Standing waves for the fractional Schrödinger equation

Ondes stationnaires de l'équation de Schrödinger fractionnaire

Giovanni Molica Bisci [a] Vicenţiu D. Rădulescu [b,c]

[a] *Dipartimento PAU, Università 'Mediterranea' di Reggio Calabria, Via Graziella, Feo di Vito, 89124 Reggio Calabria, Italy*

[b] *Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

[c] *Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania*

Abstract

We investigate the existence of multiple standing waves for a class of fractional Schrödinger equations. The nonlinear term is assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity. The proof combines variational methods and energy estimates in a suitable fractional Sobolev space.

Résumé

Nous étudions l'existence de plusieurs ondes stationnaires pour une classe d'équations de Schrödinger fractionnaires. Le terme non linéaire a un comportement sur-linéaire autour de l'origine et une croissance sous-linéaire à l'infini. La preuve combine des méthodes variationnelles avec des estimations de l'énergie dans un espace de Sobolev fractionnaire.

Version française abrégée

Soit $(-\Delta)^s$ l'opérateur de Laplace fractionnaire d'ordre $s \in (0,1)$. Nous rappelons que $(-\Delta)^s u = \mathfrak{F}^{-1}(|\xi|^{2s}(\mathfrak{F}u)(\xi))$, où $\mathfrak{F}u : \mathbb{R}^n \to \mathbb{R}$ est la transformée de Fourier de u.

Dans cette Note nous étudions le problème non linéaire

$$(-\Delta)^s u + V(x)u = \lambda(f(x,u) + \mu g(x,u)), \quad x \in \mathbb{R}^n \ (n \geq 3). \qquad (P)$$

On suppose que le potential $V : \mathbb{R}^n \to \mathbb{R}$ satisfait les hypothèses suivantes :

Email addresses: gmolica@unirc.it (Giovanni Molica Bisci), vicentiu.radulescu@math.cnrs.fr (Vicenţiu D. Rădulescu).

(p$_1$) $V \in C(\mathbb{R}^n)$ *et* $\inf_{x\in\mathbb{R}^n} V(x) > 0$;

(p$_2$) *pour tout* $M > 0$ *il existe* $r_0 > 0$ *tel que* $\lim_{|y|\to+\infty} |\{x \in B(y, r_0) : V(x) \leq M\}| = 0$.

Les fonctions continues $f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ ont un comportement sur-linéaire autour de l'origine et une croissance sous-linéaire à l'infini. Plus précisément, on suppose que f et g vérifient les conditions suivantes :

(h$_1$) *il existe* $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \setminus \{0\}$ *et* $q \in (0, 1)$ *tels que*

$$\max\{|f(x,t)|, |g(x,t)|\} \leq W(x)|t|^q \quad \textit{pour tout } (x,t) \in \mathbb{R}^n \times \mathbb{R}\,;$$

(h$_2$) $\lim_{t\to 0} \dfrac{f(x,t)}{t} = \lim_{t\to 0} \dfrac{g(x,t)}{t} = 0$ *uniformément pour* $x \in \mathbb{R}^n$;

(h$_3$) *il existe* $s_0 \in \mathbb{R}$ *tel que* $\sup_{\sigma>0} \left(\min_{|x|\leq\sigma} \int_0^{s_0} f(x,\tau)\, d\tau \right) > 0$.

Considérons l'espace de Sobolev fractionnaire avec poids $E^n_s(V)$ définit par

$$E^n_s(V) := \left\{ u \in H^s(\mathbb{R}^n);\ \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx < \infty \right\}.$$

On dit que $u \in E^n_s(V)$ est une solution du problème (P) si pour tout $v \in E^n_s(V)$

$$\int_{\mathbb{R}^n} |\xi|^{2s} \mathfrak{F}u(\xi)\mathfrak{F}v(\xi)\, d\xi + \int_{\mathbb{R}^n} V(x)u(x)v(x)\, dx = \lambda \int_{\mathbb{R}^n} f(x, u(x))v(x)\, dx + \lambda\mu \int_{\mathbb{R}^n} g(x, u(x))v(x)\, dx.$$

Le résultat principal de cette Note est le suivant.

Théorème 0.1 *Supposons que les hypothèses* (h$_1$)–(h$_3$) *et* (p$_1$)–(p$_2$) *sont satisfaites. Alors les propriétés suivantes sont vraies.*

(i) Il existe $\mu_0 > 0$ *tel que pour chaque* $\mu \in [-\mu_0, \mu_0]$ *il y a un intervalle ouvert* $\emptyset \neq \Sigma_\mu \subset (0, \infty)$ *et* $\kappa_\mu > 0$ *avec la propriété que le problème* (P) *admet au moins deux solutions non triviales* $v_{\lambda\mu}$ *et* $w_{\lambda\mu}$ *vérifiant*

$$\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leq \kappa_\mu, \qquad \textit{pour tout } \lambda \in \Sigma_\mu.$$

(ii) Il existe $\lambda_0 > 0$ *tel que pour chaque* $\lambda \in (\lambda_0, \infty)$ *il y a un intervalle ouvert* $\emptyset \neq \Lambda_\lambda \subset \mathbb{R}$ *et* $\kappa_\lambda > 0$ *avec la propriété que le problème* (P) *admet au moins deux solutions non trivialles* $v_{\lambda\mu}$ *et* $w_{\lambda\mu}$ *vérifiant* $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0 < \mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ *et* $\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leq \kappa_\lambda$, *pour tout* $\mu \in \Lambda_\lambda$.

La preuve combine le principe variationnel de Ricceri [5,6] avec le théorème du col de Ambrosetti et Rabinowitz [1].

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^n)$ functions. For every $u \in \mathcal{S}$, the fractional Laplace operator acting on u is defined by $(-\Delta)^s u = \mathfrak{F}^{-1}(|\xi|^{2s}(\mathfrak{F}u)(\xi))$, where $s \in (0,1)$ and $\mathfrak{F}u : \mathbb{R}^n \to \mathbb{R}$ denotes the Fourier transform of u.

In this Note, we study the combined effects of two nonlinear terms and two real parameters for the existence of multiple solutions to the following fractional parametric Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda(f(x,u) + \mu g(x,u)), \quad x \in \mathbb{R}^n,\ n > 2. \tag{1}$$

We assume that the potential V satisfies the following hypotheses:

(p_1) *$V \in C(\mathbb{R}^n)$ with* $\inf\limits_{x\in\mathbb{R}^n} V(x) > 0$;

(p_2) *for any $M > 0$ there exists $r_0 > 0$ such that* $\lim_{|y|\to+\infty} |\{x \in B(y, r_0) : V(x) \leq M\}| = 0$, *where $B(y, r)$ is the open ball in $\mathbb{R}^n$ with center y and radius $r > 0$ and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.*

We assume that the continuous functions f, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ have a superlinear behaviour at the origin and a sublinear decay at infinity. More precisely, we suppose that f and g satisfy the following conditions:

(h_1) *there exist $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \setminus \{0\}$ and $q \in (0, 1)$ such that*

$$\max\{|f(x,t)|, |g(x,t)|\} \leq W(x)|t|^q \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R};$$

(h_2) $\lim\limits_{s\to 0} \dfrac{f(x,t)}{t} = \lim\limits_{t\to 0} \dfrac{g(x,t)}{t} = 0$ *uniformly for each $x \in \mathbb{R}^n$*;

(h_3) *there exist $s_0 \in \mathbb{R}$ such that*

$$\sup_{\sigma>0} \left(\min_{|x|\leq\sigma} \int_0^{s_0} f(x,\tau)\, d\tau \right) > 0.$$

Let $H^s(\mathbb{R}^n)$ denote the fractional Sobolev space of all functions $u \in L^2(\mathbb{R}^n)$ such that

$$\text{the map } (x,y) \longmapsto \frac{u(x)-u(y)}{|x-y|^{(n+2s)/2}} \text{ is in } L^2(\mathbb{R}^{2n}, dxdy).$$

On $H^s(\mathbb{R}^n)$ we consider the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(x)|^2\, dx + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy \right)^{1/2}. \tag{2}$$

Consider the weighted fractional Sobolev space

$$E_s^n(V) := \left\{ u \in H^s(\mathbb{R}^n);\ \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx < \infty \right\} \tag{3}$$

endowed with the norm

$$\|u\|_{E_s^n(V)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx \right)^{1/2}.$$

We say that $u \in E_s^n(V)$ is a solution of problem (1) if for all $v \in E_s^n(V)$

$$\int_{\mathbb{R}^n} |\xi|^{2s} \mathfrak{F}u(\xi) \mathfrak{F}v(\xi)\, d\xi + \int_{\mathbb{R}^n} V(x)u(x)v(x)\, dx = \lambda \int_{\mathbb{R}^n} f(x, u(x))v(x)\, dx + \lambda\mu \int_{\mathbb{R}^n} g(x, u(x))v(x)\, dx.$$

For all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ we set $F(x,t) := \int_0^t f(x,\tau) d\tau$ and $G(x,t) := \int_0^t g(x,\tau)\, d\tau$.
The energy functional $\mathcal{J}_{\lambda,\mu} : E_s^n(V) \to \mathbb{R}$ associated to problem (1) is defined by

$$\mathcal{J}_{\lambda,\mu}(u) := \frac{1}{2} \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx \right) - \lambda \int_{\mathbb{R}^n} F(x, u(x))\, dx - \lambda\mu \int_{\mathbb{R}^n} G(x, u(x))\, dx.$$

A straightforward computation shows that $\mathcal{J}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous, coercive, bounded from below, and satisfies the Palais-Smale condition for all λ, $\mu \in \mathbb{R}$, see [4] for detailed proofs.

The main result of this Note is the following.

Theorem 0.1 *Assume that f, g and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2)*. Then the following properties hold.*

(i) There exists $\mu_0 > 0$ such that for all $\mu \in [-\mu_0, \mu_0]$ there are an open interval $\emptyset \neq \Sigma_\mu \subset (0, \infty)$ and $\kappa_\mu > 0$ with the property that problem (1) has at least two distinct nontrivial solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ satisfying

$$\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leq \kappa_\mu\,,$$

provided that $\lambda \in \Sigma_\mu$.

(ii) There exists $\lambda_0 > 0$ such that for all $\lambda \in (\lambda_0, \infty)$ there are an open interval $\emptyset \neq \Lambda_\lambda \subset \mathbb{R}$ and $\kappa_\lambda > 0$ with the property that problem (1) has at least two distinct nontrivial solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ satisfying $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0 < \mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ and $\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leq \kappa_\lambda$, provided that $\mu \in \Lambda_\lambda$.

We point out that this result is no longer valid for all $\lambda \in \mathbb{R}$. For instance, consider the functions

$$f(x,t) := \frac{\sin^2 t}{(1+|x|^\alpha)^\beta} \quad \text{and} \quad g(x,t) := \frac{\arctan^2 t}{(1+|x|^\alpha)^\beta} \qquad \forall\,(x,t) \in \mathbb{R}^n \times \mathbb{R}\,,$$

with $\alpha, \beta > 0$ such that $\alpha\beta > n \geq 3$. In such a case, the nonlocal fractional equation

$$(-\Delta)^s u + V(x)u = \frac{\lambda}{(1+|x|^\alpha)^\beta}(\sin^2 u + \mu \arctan^2 u), \quad x \in \mathbb{R}^n$$

has only the trivial solution, whenever $0 < \lambda < (1+|\mu|\pi)^{-1} S_2^{-2}$ and μ is arbitrary. Here, S_2 denotes the best Sobolev embedding constant of the embedding $E^n_s(V) \hookrightarrow L^2(\mathbb{R}^n)$. More generally, the following non-existence result holds.

Proposition 0.2 *Assume that f, g and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2)*. Assume that $f(x,t) := W(x)h(t)$ and $g(x,s) := W(x)k(t)$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, where h and k are Lipschitz continuous functions with Lipschitz constants L_h and L_k, and W satisfying hypothesis* (h_1).

Then the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda W(x)(h(u) + \mu k(u)), \quad x \in \mathbb{R}^n$$

admits only the trivial solution, provided that $\mu \in \mathbb{R}$ and

$$0 < \lambda < \frac{1}{\|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu| L_k) S_2^2}\,.$$

1. Proof of Theorem 0.1.

Fix $\sigma > 0$ and $\eta \in \mathbb{R}$. For every $\varepsilon > 0$, define $u^\eta_\varepsilon \in E^n_s(V)$ as follows

$$u^\eta_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(0,\sigma) \\ \dfrac{\eta}{\varepsilon}(\sigma - |x|) & \text{if } x \in B(0,\sigma) \setminus B(0, \sigma - \varepsilon) \\ \eta & \text{if } x \in B(0, \sigma - \varepsilon). \end{cases} \tag{4}$$

Note that $u^\eta_\varepsilon \in E^n_s(V)$ and $\|u^\eta_\varepsilon\|_\infty := \max\limits_{x \in \mathbb{R}^n} |u^\eta_\varepsilon(x)| \leq |\eta|$.

Lemma 1.1 *There exists $u_0 \in E^n_s(V)$ such that* $\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx > 0$.

Proof. By hypothesis (h$_3$), there exist $\sigma_0 > 0$ and $s_0 \in \mathbb{R}$ such that

$$\min_{|x|\le\sigma_0} \int_0^{s_0} f(x,\tau)\,d\tau > 0\,.$$

Fix $\varepsilon \in (0, \sigma_0/2)$ and denote $\omega_0 := \min_{|x|\le\sigma_0} F(x, s_0) > 0$. Let $u_\varepsilon^{s_0} \in E_s^n(V)$ be the function obtained in (4) by replacing σ with σ_0, as well as η with s_0. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} F(x, u_\varepsilon^{s_0}(x))\,dx &= \int_{|x|\le\sigma_0-\varepsilon} F(x, u_\varepsilon^{s_0}(x))\,dx + \int_{|x|\ge\sigma_0} F(x, u_\varepsilon^{s_0}(x))\,dx + \int_{\sigma_0-\varepsilon<|x|<\sigma_0} F(x, u_\varepsilon^{s_0}(x))\,dx \\
&\ge \omega_0|B_{\sigma_0/2}| - \int_{\sigma_0-\varepsilon<|x|<\sigma_0} |F(x, u_\varepsilon^{s_0}(x))|\,dx \\
&\ge \omega_0|B_{\sigma_0/2}| - \max_{|x|\in[\sigma_0/2,\sigma_0]\,,|t|\le|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|).
\end{aligned}$$

Since

$$\max_{|x|\in[\sigma_0/2,\sigma_0]\,,|t|\le|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|) \to 0 \quad \text{as } \varepsilon \to 0^+,$$

there exists $\varepsilon_0 > 0$ such that

$$\omega_0|B_{\sigma_0/2}| > \max_{|x|\in[\sigma_0/2,\sigma_0]\,,|t|\le|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon_0}|).$$

Thus $u_0 := u_{\varepsilon_0}^{s_0} \in E_s^n(V)$ is the required function. □

For every $\mu \in \mathbb{R}$, we define $L_\mu : E_s^n(V) \to \mathbb{R}$ by

$$L_\mu(u) := \int_{\mathbb{R}^n} F(x, u(x))\,dx + \mu \int_{\mathbb{R}^n} G(x, u(x))\,dx.$$

For all $\rho \in \mathbb{R}$, we set

$$\chi(\varrho) := \frac{\sup\left\{L_\mu(u) : \|u\|_{E_s^n(V)} < \sqrt{2\varrho}\right\}}{\varrho}\,.$$

Lemma 1.2 *We have* $\lim_{\varrho\to0^+} \chi(\varrho) = 0$.

Proof. Fix arbitrarily $\varepsilon > 0$ and $p \in (2, 2_s^*)$, where $2_s^* := 2n/(n-2s)$ is the Sobolev fractional critical exponent. Taking into account the growth rates of f and g, we obtain for all $u \in E_s^n(V)$

$$L_\mu(u) \le (1+|\mu|)(\varepsilon S_2^2\|u\|_{E_s^n(V)}^2 + c(\varepsilon)S_p^p\|u\|_{E_s^n(V)}^p).$$

Thus for every $\varrho > 0$ we have

$$0 \le \chi(\varrho) \le (1+|\mu|)(2\varepsilon S_2^2 + c(\varepsilon)2^{\frac{p}{2}} S_p^p \varrho^{\frac{p}{2}-1}).$$

Taking $\varrho \to 0^+$, the right-hand side of the above inequality tends to zero due to the arbitrariness of $\varepsilon > 0$, which concludes the proof. □

Proof of Theorem 0.1. (i) Let $u_0 \in E_s^n(V)$ be the element from Lemma 1.1 and set

$$\mu_0 := \frac{\int_{\mathbb{R}^n} F(x, u_0(x))\,dx}{\left|\int_{\mathbb{R}^n} G(x, u_0(x))\,dx\right| + 1}\,.$$

Let $\Phi(u) := \dfrac{1}{2}\|u\|_{E_s^n(V)}^2$ and $\Psi := L_\mu$ for every fixed $\mu \in [-\mu_0, \mu_0]$. Then $\mathcal{J}_{\lambda,\mu} = \Phi - \lambda J_\mu$.

By straightforward computation we deduce that for every $\mu \in [-\mu_0, \mu_0]$ we have

$$L_\mu(u_0) = \int\limits_{\mathbb{R}^n} F(x, u_0(x))\,dx + \mu \int\limits_{\mathbb{R}^n} G(x, u_0(x))\,dx \geq \mu_0 > 0. \tag{5}$$

Using relation (5) and Lemma 1.2, there exists $\varrho_\mu > 0$ such that

$$\varrho_\mu < \min\left\{1, \frac{\|u_0\|^2_{E^n_s(V)}}{2}\right\}$$

and

$$\chi(\varrho_\mu) := \frac{\sup\{L_\mu(u) : \ \|u\|_{E^n_s(V)} < \sqrt{2\varrho_\mu}\}}{\varrho_\mu} < \frac{L_\mu(u_0)}{\|u_0\|^2_{E^n_s(V)}}.$$

Now all the assumptions of Ricceri's variational principle [5,6] are fulfilled. Thus there are an open interval $\emptyset \neq \Sigma_\mu \subset [0, \overline{a}_\mu]$ and $\kappa_\mu > 0$ such that for all $\lambda \in \Sigma_\mu$, the functional $\mathcal{J}_{\lambda,\mu} = \Phi - \lambda L_\mu$ admits at least three distinct critical points $u^i_{\lambda,\mu} \in E^n_s(V)$ $(i \in \{1, 2, 3\})$, having norm less than κ_μ.

(ii) Define

$$c_0 := \int\limits_{\mathbb{R}^n} |G(x, u_0(x))| dx \quad \text{and} \quad \lambda_0 := \frac{\|u_0\|^2_{E^n_s(V)}}{2\int_{\mathbb{R}^n} F(x, u_0(x))\,dx}.$$

For all $\lambda > \lambda_0$ we set

$$\mu^*_\lambda := \frac{1}{1 + c_0}\left(1 - \frac{\lambda_0}{\lambda}\right)\int\limits_{\mathbb{R}^n} F(x, u_0(x))\,dx.$$

Fix $\lambda > \lambda_0$ and $\mu \in (-\mu^*_\lambda, \mu^*_\lambda) \equiv \Lambda_\lambda$. Since $J_{\lambda,\mu}$ is coercive and bounded from below, there exists $v_{\lambda\mu} \in E^n_s(V)$ such that $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) = \inf_{u \in E^n_s(V)} \mathcal{J}_{\lambda,\mu}(u)$. By straightforward computation we deduce that $\inf_{u \in E^n_s(V)} \mathcal{J}_{\lambda,\mu}(u) < 0$ for all $\lambda > \lambda_0$. Therefore $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$.

On the other hand, since $\mathcal{J}_{\lambda,\mu}$ satisfies the Palais-Smale condition and has the mountain pass geometry for all $\lambda > \lambda_0$ and $\mu \in (-\mu^*_\lambda, \mu^*_\lambda)$, we find $w_{\lambda\mu} \in E^n_s(V)$ such that $\mathcal{J}'_{\lambda,\mu}(w_{\lambda\mu}) = 0$ and

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \geq \left(\frac{1}{4} - \lambda(1 + |\mu|)c(\lambda, \mu)S^p_p \varrho^{p-2}\right)\varrho^2 > 0.$$

The mountain pass level $\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ is characterized by

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) = \inf_{g \in \Gamma} \max_{t \in [0,1]} \mathcal{J}_{\lambda,\mu}(g(t)), \tag{6}$$

where $\Gamma := \{g \in C([0, 1]; E^n_s(V)) : g(0) = 0, \ g(1) = u_0\}$.

Define the function $g_0 : [0, 1] \to E^n_s(V)$ by $g_0(t) := tu_0$. Since $g_0 \in \Gamma$, using relation (6) we obtain for all $\mu \in \Lambda_\lambda$

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \leq \max_{t \in [0,1]} \mathcal{J}_{\lambda,\mu}(tu_0) \leq \frac{1}{2}\|u_0\|^2_{E^n_s(V)} + \lambda \max_{t \in [0,1]}\left\{\left|\int\limits_{\mathbb{R}^n} F(x, tu_0(x))\,dx\right| + \mu^*_\lambda \left|\int\limits_{\mathbb{R}^n} G(x, tu_0(x))\,dx\right|\right\}.$$

Thus for all $\mu \in \Lambda_\lambda$ we have

$$\begin{aligned}\|w_{\lambda\mu}\|^2_{E^n_s(V)} &\leq 2\lambda(1 + \mu^*_\lambda)\|W\|_{L^{\frac{2}{1-q}}(\mathbb{R}^n)} S^{q+1}_2 \|w_{\lambda\mu}\|^{q+1}_{E^n_s(V)} \\ &+ \|u_0\|^2_{E^n_s(V)} + 2\lambda \max_{t \in [0,1]}\left\{\left|\int\limits_{\mathbb{R}^n} F(x, tu_0(x))\,dx\right| + \mu^*_\lambda \left|\int\limits_{\mathbb{R}^n} G(x, tu_0(x))\,dx\right|\right\}.\end{aligned}$$

Since $q+1<2$, there exists $\kappa_\lambda^1>0$ such that $\|w_{\lambda\mu}\|_{E_s^n(V)} \le \kappa_\lambda^1$ for every $\mu \in \Lambda_\lambda$. Moreover, since $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$ for every $\mu \in \Lambda_\lambda$, a similar argument as above shows the existence of $\kappa_\lambda^2 > 0$ such that $\|v_{\lambda\mu}\|_{E_s^n(V)} \le \kappa_\lambda^2$, for every $\mu \in \Lambda_\lambda$. Thus letting $\kappa_\lambda := \max\{\kappa_\lambda^1, \kappa_\lambda^2\}$, the proof of Theorem 0.1 is completed. □

Using the definition of μ_λ^*, we observe that for every $\lambda > \lambda_0$ we have

$$\mu_\lambda^* < \frac{\int_{\mathbb{R}^n} F(x,u_0(x))\,dx}{1+c_0}\,.$$

Since the right-hand side does not depend on λ, we have a uniform estimation of Λ_λ, namely

$$\Lambda_\lambda \subset \left[-(1+c_0)^{-1}\int_{\mathbb{R}^n} F(x,u_0(x))\,dx,\ (1+c_0)^{-1}\int_{\mathbb{R}^n} F(x,u_0(x))\,dx\right],$$

for all $\lambda > \lambda_0$.

We refer to Brezis [2] and Ciarlet [3] as basic references for the mathematical methods employed in this paper.

Acknowledgements. This work has been completed while V. Rădulescu was visiting the City University of Hong Kong in February 2015. He would like to express his gratitude to Professor Philippe G. Ciarlet for this invitation and for enlightened discussions.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973), 349-381.

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[3] P.G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2013.

[4] G. Molica Bisci, V. Rădulescu, Multiple entire solutions of sublinear fractional Schrödinger equations, in preparation.

[5] B. Ricceri, On a three critical points theorem, *Arch. Math. (Basel)* **75** (2000), 220-226.

[6] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, *Math. Comput. Modelling* **32** (2000), 1485-1494.