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ON A NEW FRACTIONAL SOBOLEV SPACE AND APPLICATIONS TO NONLOCAL VARIATIONAL PROBLEMS WITH VARIABLE EXPONENT

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Abstract. The content of this paper is at the interplay between function spaces $L^{p(x)}$ and $W^{k,p(x)}$ with variable exponents and fractional Sobolev spaces $W^{s,p}$. We are concerned with some qualitative properties of the fractional Sobolev space $W^{s,q(x),p(x,y)}$, where q and p are variable exponents and $s \in (0,1)$. We also study a related nonlocal operator, which is a fractional version of the nonhomogeneous $p(x)$-Laplace operator. The abstract results established in this paper are applied in the variational analysis of a class of nonlocal fractional problems with several variable exponents.

1. **Introduction and preliminary results.** Fractional Sobolev spaces and the corresponding nonlocal equations have major applications to various nonlinear problems, including phase transitions, thin obstacle problem, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, ultra-relativistic limits of quantum mechanics, multiple scattering, minimal surfaces, material science, water waves, etc. We refer to Di Nezza, Palatucci and Valdinoci [9] for a comprehensive introduction to the study of nonlocal problems. After the seminal papers by Caffarelli *et al.* [6, 7, 8], a large amount of papers were written on problems involving the fractional diffusion operator $(-\Delta)^s$ $(0 < s < 1)$. The cited results turn out to be very fruitful in order to recover an elliptic PDE approach in a nonlocal framework, and they have recently been used very often, see [1, 2, 14, 17, 20, 21]. We also refer to the recent monographs [15, 5, 10] for a thorough variational approach of nonlocal problems.

A natural question is to see what results can be recovered when the standard Laplace operator is replaced by the fractional Laplacian. On the other hand, for

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some nonhomogeneous materials (such as electrorheological fluids, sometimes referred to as "smart fluids"), the standard approach based on Lebesgue and Sobolev spaces L^p and $W^{1,p}$, is not adequate. This leads to the study of *variable exponent* Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where p is a real-valued function. Variable exponent Lebesgue spaces appeared in the literature in 1931 in the paper by Orlicz [16]. Zhikov [24] started a new direction of investigation, which created the relationship between spaces with variable exponent and variational integrals with nonstandard growth conditions. We also point out the important contributions of Marcellini [13], who studied minimization problems with (p, q)-growth, namely

$$\inf \int_\Omega F(x, |\nabla u|)dx,$$

where $t^p \leq F(x,t) \leq t^q + 1$ for all $t \geq 0$. The case corresponding to the variable exponent corresponds to $F(x,t) = t^{p(x)}$, where $p : \Omega \to (1, \infty)$ is a bounded function. We refer to [18, 19] for the abstract treatment of function spaces with variable exponent.

It is therefore a natural question to see what results can be "recovered" when the $p(x)$-Laplace operator is replaced by the fractional $p(x)$-Laplacian. As far as we know, the only result about the fractional Sobolev spaces with variable exponent and the fractional $p(x)$-Laplacian is obtained in [12]. In particular, the authors generalize $p(x)$-Laplace operator to the fractional case. They also introduce a suitable functional space to study an equation in which a fractional variable exponent operator is present.

Let Ω be a smooth open set in $\mathbb{R}^N$. For any real $s > 0$ and for any functions $q(x)$ and $p(x, y)$, we want to define the fractional Sobolev space with variable exponent. We start by fixing $0 < s < 1$ and $q : \overline{\Omega} \to \mathbb{R}$ and $p : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be two continuous functions. We assume that p is symmetric and

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) \leq p(x, y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) < \infty, \tag{P}$$

$$p((x, y) - (z, z)) = p(x, y), \quad \forall (x, y),\, (z, z) \in \Omega \times \Omega, \tag{P'}$$

and

$$1 < q^- = \min_{x\in\overline{\Omega}} q(x) \leq q(x) \leq q^+ = \max_{x\in\overline{\Omega}} q(x) < \infty. \tag{Q}$$

We define the fractional Sobolev space with variable exponents via the Gagliardo approach as follows:

$$\begin{aligned} E &= W^{s,q(x),p(x,y)}(\Omega) \\ &= \left\{ u \in L^{q(x)}(\Omega), \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < \infty, \quad \text{for some} \quad \lambda > 0 \right\}. \end{aligned}$$

Let

$$[u]^{s,p(x,y)} = \inf \left\{ \lambda > 0, \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < 1 \right\}$$

be the corresponding variable exponent Gagliardo seminorm. If we equip E with the norm

$$\|u\|_E = [u]^{s,p(x,y)} + |u|_{q(x)},$$

($L^{q(x)}(\Omega)$ and $|.|_{q(x)}$ will be introduced in the next section), then E becomes a Banach space. Let E_0 denote the closure of $C_0^\infty(\Omega)$ in E. Then E_0 is a Banach space with the norm

$$\|u\| = [u]^{s,p(x,y)}.$$

In [12], the authors prove the following basic theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0,1)$. Let $q(x)$, $p(x,y)$ be continuous variable exponents with $sp(x,y) < N$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) > p(x,x)$ for $x \in \overline{\Omega}$. Let (P) and (Q) be satisfied. Assume that $r : \overline{\Omega} \to (1,\infty)$ is a continuous function such that*

$$p^*(x) = \frac{Np(x,x)}{N - sp(x,x)} > r(x) \geq r_- > 1,$$

for $x \in \overline{\Omega}$. Then there exists a constant $C = C(N, s, p, q, r, \Omega)$ such that for every $f \in W^{s,q(x),p(x,y)}(\Omega)$,

$$|f|_{r(x)} \leq C\|f\|_E.$$

Thus, the space $W^{s,q(x),p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in (1, p^*)$. Moreover, this embedding is compact.

Remark 1.2. The above theorem remains true if we replace E by E_0.

As an application of Theorem 1.1 in [12], the authors study the existence of solutions to some nonlocal problems. Let us consider the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) = p.v. \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy.$$

In the constant exponent case it is know as the fractional p-Laplacian. On the other hand, we remark that $\mathcal{L}$ is a fractional version of the well known $p(x)$-Laplacian.

The main purpose of this paper is to present some further basic results both on the function spaces E_0, E and the fractional operator $\mathcal{L}$. Then, we study the existence of solutions to some nonlocal problems. This paper is organized as follows. In Sect. 2, we give some definitions and fundamental properties of the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. In Sect. 3, we study the reflexivity, separability, density of E_0 and E. Moreover, we prove some basic properties of operator $\mathcal{L}$. Finally, in Section 4, using a direct variational method, we give an application of our abstract results.

2. **Terminology and abstract setting.**

In this section, we recall some necessary properties of variable exponent spaces. We refer to [11, 19] and the references therein.

Consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x \in \Omega} p(x) \qquad \text{and} \qquad p^- = \inf_{x \in \Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{ u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf \left\{ \mu > 0;\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)}\, dx \leq 1 \right\}.$$

We point out that if $p(x) \equiv p \in [1,\infty)$ then the optimal choice in the above expression is $\mu = \|u\|_{L^p}$.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/q(x)=1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ then the following Hölder-type inequality holds:

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}\,. \tag{1}$$

Moreover, if $p_j \in C_+(\overline{\Omega})$ $(j=1,2,\ldots,k)$ and

$$\frac{1}{p_1(x)}+\frac{1}{p_2(x)}+\cdots+\frac{1}{p_k(x)}=1$$

then for all $u_j \in L^{p_j(x)}(\Omega)$ $(j=1,\ldots,k)$ we have

$$\left|\int_\Omega u_1u_2\cdots u_k\,dx\right| \le \left(\frac{1}{p_1^-}+\frac{1}{p_2^-}+\cdots+\frac{1}{p_k^-}\right)|u_1|_{p_1(x)}|u_2|_{p_2(x)}\cdots|u_k|_{p_k(x)}\,. \tag{2}$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u|^{p(x)}dx.$$

Proposition 2.1. *We have:*
(i) $|u|_{p(x)} < 1(=1;>1) \Leftrightarrow \rho(u) < 1(=1;>1)$.
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$.

Proposition 2.2. *If* $u, u_n \in L^{p(x)}(\Omega)$ *and* $n \in \mathbb{N}$, *then the following statements are equivalent each other:*
(1) $\lim\limits_{n\to+\infty} |u_n - u|_{p(x)} = 0$.
(2) $\lim\limits_{n\to+\infty} \rho(u_n - u) = 0$.
(3) $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to+\infty} \rho(u_n) = \rho(u)$.

From Theorems 1.6, 1.8 and 1.10 of [11], we obtain the following proposition:

Theorem 2.3. *Suppose that* (Q) *is satisfied.*
(i) If Ω *is a bounded open domain,* $(L^{q(x)}(\Omega), |.|_{q(x)})$ *is a reflexive uniformly convex and separable space.*
(ii) If Ω *is an open subset, then* $C_0^\infty(\Omega)$ *is dense in the space* $(L^{q(x)}(\Omega), |.|_{q(x)})$.

If k is a positive integer number and $p \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ \text{for all } |\alpha| \le k\}.$$

Here $\alpha = (\alpha_1,\ldots,\alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}_{x_1}\ldots\partial^{\alpha_N}_{x_N}}\,.$$

On $W^{k,p(x)}(\Omega)$ we consider the following norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}.$$

Then $W^{k,p(x)}(\Omega)$ is a reflexive and separable Banach space. Let $W_0^{k,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

3. Basic results.

3.1. The spaces $W^{s,q(x),p(x,y)}(\Omega)$.

Lemma 3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded open domain. Furthermore, assume that (P) and (Q) hold. Then $W^{s,q(x),p(x,y)}(\Omega)$ is a separable reflexive space.*

Proof. We define the operator $T : W^{s,q(x),p(x,y)}(\Omega) \to L^{q(x)}(\Omega) \times L^{p(x,y)}(\Omega \times \Omega)$ by

$$T(u) = (u(x), \frac{u(x)-u(y)}{|x-y|^{\frac{N}{p(x,y)}+s}}).$$

Clearly T is an isometry. Then, using Theorem 2.3, the rest of the proof is similar to Theorem 8.1 of [3]. $\square$

Lemma 3.2. *Let (P), $(P^{'})$ and (Q) be satisfied. If $\Omega = \mathbb{R}^N$, then the space $C^\infty(\Omega)$ is a dense subspace of $W^{s,q(x),p(x,y)}(\Omega)$.*

Proof. Let $u \in W^{s,q(x),p(x,y)}(\Omega)$ and $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that $\eta \geq 0$ in $\mathbb{R}^N$ and $supp(\eta) \subset B_1$. We also assume that $\int_{B_1} \eta(x)dx = 1$. We denote by u_ϵ the typical mollifier of u, that is,

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)u(y)dy, \quad x \in \mathbb{R}^N,$$

with $\eta_\epsilon(x) = \epsilon^{-N}\eta(\frac{x}{\epsilon})$. Since $u \in L^{q(x)}(\Omega)$, by Theorem 2.3, we know that

$$|u_\epsilon - u|_{q(x)} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3}$$

Hence, from Proposition 2.2, it suffices to prove that

$$\int_{\Omega\times\Omega} |u_\epsilon(x) - u(x) - u_\epsilon(y) + u(y)|^{p(x,y)} K(x,y)dxdy \to 0, \quad \text{as} \quad \epsilon \to 0, \tag{4}$$

where $K(x,y) = |x-y|^{-N-sp(x,y)}$. Using the Hölder inequality in combination with Tonelli's and Fubini's theorems, we obtain

$$\begin{aligned}
&\int_{\Omega\times\Omega} |u_\epsilon(x) - u(x) - u_\epsilon(y) + u(y)|^{p(x,y)} K(x,y)dxdy \\
&= \int_{\Omega\times\Omega} K(x,y)[\int_{\mathbb{R}^N} (u(x-z) - u(y-z))\eta_\epsilon(z)dz - u(x) + u(y)]^{p(x,y)} dxdy \\
&= \int_{\Omega\times\Omega} K[\int_{B_1} (u(x-\epsilon z) - u(y-\epsilon z) - u(x) + u(y))\eta(z)dz]^{p(x,y)} dxdy \\
&\leq |B_1|^{p^-+p^+-1} \int_{\Omega\times\Omega} K[\int_{B_1} |u(x-\epsilon z) - u(y-\epsilon z) - u(x) + u(y)|^{p(x,y)} \eta^{p(x,y)}(z)dz]dxdy \\
&\leq |B_1|^{p^-+p^+-1} \int_{\Omega\times\Omega\times B_1} |u(x-\epsilon z) - u(y-\epsilon z) - u(x) + u(y)|^{p(x,y)} K\eta(z)^{p(x,y)} dxdydz \\
&\leq |B_1|^{p^-+p^+-1} \int_{\Omega\times\Omega\times B_1} |u(x-\epsilon z) - u(y-\epsilon z) - u(x) + u(y)|^{p(x,y)} \times \\
&\times K(x,y)(\eta(z)^{p^+} + \eta(z)^{p^-})dxdydz
\end{aligned} \tag{5}$$

We claim that

$$\lim_{\epsilon\to 0}\int_{\Omega\times\Omega}|u(x-\epsilon z)-u(y-\epsilon z)-u(x)+u(y)|^{p(x,y)}K(x,y)dxdy=0. \tag{6}$$

Fix $z\in B_1$ and put $w=(z,z)\in\Omega\times\Omega$. We define the function $v:\Omega\times\Omega\to\mathbb{R}$ by

$$v(x,y)=(u(x)-u(y))(K(x,y))^{\frac{1}{p(x,y)}},\quad \forall (x,y)\in\Omega\times\Omega.$$

Then $v\in L^{p(x,y)}(\Omega\times\Omega)$. If $\epsilon^{'}>0$, by Theorem 2.3, there exists $g\in C_0^\infty(\Omega\times\Omega)$ with $|v-g|_{p(x,y)}<\frac{\epsilon^{'}}{3}$, so

$$\begin{aligned}&|v(.-\epsilon w)-v|_{p(x,y)}\\ \leq&|v(.-\epsilon w)-g(.-\epsilon w)|_{p(x,y)}+|g(.-\epsilon w)-g|_{p(x,y)}+|v-g|_{p(x,y)}\\ \leq&\frac{\epsilon^{'}}{3}+\frac{\epsilon^{'}}{3}+\frac{\epsilon^{'}}{3}=\epsilon^{'},\end{aligned}$$

with ϵ is sufficiently small. This proves our claim (6).

Moreover, for a.e. $z\in B_1$, there exists a positive constant c such that

$$\begin{aligned}&(\eta(z)^{p^+}+\eta(z)^{p^-})\int_{\Omega\times\Omega}|u(x-\epsilon z)-u(y-\epsilon z)-u(x)+u(y)|^{p(x,y)}Kdxdy\\ &\leq c(\eta(z)^{p^+}+\eta(z)^{p^-})(\int_{\Omega\times\Omega}|u(x-\epsilon z)-u(y-\epsilon z)|^{p(x,y)}Kdxdy\\ &+\int_{\Omega\times\Omega}|u(x)-u(y)|^{p(x,y)}K(x,y)dxdy)\\ &\leq 2c(\eta(z)^{p^+}+\eta(z)^{p^-})\int_{\Omega\times\Omega}|u(x)-u(y)|^{p(x,y)}Kdxdy\in L^\infty(B_1),\end{aligned} \tag{7}$$

for any $\epsilon>0$. Hence, using (6), (7) and the dominated convergence theorem, we infer that

$$\int_{B_1}\eta(z)^{p(x,y)}\int_{\Omega\times\Omega}|u(x-\epsilon z)-u(y-\epsilon z)-u(x)+u(y)|^{p(x,y)}K(x,y)dxdydz\to 0,$$

as $\epsilon\to 0$. From the above assertion, we obtain (4). Using this fact in combination with (3), we conclude our proof. □

4. **Properties of the fractional operator $\mathcal{L}$.** In this section, we give some basic properties of the operator $\mathcal{L}$. Let (P) and (Q) be satisfied. In the sequel, we denote by $K(x,y)=|x-y|^{-N-sp(x,y)}$.

Consider the following functionals:

$$I_1(u)=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy,\quad \forall u\in E_0,$$

$$I_2(u)=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy+\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx,\quad \forall u\in E,$$

and $L_1:E_0\to E_0^*$ such that for all $u,\varphi\in E_0$

$$\langle L_1(u),\varphi\rangle=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy.$$

Similarly, we consider $L_2 : E \to E^*$ such that

$$\langle L_2(u), \varphi\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy + \int_\Omega |u|^{q(x)-2}u\varphi, \quad \forall u,\varphi \in E.$$

Lemma 4.1. *The functional I_1 is well defined on E_0. Moreover, $I_1 \in C^1(E_0, \mathbb{R})$ with the derivative given by*

$$\langle I_1^{'}(u), \varphi\rangle = \langle L_1(u), \varphi\rangle, \quad \forall u, \varphi \in E_0,$$

Proof. The proof is standard, see [15, 19]. □

Lemma 4.2. (*i*) *L_1 is a bounded and strictly monotone operator.*
(*ii*) *L_1 is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in E_0 and $\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle \le 0$, then $u_n \to u$ in E_0.*
(*iii*) *$L_1 : E_0 \to E_0^*$ is a homeomorphism.*

Proof. (*i*) Evidently, L_1 is a bounded operator. Recall the Simon inequalities [22], which imply the strict monotonicity of L_1:

$$\begin{cases} |x-y|^p \le c_p \left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y); \; p \ge 2 \\ |x-y|^p \le C_p \left[\left(|x|^{p-2}x - |y|^{p-2}y\right)\cdot(x-y)\right]^{\frac{p}{2}} (|x|^p + |y|^p)^{\frac{2-p}{2}}; \; 1 < p < 2, \end{cases} \tag{8}$$

for all $x, y \in \mathbb{R}^N$, where $c_p = (\frac{1}{2})^{-p}$ and $C_p = \frac{1}{p-1}$.
(*ii*) Let $(u_n) \in E_0$ be a sequence such that $u_n \rightharpoonup u$ in E_0 and $\limsup_{n\to+\infty}\langle L_1(u_n) - L_1(u), u_n - u\rangle \le 0$. Then, from (*i*), we deduce that

$$\lim_{n\to+\infty} \langle L_1(u_n) - L_1(u), u_n - u\rangle = 0. \tag{9}$$

By Theorem 1.1, we obtain

$$u_n(x) \to u(x), \quad \text{a.e. } x \in \Omega. \tag{10}$$

This along with Fatou's lemma yield

$$\liminf_{n\to+\infty} \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}Kdxdy \ge \int_{\Omega\times\Omega} |u(x)-u(y)|^{p(x,y)}Kdxdy, \tag{11}$$

On the other hand, we have

$$\lim_{n\to+\infty} \langle L_1(u_n), u_n - u\rangle = \lim_{n\to+\infty} \langle L_1(u_n) - L_1(u), u_n - u\rangle = 0. \tag{12}$$

Now using Young's inequality, there exists a positive constant c such that

$$\begin{aligned} \langle L_1(u_n), u_n - u\rangle &= \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy \\ &\quad - \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))(u(x)-u(y))Kdxdy \\ &\ge \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)}K(x,y)dxdy \\ &\quad - \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p(x,y)-1}|u(x)-u(y)|K(x,y)dxdy \end{aligned} \tag{13}$$

$$\geq c \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} K(x,y)dxdy$$
$$- c \int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} K(x,y)dxdy.$$

As a consequence of (11), (12) and (13), we get

$$\lim_{n\to+\infty} \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} K(x,y)dxdy = \int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} K dxdy. \quad (14)$$

Now from (10), (14) and the Brezis-Lieb lemma [4], our result is proved.
(iii) By (i), L_1 is an injection. In view of Proposition 2.1, we obtain

$$\lim_{\|u\|\to+\infty} \frac{\langle L_1(u), u\rangle}{\|u\|} = +\infty.$$

Therefore, L_1 is coercive. Thus, in light of Minty-Browder theorem (see [23]), L_1 is a surjection. Hence L_1 has an inverse mapping $L_1^{-1} : E_0^* \to E_0$.

It remains to show that L_1^{-1} is continuous. Indeed, let $(f_n), f \in E_0^*$ such that $f_n \to f$ in E_0^*. Let $u_n = L_1^{-1}(f_n)$, $u = L_1^{-1}(f)$, then

$$L_1(u_n) = f_n \ \text{ and } \ L_1(u) = f.$$

In view of the coercivity of L_1, (u_n) is bounded in E_0. We may assume that $u_n \rightharpoonup u_0$ in E_0. It follows that

$$\lim_{n\to+\infty} \langle L_1(u_n) - L_1(u_0), u_n - u_0\rangle = \lim_{n\to+\infty} \langle f_n, u_n - u_0\rangle = 0.$$

Using the fact that L_1 is of type (S^+), we conclude that $u_n \to u_0$ in E_0. This concludes the proof. ☐

Remark 4.3. The above results still hold true if we replace L_1 by L_2.

5. **Application to nonlocal fractional problems with variable exponent.** In this section, we work under the hypotheses of Theorem 1.1. We investigate the existence of solutions of the following problem

$$\begin{cases} \mathcal{L}u(x) + |u(x)|^{q(x)-1}u(x) = \lambda |u(x)|^{r(x)-1}u(x), & \text{in } \Omega, \\ u(x) = 0 & \text{in } \partial\Omega, \end{cases} \quad (15)$$

where $\lambda > 0$, $1 < r(x) < p^-$.

We say that $u \in E_0$ is a weak solution of problem (15) if for all $v \in E_0$

$$\int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))K(x,y)dxdy +$$
$$\int_\Omega |u(x)|^{q(x)-1}u(x)v(x)dx - \lambda \int_\Omega |u(x)|^{r(x)-1}u(x)v(x)dx = 0.$$

Theorem 5.1. *For every $\lambda > 0$, problem (15) admits at least one nontrivial weak solution.*

Define the functional $J_\lambda : E_0 \to \mathbb{R}$ by

$$J_\lambda(u) = I_1(u) + \int_\Omega \frac{|u(x)|^{q(x)}}{q(x)} dx - \lambda \int_\Omega \frac{|u(x)|^{r(x)}}{r(x)} dx, \quad \forall u \in E.$$

By Lemma 4.1, $J_\lambda \in C^1(E_0, \mathbb{R})$ and

$$\langle J_\lambda^{'}(u), v\rangle = \int_{\Omega\times\Omega} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))K(x,y)dxdy$$

$$+\int_\Omega |u(x)|^{q(x)-1}u(x)v(x)dx - \lambda\int_\Omega |u(x)|^{r(x)-1}u(x)v(x)dx, \qquad \forall v\in E_0.$$

Proof of Theorem 5.1. We first show that J_λ satisfies the Palais-Smale condition. Let $(u_n)\in E_0$ be a $(PS)-$sequence of J_λ. We claim that (u_n) is bounded in E_0. Arguing by contradiction, we suppose that (u_n) is unbounded in E_0. Without loss of generality, we can assume that $\|u_n\| > 1$ for all $n\geq 1$. There exists a positive constant c such that

$$\begin{aligned} c \geq J_\lambda(u_n) &= I_1(u_n) + \int_\Omega \frac{|u_n(x)|^{q(x)}}{q(x)}dx - \lambda\int_\Omega \frac{|u_n(x)|^{r(x)}}{r(x)}dx \\ &\geq I_1(u_n) - \lambda\int_\Omega \frac{|u_n(x)|^{r(x)}}{r(x)}dx \\ &\geq \|u_n\|^{p^-} - \frac{c\lambda}{r^-}\|u_n\|^{r^+}. \end{aligned}$$

In the last inequality, we used Proposition 2.1 and Theorem 1.1. In view of $r^+ < p^-$, it follows that (u_n) is bounded in E_0. Thus, up to a subsequence and using Theorem 1.1, we can assume that

$$u_n \rightharpoonup u \ \text{ in } \ E_0, \quad u_n\to u \ \text{ in } \ L^{r(x)}(\Omega) \ \text{ and } \ u_n\to u \ \text{ in } \ L^{q(x)}(\Omega).$$

We show in what follows that

$$u_n \to u \ \text{ in } \ E_0.$$

Since (u_n) is a $(PS)-$sequence and using the above assertions, we get

$$\langle J_\lambda^{'}(u_n) - J_\lambda^{'}(u_n), u_n - u\rangle = \langle L_1(u_n) - L_1(u), u_n-u\rangle = 0.$$

Now since L_1 is an operator of type (S_+), we conclude that $u_n\to u$ in E_0, which shows that the Palais-Smale condition is satisfied.

Next, we show that J_λ is coercive. Indeed, as we have observed, for any $\lambda>0$ and $u\in E_0$ with $\|u\|>1$, we have

$$J_\lambda(u) \geq \|u\|^{p^-} - \frac{c\lambda}{r^-}\|u\|^{r^+}.$$

This implies the coercivity of J_λ. We deduce that $J_\lambda\in C^1(E_0,\mathbb{R})$ is bounded from below, coercive and satisfies the Palais-Smale condition. These facts in combination with Ekeland's variational principle show that there exists $u_\lambda \in E$ a global minimum of J_λ. It remains to prove that $u_\lambda\neq 0$. Fix $\phi\in E_0, \phi\neq 0$ and $\phi\geq 0$ in Ω. Then, for each $t\in(0,1)$, we have

$$\begin{aligned} J_\lambda(t\phi) &= I_1(t\phi) + \int_\Omega \frac{t^{q(x)}|\phi(x)|^{q(x)}}{q(x)}dx - \lambda\int_\Omega \frac{t^{r(x)}|\phi(x)|^{r(x)}}{r(x)}dx \\ &\leq t^{p^+}I_1(\phi) + t^{q^+}\int_\Omega \frac{|\phi(x)|^{q(x)}}{q(x)}dx - \lambda t^{r^-}\int_\Omega \frac{|\phi(x)|^{r(x)}}{r(x)}dx. \end{aligned}$$

Taking into account $r^- < p^+$ and $r^- < q^+$, for t small enough, we infer that

$$J_\lambda(t\phi) < 0.$$

This completes the proof of our theorem. □

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