Eigenvalue problems for anisotropic discrete boundary value problems*

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Abstract. In this paper, we prove the existence of a continuous spectrum for a family of discrete boundary value problems. The main existence results are obtained by using critical point theory. The equations studied in the paper represent a discrete variant of some recent anisotropic variable exponent problems which deserve as models in different fields of mathematical physics.

2000 Mathematics Subject Classification: 47A75, 35B38, 35P30, 34L05, 34L30.

Key words: eigenvalue problem, discrete boundary value problem, critical point, weak solution, continuous spectrum.

1 Introduction and main results

This paper is concerned with the study of the existence of solutions for the discrete boundary value problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda|u(k)|^{q(k)-2}u(k), \quad k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0\,, \end{cases} \tag{1}$$

where $T \geq 2$ is a positive integer and $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator. Here and hereafter, we denote by $\mathbb{Z}[a,b]$ the discrete interval $\{a, a+1, ..., b\}$ where a and b are integers and $a < b$. Moreover, in this paper we assume that functions $p : \mathbb{Z}[0,T] \to [2,\infty)$ and $q : \mathbb{Z}[1,T] \to [2,\infty)$ are bounded while λ is a positive constant.

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The study of discrete boundary value problems has captured special attention in the last years. We just refer to the recent results of Agarwal *et al.* [2], Yu and Guo [22], Cai and Yu [4], Zhang and Liu [23] and the references therein. The studies regarding such type of problems can be placed at the interface of certain mathematical fields such as nonlinear partial differential equations and numerical analysis. On the other hand, they are strongly motivated by their applicability in mathematical physics. We note that the problem (1) is the discrete variant of the variable exponent anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, and $p_i(x)$, $q(x)$ are continuous on $\overline{\Omega}$ such that $N > p_i(x) \geq 2$ and $q(x) > 1$ for any $x \in \overline{\Omega}$ and all $i \in \mathbb{Z}[1, N]$. Problem (2) was recently analyzed by Mihăilescu-Pucci-Rădulescu in [10, 11] (see also the studies in [8, 16, 17, 20, 21] for the case when $p_i(x)$ are constant functions). Problems like (2) have been intensively studied in the last decades since they can model various phenomena arising from the study of elastic mechanics (see, Zhikov [24]), electrorheological fluids (see, Acerbi and Mingione [1], Diening [6], Halsey [9], Ruzicka [18], Mihăilescu and Rădulescu [12, 13, 14, 15]) or image restoration (see, Chen, Levine and Rao [5]).

In this paper our goal is to use the critical point theory in order to establish the existence of a continuous spectrum of eigenvalues for problems of type (1). Our idea is to transfer the problem of the existence of solutions for problem (1) into the problem of existence of critical points for some associated energy functional. On the other hand, we point out that, to our best knowledge, discrete problems like (1), involving anisotropic exponents, have not yet been discussed. Thus, the present paper can be regarded as a contribution in this direction.

We are interested in finding week solutions for problems of type (1). For this purpose we define the function space

$$H = \{u : \mathbb{Z}[0, T+1] \to \mathbb{R};\ \text{such that}\ u(0) = u(T+1) = 0\}.$$

Clearly, H is a T-dimensional Hilbert space (see [2]) with the inner product

$$(u, v) = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u, v \in H\,.$$

This associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

By a *weak solution* for problem (1) we understand a function $u \in H$ such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) - \lambda \sum_{k=1}^{T} |u(k)|^{q(k)-2}u(k)v(k) = 0\,,$$

for any $v \in H$.

Denote for short $\max_{k\in\mathbb{Z}[a,b]} p(k)$ by $\max_{\mathbb{Z}[a,b]} p$ and $\min_{k\in\mathbb{Z}[a,b]} p(k)$ by $\min_{\mathbb{Z}[a,b]} p$.

The main results of this paper are the following.

Theorem 1. *Assume that functions p and q verify the hypothesis*

$$\max_{\mathbb{Z}[0,T]} p < \min_{\mathbb{Z}[1,T]} q \,. \tag{3}$$

Then for any $\lambda > 0$ problem (1) has a nontrivial weak solution.

Theorem 2. *Assume that functions p and q verify the hypothesis*

$$\max_{\mathbb{Z}[1,T]} q < \min_{\mathbb{Z}[0,T]} p \,. \tag{4}$$

Then there exists $\lambda^{\star\star} > 0$ such that for any $\lambda > \lambda^{\star\star}$ problem (1) has a nontrivial weak solution.

Theorem 3. *Assume that functions p and q verify the hypothesis*

$$\min_{\mathbb{Z}[1,T]} q < \min_{\mathbb{Z}[0,T]} p \,. \tag{5}$$

Then there exists $\lambda^{\star} > 0$ such that for any $\lambda \in (0, \lambda^{\star})$ problem (1) has a nontrivial weak solution.

Remark 1. We point out that if relation (5) is verified then relation (4) is fulfilled, too. Consequently, the result of Theorem 2 can be completed with the conclusion of Theorem 3. More exactly, we deduce the following corollary.

Corollary 1. *Assume that functions p and q verify the hypothesis*

$$\min_{\mathbb{Z}[1,T]} q < \min_{\mathbb{Z}[0,T]} p \,.$$

Then there exist $\lambda^{\star} > 0$ and $\lambda^{\star\star} > 0$ such that for any $\lambda \in (0, \lambda^{\star}) \cup (\lambda^{\star\star}, \infty)$ problem (1) possesses a nontrivial weak solution.

Remark 2. On the other hand, we point out that the result of Theorem 3 holds true in situations that extend relation (4) since in relation (5) we could have

$$\min_{\mathbb{Z}[1,T]} q < \min_{\mathbb{Z}[0,T]} p < \max_{\mathbb{Z}[1,T]} q \,.$$

2 Auxiliary results

From now on we will use the following notations:

$$p^- = \min_{\mathbb{Z}[0,T]} p, \quad p^+ = \max_{\mathbb{Z}[0,T]} p, \quad q^- = \min_{\mathbb{Z}[1,T]} q, \quad q^+ = \max_{\mathbb{Z}[1,T]} q \,.$$

On the other hand, it is useful to introduce other norms on H, namely

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m \right)^{1/m}, \quad \forall\, u \in H \text{ and } m \geq 2\,.$$

It can be verified (see [4]) that

$$T^{(2-m)/(2m)} \cdot |u|_2 \leq |u|_m \leq T^{1/m} \cdot |u|_2, \quad \forall\, u \in H \text{ and } m \geq 2\,. \tag{6}$$

We start with the following auxiliary result.

Lemma 1. *a) There exist two positive constants C_1 and C_2 such that*

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \cdot \|u\|^{p^-} - C_2, \quad \forall\, u \in H \quad \textit{with } \|u\| > 1\,.$$

b) There exists a positive constant C_3 such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_3 \cdot \|u\|^{p^+}, \quad \forall\, u \in H \quad \textit{with } \|u\| < 1\,.$$

c) For any $m \geq 2$ there exists a positive constant c_m such that

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \cdot \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall\, u \in H\,.$$

Proof. a) Fix $u \in H$ with $\|u\| > 1$. We define

$$\alpha_k = \begin{cases} p^+, & \text{if} \quad |\Delta u(k)| < 1 \\ p^-, & \text{if} \quad |\Delta u(k)| > 1\,, \end{cases}$$

for each $k \in \mathbb{Z}[0, T]$.

We deduce that

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &\geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{\alpha_{k-1}} \\ &\geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^-} - \sum_{\{k \in \mathbb{Z}[0,T];\ \alpha_{k-1} = p^+\}} \left(|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+} \right) \\ &\geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^-} - T\,. \end{aligned}$$

The above inequality and relation (6) imply

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq T^{(2-p^-)/2} \cdot \|u\|^{p^-} - T, \quad \forall\, u \in H \text{ with } \|u\| > 1\,.$$

Thus, we proved that a) holds true.

b) Assume $u \in H$ with $\|u\| < 1$. It follows that $|\Delta u(k)| < 1$ for each $k \in \mathbb{Z}[0,T]$. So, by (6) we deduce that

$$\begin{aligned} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} &\geq \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p^+} \\ &\geq 1/T^{(2-p^+)/2} \cdot \|u\|^{p^+}\,. \end{aligned}$$

Thus, we proved that b) holds true.

c) Since

$$|u(k)| \leq \sum_{i=0}^{k-1} |\Delta u(i)|, \quad \forall\, u \in H \text{ and } k \in \mathbb{Z}[0,T]\,,$$

we deduce that for any positive real number $m \geq 2$ there exists a positive constant $c_{m,k}$ such that

$$|u(k)|^m \leq c_{m,k} \cdot \sum_{i=0}^{k-1} |\Delta u(i)|^m, \quad \forall\, u \in H \text{ and } k \in \mathbb{Z}[0,T]\,.$$

The above information implies that there exists a positive constant c_m such that

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \cdot \sum_{k=1}^{T+1} |\Delta u(k-1)|^m, \quad \forall\, u \in H\,. \tag{7}$$

The proof of Lemma 1 is complete. □

3 Proof of the main results

For any $\lambda > 0$ the energy functional corresponding to problem (1) is defined as $J_\lambda : H \to \mathbb{R}$,

$$J_\lambda(u) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda \cdot \sum_{k=1}^{T} \frac{1}{q(k)} |u(k)|^{q(k)}\,.$$

Standard arguments assure that $J_\lambda \in C^1(H, \mathbb{R})$ and

$$\langle J_\lambda^{'}(u), v\rangle = \sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \Delta v(k-1) - \lambda \sum_{k=1}^{T} |u(k)|^{q(k)-2} u(k) v(k)\,,$$

for all $u,\ v \in H$. Thus the weak solutions of (1) coincide with the critical points of J_λ.

3.1 Proof of Theorem 1

In order to prove that J_λ has a nontrivial critical point our idea is to show that actually J_λ possesses a mountain-pass geometry. With that end in view we start by proving two auxiliary results.

Lemma 2. *There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in H$ with $\|u\| = \eta$.*

Proof. First, we point out that

$$|u(k)|^{q^-} + |u(k)|^{q^+} \geq |u(k)|^{q(k)}, \quad \forall\, k \in \mathbb{Z}[1,T] \text{ and } u \in H\,. \tag{8}$$

Using the above inequality we find

$$J_\lambda(u) \geq \frac{1}{p^+}\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} - \frac{\lambda}{q^-}(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+}), \quad \forall\, u \in H\,. \tag{9}$$

Next, we focus on the case when $u \in H$ with $\|u\| < 1$. Thus, $|\Delta u(k)| < 1$ for any $k \in \mathbb{Z}[0, T+1]$. Then using Lemma 1 c) and relation (6) we infer

$$|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+} \leq c_{q^-}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{q^-} + c_{q^+}\sum_{k=1}^{T+1}|\Delta u(k-1)|^{q^+} \leq c_{q^-}T\|u\|^{q^-} + c_{q^+}T\|u\|^{q^+}\,. \tag{10}$$

For $u \in H$ with $\|u\| < 1$ the above inequalities combined with relation (9), Lemma 1 b) and relation (6) imply

$$\begin{aligned} J_\lambda(u) &\geq \frac{C_3}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}(c_{q^-}T\cdot\|u\|^{q^-} + c_{q^+}T\cdot\|u\|^{q^+}) \\ &= (d_1 - d_2\cdot\|u\|^{q^- - p^+} - d_3\cdot\|u\|^{q^+ - p^+})\cdot\|u\|^{p^+}\,, \end{aligned}$$

where d_1, d_2 and d_3 are positive constants.

We remark that the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) = d_1 - d_2\cdot t^{q^+ - p^+} - d_3\cdot t^{q^- - p^+}$$

is positive in a neighborhood of the origin by continuity argument. We conclude that Lemma 2 holds true. □

Lemma 3. *There exists $e \in H$ with $\|e\| > \eta$ (where η is given in Lemma 2) such that $J_\lambda(e) < 0$.*

Proof. Consider the function $\psi : \mathbb{Z}[0, T+1] \to \mathbb{R}$ such that there exists k_0 an integer satisfying $0 < k_0 < T+1$ for which $\psi(k_0) = 1$ and $\psi(k) = 0$ for any $k \in \mathbb{Z}[0, T+1] \setminus \{k_0\}$. Thus, we deduce that $\psi \in H$. For each $t > 1$ we have

$$J_\lambda(t\psi) = \frac{t^{p(k_0)}}{p(k_0)} + \frac{t^{p(k_0-1)}}{p(k_0-1)} - \lambda\cdot\frac{t^{q(k_0)}}{q(k_0)} \leq \frac{2\cdot t^{p^+}}{p^-} - \lambda\cdot\frac{t^{q^-}}{q^+}\,.$$

Since $q^- > p^+$ it is clear that $\lim_{t\to\infty} J_\lambda(t\psi) = -\infty$. Then, for $t > 1$ large enough we can take $e = t\psi$ such that $\|e\| > \eta$ and $J_\lambda(e) < 0$.

The proof of Lemma 3 is complete. □

Proof of Theorem 1. By Lemmatas 2 and 3 and the mountain-pass theorem of Ambrosetti and Rabinowitz [3] we deduce the existence of a sequence $\{u_n\} \subset H$ such that

$$J_\lambda(u_n) \to \bar{c} > 0 \quad \text{and} \quad J_\lambda^{'}(u_n) \to 0 \quad \text{as } n \to \infty. \tag{11}$$

We prove that $\{u_n\}$ is bounded in H. Arguing by contradiction, we assume that passing eventually to a subsequence, still denoted by $\{u_n\}$, we have $\|u_n\| \to \infty$. Thus, we may assume that for n large enough we have $\|u_n\| > 1$.

Relation (11) and the above considerations imply that for n large enough we have

$$\begin{aligned} 1 + \bar{c} + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{q^-}\langle J_\lambda^{'}(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right)\sum_{k=1}^{T+1} |\Delta u_n(k-1)|^{p(k-1)}. \end{aligned}$$

By Lemma 1 a) and the above inequality we deduce that there exist two positive constants D_1 and D_2 such that

$$1 + \bar{c} + \|u_n\| \geq D_1 \cdot \|u_n\|^{p^-} - D_2\,,$$

for n large enough. Dividing by $\|u_n\|^{p^-}$ in the above inequality and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in H. That information combined with the fact that H is a finite dimensional Hilbert space implies that there exists a subsequence, still denoted by $\{u_n\}$, and $u_0 \in H$ such that $\{u_n\}$ converges to u_0 in H.

Then, by relation (11) we have

$$J_\lambda(u_0) = \bar{c} > 0 \quad \text{and} \quad J_\lambda^{'}(u_0) = 0.$$

We conclude that u_0 is a nontrivial weak solution of problem (1). □

3.2 Proof of Theorem 2

For any $\lambda > 0$ let J_λ be defined as above.

Now we show that J_λ possesses a nontrivial global minimum point in H. With that end in view we remark that Lemma 1 a) implies that J_λ is coercive on H. On the other hand, it is obvious that it is also weakly lower semicontinuous on the finite dimensional Hilbert space H. These two facts enable us to apply Theorem 1.2 in [19] in order to find that there exists $u_\lambda \in H$ a global minimizer of J_λ and thus a weak solution of problem (1).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and defining the function $v_0 : \mathbb{Z}[0, T+1] \to \mathbb{R}$ such that there exists an integer k_0 with $0 < k_0 < T+1$ for which

$v_0(k_0) = t_0$ and $v_0(k) = 0$ for any $k \in \mathbb{Z}[0, T+1] \setminus \{k_0\}$ we deduce that $v_0 \in H$ and

$$J_\lambda(v_0) = \frac{t_0^{p(k_0-1)}}{p(k_0-1)} + \frac{t_0^{p(k_0)}}{p(k_0)} - \frac{\lambda \cdot t_0^{q(k_0)}}{q(k_0)} \leq L_1 - L_2 \cdot \lambda\,,$$

where L_1 and L_2 are two positive constants. Thus, there exists $\lambda^{\star\star} > 0$ such that $J_\lambda(v_0) < 0$ for any $\lambda \in [\lambda^{\star\star}, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geq \lambda^{\star\star}$ and thus u_λ is a nontrivial weak solution of problem (1) for λ large enough. The proof of Theorem 2 is complete. □

3.3 Proof of Theorem 3

For any $\lambda > 0$ let J_λ be defined as above.

We show that, by using the hypothesis of Theorem 3, the functional J_λ has a nontrivial critical point by applying Ekeland's variational principle [7]. In order to do that we first prove two auxiliary results.

Lemma 4. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there exist ρ, $a > 0$ such that $J_\lambda(u) \geq a > 0$ for any $u \in H$ with $\|u\| = \rho$.*

Proof. First, let us remark that for any $u \in H$, Lemma 1 c) implies

$$c_2 \cdot \|u\| \geq |u|_2\,.$$

Combining that fact and inequality (6) we deduce that

$$c_2 \cdot T^{1/q^-} \cdot \|u\| \geq |u|_{q^-}, \quad \forall\, u \in H\,. \tag{12}$$

We fix $\rho \in (0, 1)$ such that $\rho < \min\{1, 1/(c_2 \cdot T^{1/q^-})\}$. Thus, for any $u \in H$ with $\|u\| = \rho$ we have $|u|_{q^-} < 1$. It follows that, in this case, $|u(k)| < 1$ holds for any $k \in \mathbb{Z}[0, T+1]$. Therefore

$$\sum_{k=1}^{T} |u(k)|^{q(k)} \leq |u|_{q^-}^{q^-}, \quad \forall\, u \in H \text{ with } \|u\| = \rho\,. \tag{13}$$

By relations (12) and (13) we obtain

$$\sum_{k=1}^{T} |u(k)|^{q(k)} \leq c_2^{q^-} \cdot T \cdot \|u\|^{q^-}, \quad \forall\, u \in H \text{ with } \|u\| = \rho\,.$$

By Lemma 1 b) and the above relation we deduce that for any $u \in H$ with $\|u\| = \rho$ the following inequalities hold true

$$\begin{aligned} J_\lambda(u) &\geq \frac{C_3}{p^+} \cdot \|u\|^{p^+} - \frac{\lambda \cdot c_2^{q^-} \cdot T}{q^-} \cdot \|u\|^{q^-} \\ &= (C_4 \cdot \rho^{p^+ - q^-} - \lambda \cdot C_5) \cdot \rho^{q^-}\,, \end{aligned}$$

where C_4 and C_5 are positive constants. By the above inequality and the fact that $q^- < p^- \leq p^+$ we remark that if we define

$$\lambda^\star = \frac{C_4 \cdot \rho^{p^+ - q^-}}{2 \cdot C_5} \tag{14}$$

then for any $\lambda \in (0, \lambda^\star)$ and any $u \in H$ with $\|u\| = \rho$ there exists $a = \frac{C_4 \cdot \rho^{p^+}}{2}$ such that

$$J_\lambda(u) \geq a > 0.$$

The proof of Lemma 4 is complete. □

Lemma 5. *There exists $\varphi \in H$ such that $\varphi \geq 0$, $\varphi \neq 0$, and $J_\lambda(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Since $q^- < p^-$ it follows that there exists an integer k_0 such that $0 < k_0 < T + 1$ and $q^- = q(k_0) < p^- \leq p(k_0)$. We define the function $\varphi : \mathbb{Z}[0, T+1] \to \mathbb{R}$ such that $\varphi(k_0) = 1$ and $\varphi(k) = 0$ for any $k \in \mathbb{Z}[0, T+1] \setminus \{k_0\}$. We deduce that $\varphi \in H$ and for any $t \in (0, 1)$ we have

$$J_\lambda(t \cdot \varphi) = \frac{t^{p(k_0 - 1)}}{p(k_0 - 1)} + \frac{t^{p(k_0)}}{p(k_0)} - \lambda \cdot \frac{t^{q(k_0)}}{q(k_0)} \leq \frac{2 \cdot t^{p^-}}{p^-} - \frac{\lambda \cdot t^{q^-}}{q^+} .$$

The above inequality implies

$$J_\lambda(t \cdot \varphi) < 0$$

for any $t < \delta^{1/(p^- - q^-)}$ where

$$0 < \delta < \frac{p^- \cdot \lambda}{2 \cdot q^+} .$$

The proof of Lemma 5 is complete. □

Proof of Theorem 3. Let $\lambda^\star > 0$ be defined as in (14) and $\lambda \in (0, \lambda^\star)$. By Lemma 4 it follows that on the boundary of the ball centered at the origin and of radius ρ in H, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{15}$$

On the other hand, by Lemma 5, there exists $\varphi \in H$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, relation (6) and Lemma 1 b) imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \frac{C_3}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} (c_{q^-} T \cdot \|u\|^{q^-} + c_{q^+} T \cdot \|u\|^{q^+}) .$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} J_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\epsilon) &< \inf_{B_\rho(0)} J_\lambda + \epsilon \\ J_\lambda(u_\epsilon) &< J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J_\lambda(u_\epsilon) \le \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \le \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda\,,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \ge 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\| \ge 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda^{'}(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|J_\lambda^{'}(u_\epsilon)\| \le \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda^{'}(w_n) \to 0. \tag{16}$$

Since the sequence $\{w_n\}$ is bounded in H, there exists $w \in H$ such that, up to a subsequence, $\{w_n\}$ converges to w in H. So, by (16),

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J_\lambda^{'}(w) = 0. \tag{17}$$

We conclude that w is a nontrivial weak solution for problem (1).

The proof of Theorem 3 is complete. □

Acknowledgements. M. Mihăilescu and V. Rădulescu have been supported by Grant CNCSIS PN II–79/2007 *"Procese Neliniare Degenerate şi Singulare"*. V. Rădulescu also acknowledges the support through Grants 2-CEx06-11-18/2006 and GAR-18/315/2008. S. Tersian has been supported by Grant VU-MI-02/05 of the National Research Fund in Bulgaria. First and third authors thank Department of Mathematics and Its Applications, CEU, Budapest for the opportunity to meet and discuss on discrete boundary value problems.

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