there exists $\xi_\rho = \xi_\rho(\lambda) > 0$ such that for a.a. $z \in \Omega$, $x \mapsto f(z, x, \lambda) + \xi_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$;
- (vii) for every interval $K = [x_0, \hat{x}]$ with $x_0 > 0$ and every $\lambda > \lambda' > 0$, there exists $d_K(x_0, \lambda)$ nondecreasing in λ with $d_K(x_0, \lambda) \rightarrow +\infty$ as $\lambda \rightarrow +\infty$ and $\hat{d}_K(x_0, \lambda, \lambda')$ such that

$$\begin{aligned} f(z, x, \lambda) &\geq d_K(x_0, \lambda) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in K \\ f(z, x, \lambda) - f(z, x, \lambda') &\geq \hat{d}_K(x_0, \lambda, \lambda') \quad \text{for a.a. } z \in \Omega, \text{ all } x \in K. \end{aligned}$$

Since we are interested to find positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that $f(z, x, \lambda) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$ and all $\lambda > 0$. Note that hypotheses $H(f)(ii)$, (iii) imply that

$$\lim_{x \rightarrow +\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

Therefore, $f(z, \cdot, \lambda)$ is $(p - 1)$ -superlinear near $+\infty$. However, we do not employ the Ambrosetti–Rabinowitz condition (unilateral version). We recall (see [1]), that $f(z, \cdot, \lambda)$ satisfies the (unilateral) Ambrosetti–Rabinowitz condition, if there exist $\eta = \eta(\lambda) > p$ and $M = M(\lambda) > 0$ such that

- (a) $0 < \eta F(z, x, \lambda) \leq f(z, x, \lambda)x$ for a.a. $z \in \Omega$, all $x \geq M$,
- (b) $\text{ess inf}_\Omega F(\cdot, M, \lambda) > 0$.

Integrating (1a) and using (1b), we obtain a weaker condition, namely

$$c_4 x^\eta \leq F(z, x, \lambda) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M \text{ and some } c_4 > 0. \tag{2}$$

Relation (2) implies the much weaker hypothesis $H(f)(iii)$. Next, by direct computation, we deduce that the Ambrosetti–Rabinowitz condition implies hypothesis $H(f)(iv)$. This weaker “superlinearity” condition incorporates in our setting $(p - 1)$ -superlinear nonlinearities with “slower” growth near $+\infty$, which fail to satisfy the Ambrosetti–Rabinowitz condition. Finally, we notice that hypothesis $H(f)(v)$ implies the presence of a concave nonlinearity near zero.

The following functions satisfy hypotheses $H(f)$. For the sake of simplicity, we drop the z -dependence:

$$\begin{aligned} f_1(x, \lambda) &= \lambda x^{q-1} + x^{r-1} \quad \text{for all } x \geq 0, \text{ with } 1 < q < p < r < p^*; \\ f_2(x, \lambda) &= \begin{cases} \lambda x^{q-1} - x^{\eta-1} & \text{if } x \in [0, 1] \\ x^{p-1} \left(\ln x + \frac{1}{p} \right) + \left(\lambda - \frac{1}{p} \right) x^{\nu-1} & \text{if } x > 1 \end{cases} \\ &\quad \text{with } q, \nu \in (1, p) \text{ and } \eta > p; \\ f_3(x, \lambda) &= \begin{cases} x^{q-1} & \text{if } x \in [0, \rho(\lambda)] \\ x^{r-1} + \eta(\lambda) & \text{if } \rho(\lambda) < x \end{cases} \\ &\quad \text{with } 1 < q < p < r < p^*, \eta(\lambda) = \rho(\lambda)^{p-1} - \rho(\lambda)^{r-1} \text{ and } \rho(\lambda) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+. \end{aligned}$$

Note that $f_2(\cdot, \lambda)$ does not satisfy the Ambrosetti–Rabinowitz condition.

In the analysis of problem (P), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\bar{\Omega})$. This is an ordered Banach space, with positive cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In the Sobolev space $W^{1,p}(\Omega)$, we use the norm $\|u\| = [\|u\|_p^p + \|Du\|_p^p]^{1/p}$ for all $u \in W^{1,p}(\Omega)$. To distinguish, we use $|\cdot|$ to denote the norm of \mathbb{R}^N .

We point out that the first work concerning positive solutions for problems with concave and convex nonlinearities is due to Ambrosetti, Brezis and Cerami [2]. We also refer to the books by Brezis [3] and Ciarlet [4] for related results and complements.

The main result in this Note is the following bifurcation property.

Theorem 0.1. Assume that hypotheses $H(a)$ and $H(f)$ are fulfilled. Then there exists $\lambda^* > 0$ such that

- (a) for all $\lambda \in (0, \lambda^*)$, problem (P) has at least two positive solutions $u_0, \hat{u} \in \text{int } C_+$, $u_0 \leq \hat{u}$, $u_0 \neq \hat{u}$;
- (b) for $\lambda = \lambda^*$ problem (P) has at least one positive solution $u_* \in \text{int } C_+$;
- (c) for all $\lambda > \lambda^*$ problem (P) has no positive solution.

Sketch of the proof. We introduce the following Carathéodory function:

$$\hat{f}(z, x, \lambda) = f(z, x, \lambda) + (x^+)^{p-1} \quad \text{for all } (z, x, \lambda) \in \Omega \times \mathbb{R} \times (0, +\infty).$$

Let $\hat{F}(z, x, \lambda) = \int_0^x \hat{f}(z, s, \lambda) ds$ and consider the C^1 -functional $\hat{\varphi}_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du) dz + \frac{1}{p} \|u\|_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{F}(z, u, \lambda) dz.$$

Step 1. The functional $\hat{\varphi}_\lambda$ satisfies the Cerami compactness condition.

Step 2. There exists $\lambda_+ > 0$ such that for every $\lambda \in (0, \lambda_+)$ there is $\rho_\lambda > 0$ for which we have:

$$\inf\{\hat{\varphi}_\lambda(u) : \|u\| = \rho_\lambda\} = \hat{m}_\lambda > 0 = \hat{\varphi}_\lambda(0).$$

Step 3. If $\lambda > 0$ and $u \in \text{int } C_+$, then $\varphi_\lambda(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. This is a direct consequence of $H(f)$ (iii).

Next, we introduce the following sets:

$$\mathcal{S} = \{\lambda > 0 : \text{problem (P) admits a positive solution}\},$$

$$S(\lambda) = \text{the set of positive solutions of (P)}.$$

Step 4. We have $\mathcal{S} \neq \emptyset$ and for every $\lambda \in \mathcal{S}$, $\emptyset \neq S(\lambda) \subseteq \text{int } C_+$.

Step 5. If $\lambda \in \mathcal{S}$, then $(0, \lambda] \subseteq \mathcal{S}$.

Step 6. Let $\lambda^* = \sup \mathcal{S}$. Then λ^* is finite.

Step 7. If $\lambda \in (0, \lambda^*)$, then problem (P) admits at least two distinct positive solutions $u_0, \hat{u} \in \text{int } C_+$, with $u_0 \leq \hat{u}$.

Next, we examine what happens in the critical case $\lambda = \lambda^*$. To this end, note that hypotheses $H(f)$ (ii), (v) imply that we can find $M_1, M_2 > 0$ such that

$$f(z, x, \lambda) \geq M_1 x^{q-1} - M_2 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3}$$

This unilateral growth estimate on the reaction $f(z, \cdot, \lambda)$ leads to the following auxiliary Robin problem:

$$\begin{cases} -\text{div } a(Du(z)) = M_1 u(z)^{q-1} - M_2 u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_0}(z) + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega, \\ u > 0. \end{cases} \tag{4}$$

Step 8. Problem (4) admits a unique positive solution $\bar{u} \in \text{int } C_+$.

Step 9. If $\lambda \in \mathcal{S}$, then $\bar{u} \leq u$ for all $u \in S(\lambda)$.

Step 10. We have $\lambda^* \in \mathcal{S}$ and so $\mathcal{S} = (0, \lambda^*]$. Indeed, let $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{S}$ be such that $\lambda_n \rightarrow (\lambda^*)^-$. Then we can find $u_n \in S(\lambda_n) \subseteq \text{int } C_+$ and we can assume that $\hat{\varphi}_{\lambda_n}(u_n) < 0$ for all $n \geq 1$. Therefore

$$\int_\Omega pG(Du_n) dz + \int_{\partial\Omega} \beta(z)u_n^p d\sigma - \int_\Omega pF(z, u_n, \lambda_n) dz < 0 \quad \text{for all } n \geq 1. \tag{5}$$

We also have:

$$-\langle A(u_n), u_n \rangle - \int_{\partial\Omega} \beta(z) u_n^p \, d\sigma + \int_{\Omega} f(z, u_n, \lambda_n) u_n \, dz = 0 \quad \text{for all } n \geq 1. \tag{6}$$

Adding relations (5) and (6), we obtain

$$\int_{\Omega} [pG(Du_n) - (a(Du_n), Du_n)_{\mathbb{R}^N}] \, dz + \int_{\Omega} [f(z, u_n, \lambda_n) u_n - pF(z, u_n, \lambda_n)] \, dz < 0.$$

Thus, by hypothesis $H(a)(iv)$,

$$\int_{\Omega} [f(z, u_n, \lambda_n) u_n - pF(z, u_n, \lambda_n)] \, dz < c \quad \text{for all } n \geq 1, \text{ some } c > 0. \tag{7}$$

From (7) and using hypothesis $H(f)(iv)$, we show that $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \rightarrow \infty. \tag{8}$$

Since $u_n \in S(\lambda)$ for all $n \geq 1$, we have:

$$\langle A(u_n), h \rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma - \int_{\Omega} f(z, u_n, \lambda_n) h \, dz = 0 \quad \text{for all } h \in W^{1,p}(\Omega). \tag{9}$$

Choosing $h = u_n - u_* \in W^{1,p}(\Omega)$ in (9), passing to the limit as $n \rightarrow \infty$ and using (8), we obtain $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_* \rangle = 0$, hence

$$u_n \rightarrow u_* \text{ in } W^{1,p}(\Omega). \tag{10}$$

So, if in (9) we pass to the limit as $n \rightarrow \infty$ and use (10), we obtain:

$$\langle A(u_*), h \rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h \, d\sigma = \int_{\Omega} f(z, u_*, \lambda_*) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega).$$

This shows that $u_* \geq 0$ is a solution of problem (P).

From Step 9, we have $\bar{u} \leq u_n$ for all $n \geq 1$. Hence $\bar{u} \leq u_*$ and so $u_* \in S(\lambda^*) \subseteq \text{int } C_+$. Therefore $\lambda^* \in \mathcal{S}$ and so $\mathcal{S} = (0, \lambda^*]$. \square

We refer to Papageorgiou and Rădulescu [6] for more details, complete proofs, and related problems.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973) 349–381.
 [2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave–convex nonlinearities in some elliptic problems, *J. Funct. Anal.* 122 (1994) 519–543.
 [3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.
 [4] P.G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 2013.
 [5] G. Lieberman, The natural generalization of the conditions of Ladyzhenskaya and Uraltseva for elliptic equations, *Commun. Partial Differ. Equ.* 16 (1991) 311–361.
 [6] N.S. Papageorgiou, V.D. Rădulescu, Bifurcation of positive solutions for nonlinear nonhomogeneous Robin and Neumann problems with competing nonlinearities, submitted for publication.
 [7] P. Pucci, J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, Switzerland, 2007.