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ENTIRE BOUNDED SOLUTIONS VERSUS FIXED POINTS FOR NONLINEAR ELLIPTIC EQUATIONS WITH INDEFINITE WEIGHT

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Abstract. We establish a necessary and sufficient condition for the existence of an entire distributional solution for a general class of nonlinear elliptic equations with variable potential and nondecreasing nonlinear term. Our result establishes the relationship between the Green function and the growths of the weight and of the nonlinear term. The main result also points out the connection with a fixed point problem for an integral operator.
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1. Introduction

Nonlinear elliptic problems on the whole space do not have necessarily a solution. The existence of solutions is in relationship not only with the nonlinearity but also with the growth of the variable potential involved in the problem.

In their seminal paper on entire solutions of sublinear elliptic equations, Brezis and Kamin [3] pointed out a striking phenomenon. They showed that a sublinear elliptic problem on the whole space has a solution if and only if a related *linear* partial differential equation depending only on the potential has a solution. Brezis and Kamin considered the nonlinear problem

$$-\Delta u = \rho(x)\, u^{\alpha}, \quad x \in \mathbb{R}^N \ (N \geqslant 3), \tag{1.1}$$

with $0 < \alpha < 1$, $\rho \in L^{\infty}_{\text{loc}}(\mathbb{R}^N) \setminus \{0\}$, $\rho \geqslant 0$.

The main result in [3] establishes that problem (1.1) has a bounded positive solution if and only if the linear equation

$$-\Delta u = \rho(x), \quad x \in \mathbb{R}^N$$

has a bounded solution, namely if the mapping

$$\mathbb{R}^N \ni x \longmapsto \int_{\mathbb{R}^N} \frac{\rho(y)}{|x-y|^{N-2}}\,dy$$

is in $L^\infty(\mathbb{R}^N)$. This result points out the relationship between the growth of the potential and the Green function on the whole space.

The analysis developed in [3] shows that a bounded solution of problem (1.1) exists for potentials like

$$\rho(x) = \frac{1}{1+|x|^q} \quad \text{or} \quad \rho(x) = \frac{1}{(1+|x|^2)\,|\log(2+|x|)|^q} \quad (q>2),$$

while no solution exists if

$$\rho(x) = \frac{1}{1+|x|^q} \quad \text{with } q \leqslant 2\,.$$

Brezis and Kamin [3] proved also that a stronger nonexistence result holds, provided that

$$\int_{|x|\geqslant 1} \frac{\rho(x)}{|x|^{N-2}}dx = +\infty.$$

In such a case there is no function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying

$$\begin{cases} -\Delta u = \rho(x)\,u^\alpha & \text{in } \mathbb{R}^N \\ u \geqslant 0 & \text{in } \mathbb{R}^N. \end{cases}$$

In the present paper, we are concerned with a related problem on the whole space. The main novelties in our approach are the following:

(a) We study a class of nonlinear elliptic equations with general nonlinear term, which is not only of power-type.

(b) We consider a reversed sign for the variable potential with respect to the framework considered by Brezis and Kamin [3]. Due to this assumption, we are not concerned with super-harmonic functions, and not with subharmonic functions defined on the whole Euclidean space.

(c) The methods that we develop in the present paper can be applied to larger classes of differential operators. Indeed, our proofs remain valid in the general setting of A-Laplace operators formulated by Pucci and Serrin [11, 12], namely differential operators of the type

$$\operatorname{div}\,(A(|\nabla u|)\nabla u), \tag{1.2}$$

where $A = A(\rho) \in C(0,\infty)$ and

$$\rho \longmapsto \rho A(\rho) \text{ is strictly increasing in } (0,\infty) \text{ and } \rho A(\rho) \to 0 \text{ as } \rho \to 0.$$

This framework includes the degenerate p-Laplace operator ($A(\rho) = \rho^{p-2}$, $p>1$) and the mean curvature operator ($A(\rho) = 1/\sqrt{1+\rho^2}$).

(d) Our necessary and sufficient condition for the existence of an entire distributional solution is established in terms of the growth of the variable potential and the nonlinear term, in relationship with the Green function on the whole space. The entire solution appears as the fixed point of a suitable integral operator.

Our paper extends some recent results from [10], where it is studied only the sublinear case corresponding to $f(u) = u^\rho$ with $\rho \in (0,1)$. The analysis carried out in the present paper allows to extend the results in [10] to larger classes of nonlinearities, including to those with linear behaviour like

$$f(u) = \frac{u}{\log(u+1)}, \quad f(u) = \frac{u}{\sqrt{u^2+1}} \quad \text{or} \quad f(u) = \frac{\arctan u}{u}\,.$$

We also point out that general classes of nonlinear elliptic problems on the whole space can be treated by means of our results, including problems with superlinear terms. The monotonicity assumption of the nonlinear term is necessary only in some cases. As in [9] and [10], our main result establishes a necessary and sufficient condition for the existence of bounded solutions in terms of the relationship between the Green function and the growths of the indefinite potential and the nonlinear term. This idea is used in the previous papers [4] [5], [6], [7], [8], and [13]. We also refer to the works [14], [15], [16] for a relationship between solutions of nonlinear partial differential equations and fixed points of related nonlinear operators.

Consider the nonlinear problem

$$\begin{cases} \Delta u = V(x)\, f(u) & \text{in } \mathbb{R}^N \ (N \geqslant 3) \\ u \geqslant 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N\,, \end{cases} \tag{1.3}$$

where $V \geqslant 0$ is a nontrivial weight and $f \geqslant 0$ is nondecreasing.

We point out that problem (1.3) is not of interest if $N \in \{1,2\}$. Indeed, the solution should be subharmonic and nonnegative. Thus by the Liouville theorem, u is constant provided that $N \in \{1,2\}$, which attracts $V = 0$ in $\mathbb{R}^N$.

Throughout this paper we assume that $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \setminus \{0\}$, $V \geqslant 0$, and $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Borel function.

We are interested in distributional solutions of problem (1.3), namely solutions in $\mathcal{D}'(\mathbb{R}^N)$. Thus we say that $u : \mathbb{R} \to \mathbb{R}$ is a solution of problem (1.3) if u is continuous,

$$\text{the mapping } \mathbb{R}^N \ni x \longmapsto V(x) f(u(x)) \text{ is in } L^1_{\mathrm{loc}}(\mathbb{R}^N),$$

and for all $v \in C^2_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u \Delta v dx = \int_{\mathbb{R}^N} V(x) f(u) v dx.$$

2. Case of bounded domains

The basic idea to solve problem (1.3) is to approximate it with equations on bounded domains of higher and higher measure. The most natural is to consider open balls $B(0,n)$ in $\mathbb{R}^N$ and then to take $n \to \infty$. We perform our analysis on balls of the Euclidean space even if similar existence results hold for more general open bounded sets $\Omega \subset \mathbb{R}^N$. We refer to *regular* domains Ω, which are characterized by the property that all continuous function $g : \partial\Omega \to \mathbb{R}$ admits a harmonic extension $H(g;\Omega)$ to Ω. In other words, these are domains Ω for which the problem

$$\begin{cases} \Delta H(g;\Omega) = 0 & \text{in } \Omega \\ H(g;\Omega) = g & \text{on } \partial\Omega \end{cases}$$

has a solution.

We recall that a function $u : \Omega \to \mathbb{R}$ is super-harmonic if the following properties hold:

(i) u is lower-semicontinuous;

(ii) $H(u;\omega) \leqslant u$ in ω, for all $\omega \subset\subset \Omega$.

The function $u : \Omega \to \mathbb{R}$ is sub-harmonic if $-u$ is super-harmonic.

2.1. **Formulation of the problem.** Throughout this section we assume the following hypotheses:

$$V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \setminus \{0\} \text{ and } V \geqslant 0 \text{ in } \mathbb{R}^N; \tag{2.1}$$

$$f : [0,+\infty) \to [0,+\infty) \text{ is Borel measurable, nondecreasing, nontrivial, and } f(0) = 0. \tag{2.2}$$

Let $\Omega \subset \mathbb{R}^N$ be a regular domain. We are interested in distributional solutions of the nonlinear problem

$$\begin{cases} \Delta u = V(x)\, f(u) & \text{in } \mathcal{D}'(\Omega) \\ u \geqslant 0,\ u \neq 0 & \text{in } \Omega\,. \end{cases} \tag{2.3}$$

This means that $u : \Omega \to [0,+\infty)$ is a continuous nontrivial function such that $Vf(u) \in L^1_{\mathrm{loc}}(\Omega)$ and for all $v \in C^2_c(\Omega)$

$$\int_\Omega u\Delta v dx = \int_\Omega V(x) f(u) v dx.$$

Let $\Omega \subset \mathbb{R}^N$ be a ball (or even a regular bounded domain) such that $V \neq 0$ in Ω. Assume that

$$g : \overline{\Omega} \to [0,+\infty) \text{ is a continuous notrivial function.} \tag{2.4}$$

Consider the nonlinear elliptic problem

$$\begin{cases} \Delta u = V(x)\, f(u) & \text{in } \mathcal{D}'(\Omega) \\ u = g & \text{on } \partial\Omega \\ u \geqslant 0,\ u \neq 0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

We say that $\underline{u} : \overline{\Omega} \to \mathbb{R}$ is a lower-solution of

$$\Delta u = V(x)\, f(u) \qquad \text{in } \mathcal{D}'(\Omega) \tag{2.6}$$

if $\underline{u}$ is continuous, $Vf(\underline{u}) \in L^1_{\mathrm{loc}}(\Omega)$, and for all $v \in C^2_c(\Omega)$ with $v \geqslant 0$

$$\int_\Omega u\Delta v dx \geqslant \int_\Omega V(x) f(u) v dx.$$

The function $\overline{u} : \Omega \to \mathbb{R}$ is an upper-solution of problem (2.6) if $-\overline{u}$ is a lower-solution of this problem.

The weak comparison principle asserts that if $\underline{u}$ (resp., $\overline{u}$) is lower-solution (resp., upper-solution) of problem (2.6) such that (2.2) holds and

$$\liminf_{x \to y} (\overline{u} - \underline{u})(x) \geqslant 0 \quad \text{for all } y \in \partial\Omega,$$

then $\underline{u} \leqslant \overline{u}$ in ω. We refer to Pucci and Serrin [12] for details (see also Cîrstea and Rădulescu [6]).

2.2. **The main existence result.** In this section, we prove an existence and uniqueness property, which extends Theorem 5 in Cîrstea and Rădulescu [5] and Theorem A.1 in Cîrstea and Rădulescu [6]. We state the main result in the framework of balls, even if it still holds for bounded regular domains $\Omega \subset \mathbb{R}^N$ that have a Green function.

Let G be the Green function of Ω. Then for all $y \in \Omega$

$$-\Delta G(\cdot, y) = \delta_y \quad \text{in } \Omega,$$

where δ_y is the Dirac mass at y.

If $u : \Omega \to \mathbb{R}$ is a Borel measurable function, we define the integral operator

$$G_\Omega u(x) := \int_\Omega G(x,y)u(y)dy$$

for all $x \in \Omega$ such that the integral makes sense.

Cf. Armitage and Gardiner [1] for all Borel function $u \geqslant 0$, the following alternative holds:

(i) either $G_\Omega u(x) = +\infty$ for all $x \in \Omega$

or

(ii) 0 is the only nonnegative harmonic function bounded above by $G_\Omega u$ in Ω.

The operator G_Ω has the following additional properties:

(G_1) G_Ω is compact as defined on the space of bounded Borel functions;

(G_2) $G_\Omega u$ is a bounded Borel function, provided that u is bounded. Moreover,

$$\lim_{x \to y} G_\Omega u(x) = 0 \quad \text{for all } y \in \partial\Omega.$$

This operator will play a central role in the construction of the solution of problem (2.5). This solution will be obtained as the fixed point of an operator that involves both G_Ω and the harmonic extension operator.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a ball. Assume that hypotheses* (2.1), (2.2), (2.4) *are fulfilled, and $V \neq 0$ in Ω. Then problem* (2.5) *has a unique distributional solution.*

Proof. Let us first show that if a solution exists then it must be unique. Assume that u_1 and u_2 are distributional solutions of problem (2.5) and denote

$$\omega := \{x \in \Omega;\ u_1(x) < u_2(x)\}.$$

Set $u := u_1 - u_2$. Assuming by contradiction that $\omega \neq \emptyset$ we have $u > 0$ in ω and

$$\Delta u = V(x)(f(u_1) - f(u_2)) \leqslant 0 \quad \text{in } \mathcal{D}'(\omega).$$

Thus u is a superharmonic function in ω satisfying

$$\liminf_{x \to y} u(x) \geqslant 0 \quad \text{for all } y \in \partial\Omega.$$

Using the maximum principle, we obtain $u \geqslant 0$. This contradicts the definition of ω. We deduce that $\omega = \emptyset$, hence $u_1 \geqslant u_2$ in Ω. A similar argument shows that $u_1 \leqslant u_2$ in Ω. This shows that $u_1 = u_2$.

In order to establish the existence of the solution, we see that 0 is a lower-solution and n is an upper-solution (for n large enough). Thus a solution of problem (2.5) exists but we cannot guarantee that it is nontrivial. In fact, as remarked by Pucci and Serrin [12], the *dead core phenomenon* can occur, namely the solution exists, is nonnegative but it vanishes in an interior region. This is characteristic to the sublinear framework, which is including in the setting of the present paper. In such a case, the maximum principle does not hold. In fact, as established by Pucci and Serrin [11, 12], the maximum principle holds for operators like in (1.2) if and only if $f > 0$ on some neighbourhood $(0, \delta)$ and

$$\int_0^\delta \frac{ds}{H^{-1}(F(s))} = +\infty,$$

where $F(s) := \int_0^t f(t)dt$ and

$$H(\rho) := \rho^2 A(\rho) - \int_0^\rho sA(s)ds, \qquad \rho \geqslant 0.$$

The following auxiliary result extends Lemma 1 in El Mabrouk [10] to more general nonlinearities. This result asserts that solutions of problem (2.5) are fixed points of the integral operator

$$Tu(x) = H(g;\omega)(x) - \int_\Omega G(x,y)V(y)f(u(y))dy$$

defined on the space of locally bounded nonnegative Borel functions.

Lemma 2.2. *Let $u : \Omega \to \mathbb{R}$ be a locally bounded nonnegative Borel function. Then u is a solution of problem* (2.5) *if and only if for all open regular subset $\omega \subset\subset \Omega$*

$$u(x) + \int_\omega G(x,y)V(y)f(u(y))dy = H(g;\omega)(x) \quad \textit{for all } x \in \omega.$$

Proof of Lemma 2.2. By the uniqueness of the harmonic extension, it is enough to show that the function

$$v(x) := u(x) + \int_\omega G(x,y)V(y)f(u(y))dy \quad \text{for all } x \in \omega \tag{2.7}$$

is harmonic, for all open regular set $\omega \subset\subset \Omega$. Fix such a subset ω. We first observe that since u is bounded, then v is a bounded function. Using now the definition of v we deduce that v is continuous if and only if u is continuous. Fix $\varphi \in C^2_{\mathrm{loc}}(\omega)$. Multiplying by $\Delta\varphi(x)$ in the expression of v and integrating in ω, we find

$$\int_\omega v\Delta\varphi dx = \int_\omega u\Delta\varphi dx - \int_\omega V(x)f(u(x))\varphi dx.$$

This relation shows that v is harmonic in ω if and only if u is a solution of problem (2.5). The proof of Lemma 2.2 is concluded. □

Returning to the proof of Theorem 2.1, set $M := \|g\|_{L^\infty(\partial\Omega)}$ and define the bounded function $\zeta : \mathbb{R} \to \mathbb{R}$ by

$$\zeta(s) = \begin{cases} \min\{f(s), f(M)\} & \text{if } s \geqslant 0 \\ 0 & \text{if } s < 0. \end{cases}$$

Define the functions $u_n : \omega \to \mathbb{R}$ by $u_0 = 0$ and

$$u_n = H(g;\omega) - z_n \quad \text{for all } n \geqslant 1,$$

where

$$z_n(x) := G_\omega(V\zeta(u_{n-1}))(x) = \int_\omega G(x,y)V(y)\zeta(u_{n-1}(y))dy \quad \text{for all } n \geqslant 1.$$

Since $\zeta \in L^\infty(\mathbb{R})$, we obtain that (z_n) is a bounded sequence. Using the fact that the operator G_ω is compact, we can assume up to a subsequence that (z_n) converges in $L^\infty(\Omega)$ to $u : \omega \to \mathbb{R}$. Taking now $n \to \infty$ in the definition of u_n we obtain

$$u(x) = H(g;\omega)(x) - \int_\omega G(x,y)V(y)f(u(y))dy,$$

for all open regular set $\omega \subset\subset \Omega$. By Lemma 2.2 we obtain that u is a distributional solution of problem (2.5). □

The hypothesis that f is nondecreasing has been crucial to prove the uniqueness of the solution in Theorem 2.1. This result establishes the existence of a unique *nonnegative* solution. In view of the compact support property related to the maximum principle (see Pucci and Serrin [12]), the fact that this solution is *positive* strongly depends on the growth of the nonlinear term f. In fact, if f has a superlinear behaviour (say, $f(u) = u^p$ with $p > 1$) then the unique solution of problem (2.5) is *positive*. If f has a sublinear decay (say, $f(u) = u^p$ with $0 < p < 1$) then this solution is nonnegative but not necessarily positive. Indeed, let us consider $\Omega = B(0,1) \subset \mathbb{R}^N$ $(N \geqslant 3)$, $f(u) = u^{\gamma/(\gamma+2)}$ $(\gamma > 0)$, $V = 1$, and

$$g = \left[(\gamma+2)^2 + (N-2)(\gamma+2)\right]^{-(\gamma+2)/2}.$$

Then problem (2.5) admits the unique solution $u(x) = g\,|x|^{\gamma+2} \geqslant 0$, which vanishes at the origin.

3. Case of the whole space

In this section, we analyze the existence of a solution of problem (1.3). This is performed in relationship with the growth of the potential V combined with the Green function of the whole space. As in [5], the main idea is to approximate problem (1.3) with problems on balls $B(0,n)$ and then to study the asymptotic behaviour as $n \to \infty$.

We start with an useful result, which extends to our subharmonic setting a classical property pf harmonic functions. The Baire theorem implies that if u is the pointwise limit of a sequence of harmonic functions on Ω, then u is harmonic on a dense subset of Ω. The following result establishes a related property for a sequence of nonnegative solutions of problem (2.5). The proof makes use of the characterization property established in Lemma 2.2.

Lemma 3.1. *Assume that u is the pointwise limit of a sequence of nonnegative, locally uniformly bounded solutions of equation* (2.5)*. Then u is also a solution of equation* (2.5)*.*

Proof. Let $\omega \subset\subset \Omega$ an arbitrary open regular set. Applying Lemma 2.2 we obtain for all $x \in \omega$

$$u_n(x) + \int_\omega G(x,y)V(y)f(u_n(y))dy = H(u_n|_{\partial\Omega};\omega)(x). \tag{3.1}$$

Set $v_n := H(u_n|_{\partial\Omega};\omega)$. Then (v_n) is a bounded sequence of harmonic functions on ω, which converges pointwise. Thus (v_n) converges to a harmonic function. Passing to the limit as $n \to \infty$ in relation (3.1) we obtain for all $x \in \omega$

$$u(x) + \int_\omega G(x,y)V(y)f(u(y))dy = H(u|_{\partial\Omega};\omega)(x),$$

hence is a solution of problem (2.5). □

Consider the nonlinear elliptic problem

$$\begin{cases} \Delta u = V(x)\, f(u) & \text{in } \mathcal{D}'(B(0,n)) \\ u = 1 & \text{if } |x| = n \\ u \geqslant 0 & \text{in } B(0,n). \end{cases} \tag{3.2}$$

As established in Theorem 2.1, problem (3.2) has a unique solution u_n. By the maximum principle we have $u_n \leqslant 1$. Set

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } |x| \leqslant n \\ 1 & \text{if } |x| > n. \end{cases}$$

Using again the maximum principle we deduce that $(\tilde{u}_n)$ is a nonincreasing sequence of nonnegative functions.

Proposition 3.2. *Let*

$$u(x) := \inf\{\tilde{u}_n(x);\ n \geqslant 1\}. \tag{3.3}$$

Then the following properties hold.

(i) We have

$$\Delta u = V(x)f(u) \quad \textit{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.4}$$

(ii) The following alternative is true: either

(a) $u = 0$

or

(b) $\sup_{x\in\mathbb{R}^N} u(x) = 1$ *and for all* $x \in \mathbb{R}^N$

$$u(x) + \int_{\mathbb{R}^N} G(x,y)V(y)f(u(y))dy = 1. \tag{3.5}$$

Proof. (i) By definition, u is the pointwise limit of a sequence (u_n) of nonnegative bounded functions satisfying equation (2.6). Applying Lemma 3.1 we deduce that u solves the same equation.

(ii) If $\sup_{x\in\mathbb{R}^N} u(x) = 0$ then assertion (a) holds.

Assume that $\sup_{x\in\mathbb{R}^N} u(x) = M \in (0,1]$. We first deduce that $M = 1$. For this purpose it is enough to show that

$$u \leqslant M u_n \quad \text{for all } n \geqslant 1. \tag{3.6}$$

Consider the boundary value problems

$$\begin{cases} \Delta v = V(x)\, f(v) & \text{in } \mathcal{D}'(B(0,n)) \\ v = \dfrac{u}{M} & \text{if } |x| = n \\ v \geqslant 0 & \text{in } B(0,n) \end{cases}$$

and

$$\begin{cases} \Delta w = V(x)\, f(w) & \text{in } \mathcal{D}'(B(0,n)) \\ w = u & \text{if } |x| = n \\ w \geqslant 0 & \text{in } B(0,n). \end{cases}$$

By Theorem 2.1, these problems have unique solution. We first obtain

$$\begin{cases} \Delta(v - u_n) = V(x)\,(f(v) - f(u_n)) & \text{in } \mathcal{D}'(B(0,n)) \\ v - u_n \leqslant 0 & \text{if } |x| = n. \end{cases}$$

Since f is increasing we deduce by the maximum principle that

$$v \leqslant u_n \quad \text{in } B(0,n). \tag{3.7}$$

On the other side we have

$$\begin{cases} \Delta(Mv - w) \leqslant V(x)\,(f(Mv) - f(w)) & \text{in } \mathcal{D}'(B(0,n)) \\ Mv - w = 0 & \text{if } |x| = n. \end{cases}$$

Using again the maximum principle in combination with the information that $M \in (0,1]$ we find

$$Mv \geqslant w \quad \text{in } B(0,n). \tag{3.8}$$

Combining relations (3.7) and (3.8), we obtain (3.6).

It remains to prove relation (3.5). Since u solves (3.4), we find by Lemma 2.2 that for all $x \in B(0,n)$

$$u(x) + \int_{B(0,n)} G_{B(0,n)}(x,y)V(y)f(u(y))dy = H(u; B(0,n))(x) \leqslant 1\,. \tag{3.9}$$

Let (z_n) be the sequence of harmonic bounded functions defined by

$$z_n := H(u; B(0,n)) \quad \text{in } B(0,n).$$

Then $z_n \leqslant z_{n+1} \leqslant 1$ in $B(0,n)$, hence $z : \mathbb{R}^N \to \mathbb{R}$ defined by

$$z(x) := \sup_n z_n(x) \quad \text{for all } x \in \mathbb{R}^N$$

is a harmonic bounded function. Using the Liouville theorem we have $z \equiv C \leqslant 1$. Since $\sup_{\mathbb{R}^N} u = 1$, we deduce that $C = 1$ hence $z(x) = 1$ for all x.

On the other hand, using the properties of the Green function we have

$$\sup_n G_{B(0,n)}(x,y) = G_{\mathbb{R}^N}(x,y).$$

Passing now to the limit as $n \to \infty$ in relation (3.9) we obtain (3.5). □

Conversely, we prove that if problem (3.4) admits a nonnegative nontrivial bounded solution, then the function u defined by relation (3.3) is nontrivial and it is a fixed point of the integral operator

$$Sv(x) := 1 - \int_{\mathbb{R}^N} G(x,y)V(y)f(v(y))dy.$$

More precisely, the following property holds.

Proposition 3.3. *Assume that problem* (3.4) *admits a nonnegative nontrivial bounded solution. Then the function u defined by* (3.3) *satisfies relation* (3.5).

Proof. Recall that

$$u(x) := \inf\{\tilde{u}_n(x);\ n \geqslant 1\},$$

where u_n is the unique solution of the Dirichlet problem (3.2). Using Lemma 2.2 we have for all $x \in B(0,n)$

$$u_n(x) + \int_{B(0,n)} G_{B(0,n)}(x,y)V(y)f(u_n(y))dy = 1.$$

This shows that in order to conclude the proof, it is enough to show that

$$\int_{B(0,n)} G_{B(0,n)}(x,y)V(y)f(u_n(y))dy \to \int_{\mathbb{R}^N} G_{\mathbb{R}^N}(x,y)V(y)f(u(y))dy \quad \text{as } n \to \infty. \tag{3.10}$$

Let U be a bounded solution of problem (3.4). By rescaling, we can assume without loss of generality that $1 < \sup_{\mathbb{R}^N} U < +\infty$. In order to prove (3.10), we consider the auxiliary problem

$$\begin{cases} \Delta z_n = V(x)\, f(z_n) & \text{in } \mathcal{D}'(B(0,n)) \\ z_n = \sup U & \text{if } |x| = n \\ z_n \geqslant 0 & \text{in } B(0,n). \end{cases}$$

Using Lemma 2.2 we obtain for all $x \in \mathbb{R}^N$

$$z_n(x) + \int_{B(0,n)} G(x,y)V(y)f(z_n(y))dy = \sup U.$$

Using now Proposition 3.2 we have

$$\int_{B(0,n)} G(x,y)V(y)f(z_n(y))dy = \sup U - z_n(x) \to \int_{\mathbb{R}^N} G(x,y)V(y)f(z(y))dy,$$

where $z := \inf z_n$. We point out that $z \neq 0$, since $z_n \geqslant u_n$ by the maximum principle. Using now the Lebesgue dominated convergence theorem we obtain (3.10). □

Combining Propositions 3.2 and 3.3 we obtain the main result of this paper, which establishes a characterization property for the existence of bounded solutions of problem (1.3).

Theorem 3.4. *Let*

$$\tilde{u}_n(x) := \begin{cases} u_n(x) & \text{if } |x| \leqslant n \\ 1 & \text{if } |x| > n, \end{cases}$$

where u_n is the unique solution of the problem

$$\begin{cases} \Delta u = V(x)\, f(u) & \text{in } \mathcal{D}'(B(0,n)) \\ u = 1 & \text{if } |x| = n \\ u \geqslant 0 & \text{in } B(0,n). \end{cases}$$

Define

$$u(x) := \inf\{\tilde{u}_n(x);\ n \geqslant 1\}.$$

Then problem (1.3) *admits a bounded distributional solution if and only if*

$$\sup_{x \in \mathbb{R}^N} u(x) = 1.$$

In such a case, u satisfies the additional property

$$u(x) + \int_{\mathbb{R}^N} G(x,y)V(y)f(u(y))dy = 1 \quad \textit{for all } x \in \mathbb{R}^N.$$

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