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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR DEGENERATE KIRCHHOFF-TYPE SCHRÖDINGER-CHOQUARD EQUATIONS

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Abstract. In this article we study a class of Kirchhoff-type Schrödinger-Choquard equations involving the fractional p-Laplacian. By means of Kajikiya's new version of the symmetric mountain pass lemma, we obtain the existence of infinitely many solutions which tend to zero under a suitable value of λ. The main feature and difficulty of our equations arise in the fact that the Kirchhoff term M could vanish at zero, that is, the problem is *degenerate*. To our best knowledge, our result is new even in the framework of Schrödinger-Choquard problems.

1. Introduction and statement of main result

In this article, we consider a class of Kirchhoff-type Schrödinger-Choquard equations involving the fractional p-Laplacian of the form

$$M(\|u\|_s^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \lambda f(x,u) + (\mathcal{K}_\mu * |u|^{p^*_{\mu,s}})|u|^{p^*_{\mu,s}-2}u, \tag{1.1}$$

in $\mathbb{R}^N$, where hereafter $\mathcal{K}_\mu(x) = |x|^{-\mu}$,

$$\begin{aligned} \|u\|_s &= \Big([u]_s^p + \int_{\mathbb{R}^N} V(x)|u|^p dx\Big)^{1/p} [u]_s \\ &= \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy\Big)^{1/p}, \end{aligned} \tag{1.2}$$

$M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a Kirchhoff function, $V : \mathbb{R}^N \to \mathbb{R}^+$ is a scalar potential, $p^*_{\mu,s} = (pN - p\mu/2)/(N-ps)$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $(-\Delta)_p^s$ is the associated fractional operator which, up to a normalization constant, is defined as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N,$$

along functions $\varphi \in C_0^\infty(\mathbb{R}^N)$. Henceforward $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$.

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Nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [2]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media (for more details see for example [2, 8, 9] and the references therein). Indeed, the literature on nonlocal fractional operators and on their applications is quite large, see for example the recent monograph [27], the extensive paper [15] and the references cited there. The literature on non-local operators and their applications is quite large, here we just quote a few, see [3, 11, 12, 26, 28, 29, 35, 36, 37, 38, 42, 43].

This paper is motivated by some works appeared in recent years. On the one hand, the following Choquard or nonlinear Schrödinger-Newton equation

$$-\Delta u+V(x)u=(\mathcal{K}_\mu * u^2)u+\lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

was elaborated by Pekar [33] in the framework of quantum mechanics. Subsequently, it was adopted as an approximation of the Hartree-Fock theorey, see [7]. Recently, *Penrose* [34] settled it as a model of self-gravitational collapse of a quantum mechanical wave function. The first investigations for existence and symmetry of the solutions to (1.3) go back to the works of Lieb [21] and Lions [23]. Equations of type (1.3) have been extensively studied, see e.g. [1, 30, 31, 46]. For the critical case in the sense of Hardy-Littlewood-Sobolev inequality, we refer the interested reader to [18] for recent existence results in a smooth bounded domain of $\mathbb{R}^N$. In the setting of the fractional Laplacian, Wu [45] investigated existence and stability of solutions for the equations

$$(-\Delta)^s u+\omega u=(\mathcal{K}_\mu * |u|^q)|u|^{q-2}u+\lambda f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $q=2$, $\lambda=0$ and $\mu\in(N-2s,N)$. Subsequently, D'Avenia and Squassina in [13] studied some properties of solutions for (1.4) with $\lambda=0$, such as regularity, existence, multiplicity, nonexistence, symmetry as well as decays properties. In particular, they claimed the nonexistence of solutions as $q\in(2-\mu/N,2^*_{\mu,s})$. In the critical case that corresponds to $q=2^*_{\mu,s}$, Mukherjee and Sreenadh [32] obtained existence and multiplicity results for solutions of (1.4) as $\omega=0$ and $f(x,u)=u$ in a smooth bounded domain of $\mathbb{R}^N (N\geq 3)$.

On the other hand, Lü [24] studied the following Kirchhoff-type equation

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+V_\lambda(x)u=(\mathcal{K}_\mu * u^q)|u|^{q-2}u \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where $a\in\mathbb{R}^+$, $b\in\mathbb{R}^+_0$ are given numbers, $V_\lambda(x)=1+\lambda g(x)$, $\lambda\in\mathbb{R}^+$ is a parameter and $g(x)$ is a continuous potential function on $\mathbb{R}^3$, $q\in(2,6-\mu)$. By using the Nehari manifold and the concentration compactness principle, the author obtained the existence of ground state solutions for (1.5) if the parameter λ is large enough. Indeed, the study of Kirchhoff-type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. More precisely, Kirchhoff established a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.6}$$

where ρ, p_0, h, E, L are constants which represent some physical meanings respectively. Equation (1.6) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Recently, Fiscella and Valdinoci [17] proposed a steady-state Kirchhoff model involving the fractional Laplacian by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see [17, Appendix A] for more details. Very recently, by using the mountain pass theorem and the Ekeland variational principle, Pucci *et al.* [39] studied the existence of solutions for problem (1.1) under some appropriate assumptions.

Motivated by the above works, we are interested in the existence of infinitely many solutions for problem (1.1) in a possibly degenerate Kirchhoff setting. It is worth mentioning that the method exploited in this paper, is Kajikiya's new version of the symmetric mountain pass lemma, which was first used to study Kirchhoff-type fractional p-Laplacian problems in [6, 44]. As far as we know, there is no result to investigate multiplicity of solutions for (1.1) in the literature.

Throughout the paper, without explicit mention, we assume

- (A1) $V:\mathbb{R}^N\to\mathbb{R}^+$ is a continuous function and there exists $V_0>0$ such that $\inf_{\mathbb{R}^N} V\geq V_0$.

$M:\mathbb{R}^+_0\to\mathbb{R}^+_0$ is assumed to be continuous and to satisfy

- (A2) For any $\tau>0$ there exists $m=m(\tau)>0$ such that $M(t)\geq m$ for all $t\geq\tau$.
- (A3) There exists $\theta\in[1,p^*_s/p)$ if $p\geq 2$ and $\theta\in[1,\min\{p^*_s/p,p^*_{\mu,s}\})$ if $1<p<2$, where $p^*_s=Np/(N-sp)$, such that $tM(t)\leq\theta\mathscr{M}(t)$ for all $t\in\mathbb{R}^+_0$, where $\mathscr{M}(t)=\int_0^t M(\tau)d\tau$.
- (A4) There exists $m_0>0$ such that $M(t)\geq m_0t^{\theta-1}$ for all $t\in[0,1]$.

A prototype for M, due to Kirchhoff, is given by

$$M(t)=a+b\theta t^{\theta-1}\quad\text{for } t\in\mathbb{R}^+_0,\ a,\,b\geq 0,\ a+b>0. \tag{1.7}$$

When $M(t)\geq c>0$ for all $t\in\mathbb{R}^+_0$, Kirchhoff equations like (1.1) are said to be *non-degenerate* and this happens for example if $a>0$ in the model case (1.7). While, if $M(0)=0$ but $M(t)>0$ for all $t\in\mathbb{R}^+$, Kirchhoff equations as (1.1) are called *degenerate*. Of course, for (1.7) this occurs when $m_0=0$.

Concerning the function $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$, we suppose that

- (A5) There exist $q\in(1,\theta p)$ and a nonnegative function $w\in L^{\vartheta}(\mathbb{R}^N)$ such that $|f(x,t)|\leq w(x)t^{q-1}$ for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}^+$, where $\vartheta=p^*_s/(p^*_s-q)$.
- (A6) There exist $\xi\in(1,p)$, $\delta>0$, $a_0>0$ and a nonempty open subset Ω of $\mathbb{R}^N$ such that
$$F(x,t)\geq a_0t^{\xi}\quad\text{for all } (x,t)\in\Omega\times(0,\delta).$$

A simple example of f, verifying (A5)-(A6), is $f(x,t)=(1+|x|^2)^{(l-2)/2}(t^+)^{l-1}$ with $1<l<p$, where $t^+=\max\{t,0\}$.

The natural solution space for (1.1) is $W^{s,p}_V(\mathbb{R}^N)$, which is the completion of $C^\infty_0(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_s$ introduced in (1.2).

Equation (1.1) has a variational structure and its associated energy functional is $\mathcal{J}_\lambda:W^{s,p}_V(\mathbb{R}^N)\to\mathbb{R}$, defined by

$$\mathcal{J}_\lambda(u)=\frac{1}{p}\mathscr{M}(\|u\|_s^p)-\lambda\int_{\mathbb{R}^N}F(x,u)dx-\frac{1}{2p^*_{\mu,s}}\iint_{\mathbb{R}^{2N}}\frac{|u(x)|^{p^*_{\mu,s}}|u(y)|^{p^*_{\mu,s}}}{|x-y|^{\mu}}\,dx\,dy\,.$$

Under the assumption (A5), $\mathcal{J}_\lambda$ is of class $C^1(W_V^{s,p}(\mathbb{R}^N),\mathbb{R})$. We say that $u \in W_V^{s,p}(\mathbb{R}^N)$ is a (weak) solution of problem (1.1), if

$$\begin{aligned} &M(\|u\|_s^p)\Big(\langle u,\varphi\rangle_{s,p} + \int_{\mathbb{R}^N} V|u|^{p-2}u\varphi\,dx\Big)\\ &= \lambda\int_{\mathbb{R}^N} f(x,u)\varphi\,dx + \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^{p_{\mu,s}^*})|u|^{p_{\mu,s}^*-2}u\varphi\,dx, \end{aligned}$$

$$\langle u,\varphi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{\big[|u(x)-u(y)|^{p-2}(u(x)-u(y))\big]\cdot\big[\varphi(x)-\varphi(y)\big]}{|x-y|^{N+ps}}\,dx\,dy,$$

for all $\varphi \in W_V^{s,p}(\mathbb{R}^N)$. From now on, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\big(W_V^{s,p}(\mathbb{R}^N)\big)'$ and $W_V^{s,p}(\mathbb{R}^N)$. Clearly, the critical points of $\mathcal{J}_\lambda$ are exactly the weak solutions of problem (1.1).

The main result of our article is the following.

Theorem 1.1. *Let $M(0)=0$, $N\in(ps,\infty)$ with $s\in(0,1)$ and $p\in(1,\infty)$. Assume that M and V satisfy assumptions* (A1)–(A6). *Then, there exists $\overline{\lambda}>0$ such that for any $\lambda\in(0,\overline{\lambda})$ equation* (1.1) *admits a sequence of solutions $\{u_n\}_n$ in E with $\mathcal{J}_\lambda(u_n)<0$, $\mathcal{J}_\lambda(u_n)\to 0$ and $\{u_n\}_n$ converges to zero as $n\to\infty$.*

Remark 1.2. With the help of the Ekeland variational principle, Pucci *et al.* [39] just obtained the existence of a nontrivial nonnegative solution for problem (1.1) under the same hypotheses as Theorem 1.1, see [39, Theorem 1.3] for more details. Therefore, our result and approach are completely different from those developed by Pucci *et al.* in [39].

The proof of Theorem 1.1 is mainly based on the application of the symmetric mountain pass lemma, introduced by Kajikiya in [19]. For this, we need a truncation argument which allow us to control from below functional $\mathcal{J}_\lambda$. Furthermore, as usual in elliptic problems involving critical nonlinearities, we must pay attention to the lack of compactness at critical level corresponding to the space $L^{p_s^*}(\mathbb{R}^N)$. To overcome this difficulty, we fix parameter λ under a suitable threshold strongly depending on assumptions (A3) and (A4).

The paper is organized as follows. In Section 2 we discuss the variational formulation of the equation (1.1) and introduce some topological notions. In Section 3 we prove the Palais-Smale condition for the functional $\mathcal{J}_\lambda$. In Section 4 we introduce a truncation argument for our functional. In Section 5 we prove Theorem 1.1.

2. Preliminaries

We first provide some basic functional setting that will be used in the next sections. The critical exponent p_s^* is defined as $Np/(N-ps)$. Let $L^p(\mathbb{R}^N,V)$ denote the Lebesgue space of real valued functions, with $V(x)|u|^p\in L^1(\mathbb{R}^N)$, equipped with norm

$$\|u\|_{p,V} = \Big(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\Big)^{1/p} \quad \text{for all } u\in L^p(\mathbb{R}^N,V).$$

The embedding $W_V^{s,p}(\mathbb{R}^N)\hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu\in[p,p_s^*]$ by [14, Theorem 6.7], namely there exists a positive constant C_ν such that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C_\nu\|u\|_s \quad \text{for all } u\in W_V^{s,p}(\mathbb{R}^N).$$

Next, we recall the Hardy-Littlewood-Sobolev inequality, see [22, Theorem 4.3]. Hereafter we denote by $\|\cdot\|_q$ the norm of Lebesgue space $L^q(\mathbb{R}^N)$.

Theorem 2.1. *Assume that* $1 < r,\ t < \infty$, $0 < \mu < N$ *and*

$$\frac{1}{r} + \frac{1}{t} + \frac{\mu}{N} = 2.$$

Then there exists $C(N, \mu, r, t) > 0$ *such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)| \cdot |v(y)|}{|x-y|^\mu}\, dx\, dy \le C(N, \mu, r, t)\|u\|_r \|v\|_t$$

for all $u \in L^r(\mathbb{R}^N)$ *and* $v \in L^t(\mathbb{R}^N)$.

Note that, by the Hardy-Littlewood-Sobolev inequality, the integral

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu}\, dx\, dy$$

is finite, whenever $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{N} = 2, \quad \text{that is } t = \frac{2N}{2N-\mu}.$$

Hence, by the fractional Sobolev embedding theorem, if $u \in W_V^{s,p}(\mathbb{R}^N)$ this occurs provided that $tq \in [p, p_s^*]$. Thus, q has to satisfy

$$\tilde{p}_{\mu,s} = \frac{(N-\mu/2)p}{N} \le q \le \frac{(N-\mu/2)p}{N-sp} = p_{\mu,s}^*.$$

Hence, $\tilde{p}_{\mu,s}$ is called the lower critical exponent and $p_{\mu,s}^*$ is said to be the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

By the Hardy-Littlewood-Sobolev inequality, there exists $\widetilde{C}(N, \mu) > 0$ such that

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_{\mu,s}^*} |u(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\, dx\, dy \le \widetilde{C}(N, \mu)\|u\|_{p_s^*}^{2p_{\mu,s}^*},$$

for all $u \in W_V^{s,p}(\mathbb{R}^N)$. In other words, there exists $C(N, \mu) = \widetilde{C}(N, \mu) C_{p_s^*}^{2p_{\mu,s}^*} > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^{p_{\mu,s}^*})|u|^{p_{\mu,s}^*} dx &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_{\mu,s}^*} |u(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\, dx\, dy \\ &\le C(N, \mu)\|u\|_s^{2p_{\mu,s}^*}, \end{aligned} \tag{2.1}$$

for all $u \in W_V^{s,p}(\mathbb{R}^N)$. Let us define

$$S^* = \inf_{u \in W_V^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_s^p}{\left(\int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u|^{p_{\mu,s}^*})|u|^{p_{\mu,s}^*} dx\right)^{p/2p_{\mu,s}^*}}. \tag{2.2}$$

Clearly, $S^* > 0$.

Theorem 2.2 (see [39, Theorem 2.3])**.** *Let* $\{u_n\}_n$ *be a bounded sequence in* $L^{p_s^*}(\mathbb{R}^N)$ *such that* $u_n \to u$ *a.e. in* $\mathbb{R}^N$ *as* $n \to \infty$. *Then*

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{p_{\mu,s}^*} |u_n(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\, dx\, dy \\ &- \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u(x)|^{p_{\mu,s}^*} |u_n(y)-u(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\, dx\, dy \\ &\to \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{p_{\mu,s}^*} |u(y)|^{p_{\mu,s}^*}}{|x-y|^\mu}\, dx\, dy \quad \text{as } n \to \infty. \end{aligned}$$

To handle the degenerate Kirchhoff coefficient we need appropriate lower and upper bounds for M, given by (A2) and (A3). Indeed, condition (A3) implies that $M(t) > 0$ for any $t > 0$ and consequently by (A2) for all $t \in (0, 1]$ we have $M(t)/\mathscr{M}(t) \le \theta/t$. Thus, integrating on $[t, 1]$, with $0 < t < 1$, we get

$$\mathscr{M}(t) \ge \mathscr{M}(1)t^\theta, \tag{2.3}$$

and (2.3) holds for all $t \in [0, 1]$ by continuity. Hence, (A4) is a stronger request. Furthermore (2.3) is compatible with (A4), since integrating (A4) we have $\mathscr{M}(t) \ge m_0 t^\theta/\theta$ for any $t \in [0, 1]$, from which $\mathscr{M}(1) \ge m_0/\theta$.

Similarly, for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \mathscr{M}(\varepsilon)/\varepsilon^\theta > 0$ such that

$$\mathscr{M}(t) \le \delta_\varepsilon t^\theta \quad \text{for any } t \ge \varepsilon. \tag{2.4}$$

To prove the multiplicity result stated in Theorem 1.1, we will use some topological results introduced by Krasnoselskii in [20]. For the sake of completeness and for reader's convenience, we recall here some basic notions on the Krasnoselskii's genus. Let X be a Banach space and let us denote by Σ the class of all closed subsets $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Let $A \in \Sigma$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as being the least positive integer n such that there is an odd mapping $\phi \in C(A, \mathbb{R}^n)$ such that $\phi(x) \neq 0$ for any $x \in A$. If n does not exist, we set $\gamma(A) = \infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

In the sequel we recall only the properties of the genus that will be used throughout this work. More information on this subject may be found in the references [19, 20, 40].

Proposition 2.3. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

1. *If there exists an odd continuous mapping from A to B, then $\gamma(A) \le \gamma(B)$.*
2. *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*
3. *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \ge \gamma(A) - \gamma(B)$.*
4. *Then n-dimensional sphere S^n has a genus of $n + 1$ by the Borsuk-Ulam theorem.*
5. *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$, with $N_\delta(A) = \{x \in X : \operatorname{dist}(x, A) \le \delta\}$.*

We conclude this section by recalling the symmetric mountain-pass lemma introduced by Kajikiya in [19]. The proof of Theorem 1.1 is based on the application of the following result.

Lemma 2.4. *Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose that the following conditions hold.*

1. *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais-Smale condition, i.e. for some $\bar{c} > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $\lim_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence;*
2. *For each $n \in \mathbb{N}$, there exists $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$.*

Then either (i) or (ii) below holds.

- (i) *There exists a sequence $\{u_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) < 0$ and $\{u_n\}_n$ converges to zero.*

(ii) *There exist two sequences* $\{u_n\}_n$ *and* $\{v_n\}_n$ *such that* $J'(u_n) = 0$, $J(u_n) = 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$; $J'(v_n) = 0$, $J(v_n) < 0$, $\lim_{n\to\infty} J(v_n) = 0$, *and* $\{v_n\}_n$ *converges to a non-zero limit.*

3. The Palais-Smale condition

Throughout this paper, we consider $N > ps$ with $s \in (0, 1)$ and $p \in (1, \infty)$, $M(0) = 0$ and we assume M and V satisfy (A1)–(A4), without further mentioning.

To apply Lemma 2.4, we discuss now the compactness property for the functional $\mathcal{J}_\lambda$, given by the Palais-Smale condition. We recall that $\{u_n\}_n \subset W^{s,p}_V(\mathbb{R}^N)$ is a Palais-Smale sequence for $\mathcal{J}_\lambda$ at level $c \in \mathbb{R}$ if

$$\mathcal{J}_\lambda(u_n) \to c \quad \text{and} \quad \mathcal{J}'_\lambda(u_n) \to 0 \quad \text{in } (W^{s,p}_V(\mathbb{R}^N))' \text{ as } n \to \infty. \tag{3.1}$$

We say that $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition at level c if any Palais-Smale sequence $\{u_n\}_n$ at level c admits a convergent subsequence in $W^{s,p}_V(\mathbb{R}^N)$.

Before going to prove Theorem 1.1, we first give some auxiliary lemmas.

Lemma 3.1 (see [39, Lemma 4.1])**.** *If* (A5) *holds, then there exist* $\rho \in (0, 1]$ *and* $\lambda_0 = \lambda_0(\rho) > 0$, $\ell = \ell(\rho)$, *such that* $\mathcal{J}_\lambda(u) \geq \ell > 0$ *for any* $u \in W^{s,p}_V(\mathbb{R}^N)$, *with* $\|u\|_s = \rho$, *and for all* $\lambda \leq \lambda_0$.

Set

$$c_\lambda = \inf\{\mathcal{J}_\lambda(u) : u \in \overline{B_\rho}\},$$

where $B_\rho = \{u \in W^{s,p}_V(\mathbb{R}^N) : \|u\|_s < \rho\}$ and $\rho \in (0, 1]$ is the number determined in Lemma 3.1.

Lemma 3.2 (see [39, Lemma 4.2])**.** *If* (A5) *and* (A6) *hold, then* $c_\lambda < 0$ *for each* $\lambda \in (0, \lambda_0]$.

Lemma 3.3. *If* (A5) *and* (A6) *hold, then there exists* $\lambda^* > 0$ *such that, up to a subsequence,* $\{u_n\}_n$ *strongly converges to some* u_λ *in* $W^{s,p}_V(\mathbb{R}^N)$ *for all* $\lambda \in (0, \lambda^*]$.

Proof. Because of the degenerate nature of (1.1), two situations must be considered: either $\inf_{n\in\mathbb{N}} \|u_n\|_s = d_\lambda > 0$ or $\inf_{n\in\mathbb{N}} \|u_n\|_s = 0$. For this, we divide the proof in two cases.

(i) *Case* $\inf_{n\in\mathbb{N}} \|u_n\|_s = d_\lambda > 0$. By Lemmas 3.1 and 3.2 and the Ekeland variational principle, applied in $\overline{B}_\rho$, there exists a sequence $(u_n)_n \subset B_\rho$ such that

$$c_\lambda \leq \mathcal{J}_\lambda(u_n) \leq c_\lambda + 1/n \quad \text{and} \quad \mathcal{J}_\lambda(v) \geq \mathcal{J}_\lambda(u_n) - \|u_n - v\|_s/n \tag{3.2}$$

for all $n \in \mathbb{N}$ and for any $v \in \overline{B}_\rho$. Fixed $n \in \mathbb{N}$, for all $v \in S_V$, where $S_V = \{u \in W^{s,p}_V(\mathbb{R}^N) : \|u\|_s = 1\}$, and for all $\sigma > 0$ so small that $u_n + \sigma v \in \overline{B}_\rho$, we have

$$\mathcal{J}_\lambda(u_n + \sigma v) - \mathcal{J}_\lambda(u_n) \geq -\frac{\sigma}{n}$$

by (3.2). Since $\mathcal{J}_\lambda$ is Gâteaux differentiable in $W^{s,p}_V(\mathbb{R}^N)$, we get

$$\langle \mathcal{J}'_\lambda(u_n), v\rangle = \lim_{\sigma\to 0} \frac{\mathcal{J}_\lambda(u_n + \sigma v) - \mathcal{J}_\lambda(u_n)}{\sigma} \geq -\frac{1}{n}$$

for all $v \in S_V$. Hence

$$|\langle \mathcal{J}'_\lambda(u_n), v\rangle| \leq \frac{1}{n},$$

since $\upsilon \in S_V$ is arbitrary. Consequently, $\mathcal{J}_\lambda'(u_n) \to 0$ in $(W_V^{s,p}(\mathbb{R}^N))'$ as $n \to \infty$ and clearly, up to a subsequence, the bounded sequence $(u_n)_n$ weakly converges to some $u_\lambda \in \overline{B}_\rho$ and has the following properties

$$\begin{gathered} u_n \rightharpoonup u_\lambda \quad \text{weakly in } W_V^{s,p}(\mathbb{R}^N), \quad \|u_n\|_s \to \alpha_\lambda, \\ \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx \to \kappa_\lambda, \\ u_n \to u_\lambda \text{ a.e. in } \mathbb{R}^N, \quad |u_n|^{p^*_{\mu,s}} \rightharpoonup |u_\lambda|^{p^*_{\mu,s}} \text{ weakly in } L^{\frac{p^*_s}{p^*_{\mu,s}}}(\mathbb{R}^N), \\ |u_n|^{p^*_{\mu,s}-2}u_n \rightharpoonup |u_\lambda|^{p^*_{\mu,s}-2}u_\lambda \quad \text{weakly in } L^{\frac{p^*_s}{p^*_{\mu,s}-1}}(\mathbb{R}^N), \end{gathered} \tag{3.3}$$

as $n \to \infty$, by [10, Lemma 2.1]. Clearly $\alpha_\lambda > 0$ since we are in the case in which $d_\lambda > 0$. Therefore $M(\|u_n\|_s^p) \to M(\alpha_\lambda^p) > 0$ as $n \to \infty$, by continuity of M and the fact that 0 is the unique zero of M.

By the Hardy-Littlewood-Sobolev inequality and that $p^*_s/p^*_{\mu,s} = 2N/(2N-\mu)$, the Riesz potential defines a linear continuous map $\mathcal{K}_\mu * (\,\cdot\,) : L^{\frac{p^*_s}{p^*_{\mu,s}}}(\mathbb{R}^N) \to L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Since $|u_n|^{p^*_{\mu,s}} \rightharpoonup |u_\lambda|^{p^*_{\mu,s}}$ weakly in $L^{\frac{p^*_s}{p^*_{\mu,s}}}(\mathbb{R}^N)$, then as $n \to \infty$

$$\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}} \rightharpoonup \mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}} \quad \text{weakly in } L^{\frac{2N}{\mu}}(\mathbb{R}^N). \tag{3.4}$$

Note that for any subset $U \subset \mathbb{R}^N$, it holds

$$\begin{aligned} \int_U ||u_n|^{p^*_{\mu,s}-2}u_n u_\lambda|^{\frac{2N}{2N-\mu}} dx &\le \int_U |u_n|^{(p^*_{\mu,s}-1)\frac{p^*_s}{p^*_{\mu,s}}} |u_\lambda|^{\frac{p^*_s}{p^*_{\mu,s}}} dx \\ &\le \|u_n\|_{p^*_s}^{\frac{p^*_{\mu,s}-1}{p^*_{\mu,s}}} \Big(\int_U |u_\lambda|^{p^*_s} dx\Big)^{\frac{1}{p^*_{\mu,s}}} \\ &\le C\Big(\int_U |u_\lambda|^{p^*_s} dx\Big)^{\frac{1}{p^*_{\mu,s}}}. \end{aligned}$$

This and $u_\lambda \in L^{p^*_s}(\mathbb{R}^N)$ imply that the sequence $\{||u_n|^{p^*_{\mu,s}-2}u_n u_\lambda|^{\frac{2N}{2N-\mu}}\}_n$ is equi-integrable in $L^1(\mathbb{R})$. Moreover, $|u_n|^{p^*_{\mu,s}-2}u_n u_\lambda \to |u_\lambda|^{p^*_{\mu,s}}$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Hence the Vitali convergence theorem yields that $|u_n|^{p^*_{\mu,s}-2}u_n u_\lambda \to |u_\lambda|^{p^*_{\mu,s}}$ strongly in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$. Thus,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}-2}u_n u_\lambda \, dx = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}} dx \tag{3.5}$$

by (3.4). Similarly,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}-2}u_\lambda u_n \, dx = \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}} dx. \tag{3.6}$$

For any subset $U \subset \mathbb{R}^N$, by (A5) we have

$$\int_U |f(x,u_n)(u_n - u_\lambda)| dx \le \|w\|_{L^{\frac{p^*_s}{p^*_s-q}}(U)} \|u_n - u_\lambda\|_{p^*_s} \le C\|w\|_{L^{\frac{p^*_s}{p^*_s-q}}(U)}.$$

It follows from $w \in L^{\frac{p^*_s}{p^*_s-q}}(\mathbb{R}^N)$ that sequence $\{f(x,u_n)(u_n - u_\lambda)\}_n$ is equi-integrable in $L^1(\mathbb{R}^N)$. Clearly, $f(x,u_n)(u_n - u_\lambda) \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Hence the Vitali convergence theorem implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u_\lambda)dx=0. \tag{3.7}$$

A similar argument shows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_\lambda)(u_n-u_\lambda)dx=0, \tag{3.8}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_\lambda\,dx=\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\,dx. \tag{3.9}$$

Furthermore, by (A6) for equi-integrability, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n)dx=\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_\lambda)dx. \tag{3.10}$$

Let us now introduce, for simplicity, for all $v\in W_V^{s,p}(\mathbb{R}^N)$ the linear functional $L(v)$ on $W_V^{s,p}(\mathbb{R}^N)$ defined by

$$\begin{aligned}\langle L(v),w\rangle&=\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}}\,dx\,dy\\&\quad+\int_{\mathbb{R}^N}V(x)|v|^{p-2}vwdx\\&=\langle v,w\rangle_{s,p}+\int_{\mathbb{R}^N}V(x)|v|^{p-2}vwdx\end{aligned}$$

for all $w\in W_V^{s,p}(\mathbb{R}^N)$. The Hölder inequality gives

$$|\langle L(v),w\rangle|\le[v]_s^{p-1}[w]_s+\|v\|_{p,V}^{p-1}\|w\|_{p,V}\le\|v\|_s^{p-1}\|w\|_s.$$

Thus, for each $v\in W_V^{s,p}(\mathbb{R}^N)$, the linear functional $L(v)$ is continuous on $W_V^{s,p}(\mathbb{R}^N)$. Hence, the weak convergence of $\{u_n\}_n$ in $W_V^{s,p}(\mathbb{R}^N)$ gives that

$$\lim_{n\to\infty}\langle L(u_\lambda),u_n-u_\lambda\rangle=0. \tag{3.11}$$

Since $\{u_n\}_n$ in bounded in $W_V^{s,p}(\mathbb{R}^N)$, then $\{L(u_n)\}_n$ is bounded in $(W_V^{s,p}(\mathbb{R}^N))'$. Hence there exist some functional $\xi\in(W_V^{s,p}(\mathbb{R}^N))'$ and a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, such that

$$\lim_{n\to\infty}\langle L(u_n),v\rangle=\langle\xi,v\rangle \tag{3.12}$$

for any $v\in W_V^{s,p}(\mathbb{R}^N)$. Then, $\langle\mathcal{J}_\lambda'(u_n),u_\lambda\rangle\to0$, (3.11) and (3.12) give

$$M(\alpha_\lambda^p)\langle\xi,u_\lambda\rangle=\lambda\int_{\mathbb{R}^N}f(x,u_\lambda)u_\lambda\,dx+\int_{\mathbb{R}^N}(\mathcal{K}*|u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}}dx. \tag{3.13}$$

By (3.13) and (A6), we have $\langle\xi,u_\lambda\rangle\ge0$.

Since $\{u_n\}_n$ is a (PS) sequence, we deduce from Theorem 2.2, (3.4)-(3.13) that

$$\begin{aligned}o(1)&=\langle\mathcal{J}_\lambda'(u_n)-\mathcal{J}_\lambda'(u_\lambda),u_n-u_\lambda\rangle\\&=M(\|u_n\|_s^p)\|u_n\|_s^p-M(\|u_n\|_s^p)\langle L(u_n),u_\lambda\rangle-M(\|u\|_s^p)\langle L(u_\lambda),u_n-u_\lambda\rangle\\&\quad-\lambda\int_{\mathbb{R}^N}[f(x,u_n)-f(x,u_\lambda)](u_n-u_\lambda)dx\\&\quad-\int_{\mathbb{R}^N}\Big[(\mathcal{K}_\mu*|u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}-2}u_n\end{aligned}$$

$$
\begin{aligned}
&\quad - (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}-2}u_\lambda\Big](u_n - u_\lambda)dx \\
&= M(\alpha_\lambda^p)[\alpha_\lambda^p - \langle \xi, u_\lambda\rangle] - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx + o(1) \\
&= M(\|u_n\|_s^p)\langle L(u_n) - L(u_\lambda), u_n - u_\lambda\rangle \\
&\quad - \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx + o(1).
\end{aligned} \tag{3.14}
$$

Hence the above inequality yields

$$
\begin{aligned}
&\lim_{n\to\infty} M(\|u_n\|_s^p)\langle L(u_n) - L(u_\lambda), u_n - u_\lambda\rangle \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx.
\end{aligned} \tag{3.15}
$$

Now, since $\{u_n\}_n$ is a minimizing $(PS)_{c_\lambda}$ sequence, by (3.15) we also get as $n \to \infty$

$$
\begin{aligned}
c_\lambda &= \frac{1}{p}\mathscr{M}(\|u_n\|_s^p) - \lambda \int_{\mathbb{R}^N} F(x, u_n)dx - \frac{1}{2p^*_{\mu,s}} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}} dx + o(1) \\
&\geq \frac{1}{\theta p} M(\|u_n\|_s^p)\|u_n\|_s^p - \lambda \int_{\mathbb{R}^N} F(x, u_n)dx \\
&\quad - \frac{1}{2p^*_{\mu,s}} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}} dx + o(1) \\
&= \frac{1}{\theta p} M(\|u_n\|_s^p)\langle L(u_n) - L(u_\lambda), u_n - u_\lambda\rangle + \frac{1}{\theta p} M(\|u_n\|_s^p)\langle L(u_n), u_\lambda\rangle \\
&\quad - \lambda \int_{\mathbb{R}^N} F(x, u_n)dx - \frac{1}{2p^*_{\mu,s}} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}} dx + o(1) \\
&= \frac{1}{\theta p} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx + \frac{1}{\theta p} M(\|u_n\|_s^p)\langle L(u_n), u_\lambda\rangle \\
&\quad - \lambda \int_{\mathbb{R}^N} F(x, u_n)dx - \frac{1}{2p^*_{\mu,s}} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}} dx + o(1).
\end{aligned} \tag{3.16}
$$

Note that $M(\|u_n\|_s^p)\langle L(u_n), u_\lambda\rangle = M(\alpha_\lambda^p)\langle \xi, u_\lambda\rangle + o(1)$. Then by (3.16) we get

$$
M(\|u_n\|_s^p)\langle L(u_n), u_\lambda\rangle = \lambda \int_{\mathbb{R}^N} f(x, u_\lambda)u_\lambda \, dx + \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}} dx + o(1).
$$

Inserting this equality in (3.16) and using Theorem 2.2, we deduce

$$
\begin{aligned}
c_\lambda &\geq \frac{1}{\theta p} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx \\
&\quad - \frac{1}{2p^*_{\mu,s}} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}} dx \\
&\quad + \frac{1}{\theta p} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}} dx + \frac{\lambda}{\theta p} \int_{\mathbb{R}^N} f(x, u_\lambda)u_\lambda \, dx \\
&\quad - \lambda \int_{\mathbb{R}^N} F(x, u_n)dx + o(1) \\
&\geq \frac{1}{\theta p} \int_{\mathbb{R}^N} (\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}} dx
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2p^*_{\mu,s}}\Big(\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n|^{p^*_{\mu,s}})|u_n|^{p^*_{\mu,s}}dx - \int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_\lambda|^{p^*_{\mu,s}})|u_\lambda|^{p^*_{\mu,s}}dx\Big)\\
&+ \frac{\lambda}{\theta p}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)dx + o(1)\\
\geq{}& \big(\frac{1}{\theta p} - \frac{1}{2p^*_{\mu,s}}\big)\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx\\
&+ \frac{\lambda}{\theta p}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_n)dx + o(1),
\end{aligned}
$$

thanks to $\theta p < 2p^*_{\mu,s}$. Hence

$$
\begin{aligned}
c_\lambda \geq{}& \big(\frac{1}{\theta p} - \frac{1}{2p^*_{\mu,s}}\big)\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx\\
&+ \frac{\lambda}{\theta p}\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\,dx - \lambda\int_{\mathbb{R}^N} F(x,u_\lambda)dx + o(1).
\end{aligned} \tag{3.17}
$$

Clearly, $|F(x,u_\lambda)| \leq w(x)|u_\lambda|^q$ for all $x \in \mathbb{R}^N$ and for all $\lambda \in (0,\lambda_0]$ thanks to (A5). In view of the choice of ρ in Lemma 3.1, we know that ρ is independent of λ. Thus, $\{u_\lambda\}_{\lambda\in(0,\lambda_0]}$ is uniformly bounded in $W^{s,p}_V(\mathbb{R}^N)$. Furthermore, there exists $C > 0$, which does not depend on λ, such that $\int_{\mathbb{R}^N} F(x,u_\lambda)dx \leq C$ and $\big|\int_{\mathbb{R}^N} f(x,u_\lambda)u_\lambda\,dx\big| \leq C$. Hence by (3.17), we deduce

$$
\big(\frac{1}{\theta p} - \frac{1}{2p^*_{\mu,s}}\big)\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx \leq c_\lambda + 2C\lambda + o(1).
$$

This and Lemma 3.2 imply that

$$
\lim_{\lambda\to 0}\kappa_\lambda = \lim_{\lambda\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx = 0. \tag{3.18}
$$

Let us now recall the well-known Simon inequalities. We refer to [41, Formula 2.2] (see also [16, p. 713]). There exist positive numbers C_p and $\widetilde{C}_p$, depending only on p, such that

$$
|\xi - \eta|^p \leq \begin{cases} C_p(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) & \text{for } p \geq 2\\ \widetilde{C}_p\big[(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) & \\ \times(\xi - \eta)\big]^{p/2}(|\xi|^p + |\eta|^p)^{(2-p)/2} & \text{for } 1 < p < 2, \end{cases} \tag{3.19}
$$

for all $\xi,\eta \in \mathbb{R}^N$. According to the Simon inequality, we divide the discussion into two cases.

Case $p \geq 2$. By (2.1) and (3.19), we have

$$
\begin{aligned}
&\langle L(u_n) - L(u_\lambda), u_n - u_\lambda\rangle\\
&\geq \frac{1}{C_p}\|u_n - u_\lambda\|_s^p\\
&\geq \frac{C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)}{C_p}\Big(\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx\Big)^{p/2p^*_{\mu,s}}.
\end{aligned} \tag{3.20}
$$

Combining (3.20) with (3.18), we have

$$
C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}M(\alpha_\lambda^p)\lim_{n\to\infty}\big(\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n - u_\lambda|^{p^*_{\mu,s}})|u_n - u_\lambda|^{p^*_{\mu,s}}dx\big)^{p/2p^*_{\mu,s}}
$$

$$\le \lim_{n\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n-u_\lambda|^{p^*_{\mu,s}})|u_n-u_\lambda|^{p^*_{\mu,s}}dx = \kappa_\lambda.$$

Hence,

$$C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}M(\alpha_\lambda^p)\kappa_\lambda^{p/2p^*_{\mu,s}} \le \kappa_\lambda. \tag{3.21}$$

Define

$$\lambda^* = \begin{cases}\inf\{\lambda\in(0,\lambda_0] : \kappa_\lambda>0\}, & \text{if } \kappa_\lambda\not\equiv 0,\\ \lambda_0, & \text{if } \kappa_\lambda\equiv 0.\end{cases}$$

If $\kappa_\lambda\not\equiv 0$, then $\lambda^*=\inf\{\lambda\in(0,\lambda_0]:\kappa_\lambda>0\}>0$. Otherwise, there exists a sequence $(\lambda_k)_k$, with $\kappa_{\lambda_k}>0$, such that $\lambda_k\to 0$ as $k\to\infty$. Thus, (3.21) implies that

$$\kappa_{\lambda_k}^{1-\frac{p}{2p^*_{\mu,s}}} \ge C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}M(\alpha_\lambda^p),$$

which means that $\alpha_{\lambda_k}\to 0$ as $k\to\infty$. Without loss of generality, we assume that $\alpha_{\lambda_k}\in(0,1]$ for all $k\ge 1$. Then

$$\begin{aligned} M(\alpha^p_{\lambda_k})\alpha^p_{\lambda_k} &\ge M(\alpha^p_{\lambda_k})[\alpha^p_{\lambda_k}-\langle\xi,u_{\lambda_k}\rangle]\\ &= \delta_{\lambda_k}\ge\Big[C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}M(\alpha^p_{\lambda_k})\Big]^{1/(1-p/2p^*_{\mu,s})}. \end{aligned}$$

Hence, (A4) gives

$$\alpha_{\lambda_k}^{p-\frac{p^2}{2p^*_{\mu,s}}} \ge C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}m_0^{\frac{p}{2p^*_{\mu,s}}}\alpha_{\lambda_k}^{\frac{p^2(\theta-1)}{2p^*_{\mu,s}}},$$

that is,

$$\alpha_{\lambda_k}^{p-\frac{\theta p}{2p^*_{\mu,s}}} \ge C^{-\frac{p}{2p^*_{\mu,s}}}(N,\mu)C_p^{-1}m_0^{\frac{p}{2p^*_{\mu,s}}}.$$

This is impossible, since $\theta<p^*_s/p<2p^*_{\mu,s}$. Therefore, $\kappa_\lambda=0$ for all $\lambda\in(0,\lambda^*]$, that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u_n-u_\lambda|^{p^*_{\mu,s}})|u_n-u_\lambda|^{p^*_{\mu,s}}dx = 0 \tag{3.22}$$

for all $\lambda\in(0,\lambda^*]$. This, (3.15) and inequality (3.19), give that $u_n\to u_\lambda$ strongly in $W^{s,p}_V(\mathbb{R}^N)$ as $n\to\infty$.

Also for the case $1<p<2$, we can argue as above, to obtain the existence of a threshold $\lambda^*>0$ such that $u_n\to u_\lambda$ strongly in $W^{s,p}_V(\mathbb{R}^N)$ as $n\to\infty$ for all $\lambda\in(0,\lambda^*]$.

(ii) *Case* $\inf_{n\in\mathbb{N}}\|u_n\|_s=0$. If 0 is an isolated point for the real sequence $\{\|u_n\|_s\}_n$, then there is a subsequence $\{u_{n_k}\}_k$ such that

$$\inf_{k\in\mathbb{N}}\|u_{n_k}\|_s=d>0,$$

and we can proceed as before. Otherwise, 0 is an accumulation point of the sequence $\{\|u_n\|_s\}_n$ and so there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ such that $u_{n_k}\to 0$ strongly in $W^{s,p}_V(\mathbb{R}^N)$ as $n\to\infty$.

In conclusion, $\mathcal{J}_\lambda$ satisfies the (PS) condition in $W^{s,p}_V(\mathbb{R}^N)$ at the level c_λ in all the possible cases. □

4. A truncation argument

We note that our functional $\mathcal{J}_\lambda$ is not bounded from below in $W^{s,p}_V(\mathbb{R}^N)$. Indeed, if $\varepsilon = 1$, it follows from (2.4) that

$$\mathscr{M}(t) \leq \mathscr{M}(1)t^\theta \quad \text{for all } t \geq 1. \tag{4.1}$$

Furthermore, $F(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ by (A5) and (A6). Let $u \in C_0^\infty(\mathbb{R}^N)$, with $u \geq 0$ a.e. in $\mathbb{R}^N$ such that $\|u\|_s = 1$. Then for all $t \geq 1$, we have

$$\begin{aligned} \mathcal{J}_\lambda(tu) &\leq \frac{1}{p}\mathscr{M}(1)t^{\theta p}\|u\|_s^{\theta p} - \frac{t^{2p^*_{\mu,s}}}{2p^*_{\mu,s}}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * u^{p^*_{\mu,s}})u^{p^*_{\mu,s}}dx \\ &\leq \frac{1}{p}\mathscr{M}(1)t^{\theta p}\|u\|_s^{\theta p} - \frac{t^{2p^*_{\mu,s}}}{2p^*_{\mu,s}}(S^*)^{-p^*_{\mu,s}}\|u\|_s^{2p^*_{\mu,s}} \\ &= \frac{1}{p}\mathscr{M}(1)t^{\theta p} - \frac{1}{2p^*_{\mu,s}}(S^*)^{-2p^*_{\mu,s}}t^{2p^*_{\mu,s}}. \end{aligned} \tag{4.2}$$

Hence, $\mathcal{J}_\lambda(tu) \to -\infty$ as $t \to \infty$, since $\theta < p^*_s/p < p^*_{\mu,s}$.

For this in the sequel we introduce a truncation like in [4], to get a special lower bound which will be worth to construct critical values for $\mathcal{J}_\lambda$. Let us denote

$$\mathcal{G}_\lambda(t) = \frac{\mathscr{M}(1)}{p}t^{p\theta} - \lambda C_w^q\, t^q - \frac{C(N,\mu)}{2p^*_{\mu,s}}t^{2p^*_{\mu,s}}$$

where

$$C_w^q := C_{p^*_s}^q \|w\|_{L^{\frac{p^*_s}{p^*_s - q}}(\mathbb{R}^N)},$$

while $C(N,\mu)$ is defined in (2.1). Denoting $m = m(1)$ the constant given by (A3) with $\tau = 1$, we can take $R_1 \in (0,1)$ sufficiently small such that

$$\frac{m}{p\theta}R_1^p \geq \frac{\mathscr{M}(1)}{p}R_1^{p\theta} > \frac{C(N,\mu)}{2p^*_{\mu,s}}R_1^{2p^*_{\mu,s}}, \tag{4.3}$$

since $p \leq p\theta < p^*_{\mu,s}$, and we define

$$\lambda_* = \frac{1}{2\,C_w^q\,R_1^q}\Big(\frac{\mathscr{M}(1)}{p}R_1^{p\theta} - \frac{C(N,\mu)}{2p^*_{\mu,s}}R_1^{2p^*_{\mu,s}}\Big), \tag{4.4}$$

so that $\mathcal{G}_{\lambda_*}(R_1) > 0$. From this, we consider

$$R_0 = \max\{t \in (0,R_1) :\ \mathcal{G}_{\lambda_*}(t) \leq 0\}.$$

Since by $q < p$ we have $\mathcal{G}_\lambda(t) \leq 0$ for t near to 0 and since also $\mathcal{G}_{\lambda_*}(R_1) > 0$, it easily follows that $\mathcal{G}_{\lambda_*}(R_0) = 0$.

We can choose $\psi \in C_0^\infty([0,\infty),[0,1])$ such that $\psi(t) = 1$ if $t \in [0,R_0]$ and $\psi(t) = 0$ if $t \in [R_1,\infty)$. Thus, we consider the truncated functional

$$\mathcal{I}_\lambda(u) = \frac{1}{p}\mathscr{M}(\|u\|_s^p) - \lambda\int_{\mathbb{R}^N}F(x,u)dx - \psi(\|u\|_s)\frac{1}{2p^*_{\mu,s}}\int_{\mathbb{R}^N}(\mathcal{K}_\mu * |u|^{p^*_{\mu,s}})|u|^{p^*_{\mu,s}}dx.$$

It immediately follows that $\mathcal{I}_\lambda(u) \to \infty$ as $\|u\|_s \to \infty$, by (A2) and (A3). Hence, $\mathcal{I}_\lambda$ is coercive and bounded from below.

Now, we prove a local Palais-Smale and a topological result for the truncated functional $\mathcal{I}_\lambda$.

Lemma 4.1. *There exists $\overline{\lambda} > 0$ such that, for any $\lambda \in (0,\overline{\lambda})$*

(i) *if* $\mathcal{I}_\lambda(u) \le 0$ *then* $\|u\|_s < R_0$ *and* $\mathcal{J}_\lambda(v) = \mathcal{I}_\lambda(v)$ *for any* v *in a sufficiently small neighborhood of* u*;*
(ii) $\mathcal{I}_\lambda$ *satisfies the* $(PS)_{c_\lambda}$ *condition on* $W_V^{s,p}(\mathbb{R}^N)$*.*

Proof. Considering λ_0, λ^* and λ_* given respectively by Lemma 3.1, 3.3 and (4.4), we choose $\overline{\lambda}$ sufficiently small such that $\overline{\lambda} \le \min\{\lambda_0, \lambda^*, \lambda_*\}$. Let $\lambda < \overline{\lambda}$.

For proving (i) we assume that $\mathcal{I}_\lambda(u) \le 0$. When $\|u\|_s \ge 1$, by using (A2), (A3) with $\tau = 1$ and $\lambda < \lambda_*$, we see that

$$\mathcal{I}_\lambda(u) \ge \frac{m}{p\theta}\|u\|_s^p - \frac{\lambda_*}{q} C_w^q \|u\|_s^q > 0,$$

where the last inequality follows by $q < p$ and because by $\mathcal{G}_{\lambda_*}(R_1) > 0$ and (4.3) we have

$$\frac{m}{p\theta} R_1^p - \frac{\lambda_*}{q} C_w^q R_1^q > 0.$$

Thus, we get the contradiction $0 \ge \mathcal{I}_\lambda(u) > 0$. Similarly, when $R_1 \le \|u\|_s < 1$, by using (2.3), (4.3) and $\lambda < \lambda_*$, we obtain

$$\mathcal{I}_\lambda(u) \ge \frac{\mathscr{M}(1)}{p}\|u\|_s^{p\theta} - \frac{\lambda_*}{q} C_w^q \|u\|_s^q > 0,$$

where the last inequality follows by $q < p \le p\theta$ and because by $\mathcal{G}_{\lambda_*}(R_1) > 0$ we have

$$\frac{\mathscr{M}(1)}{p} R_1^{p\theta} - \frac{\lambda_*}{q} C_w^q R_1^q > 0.$$

We get again the contradiction $0 \ge \mathcal{I}_\lambda(u) > 0$. When $\|u\|_s < R_1$, since $\phi(t) \le 1$ for any $t \in [0, \infty)$ and $\lambda < \lambda_*$, we have

$$0 \ge \mathcal{I}_\lambda(u) \ge \mathcal{G}_\lambda(\|u\|_s) \ge \mathcal{G}_{\lambda_*}(\|u\|_s),$$

and this yields $\|u\|_s \le R_0$, by definition of R_0. Furthermore, for any $u \in B(0, R_0/2)$ we have $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$.

Arguing exactly as Lemma 3.3 we know that $\mathcal{I}_\lambda$ satisfies the $(PS)_{c_\lambda}$ condition on $W_V^{s,p}(\mathbb{R}^N)$ for $\lambda < \lambda^*$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *For any* $\lambda > 0$ *and* $n \in \mathbb{N}$*, there exists* $\varepsilon = \varepsilon(\lambda, n) > 0$ *such that*

$$\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \ge n,$$

where $\mathcal{I}_\lambda^{-\varepsilon} = \{u \in W_V^{s,p}(\mathbb{R}^N) : \mathcal{I}_\lambda(u) \le -\varepsilon\}$.

Proof. Fix $\lambda > 0$, $n \in \mathbb{N}$. Let Y_n be a n-dimensional subspace of $W_V^{s,p}(\mathbb{R}^N)$. For any $u \in Y_n$, $u \ne 0$ write $u = r_n\phi$ with $\phi \in Y_n$, $\|\phi\|_s = 1$ and $\bar{\phi} = \int_\Omega |\phi|^\xi \, dx > 0$. Then, by (A6) and continuity of M, for all r_n, with $0 < r_n < \min\{1, \delta\}$, we have

$$\begin{aligned}
\mathcal{I}_\lambda(u) &= \frac{1}{p}\mathscr{M}(\|u\|_s^p) - \lambda\int_\Omega F(x,u)dx - \frac{1}{2p_{\mu,s}^*}\int_{\mathbb{R}^N} (\mathcal{K}_\mu * u^{p_{\mu,s}^*}) u^{p_{\mu,s}^*} dx \\
&\le \frac{1}{p}\Big(\sup_{0\le s\le r_n^p} M(s)\Big)\|\phi\|_s^p r_n^p - \lambda\Big(a_0\int_{B(x_0,2R)} |\varphi|^\xi dx\Big) r_n^\xi \\
&\quad - \frac{1}{2p_{\mu,s}^*}(S^*)^{-2p_{\mu,s}^*} r_n^{2p_{\mu,s}^*} \\
&= \frac{1}{p} M_1 r_n^p - \lambda a_0 \bar{\phi}\, r_n^\xi - \frac{1}{2p_{\mu,s}^*}(S^*)^{-2p_{\mu,s}^*} r_n^{2p_{\mu,s}^*} = \varepsilon_n,
\end{aligned}$$

where $M^* = \max_{\tau\in[0,R_0]} M(\tau) < \infty$, here we use all the norms are equivalent for finite dimensional Y_n. Hence, we can choose r_n so small that $\mathcal{I}_\lambda(u) < \varepsilon_n < 0$. Let

$$S_{r_n} = \left\{u \in W_V^{s,p}(\mathbb{R}^N); \|u\| = r_n\right\}.$$

Then $S_{r_n}\cap Y_n \subset \mathcal{I}_\lambda^{\varepsilon_m}$. Then, by Proposition 2.3 (2), we have $\gamma(\mathcal{I}_\lambda^{\varepsilon_m}) \geq \gamma(S_{r_n}\cap Y_n) \geq n$. Therefore, we can denote $\Gamma_n = \{A \in \Sigma; \gamma(A) \geq n\}$ and let

$$c_m := \inf_{A\in\Gamma_m} \sup_{u\in A} \widetilde{\mathcal{I}_\lambda}(u), \tag{4.5}$$

then

$$-\infty < c_n \leq \varepsilon_n < 0, \quad n \in \mathbb{N}, \tag{4.6}$$

because $\widetilde{\mathcal{I}_\lambda}^{\varepsilon_n} \in \Gamma_n$ and $\widetilde{\mathcal{I}_\lambda}$ is bounded from below. □

5. Proof of Theorem 1.1

Here we define for any $n \in \mathbb{N}$ the sets

$$\Sigma_n = \left\{A \subset W_V^{s,p}(\mathbb{R}^N) \setminus \{0\} : A \text{ is closed}, A = -A \text{ and } \gamma(A) \geq n\right\},$$
$$K_c = \left\{u \in W_V^{s,p}(\mathbb{R}^N) : \mathcal{I}_\lambda'(u) = 0 \text{ and } \mathcal{I}_\lambda(u) = c\right\},$$

and the number

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \mathcal{I}_\lambda(u). \tag{5.1}$$

Before proving our main result, we state some crucial properties of the family of numbers $\{c_n\}_{n\in\mathbb{N}}$.

Lemma 5.1. *For any $\lambda > 0$ and $n \in \mathbb{N}$, the number c_n is negative.*

Proof. Let $\lambda > 0$ and $n \in \mathbb{N}$. By Lemma 4.2, there exists $\varepsilon > 0$ such that $\gamma(\mathcal{I}_\lambda^{-\varepsilon}) \geq n$. Since also $\mathcal{I}_\lambda$ is continuous and even, $\mathcal{I}_\lambda^{-\varepsilon} \in \Sigma_n$. From $\mathcal{I}_\lambda(0) = 0$ we have $0 \notin \mathcal{I}_\lambda^{-\varepsilon}$. Furthermore $\sup_{u\in\mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon$. In conclusion, remembering also that $\mathcal{I}_\lambda$ is bounded from below, we get

$$-\infty < c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \mathcal{I}_\lambda(u) \leq \sup_{u\in\mathcal{I}_\lambda^{-\varepsilon}} \mathcal{I}_\lambda(u) \leq -\varepsilon < 0.$$

As desired. □

Lemma 5.2. *Let $\lambda \in (0, \overline{\lambda})$, with $\overline{\lambda}$ given in Lemma 4.1. Then, all c_n given by* (3.2) *are critical values for $\mathcal{I}_\lambda$ and $c_n \to 0$ as $n \to \infty$.*

The proof of the above lemma is similar to that of [44, Lemma 4.7], so we omit it.

Proof of Theorem 1.1. By Lemma 4.1, $\mathcal{I}_\lambda(u) = \mathcal{J}_\lambda(u)$ if $\mathcal{I}_\lambda(u) < 0$. Then, by Lemmas 4.1, 4.2, 5.1 and 5.2, one can see that all the assumptions of the new version of symmetric mountain pass lemma due to Kajikiya in [19] are satisfied. Hence, the proof is complete. □

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