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Existence of solutions for perturbed fractional p-Laplacian equations

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Abstract

The purpose of this paper is to investigate the existence of weak solutions for a perturbed nonlinear elliptic equation driven by the fractional p-Laplacian operator as follows:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where λ is a real parameter, $(-\Delta)_p^s$ is the fractional p-Laplacian operator with $0 < s < 1 < p < \infty$, $p < r < \min\{q, p_s^*\}$ and $V, a, b : \mathbb{R}^N \to (0, \infty)$ are three positive weights. Using variational methods, we obtain nonexistence and multiplicity results for the above-mentioned equations depending on λ and according to the integrability properties of the ratio a^{q-p}/b^{r-p}. Our results extend the previous work of Autuori and Pucci (2013) [5] to the fractional p-Laplacian setting. Furthermore, we weaken one of the conditions used in their paper. Hence the results of this paper are new even in the fractional Laplacian case.

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1. Introduction

In this paper we deal with the following one-parameter elliptic equations:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N > ps$ with $s \in (0,1)$ and $(-\Delta)_p^s$ is the fractional p-Laplace operator which, up to normalization factors, may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. As for some recent results on the fractional p-Laplacian, we refer to for example [18,19,22] and the references therein.

Notice that when $p = 2$, problem (1.1) reduces to the following fractional Laplacian equations:

$$(-\Delta)^s u + V(x)u = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, \tag{1.2}$$

which can been seen as the fractional form of the following classical stationary Schrödinger equations:

$$-\Delta u + V(x)u = \lambda a(x)|u|^{r-2}u - b(x)|u|^{q-2}u \text{ in } \mathbb{R}^N. \tag{1.3}$$

For standing wave solutions of fractional Schrödinger equations in $\mathbb{R}^N$, we refer to [11,14,16, 20,21,26] and the references therein. Especially, models governed by unbounded potentials involving fractional Schrödinger equations have been investigated in the last years, see for instance [13,32].

However, all these papers deal with problems which are not directly comparable to problem (1.1). In fact, the present paper is inspired by the following works: *Alama* and *Tarantello* in [2] studied the following Dirichlet problem with indefinite weights:

$$-\Delta u - \lambda u = \omega(x)u^{q-1} - h(x)u^{r-1} \quad \text{in } \Omega,$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, the coefficients ω, $h \in L^1(\Omega)$ are nonnegative and $2 < q < r$. They first showed that the existence, nonexistence and multiplicity results depend on λ and the integrability of the ratio w^{r-1}/h^{q-1}. In [28], *Pucci* and *Rădulescu* first considered the following related problem in the whole space:

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + u^{p-1} = \lambda u^{q-1} - h(x)u^{r-1} \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $h > 0$ satisfies

$$0 < \int\limits_{\mathbb{R}^N} h(x)^{q/(q-r)} dx < \infty,$$

$\lambda > 0$ is a parameter and $2 \le p < q < \min\{r, p^*\}$, with $p^* = Np/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \le p$. They obtained that the nonexistence and multiplicity of nontrivial solutions are corresponding to the smallness and the largeness of λ respectively. In [6], *Autuori* and *Pucci* extended the above result by studying the quasilinear elliptic equation:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + a(x)u^{p-2}u = \lambda\omega(x)u^{q-2}u - h(x)u^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $\max\{2, p\} < q < \min\{r, p^*\}$, the coefficients ω and h are related by integrability condition

$$\int_{\mathbb{R}^N} \left[\frac{\omega^r(x)}{h^q(x)}\right]^{1/(r-q)} dx \in \mathbb{R}^+. \tag{1.6}$$

Moreover, they proposed two open questions: the relaxation of $\max\{2, p\} < q$ and the replacement of (1.6) by the assumption that $\omega\,(\omega/h)^{(q-p)/(r-q)}$ is in $L^{N/p}(\mathbb{R}^N)$. Note that thanks to $q < p^*$, this latter request is weaker than (1.6) which already appeared in [2]. After that, *Autuori* and *Pucci* in [5] turned to study the following elliptic equation involving fractional Laplacian:

$$(-\Delta)^s u + a(x)u = \lambda\omega(x)|u|^{p-2}u - h(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$, $2 < q < \min\{r, 2_s^*\}$, $2_s^* := 2N/(N-2s)$. The coefficients ω and h are related by condition (1.6). By using variational methods, the authors obtained the existence and multiplicity of entire solutions of (1.7). Similarly, two open problems mentioned above in [6] can be applied to the fractional setting. Recently, *Pucci* and *Zhang* [29] solved the above open problems for a class of quasilinear elliptic equations in the setting of variable exponents. More recently, *Pucci et al.* in [31] also gave a positive answer to these open problems in the context of Kirchhoff problems involving the fractional p-Laplacian. In this article, we would like to take a quite different approach to conquer these problems in the setting of the fractional p-Laplacian.

To this end, we suppose that the nonlinear terms in (1.1) are related to the main elliptic part by the request that

(H1) *$p < r < \min\{q, p_s^*\}$, where $p_s^* = Np/(N-ps)$ is the fractional Sobolev critical exponent;*
(H2) *$V : \mathbb{R}^N \to \mathbb{R}^+$ is a positive weight and there exists a constant $V_0 > 0$ such that $V(x) \ge V_0$ for all $x \in \mathbb{R}^N$;*
(H3) *$a : \mathbb{R}^N \to \mathbb{R}^+$ is a positive weight satisfying $a \in L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N) \bigcap L^1_{\mathrm{loc}}(\mathbb{R}^N)$;*
(H4) *$b : \mathbb{R}^N \to \mathbb{R}^+$ is of class $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $a(x)$ and $b(x)$ are related by the condition*

$$0 < \int_{\mathbb{R}^N} \left[\frac{a(x)^{(q-p)/(q-r)}}{b(x)^{(r-p)/(q-r)}}\right]^{\frac{N}{ps}} dx < \infty,$$

i.e.,

$$a\,(a/b)^{(r-p)/(q-r)} \in L^{N/ps}(\mathbb{R}^N).$$

We first give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W$ is a weak solution of problem (1.1), if

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx$$

$$= \lambda \int_{\mathbb{R}^N} a(x)|u(x)|^{r-2}u(x)\varphi(x)dx - \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)\varphi(x)dx,$$

for any $\varphi \in W$, where space W will be introduced in Section 2.

Now we are in a position to state the main result of this paper as follows.

Theorem 1.1. *Suppose that the assumptions* (H1)–(H4) *are satisfied. Then there exist constants* λ_0 *and* λ^* *with* $\lambda^* \geq \lambda_0 > 0$ *such that problem* (1.1) *has*

(i) *only the trivial weak solution if* $\lambda < \lambda_0$*;*
(ii) *two nontrivial weak solutions if* $\lambda > \lambda^*$.

Remark 1.1. (a) For the related results about concave–convex problems, we refer to the seminal paper [3] for the semilinear case in bounded domains. We also refer to [38] for the fractional Laplacian case in bounded domains and to [37] for the fractional p-Laplacian case in $\mathbb{R}^N$.

(b) Under the conditions (H1) and (H3), the condition (H4) is more general than that in [5, (1.4)] for fractional Laplacian problems, see [2] for further details. Consequently, we obtain the corresponding results by replacing condition [5, (1.4)] with (H4). Hence, in this sense our main result is new even in the case of the fractional Laplacian. Of course, from Theorem 1.1 we know that it still remains an open problem to verify whether $\lambda_0 = \lambda^*$ in the non-local setting.

Finally, we would like to point out that in recent years, a great attention has been focused on the study of fractional and non-local operators of elliptic type. This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [4,9,20,23]. The literature on fractional and non-local operators of elliptic and their applications is quite large, for example, we refer the reader to [7,10,12,17,25,27,33–35] and the references therein. For the basic properties of fractional Sobolev spaces with applications to partially differential equations, we refer the reader to [15,24] and the references therein.

This article is organized as follows. In Section 2, we will introduce the working space W and give some necessary definitions and properties, which will be used in the sequel. In Section 3, using critical point theory, we will prove the main result.

2. The functional framework

In this section, we first give some basic results of fractional Sobolev space that will be used in the next section. Let $0 < s < 1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows:

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\},$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|^p_{L^p(\mathbb{R}^N)} + \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

Then it is easy to see that $(W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)})$ is a uniformly convex Banach space and the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous for any $\nu \in [p, p_s^*]$ by Theorem 6.7 of [15], that is, there exists a positive constant C_* such that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \le C_* \|u\|_{W^{s,p}(\mathbb{R}^N)} \text{ for all } u \in W^{s,p}(\mathbb{R}^N). \tag{2.1}$$

The space E denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_E = \left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int\limits_{\mathbb{R}^N} V(x)|u(x)|^p dx \right)^{\frac{1}{p}}.$$

Then $(E.\|\cdot\|_E)$ is a uniformly convex Banach space, see [30, Lemma 10].

Lemma 2.1. *(See [30, Lemma 1].) Let* (H2) *hold. Then the embeddings $E \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ are continuous with*

$$\min\{1, V_0\}\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \le \|u\|^p_E, \tag{2.2}$$

for all $u \in W$ and $\nu \in [p, p_s^*]$. Moreover, for any $R > 0$ and $\nu \in [1, p_s^*)$ the embedding $E \hookrightarrow L^\nu(B_R(0))$ is compact.

$L^r(\mathbb{R}^N, a)$ is defined by a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that for any function u in $L^r(\mathbb{R}^N, a)$

$$\int\limits_{\mathbb{R}^N} a(x)|u(x)|^r dx < \infty,$$

equipped with the norm

$$\|u\|_{L^r(\mathbb{R}^N,a)} = \left(\int\limits_{\mathbb{R}^N} a(x)|u(x)|^r dx \right)^{\frac{1}{r}},$$

then $L^r(\mathbb{R}^N, a)$ is a uniformly convex Banach space by Proposition A.6 of [6]. The space $L^q(\mathbb{R}^N, b)$ is defined in a similar way. We shall work on the space W defined as follows: W is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_W = \|u\|_E + \|u\|_{L^q(\mathbb{R}^N,b)}.$$

Lemma 2.2. $(W, \|\cdot\|_W)$ *is a reflexive Banach space.*

Proof. We only need to verify that $(W, \|\cdot\|_W)$ is reflexive. For this, we define a product space $X = E \times L^q(\mathbb{R}^N, b)$, endowed with the norm

$$\|u\|_X = \|u\|_E + \|u\|_{L^q(\mathbb{R}^N,b)}.$$

Then $(X, \|\cdot\|_X)$ is a reflexive Banach space by Theorem 1.23 (ii) of [1], since E and $L^q(\mathbb{R}^N, b)$ are two uniformly convex Banach spaces. We define an operator $\Gamma : (W, \|\cdot\|_W) \to (X, \|\cdot\|_X)$ satisfying $\Gamma(u) = (u, u)$ for all $u \in W$. Then Γ is well defined, linear and isometric. Therefore, $\Gamma(W)$ is a closed subspace of X, and so $\Gamma(W)$ is reflexive by Theorem 1.22 (ii) of [1]. Consequently, $(W, \|\cdot\|_W)$ is reflexive. $\square$

From now on, $B_R(0)$ will denote the ball in $\mathbb{R}^N$ of center zero and radius $R > 0$.

Theorem 2.1. *Suppose that* (H2) *and* (H3) *are fulfilled,* $b : \mathbb{R}^N \to \mathbb{R}^+$ *is a positive weight. If* $\{v_j\}$ *is a bounded sequence in* W*, then there exists* $v \in W \bigcap L^r(\mathbb{R}^N, a)$ *such that up to a subsequence,*

$$v_j \to v \ \ \textit{strongly in } L^r(\mathbb{R}^N, a)$$

as $j \to \infty$*, for any* $r \in (p, p_s^*)$.

Proof. The proof is similar to that of [31, Lemma 2.3] and that of [29, Lemma 2.6], here we would like to give a detailed treatment for the reader's convenience. Since $\{v_j\}_j$ is bounded in W, by Lemma 2.1 we have $\{v_j\}_j$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$. Then by the reflexivity of W, up to a subsequence, we get that $v_j \rightharpoonup v$ weakly in $W \bigcap L^{p_s^*}(\mathbb{R}^N)$ as $j \to \infty$. Next we prove that

$$v_j \to v \ \ \text{strongly in } L^r(\mathbb{R}^N, a).$$

Now for any $\varepsilon > 0$, there exists $R_1 > 0$ such that

$$\int\limits_{\mathbb{R}^N \setminus B_R(0)} |a(x)|^{\frac{p_s^*}{p_s^*-r}} dx < \varepsilon \ \ \text{for all } R \geq R_1, \tag{2.3}$$

since $a \in L^{p_s^*/(p_s^*-r)}(\mathbb{R}^N)$ by assumption (H3). Fix $R_1 > 0$, we have

$$v_j \rightharpoonup v \ \ \text{weakly in } \ W^{s,p}(B_{R_1}(0)) \bigcap L^{p_s^*}(B_R(0)),$$

by Theorem 6.7 of [15]. Since $p < r < p_s^*$, by Corollary 7.2 of [15], we obtain $v_j \to v$ strongly in $L^r(B_{R_1}(0))$. Without loss of generality, we assume that $u_n \to u$ a.e. in $B_{R_1}(0)$. Thus, $a(x)|v_j - v|^r \to 0$ a.e. in $B_{R_1}(0)$. For each measurable subset $U \subset B_{R_1}(0)$, we have

$$\int_U a(x)|v_j(x) - v(x)|^r dx \le \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(U)} \|v_j - v\|^r_{L^{p_s^*}(U)} \le C\|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(U)}.$$

Since $a \in L^{p_s^*/(p_s^*-r)}(\mathbb{R}^N)$, we obtain that $\{a(x)|v_j(x) - v(x)|^r\}_j$ is equi-integrable and uniformly bounded in $B_{R_1}(0)$. Then the Vitali convergence theorem implies

$$\lim_{j\to\infty} \int_{B_{R_1(0)}} a(x)|v_j(x) - v(x)|^r dx = 0.$$

Hence, for above $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\int_{B_{R_1(0)}} a(x)|v_j(x) - v(x)|^r dx < \varepsilon, \quad \text{for all } j \ge N_1. \tag{2.4}$$

Combining (2.3) and (2.4), for all $j \ge N_1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|v_j(x) - v(x)|^r dx \\
&= \int_{B_{R_1(0)}} a(x)|v_j(x) - v(x)|^r dx + \int_{\mathbb{R}^N \setminus B_{R_1(0)}} a(x)|v_j(x) - v(x)|^r dx \\
&< \varepsilon + \left(\int_{\mathbb{R}^N \setminus B_{R_1(0)}} |a(x)|^{\frac{p_s^*}{p_s^*-r}} \right)^{r/q} \left(\int_{\mathbb{R}^N \setminus B_{R_1(0)}} |v_j(x) - v(x)|^{p_s^*} dx \right)^{r/p_s^*} \\
&\le C\varepsilon,
\end{aligned}$$

where C denotes various positive constants. Therefore, $v_j \to v$ strongly in $L^r(\mathbb{R}^N, a)$. □

3. Proof of Theorem 1.1

For $u \in W$, we define

$$J(u) = \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u(x)|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u(x)|^q dx,$$

$$H(u) = \frac{\lambda}{r} \int_{\mathbb{R}^N} a(x)|u(x)|^r dx,$$

and

$$I(u) = J(u) - H(u).$$

Obviously, the energy functional $I : W \to \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 3.1. *Under the conditions* (H1)–(H4)*, the functional I is coercive in W.*

Proof. For any $k_1, k_2 > 0$ and $0 < \alpha < \beta$

$$k_1 t^{\alpha} - k_2 t^{\beta} \leq k_1 (k_1/k_2)^{\alpha/(\beta-\alpha)} \quad \text{for all } t \geq 0. \tag{3.1}$$

Indeed, since the function $t \in [0, \infty) \to t^{\theta}$ is increasing for any $\theta > 0$, it follows that

$$k_1 - k_2 t^{\beta-\alpha} < 0, \ \forall t > (k_1/k_2)^{1/(\beta-\alpha)}$$

and

$$k_1 t^{\alpha} - k_2 t^{\beta} \leq k_1 t^{\alpha} \leq k_1 (k_1/k_2)^{\alpha/(\beta-\alpha)}, \ \ \forall t \in \left[0, \ (k_1/k_2)^{1/(\beta-\alpha)}\right].$$

The above two inequalities imply that (3.1) holds.

Taking $k_1 = \lambda a(x)/r, k_2 = b(x)/q, \alpha = r - p, \beta = q - p$ and $t = |u(x)|$ in (3.1), for all $x \in \mathbb{R}^N$, we get

$$\begin{aligned}
&\frac{\lambda a(x)}{r}|u(x)|^{r-p} - \frac{b(x)}{2q}|u(x)|^{q-p} \\
&\quad \leq (\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \left(\frac{a(x)^{q-p}}{b(x)^{r-p}}\right)^{1/(q-r)}.
\end{aligned} \tag{3.2}$$

Since $\left(a^{q-p}/b^{r-p}\right)^{1/(q-r)} \in L^{N/(ps)}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\begin{aligned}
&(\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \left(\int_{\mathbb{R}^N \setminus B_R(0)} \left(\frac{a(x)^{q-p}}{b(x)^{r-p}}\right)^{N/(ps(q-r))} dx\right)^{ps/N} \\
&\quad < \frac{\min\{1, V_0\}}{2C_*^p}\varepsilon, \ \text{where } C_* \text{ denotes the embedding constant of } W^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N).
\end{aligned} \tag{3.3}$$

So, by (3.2) and (3.3) we obtain

$$\begin{aligned}
I(u) &= \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u(x)|^q dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} a(x)|u(x)|^r dx \\
&\geq \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx + \frac{1}{2q}\int_{\mathbb{R}^N} b(x)|u(x)|^q dx
\end{aligned}$$

$$- (\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \int\limits_{\mathbb{R}^N \setminus B_R(0)} \left(\frac{a(x)^{q-p}}{b(x)^{r-p}}\right)^{1/(q-r)} |u(x)|^p dx$$

$$+ \int\limits_{B_R(0)} \frac{b(x)}{2q}|u(x)|^q - \frac{\lambda a(x)}{r}|u(x)|^r dx. \tag{3.4}$$

For fixed $R > 0$ and for any $\delta > 0$ and $M > 0$, decompose $B_R(0) = X \bigcup Y \bigcup Z$ with X, Y, Z measurable sets defined as follows:

$$\begin{cases} X = \{x \in B_R(0) : a(x) < M \ \text{ and } \ b(x) > \delta\}, \\ Y = \{x \in B_R(0) : a(x) < M \ \text{ and } \ b(x) \le \delta\}, \\ Z = \{x \in B_R(0) : a(x) \ge M\}. \end{cases} \tag{3.5}$$

We apply (3.1) to derive

$$\int\limits_X \frac{\lambda a(x)}{r}|u(x)|^r - \frac{b(x)}{2q}|u(x)|^q dx$$

$$\le (\lambda/r)^{q/(q-r)} (2q)^{r/(q-r)} \int\limits_X \frac{a(x)^{q/(q-r)}}{b(x)^{r/(q-r)}} dx \le C_1, \tag{3.6}$$

and

$$\int\limits_{Y \bigcup Z} \frac{\lambda a(x)}{r}|u(x)|^r - \frac{b(x)}{2q}|u(x)|^q dx$$

$$\le (\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \int\limits_{Y \bigcup Z} \frac{a(x)^{(q-p)/(q-r)}}{b(x)^{(r-p)/(q-r)}}|u(x)|^p dx \tag{3.7}$$

$$\le (\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \left(\int\limits_{Y \bigcup Z} \left[\frac{a(x)^{(q-p)/(q-r)}}{b(x)^{(r-p)/(q-r)}}\right]^{N/ps} dx \right)^{ps/N} \|u\|^p_{L^{p_s^*}(B_R(0))},$$

where $C_1 = C_1(M, \delta, R)$ is a constant. Since $a, b \in L^1_{\text{loc}}(\mathbb{R}^N)$, we have $|Z| \to 0$ as $M \to \infty$ and for fixed M, $|Y| \to 0$ as $\delta \to 0$. Thus, we can choose M sufficiently large and then $\delta > 0$ sufficiently small such that

$$(\lambda/r)^{(q-p)/(q-r)}(2q)^{(r-p)/(q-r)} \left(\int\limits_{Y \bigcup Z} \left[\frac{a(x)^{(q-p)/(q-r)}}{b(x)^{(r-p)/(q-r)}}\right]^{N/(ps)} dx \right)^{ps/N}$$

$$< \frac{\min\{1, V_0\}}{2C_*^p}\varepsilon. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8), we conclude

$$\int_{B_R(0)} \frac{\lambda a(x)}{r}|u(x)|^r - \frac{b(x)}{2q}|u(x)|^q dx \le C_1 + \frac{\varepsilon}{2}\|u\|_E^p \tag{3.9}$$

By (3.4) and (3.9), we obtain

$$I(u) \ge \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx + \frac{1}{2q}\int_{\mathbb{R}^N} b(x)|u(x)|^q dx$$
$$- \varepsilon\|u\|_E^p - C_1$$

Taking $\varepsilon = 1/(2p)$ and using the following inequality:

$$\xi^l \ge \xi - 1 \quad \text{for all } \xi \ge 0 \text{ and } l \ge 1,$$

we get

$$\begin{aligned} I(u) &\ge \frac{1}{2p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{2p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx + \frac{1}{2q}\int_{\mathbb{R}^N} b(x)|u(x)|^q dx - C_1 \\ &\ge \min\left\{\frac{1}{2p}, \frac{1}{2q}\right\}\|u\|_W - \frac{1}{2p} - \frac{1}{2q} - C_1. \end{aligned}$$

Hence, I is coercive in W. □

Lemma 3.2. *The functional $J : W \to \mathbb{R}$ is convex and of class C^1 and*

$$\begin{aligned} \langle J'(u), v\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)v(x)dx + \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)v(x)dx, \end{aligned}$$

for all $u, v \in W$. Moreover, J is weakly lower semi-continuous in W.

Proof. It is easy to see that J is Gâteaux-differentiable in W and for all $u, v \in W$

$$\begin{aligned} \langle J'(u), v\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)v(x)dx + \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)v(x)dx. \end{aligned}$$

Now, let $\{u_n\}_n \subset W$, $u \in W$ satisfy $u_n \to u$ strongly in W as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then the sequence

$$\left\{ \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}} \right\}_n \quad \text{is bounded in } L^{p'}(\mathbb{R}^{2N})$$

as well as

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}} \longrightarrow \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{(N+ps)/p'}} \quad \text{a.e. in } \mathbb{R}^{2N},$$

as $n \to \infty$. Thus, the Brézis–Lieb lemma (see [8]) implies

$$\begin{aligned} &\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left| \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}} \right|^{p'} dxdy \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} - \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \right) dxdy. \end{aligned} \tag{3.10}$$

Since $u_n \to u$ strongly in W, it is easy to see that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left(\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} - \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \right) dxdy = 0.$$

It follows from (3.10) that

$$\begin{aligned} &\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left| \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u(x)-u(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{(N+ps)/p'}} \right|^{p'} dxdy \\ &= 0. \end{aligned} \tag{3.11}$$

Similarly, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x) \left| |u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x) \right|^{p'} dx = 0 \tag{3.12}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} b(x) \left| |u_n(x)|^{q-2}u_n(x) - |u(x)|^{q-2}u(x) \right|^{q'} dx = 0. \tag{3.13}$$

Combining (3.11)–(3.13) with the Hölder inequality, we have

$$\|J'(u_n) - J'(u)\|_{W'} = \sup_{\varphi \in W,\ \|\varphi\|_W = 1} |\langle J'(u_n) - J'(u), \varphi\rangle|$$
$$\longrightarrow 0, \quad \text{as } n \to \infty.$$

Hence, $J \in C^1(W, \mathbb{R})$. Obviously, J is a convex functional in W. Furthermore, J is weakly lower semi-continuous in W by Corollary 3.9 of [6]. □

Lemma 3.3. *$H \in C^1(W, \mathbb{R})$ and for any fixed $u \in W$, we have*

$$\langle H'(u(x)), v(x)\rangle = \lambda \int_{\mathbb{R}^N} a(x)|u(x)|^{r-2}u(x)v(x)dx,$$

and $H'(u) \in W'$. If $v_n \rightharpoonup v$ weakly in W, then $\langle H'(u), v_n\rangle \to \langle H'(u), v\rangle$. Moreover, under the condition (H1)*, functional H is weakly continuous in W.*

Proof. We only need to prove that H is weakly continuous in W and is of class C^1. Let $\{u_n\}_n \subset W$, $u \in W$ satisfy $u_n \rightharpoonup u$ weakly in W as $n \to \infty$. By Theorem 2.1, without loss of generality, we assume that $u_n \to u$ strongly in $L^r(\mathbb{R}^N, a)$ for $p < r < p_s^*$ and a.e. in $\mathbb{R}^N$. Then a similar argument as in Lemma 3.2 gives that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x)|u_n(x)|^r dx = \int_{\mathbb{R}^N} a(x)|u_n(x)|^r dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x)\left||u_n(x)|^{r-2}u_n(x) - |u(x)|^{r-2}u(x)\right|^{r'} dx = 0.$$

So it is easy to verify that H is weakly continuous in W and is of class C^1. □

Combining Lemma 3.2 and Lemma 3.3, we get that $I \in C^1(W, \mathbb{R})$ and I is weakly semi-continuous in W. To solve problem (1.1), we first consider

$$\begin{aligned}\lambda^* &:= \inf_{u\in\mathcal{M}} \left(\frac{r}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{r}{p}\int_{\mathbb{R}^N} V(x)|u(x)|^p dx + \frac{r}{q}\int_{\mathbb{R}^N} b(x)|u(x)|^q dx \right)\\ &= \inf_{u\in\mathcal{M}} rJ(u),\end{aligned}$$

where $\mathcal{M} = \left\{u \in W : \int_{\mathbb{R}^N} a(x)|u(x)|^r dx = 1\right\}$.

Lemma 3.4. *$\inf_{u\in\mathcal{M}} J(u)$ is achieved at some $u_0 \in \mathcal{M}$ and $\lambda^* := r \inf_{u\in\mathcal{M}} J(u) = rJ(u_0) > 0$. Particularly, $J(|u_0|) = J(u_0)$, that is, $|u_0|$ is also a global minimizer of $\inf_{u\in\mathcal{M}} J(u)$.*

Proof. Let $\{u_n\}$ be a minimizing sequence. Then

$$J(u_n) \to \inf_{\mathcal{M}} J \ \text{ as } n \to \infty.$$

Hence $\{u_n\}$ is bounded in W. By Theorem 2.1, $\{u_n\}$ has a convergent subsequence in $L^r(\mathbb{R}^N, a)$, i.e., up to a subsequence, $u_n \to u_0$ strongly in $L^r(\mathbb{R}^N, a)$. Then we have

$$1 = \int_{\mathbb{R}^N} a(x)|u_n(x)|^r dx \to \int_{\mathbb{R}^N} a(x)|u_0(x)|^r dx \ \text{ as } n \to \infty.$$

Thus, $u_0 \in \mathcal{M}$. Since J is weakly lower semi-continuous, we have

$$\inf_{\mathcal{M}} J \le J(u_0) \le \lim_{n\to\infty} J(u_n) = \inf_{\mathcal{M}} J.$$

Therefore, $J(u_0) = \inf_{\mathcal{M}} J$. If $\lambda^* = 0$, then

$$J(u_n) \to 0 \ \text{ as } n \to \infty.$$

Thus

$$\int_{\mathbb{R}^N} a(x)|u_n(x)|^r dx \to 0 \ \text{ as } n \to \infty,$$

which contradicts with the fact $u_n \in \mathcal{M}$. So, $\lambda^* > 0$.

Finally, we show that $J(|u_0|) = J(u_0)$. For $v \in W$, we have $|v| \in W$. Indeed, we have

$$\iint_{\mathbb{R}^{2N}} \frac{||v(x)| - |v(y)||^p}{|x-y|^{N+ps}} dxdy \le \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} dxdy.$$

Thus, it is easy to obtain that $|v| \in W$. Further, $|u_0| \in W$ and $J(|u_0|) \le J(u_0)$. This gives $J(|u_0|) = J(u_0)$, due to the minimality of u_0. □

Theorem 3.1. *For all $\lambda > \lambda^*$, there exists a global nontrivial minimizer $u^* \in W$ of I with $I(u^*) < 0$.*

Proof. For all $\lambda > 0$ the functional I is weakly semi-continuous, bounded below and coercive in the reflexive Banach space W by Lemma 3.1–Lemma 3.3 and Lemma 2.1. Hence, Theorem 1.2 of [36] implies that for all $\lambda > 0$ there exists a global minimizer $u^* \in W$ of I, that is

$$I(u^*) = \inf_{u\in W} I(u).$$

Obviously, u^* is a weak solution of (1.1). Next, we show that $u^* \neq 0$. Since $\lambda > \lambda^*$, by Lemma 3.4 there exists a function $u_0 \in W$ with $\|u_0\|_{L^r(\mathbb{R}^N,a)} = 1$ such that

$$\lambda\|u_0\|^r_{L^r(\mathbb{R}^N,a)} = \lambda > \lambda^* = rJ(u_0) = \frac{r}{p}\iint\limits_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{r}{p}\int\limits_{\mathbb{R}^N} V(x)|u_0(x)|^p dx + \frac{r}{q}\int\limits_{\mathbb{R}^N} b(x)|u_0(x)|^q dx.$$

This means that

$$I(u_0) = \frac{1}{p}\iint\limits_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u_0(x)|^p dx + \frac{1}{q}\int\limits_{\mathbb{R}^N} b(x)|u_0(x)|^q dx - \frac{\lambda}{r}\int\limits_{\mathbb{R}^N} a(x)|u_0|^r dx < 0.$$

Consequently, $I(u^*) = \inf_{u\in W} I(u) \le I(u_0) < 0$. □

Next, we show that if $\lambda > \lambda^*$ problem (1.1) admits a second nontrivial weak solution $e \neq u$ via variational methods. We start by recalling a modification of the mountain pass theorem of Ambrosetti and Rabinowitz, see Theorem A.3 of [6].

Theorem 3.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces such that X can be continuously embedded into Y. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional with $\Psi(0) = 0$. Suppose that there exist $\rho, \alpha > 0$ and $e \in X$ such that $\|e\|_Y > \rho$, $\Phi(e) < \alpha$ and $\Phi(u) \ge \alpha$ for all $u \in X$ with $\|u\|_Y = \rho$. Then there exists a sequence $\{u_n\} \subset X$ such that for all n*

$$c \le \Phi(u_n) \le c + \frac{1}{n} \text{ and } \|\Phi'(u_n)\|_{X'} \le \frac{2}{n},$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.5. *Suppose that assumptions* (H1)–(H4) *are satisfied. Then for each $e \in W\backslash\{0\}$ and $\lambda > 0$ there exist $\rho \in (0, \|e\|_E)$ and $\alpha > 0$ such that*

$$I(u) \ge \alpha > 0,$$

for all $u \in W$ with $\|u\|_E = \rho$.

Proof. Let $u \in W$. By (2.1), $a \in L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)$ and the Hölder inequality, we obtain

$$\int\limits_{\mathbb{R}^N} a(x)|u(x)|^r dx \le \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \|u\|^r_{L^{p_s^*}(\mathbb{R}^N)} \le C_*^r \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \|u\|^r_{W^{s,p}(\mathbb{R}^N)}.$$

Then

$$\begin{aligned} I(u) &\geq \frac{1}{p}\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int\limits_{\mathbb{R}^N} V(x)|u(x)|^p dx - \lambda C_*^r \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \|u\|^r_{W^{s,p}(\mathbb{R}^N)} \\ &\geq \frac{1}{p}\|u\|_E^p - \lambda C_*^r \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \|u\|^r_{W^{s,p}(\mathbb{R}^N)} \\ &\geq \|u\|_E^p \left(\frac{1}{p} - \lambda\left(\frac{C_*^p}{\min\{1,V_0\}}\right)^{r/p} \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \|u\|_E^{r-p}\right). \end{aligned}$$

Now, let $\|u\|_{W^{s,p}(\mathbb{R}^N)} = \rho > 0$ and $0 < \rho < \left\{\|e\|_E, \left(\lambda p \left(\frac{C_*^p}{\min\{1,V_0\}}\right)^{r/p} \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)}\right)^{1/(p-r)}\right\}$,

so that

$$I(u) \geq \rho^p \left(\frac{1}{p} - \lambda\left(\frac{C_*^p}{\min\{1,V_0\}}\right)^{r/p} \|a\|_{L^{\frac{p_s^*}{p_s^*-r}}(\mathbb{R}^N)} \rho^{r-p}\right) =: \alpha > 0.$$

Thus, the lemma is proved. □

Theorem 3.3. *Suppose that assumptions* (H1)–(H4) *are satisfied. Then for all* $\lambda > \lambda^*$ *problem* (1.1) *exists a nontrivial weak solution* $u \in W$ *such that* $I(u) > 0$.

Proof. By Theorem 3.1, for all $\lambda > \lambda^*$ there exists a nontrivial weak solution $u^* \in W$, which is a global minimizer for I in W and $I(u^*) < 0$. Taking $e = u^*$ in Lemma 3.5, we know that I satisfies the geometrical structure of Theorem 3.2. Thus, for all $\lambda > \lambda^*$ there exists a sequence $\{u_n\} \subset W$ such that

$$I(u_n) \to c \ \text{ and } \ \|I'(u_n)\|_{W'} \to 0 \ \text{ as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \ \text{ and } \ \Gamma = \{\gamma \in C([0,1];W) : \gamma(0) = 0, \gamma(1) = u^*\}.$$

Since I is coercive in W by Lemma 3.1, the sequence $\{u_n\}$ is bounded in W. By the reflexivity of W, up to a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in W. Then $\langle I'(u_n), u_n - u\rangle \to 0$. Thus, we obtain

$$\begin{aligned} &\langle I'(u_n), u_n - u\rangle \\ &= \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n(x)-u(x)-u_n(y)+u(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int\limits_{\mathbb{R}^N} V(x)|u_n(x)|^{p-2}u_n(x)(u_n(x)-u(x))dx + \int\limits_{\mathbb{R}^N} b(x)|u_n(x)|^{q-2}u_n(x)(u_n(x)-u(x))dx \end{aligned}$$

$$-\lambda \int_{\mathbb{R}^N} a(x)|u_n(x)|^{r-2}u_n(x)(u_n(x)-u(x))dx \to 0, \tag{3.14}$$

as $n \to \infty$. By Theorem 2.1, up to a subsequence,

$$u_n \to u \ \text{ strongly in } L^r(\mathbb{R}^N, a) \ \text{ and } \ \text{a.e. in } \mathbb{R}^N,$$

as $n \to \infty$. Thus, $\int_{\mathbb{R}^N} a(x)|u_n(x)|^{r-2}u_n(x)(u_n(x)-u(x))dx \to 0$ as $n \to \infty$. Furthermore, (3.14) and the weak convergence of u_n in W imply that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n(x)-u(x)-u_n(y)+u(y))}{|x-y|^{N+ps}}dxdy \\
&-\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_n(x)-u(x)-u_n(y)+u(y))}{|x-y|^{N+ps}}dxdy \\
&\longrightarrow 0,
\end{aligned} \tag{3.15}$$

as $n \to \infty$ and

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(x)\left[|u_n(x)|^{p-2}u_n(x)-|u(x)|^{p-2}u(x)\right](u_n(x)-u(x))dx \\
&+\int_{\mathbb{R}^N} b(x)\left[|u_n(x)|^{q-2}u_n(x)-|u(x)|^{q-2}u(x)\right](u_n(x)-u(x))dx \to 0,
\end{aligned} \tag{3.16}$$

as $n \to \infty$. Note that there are the well-known vector inequalities:

$$\begin{aligned}
&\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta) \ge C_p|\xi-\eta|^p, \quad p \ge 2; \\
&\left(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\right)(\xi-\eta) \ge \widetilde{C}_p\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, \quad 1<p<2,
\end{aligned} \tag{3.17}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_p, $\widetilde{C}_p$ are positive constants depending only on p. By (3.17) and (3.15), we obtain for $p \ge 2$

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} |u_n(x)-u_n(y)-u(x)+u(y))|^p|x-y|^{-(N+ps)}dxdy \\
&\le C_p^{-1}\iint_{\mathbb{R}^{2N}} \left[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u(x)-u(y)|^{p-2}(u(x)-u(y))\right] \\
&\qquad \times (u_n(x)-u(x)-u_n(y)+u(y))\,|x-y|^{-(N+ps)}dxdy \\
&\longrightarrow 0,
\end{aligned} \tag{3.18}$$

as $n \to \infty$ and for $1 < p < 2$

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y) - u(x) + u(y))|^p |x-y|^{-(N+ps)} dxdy \\
&\le \widetilde{C}_p^{-p/2} \Bigg\{ \iint_{\mathbb{R}^{2N}} \Big[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y)) \Big] \\
&\qquad \times (u_n(x) - u(x) - u_n(y) + u(y)) \, |x-y|^{-(N+ps)} dxdy \Bigg\}^{p/2} \\
&\qquad \times \Bigg\{ \iint_{\mathbb{R}^{2N}} (|u_n(x) - u_n(y)| + |u(x) - u(y)|)^p |x-y|^{-(N+ps)} dxdy \Bigg\}^{(2-p)/2} \\
&\le C \Bigg\{ \iint_{\mathbb{R}^{2N}} \Big[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y)) \Big] \\
&\qquad \times (u_n(x) - u(x) - u_n(y) + u(y)) \, |x-y|^{-(N+ps)} dxdy \Bigg\}^{p/2} \\
&\longrightarrow 0,
\end{aligned}
\tag{3.19}
$$

as $n \to \infty$, where $C > 0$ is a constant. Similarly, using (3.16), we can deduce

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n(x) - u(x)|^p dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)|u_n(x) - u(x)|^q dx = 0. \tag{3.20}
$$

Combining (3.18), (3.19) with (3.20), we get that $u_n \to u$ strongly in W as $n \to \infty$.

With a similar discussion as in Lemma 3.2, we have

$$
\begin{aligned}
&\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left| \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}} \right|^{p'} dxdy \\
&= 0,
\end{aligned}
\tag{3.21}
$$

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x) \left| |u_n(x)|^{p-2} u_n(x) - |u(x)|^{p-2} u(x) \right|^{p'} dx = 0, \tag{3.22}
$$

and

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} b(x) \left| |u_n(x)|^{q-2} u_n(x) - |u(x)|^{q-2} u(x) \right|^{q'} dx = 0. \tag{3.23}
$$

Clearly, for any $\varphi \in W$

$$
\begin{aligned}
\langle I'(u_n), \varphi\rangle = & \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dxdy \\
& + \int\limits_{\mathbb{R}^N} V(x)|u_n(x)|^{p-2}u_n(x)\varphi(x)dx + \int\limits_{\mathbb{R}^N} b(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)dx \\
& - \lambda \int\limits_{\mathbb{R}^N} a(x)|u_n(x)|^{r-2}u_n(x)\varphi(x)dx \\
& \longrightarrow 0,
\end{aligned}
$$

as $n \to \infty$. By (3.21)–(3.23), we get

$$
\langle I'(u), \varphi\rangle = 0
$$

for all $\varphi \in W$, that is, u is a critical point of I in W. Moreover, by the continuity of I and strong convergence of $u_n \to u$ in W, we obtain

$$
I(u) = \lim_{n\to\infty} I(u_n) = c > 0.
$$

This completes the proof. □

Proof of Theorem 1.1. (i) We only need to verify that if $u \in W \setminus \{0\}$ is a weak solution for problem (1.1), then there exists $\lambda_0 > 0$ such that $\lambda \geq \lambda_0$. For this, taking $\varphi = u$, we obtain

$$
\begin{aligned}
& \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int\limits_{\mathbb{R}^N} V(x)|u(x)|^p dx + \int\limits_{\mathbb{R}^N} b(x)|u(x)|^q dx \\
& = \lambda \int\limits_{\mathbb{R}^N} a(x)|u(x)|^r dx.
\end{aligned}
\tag{3.24}
$$

Then $0 < \min\{1, V_0\}\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \leq \lambda\|u\|^r_{L^r(\mathbb{R}^N,a)}$ by (3.24). So that $\lambda > 0$. By applying the inequality (3.1), we have

$$
\lambda a(x)|u|^{r-p} - b(x)|u|^{q-p} \leq \lambda^{(q-p)/(q-r)} \left[\frac{a(x)^{q-p}}{b(x)^{r-p}}\right]^{1/(q-r)} \quad \text{for all } x \in \mathbb{R}^N. \tag{3.25}
$$

It follows from (3.24), (3.25) and (2.1) that

$$
\begin{aligned}
& \min\{1, V_0\}\|u\|^p_{W^{s,p}(\mathbb{R}^N)} \\
& \leq \lambda^{(q-p)/(q-r)} \int\limits_{\mathbb{R}^N} \left[\frac{a(x)^{q-p}}{b(x)^{r-p}}\right]^{1/(q-r)} |u|^p dx \\
& \leq C_*^p \lambda^{(q-p)/(q-r)} \left\| \left[\frac{a(x)^{q-p}}{b(x)^{r-p}}\right]^{1/(q-r)} \right\|_{L^{\frac{N}{ps}}(\mathbb{R}^N)} \|u\|^p_{W^{s,p}(\mathbb{R}^N)}.
\end{aligned}
$$

Thus,

$$\left(\min\{1, V_0\} - C_*^p \lambda^{(q-p)/(q-r)} \left\| \left[\frac{a(x)^{q-p}}{b(x)^{r-p}}\right]^{1/(q-r)} \right\|_{L^{\frac{N}{ps}}(\mathbb{R}^N)}\right) \|u\|^p_{W^{s,p}(\mathbb{R}^N)} \le 0,$$

which implies that

$$\lambda \ge \left(\frac{\min\{1, V_0\}}{C_*^p \left\| \left[\frac{a(x)^{q-p}}{b(x)^{r-p}}\right]^{1/(q-r)} \right\|_{L^{\frac{N}{ps}}(\mathbb{R}^N)}} \right)^{(q-r)/(q-p)} := \lambda_0.$$

Therefore, if u is a nontrivial weak solution of problem (1.1), then $\lambda \ge \lambda_0$.

(ii) Theorem 3.1 and Theorem 3.3 assure that for all $\lambda > \lambda^*$ problem (1.1) admits two nontrivial weak solutions in W in which one has negative energy and another has positive energy. □

Corollary 3.4. *Suppose that all the assumptions of Theorem 1.1 are satisfied. Then problem (1.1) has at least two nontrivial nonnegative weak solutions in W in which one has negative energy and another has positive energy.*

Proof. First, for $v \in W$, we have $v^+ \in W$, where $v^+ = \max\{v, 0\} = (|v| + v)/2$. Indeed, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|v^+(x) - v^+(y)|^p}{|x-y|^{N+ps}} dxdy \\
&= \iint_{\mathbb{R}^{2N}} \left| \frac{|v(x)| - |v(y)| + v(x) - v(y)}{2} \right|^p |x-y|^{-N-ps} dxdy \\
&\le \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^p}{|x-y|^{N+ps}} dxdy.
\end{aligned}$$

So, $v^+ \in W$. Similarly, $v^- = \max\{-v, 0\}$ is also in W. Now, we define

$$I^+(u) = J(u) - H(u^+),$$

where $H(u^+) = (\lambda/r) \int_{\mathbb{R}^N} a(x)|u^+(x)|^r dx$. Then, I^+ is well defined on W and of class C^1 and

$$\begin{aligned}
&\langle (I^+(u))', \varphi \rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx + \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)\varphi(x)dx
\end{aligned}$$

$$-\lambda \int_{\mathbb{R}^N} a(x)|u^+(x)|^{r-2}u^+(x)\varphi(x)dx. \tag{3.26}$$

Moreover, I^+ is coercive, weakly lower semi-continuous in W and $I^+(0) = 0$. Notice that all critical points of I^+ are nonnegative. Indeed, if u is a critical point of I^+, then by (3.26) we have

$$\begin{aligned}
&\langle (I^+(u))', u^- \rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}} dxdy \\
&\quad + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)u^+(x)dx + \int_{\mathbb{R}^N} b(x)|u(x)|^{q-2}u(x)u^-(x)dx \\
&\quad - \lambda \int_{\mathbb{R}^N} a(x)|u^+(x)|^{r-2}u^+(x)u^-(x)dx = 0.
\end{aligned} \tag{3.27}$$

It follows from (3.27) that $u^- = 0$ a.e. in $\mathbb{R}^N$. Thus, $u \geq 0$ a.e. in $\mathbb{R}^N$.

Similar to Theorem 3.1, there exists $0 \leq u^{**} \in W$ such that

$$I^+(u^{**}) = \inf_{u \in W} I^+(u).$$

It is easy to see that $u^{**} \neq 0$ and $I^+(u^{**}) < 0$. Hence, we get a nontrivial nonnegative weak solution u^{**} of (1.1) with negative energy.

Next, we prove that (1.1) admits a nontrivial nonnegative weak solution in W with positive energy. Clearly, I^+ satisfies Lemma 3.5. We know that I^+ satisfies the geometrical structure of Theorem 3.2 by taking $e = u^{**}$. Similar to Theorem 3.3, we get that I^+ has a critical point $0 \leq u_+$ in W satisfying $I^+(u_+) > 0$. Therefore, this corollary is proved. □

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