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Asymmetric, Noncoercive, Superlinear $(p, 2)$-Equations

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We examine a nonlinear nonhomogeneous Dirichlet problem driven by the sum of a p-Laplacian ($p \geqslant 2$) and a Laplacian (a $(p, 2)$-equation). The reaction term is asymmetric and it is superlinear in the positive direction and sublinear in the negative direction. The superlinearity is not expressed using the Ambrosetti-Rabinowitz condition, while the asymptotic behavior as $x \to -\infty$ permits resonance with respect to any nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Using variational methods based on the critical point theory and Morse theory (critical groups), we prove a multiplicity theorem producing three nontrivial solutions.

Keywords: $(p, 2)$-equation, asymmetric reaction, superlinear growth, multiple solutions, nonlinear regularity, critical groups

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Dirichlet problem driven by the sum of a p-Laplacian ($p \geqslant 2$) and a Laplacian (a $(p, 2)$-equation):

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega, \ u|_{\partial\Omega} = 0, \ 2 \leqslant p. \tag{1}$$

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By Δ_p we denote the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

In this problem the reaction term $f(z,x)$ is a measurable function which is C^1 in the $x \in \mathbb{R}$ variable and exhibits an asymmetric behavior as $x \to \pm\infty$. More precisely, $x \longmapsto f(z,x)$ is $(p-1)$-superlinear near $+\infty$, but it is $(p-1)$-sublinear near $-\infty$. The superlinearity in the positive direction is not expressed using the Ambrosetti-Rabinowitz condition (the AR-condition for short). Instead we employ a weaker condition which incorporates in our framework superlinear nonlinearities with slower growth near $+\infty$ which fail to satisfy the AR-condition. In the negative direction where $f(z,\cdot)$ is sublinear, our hypothesis permits resonance with respect to any nonprincipal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, problem (1) is asymmetric, superlinear and at resonance.

Recently such problems were studied by Recova and Rumbos [26], [27] for semilinear Dirichlet problems driven by the Laplacian and with more restrictive conditions on the reaction term (see Theorem 1.1 of [26] and Theorem 1.2 of [27]). We also mention the semilinear works of de Paiva and Presoto [20] (they study a parametric equation driven by the Laplacian) and Motreanu, Motreanu and Papageorgiou [15] (they study an equation driven by the Laplacian, no resonance is allowed as $x \to -\infty$ and they produce only two nontrivial solutions). For equations driven by the p-Laplacian, we mention the work of Motreanu, Motreanu and Papageorgiou [16], who deal with a parametric problem involving concave nonlinearities.

We mention that $(p,2)$-equations arise in many physical applications. We refer to the works of Benci, D'Avenia, Fortunato and Pisani [3] (quantum physics) and Cherfils and Ilyasov [4] (diffusion problems). Recently there have been some existence and multiplicity results for such equations under different settings. We mention the works of Cingolani and Degiovanni [5], Mugnai and Papageorgiou [18], Papageorgiou and Rădulescu [21], Papageorgiou and Smyrlis [23] and Papageorgiou and Winkert [24].

Our approach combines variational methods based on the critical point theory with Morse theory (critical groups).

2. Mathematical Background

Let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Let $\varphi \in C^1(X,\mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1+||u_n||)\varphi'(u_n) \to 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence."

This is a compactness-type condition on the functional φ and it is more general than the more common Palais-Smale condition. The C-condition leads to a deformation theorem from which one can derive the min-max theory for the critical values of φ. Prominent in this theory is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [2], which we state here in a slightly more general form (see, for example, Gasinski and Papageorgiou [9, p. 648]).

Theorem 2.1. *Let X be a Banach space and assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = m_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$. Then $c \geqslant m_\rho$ and c is a critical value of φ.

In the analysis of problem (1), we will use the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$. Since $p \geqslant 2$, we have $W_0^{1,p}(\Omega) \subseteq H_0^1(\Omega)$. We will also use the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\}.$$

Here $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$.

We will also need some facts about the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0,\ 1 < p < \infty. \tag{2}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$ which is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. There exists a smallest eigenvalue $\hat{\lambda}_1(p) > 0$ which has the following properties:

- $\hat{\lambda}_1(p)$ is isolated (that is, there exists $\epsilon > 0$ such that the open interval $(\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon)$ contains no eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$).
- $\hat{\lambda}_1(p)$ is simple (that is, if $\hat{u}, \hat{v} \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to the eigenvalue $\hat{\lambda}_1(p)$, then $\hat{u} = \xi\hat{v}$ for some $\xi \in \mathbb{R}\backslash\{0\}$).
- $$\hat{\lambda}_1(p) = \inf\left[\frac{||Du||_p^p}{||u||_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right]. \tag{3}$$

The infimum in (3) is realized at the corresponding one-dimensional eigenspace. From (3) it is clear that the elements of this eigenspace do not change sign. Let $\hat{u}_1(p)$ be the L^p-normalized (that is, $||\hat{u}_1(p)||_p = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1(p)$. The nonlinear regularity theory (see Lieberman [12]) and the nonlinear maximum principle (see Pucci and Serrin [25]), imply that $\hat{u}_1(p) \in \operatorname{int} C_+$.

The Ljusternik-Schnirelmann minimax scheme gives a whole strictly increasing sequence $\{\hat{\lambda}_k(p)\}_{k\geqslant 1}$ of distinct eigenvalues such that $\hat{\lambda}_k(p) \to +\infty$. However, we do not know if this sequence exhausts the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. This is the case if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equations).

The following lemma can be found in Motreanu, Motreanu and Papageorgiou [17, p. 305].

Lemma 2.2. *Assume that $\vartheta \in L^\infty(\Omega)$ satisfies $\vartheta(z) \leqslant \hat{\lambda}_1(p)$ $(1 < p < \infty)$ for almost all $z \in \Omega$, with strict inequality on a set of positive measure. Then there exists $\hat{c} > 0$ such that*

$$||Du||_p^p - \int_\Omega \vartheta(z)|u|^p dz \geqslant \hat{c}||Du||_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The same results are also true for the following weighted version of problem (2):

$$-\Delta_p u(z) = \tilde{\lambda} m(z)|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0,$$

with $m \in L^\infty(\Omega), m \geqslant 0, m \not\equiv 0$. In this case

$$\tilde{\lambda}_1(p,m) = \inf\left[\frac{||Du||_p^p}{\int_\Omega m(z)|u|^p dz} : u \in W_0^{1,p}(\Omega), u \neq 0\right].$$

We have the following monotonicity property for the map $m \to \tilde{\lambda}_1(p,m)$.

Proposition 2.3. *Assume that $m, m' \in L^\infty(\Omega)$, $0 \leqslant m(z) \leqslant m'(z)$ for almost all $z \in \Omega$ and $m \not\equiv m'$. Then $\tilde{\lambda}_1(p,m') < \tilde{\lambda}_1(p,m)$.*

We mention that only the first eigenvalue has eigenfunctions of constant sign. All the other eigenvalues have nodal (that is, sign-changing) eigenfunctions. For further details on these and related issues, we refer to Gasinski and Papageorgiou [9].

For $1 < p < \infty$, let $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be the map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

When $p = 2$, we write $A_2 = A$ and we have $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. For $p \neq 2$, A_p is nonlinear and $(p-1)$-homogeneous. Also we have (see Gasinski and Papageorgiou [9, p. 746]).

Proposition 2.4. *The map $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(1 < p < \infty)$ is continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is*

$$\text{“}u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0 \\ \Rightarrow u_n \to u \quad \text{in } W_0^{1,p}(\Omega).\text{”}$$

Next we recall some basic facts about critical groups (Morse theory). For further details we refer to the book of Motreanu, Motreanu and Papageorgiou [17] (see also Cingolani, Degiovanni and Vannella [6] and Cingolani and Vannella [7].

So, let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We define the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\}, \qquad K_\varphi = \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the topological pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{0\}) \ \text{ for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood U of u.

Suppose that φ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \ \text{ for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [9, p. 628]), implies that the above definition is independent of the level $c < \inf \varphi(K_\varphi)$.

Suppose that $\varphi \in C^1(X, \mathbb{R})$, satisfies the C-condition and K_φ is finite. We define

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \ \text{ for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$

$$P(t, \infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \ \text{ for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{4}$$

where $Q(t) = \sum_{k\geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally we fix our notation. By $||\cdot||$ we denote the norm of $W_0^{1,p}(\Omega)$. From the Poincaré inequality (see, for example, Gasinski and Papageorgiou [9, p. 216]), we have

$$||u|| = ||Du||_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $x \in \mathbb{R}$. We define $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we set

$$u^\pm(\cdot) = u(\cdot)^\pm.$$

We know that $u^\pm \in W_0^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$. Given a measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, we set

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Then $z \longmapsto N_g(u)(z) = g(z,u(z))$ is measurable. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leqslant p \end{cases}$ the critical Sobolev exponent.

3. Multiplicity Theorem

In this section we prove a multiplicity theorem for problem (1) producing three nontrivial solutions. Our hypotheses on the reaction term $f(z,x)$ are the following:

H : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z,0) = 0, f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)(1+|x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+,\ p < r < p^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) if $\xi(z,x) = f(z,x)x - pF(z,x)$, then there exists $\gamma_0 \in L^1(\Omega)$ such that

$$\xi(z,x) \leqslant \xi(z,y) + \gamma_0(z) \quad \text{for almost all } z \in \Omega,\ \text{all } 0 \leqslant x \leqslant y;$$

(iv) there exist functions $\eta, \hat{\eta} \in L^\infty(\Omega)$ and $c_0 > 0$ such that

$$\eta(z) \geqslant \hat{\lambda}_1(p) \text{ for almost all } z \in \Omega, \text{ strictly on a set of positive measure,}$$

$$\begin{aligned}\eta(z) &\leqslant \liminf_{x\to+\infty} \frac{f(z,x)}{|x|^{p-2}x} \\ &\leqslant \limsup_{x\to+\infty} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \hat{\eta}(z) \text{ uniformly for almost all } z \in \Omega;\end{aligned}$$

$$-c_0 \leqslant f(z,x)x - pF(z,x) \text{ for almost all } z \in \Omega, \text{ all } x \leqslant 0;$$

(v) $f'_x(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x}$ uniformly for almost all $z \in \Omega$, $f'_x(z,0) \leqslant \hat{\lambda}_1(2)$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure;

(vi) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that $f(z,x) + \hat{\xi}_\rho x^{p-1} \geqslant 0$ for almost all $z \in \Omega$, all $0 \leqslant x \leqslant \rho$.

Remark 3.1. Hypothesis H(ii) implies that for almost all $z \in \Omega$, the primitive $F(z,\cdot)$ is p-superlinear near $+\infty$. This fact and hypothesis H(iii), imply that for almost all $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$ (see Li and Yang [13, Lemma 2.4]). Hypothesis H(iii) replaces the AR-condition which says that there exist $q > p$ and $M > 0$ such that

$$0 < qF(z,x) \leqslant f(z,x)x \text{ for almost all } z \in \Omega, \text{ all } x \geqslant M \tag{5a}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, M) \tag{5b}$$

An easy integration of (5a) and the use of (5b), imply the weaker condition

$$c_1 x^q \leqslant F(z,x) \text{ for almost all } z \in \Omega, \text{ all } x \geqslant M \text{ with } c_1 > 0. \tag{6}$$

So, the AR-condition restricts $F(z,\cdot)$ to have at least q-polynomial growth near $+\infty$. With H(iii) we avoid this (see the examples which follow). Condition H(iii) also extends earlier ones used by Li and Yang [13] and Miyagaki and Souto [14]. Hypothesis H(iv) implies that for almost all $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-sublinear near $-\infty$. Note that this hypothesis does not exclude resonance with respect to a nonprincipal eigenvalue.

Example 3.2. The following functions satisfy hypotheses H. For the sake of simplicity we drop the z-dependence:

$$f_1(x) = \begin{cases} \eta|x|^{p-2}x + (\eta - \vartheta) & \text{if } x < -1 \\ \vartheta x & \text{if } -1 \leqslant x \leqslant 1 \\ x^{r-1} + (\vartheta - 1) & \text{if } 1 \leqslant x, \end{cases}$$

$$f_2(x) = \begin{cases} \eta|x|^{p-2}x + (\eta - \vartheta) & \text{if } x < -1 \\ \vartheta x & \text{if } -1 \leqslant x \leqslant 1 \\ x^{p-1}\big(\ln x + \frac{1}{p}\big) + \big(\vartheta - \frac{1}{p}\big) & \text{if } 1 \leqslant x, \end{cases}$$

with $\vartheta < \hat{\lambda}_1(2)$. Note that f_2 does not satisfy the AR-condition (see (6)).

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega F(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have $\varphi \in C^2(W_0^{1,p}(\Omega))$.

Proposition 3.3. *If hypotheses H hold, then the functional φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \text{ all } n \geqslant 1 \tag{7}$$

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty\,. \tag{8}$$

From (8) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z,u_n)hdz\right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \tag{9}$$
$$\text{for all } h \in W_0^{1,p}(\Omega) \text{ with } \epsilon_n \to 0^+.$$

Recall that $u_n = u_n^+ - u_n^-$ for all $n \geqslant 1$. So, we have

$$\begin{aligned}
&\frac{1}{p}||Du_n^+||_p^p + \frac{1}{2}||Du_n^+||_2^2 \\
&= \frac{1}{p}||Du_n||_p^p + \frac{1}{2}||Du_n||_2^2 - \frac{1}{p}||Du_n^-||_p^p - \frac{1}{2}||Du_n^-||_2^2 \\
&\qquad + \int_\Omega F(z,u_n)dz - \int_\Omega F(z,u_n)dz \\
&= \varphi(u_n) - \frac{1}{p}||Du_n^-||_p^p - \frac{1}{2}||Du_n^-||_2^2 + \int_\Omega F(z,u_n)dz \\
&\leqslant M_1 + \frac{1}{p}\left[\int_\Omega pF(z,u_n)dz - ||Du_n^-||_p^p - ||Du_n^-||_2^2\right] \text{ for all } n \geqslant 1 \qquad (10)\\
&\qquad \text{(see (7) and recall } p \geqslant 2).
\end{aligned}$$

In (9) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\left| ||Du_n^-||_p^p + ||Du_n^-||_2^2 - \int_\Omega f(z,-u_n^-)(-u_n^-)dz \right| \leqslant \epsilon_n \text{ for all } n \geqslant 1,$$

$$\Rightarrow \quad -||Du_n^-||_p^p - ||Du_n^-||_2^2 \leqslant \epsilon_n - \int_\Omega f(z,-u_n^-)(-u_n^-)dz \text{ for all } n \geqslant 1. \tag{11}$$

We return to (10) and use (11). Then

$$\frac{1}{p}||Du_n^+||_p^p + \frac{1}{2}||Du_n^+||_2^2 \leqslant M_2 + \frac{1}{p}\int_\Omega \left[pF(z,u_n) - f(z,-u_n^-)(-u_n^-)\right]dz \quad (12)$$

for some $M_2 > 0$, all $n \geqslant 1$

We have

$$pF(z,u_n) = pF(z,u_n^+) + pF(z,-u_n^-) \text{ for all } n \geqslant 1 \quad (13)$$

and from hypothesis H(iv), we have

$$pF(z,-u_n^-) - f(z,-u_n^-)(-u_n^-) \leqslant c_0 \text{ for almost all } z \in \Omega, \text{ all } n \geqslant 1. \quad (14)$$

Returning to (12) and using (13) and (14) we obtain

$$\begin{aligned}&\frac{1}{p}||Du_n^+||_p^p + \frac{1}{2}||Du_n^+||_2^2 \leqslant M_3 + \int_\Omega F(z,u_n^+)dz \\ &\qquad \text{with } M_3 = M_2 + c_0|\Omega|_N > 0, \text{ for all } n \geqslant 1, \\ &\Rightarrow \varphi(u_n^+) \leqslant M_3 \text{ for all } n \geqslant 1. \end{aligned} \quad (15)$$

In (9) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and have

$$-||Du_n^+||_p^p - ||Du_n^+||_2^2 + \int_\Omega f(z,u_n^+)u_n^+dz \leqslant \epsilon_n \text{ for all } n \geqslant 1. \quad (16)$$

From (15) and since $p \geqslant 2$, we have

$$||Du_n^+||_p^p + ||Du_n^+||_2^2 - \int_\Omega pF(z,u_n^+)dz \leqslant pM_3 \text{ for all } n \geqslant 1. \quad (17)$$

Adding (16) and (17), we obtain

$$\begin{aligned}&\int_\Omega \left[f(z,u_n^+)u_n^+ - pF(z,u_n^+)\right]dz \leqslant M_4 \text{ for some } M_4 > 0, \text{ all } n \geqslant 1, \\ \Rightarrow \quad &\int_\Omega \xi(z,u_n^+)dz \leqslant M_4 \text{ for all } n \geqslant 1. \end{aligned} \quad (18)$$

Claim 3.4. $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ *is bounded.*

We argue indirectly. So, suppose that $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. Then we may assume that $||u_n^+|| \to \infty$ as $n \to \infty$. We set $y_n = \frac{u_n^+}{||u_n^+||} n \geqslant 1$. We have

$$||y_n|| = 1 \quad \text{and} \quad y_n \geqslant 0 \text{ for all } n \geqslant 1.$$

Hence we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^r(\Omega) \text{ as } n \to \infty, \ y \geqslant 0. \quad (19)$$

Suppose that $y \neq 0$. Then $|\{y > 0\}|_N = 0$ (recall that $y \geqslant 0$, see (14)) and we have

$$u_n^+(z) \to +\infty \text{ for almost all } z \in \{y > 0\}.$$

Hypothesis H(ii) implies that

$$\frac{F(z, u_n^+(z))}{||u_n^+||^p} = \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \to +\infty \text{ for almost all } z \in \{y > 0\}.$$

This fact and Fatou's lemma (see hypothesis H(ii)), imply that

$$\lim_{n\to\infty} \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^p} dz = +\infty. \tag{20}$$

Since $\xi(z, 0) = 0$ for almost all $z \in \Omega$, from hypothesis H(iii) we have

$$\begin{aligned}
pF(z, u_n^+) &\leqslant f(z, u_n^+)u_n^+ + \gamma_0(z) \text{ for almost all } z \in \Omega,\\
\Rightarrow \int_\Omega pF(z, u_n^+)dz &\leqslant \int_\Omega f(z, u_n^+)u_n^+ dz + ||\gamma_0||_1\\
&\leqslant M_5 + ||Du_n^+||_p^p + ||Du_n^+||_2^2\\
&\qquad \text{for some } M_5 > 0, \text{ all } n \geqslant 1 \text{ (see (16))}\\
&\leqslant M_5 + ||Du_n^+||_p^p + \frac{p}{2}||Du_n^+||_2^2 \text{ since } p \geqslant 2\\
\Rightarrow \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^p} dz &\leqslant \frac{M_5}{p||u_n^+||^p} + \frac{1}{p}||Dy_n||_p^p + \frac{1}{2||u_n^+||^{p-2}}||Dy_n||_2^2\\
&\leqslant M_6 \text{ for some } M_6 > 0, \text{ all } n \geqslant 1.
\end{aligned} \tag{21}$$

Comparing (20) and (21) we reach a contradiction.

So, suppose that $y = 0$. For $k \geqslant 1$, we set

$$v_n = (pk)^{1/p} y_n \in W_0^{1,p}(\Omega).$$

We have

$$v_n \to 0 \text{ in } L^r(\Omega) \text{ (see (19) and recall that } y = 0).$$

Hypothesis H(i) implies that

$$|F(z, x)| \leqslant c_1(1 + |x|^r) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ with } c_1 > 0.$$

Using the Krasnoselskii theorem (see, for example, Gasinski and Papageorgiou [9, p. 407]), we have

$$\int_\Omega F(z, v_n)dz \to 0 \text{ as } n \to \infty. \tag{22}$$

Since $||u_n^+|| \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$0 < (pk)^{1/p} \frac{1}{||u_n^+||} \leqslant 1 \quad \text{for all } n \geqslant n_0. \tag{23}$$

Let $\hat{\varphi}(u) = \frac{1}{p}||Du||_p^p - \int_\Omega F(z,u)dz$ for all $u \in W_0^{1,p}(\Omega)$. Let $t_n \in [0,1]$ be such that

$$\hat{\varphi}(t_n u_n^+) = \max[\hat{\varphi}(tu_n^+) : 0 \leqslant t \leqslant 1]. \tag{24}$$

From (23) and (24) we see that

$$\begin{aligned}
\hat{\varphi}(t_n u_n^+) &\geqslant \hat{\varphi}(v_n) \\
&= k||Dy_n||_p^p - \int_\Omega F(z,v_n)dz \\
&= k - \int_\Omega F(z,v_n)dz \quad \text{for all } n \geqslant n_0, \\
\Rightarrow \quad \hat{\varphi}(t_n, u_n^+) &\geqslant \frac{k}{2} \quad \text{for all } n \geqslant n_1 \geqslant n_0 \text{ (see (22))}.
\end{aligned}$$

But $k > 0$ is arbitrary. So, we infer that

$$\hat{\varphi}(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{25}$$

We have

$$\hat{\varphi}(0) = 0 \quad \text{and} \quad \hat{\varphi}(u_n^+) \leqslant M_3 \quad \text{for all } n \geqslant 1 \text{ (see (15) and note that } \hat{\varphi} \leqslant \varphi).$$

Because of (25), we see that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0,1) \quad \text{for all } n \geqslant n_2.$$

Then from (24) it follows that

$$\begin{aligned}
&\frac{d}{dt}\hat{\varphi}(tu_n^+)\Big|_{t=t_n} = 0 \quad \text{for all } n \geqslant n_2, \\
\Rightarrow \quad &\left\langle \hat{\varphi}'(t_n u_n^+), u_n^+ \right\rangle = 0 \quad \text{for all } n \geqslant n_2 \text{ (by the chain rule)}, \\
\Rightarrow \quad &\left\langle \hat{\varphi}'(t_n u_n^+), t_n u_n^+ \right\rangle = 0 \quad \text{for all } n \geqslant n_2, \\
\Rightarrow \quad &||D(t_n u_n^+)||_p^p - \int_\Omega f(z, t_n u_n^+)(t_n u_n^+)dz \quad \text{for all } n \geqslant n_2.
\end{aligned} \tag{26}$$

Hypothesis H(iii) implies that

$$\int_\Omega \xi(z, t_n u_n^+)dz \leqslant \int_\Omega \xi(z, u_n^+)dz + ||\gamma_0||_1 \leqslant M_7 \tag{27}$$
$$\text{for some } M_7 > 0, \text{ all } n \geqslant n_2 \text{ (see (18) and recall } t_n \in (0,1))$$

We return to (26) and use (27). Then

$$||D(t_n u_n^+)||_p^p \leqslant M_7 + \int_\Omega pF(z, t_n u_n^+)dz \quad \text{for all } n \geqslant n_2,$$

$$\Rightarrow \quad \hat{\varphi}(t_n u_n^+) \leqslant \frac{M_7}{p} \quad \text{for all } n \geqslant n_2. \tag{28}$$

Comparing (25) and (28) we reach a contradiction.

This proves the Claim.

From (9) and using the Claim, we have that

$$\left| \left\langle A_p(-u_n^-), h \right\rangle + \left\langle A(-u_n^-), h \right\rangle - \int_\Omega f(z, -u_n^-)hdz \right| \leqslant M_8 ||h|| \tag{29}$$

$$\text{for some } M_8 > 0, \text{ all } n \geqslant 1.$$

Suppose that $||u_n^-|| \to \infty$ and set $w_n = \frac{u_n^-}{||u_n^-||} n \geqslant 1$. Then

$$||w_n|| = 1 \quad \text{and} \quad w_n \geqslant 0 \quad \text{for all } n \geqslant 1.$$

So, we may assume that

$$w_n \xrightarrow{w} w \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad w_n \to w \quad \text{in } L^p(\Omega), \ w \geqslant 0. \tag{30}$$

From (29) we have

$$\left| \langle A_p(-w_n), h \rangle + \frac{1}{||w_n||^{p-2}} \langle A(-w_n), h \rangle - \int_\Omega \frac{N_f(-u_n^-)}{||u_n^-||^{p-1}} hdz \right| \leqslant \frac{M_8 ||h||}{||u_n^-||^{p-1}} \tag{31}$$

$$\text{for all } n \geqslant 1$$

Hypotheses H(i),(iv) imply that

$$|f(z,x)| \leqslant c_2(1 + |x|^{p-1}) \quad \text{for almost all } z \in \Omega, \text{ all } x \leqslant 0 \text{ and some } c_2 > 0,$$

$$\Rightarrow \left\{ \frac{N_f(-u_n^-)}{||u_n^-||^{p-1}} \right\}_{n \geqslant 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).$$

Using this fact and hypothesis H(iv) we have, at least for a subsequence, that

$$\frac{N_f(-u_n^-)}{||u_n^-||^{p-1}} \xrightarrow{w} -\vartheta w^{p-1} \quad \text{in } L^{p'}(\Omega) \text{ with } \eta(z) \leqslant \vartheta(z) \leqslant \hat{\eta}(z) \text{ for almost all } z \in \Omega \tag{32}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 16). In (31) we use $h = w_n - w \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (30) and (32). We obtain

$$\lim_{n \to \infty} \langle A_p(w_n), w_n - w \rangle = 0 \quad (\text{recall } p \geqslant 2),$$

$$\Rightarrow w_n \to w \quad \text{in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.4), hence } ||w|| = 1, w \geqslant 0. \tag{33}$$

Therefore, if in (31) we pass to the limit as $n \to \infty$ and use (32) and (33), then

$$\begin{aligned} &\langle A_p(w), h\rangle = \int_\Omega \vartheta(z) w^{p-1} h dz \ \text{ for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow\ &-\Delta_p w(z) = \vartheta(z) w(z)^{p-1} \ \text{ for almost all } z \in \Omega,\ w|_{\partial\Omega} = 0. \end{aligned} \tag{34}$$

Recall that

$$\hat{\lambda}_1(p) \leqslant \eta(z) \leqslant \vartheta(z) \ \text{ for almost all } z \in \Omega$$

and the first inequality is strict on a set of positive measure. So, using Proposition 2.3, we have

$$\tilde{\lambda}_1(p, \vartheta) < \tilde{\lambda}_1(p, \hat{\lambda}_1) = 1.$$

Then returning to (34) we infer that $w(\cdot)$ must be nodal or zero, a contradiction (see (33)). Therefore

$$\begin{aligned} &\{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}, \\ \Rightarrow\ &\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see the Claim)}. \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \ \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \ \text{ in } L^r(\Omega). \tag{35}$$

In (9) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (35). Then

$$\begin{aligned} &\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle] = 0, \\ \Rightarrow\ &\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle] \leqslant 0 \ \text{ (since } A \text{ is monotone)}, \\ \Rightarrow\ &\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leqslant 0, \\ \Rightarrow\ &u_n \to u \ \text{ in } W_0^{1,p}(\Omega) \ \text{ (see Proposition 2.4)}. \end{aligned}$$

This proves that φ satisfies the C-condition. □

Having established that φ satisfies the C-condition, we can compute the critical groups of φ at infinity.

Proposition 3.5. *If hypotheses H hold and $\varphi(K_\varphi)$ is bounded below, then $C_k(\varphi, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $\varphi_c = \varphi|_{C_0^1(\overline{\Omega})}$. From the nonlinear regularity theory (see Lieberman [12]), we have that $K_{\varphi_c} = K_\varphi = K$. Moreover, since $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$, from Palais [19, Theorem 16], we have

$$H_k(W_0^{1,p}(\Omega), \dot{\varphi}^a) = H_k(C_0^1(\overline{\Omega}), \dot{\varphi}_c^a) \ \text{ for all } a \in \mathbb{R},\ \text{ all } k \in \mathbb{N}, \tag{36}$$

with $\dot{\varphi}^a = \{u \in W_0^{1,p}(\Omega) : \varphi(u) < a\}$, $\dot{\varphi}_c^a = \{u \in C_0^1(\overline{\Omega}) : \varphi_c(u) < a\}$.

Choosing $a < \inf \varphi(K) = \inf \varphi_c(K)$ (this is possible by hypothesis), we have

$$H_k(W_0^{1,p}(\Omega), \dot{\varphi}^a) = H_k(W_0^{1,p}(\Omega), \varphi^a) = C_k(\varphi, \infty) \text{ for all } k \in \mathbb{N}_0, \tag{37}$$

$$H_k(C_0^1(\overline{\Omega}), \dot{\varphi}_c^a) = H_k(C_0^1(\overline{\Omega}), \varphi^a) = C_k(\varphi_c, \infty) \text{ for all } k \in \mathbb{N}_0 \tag{38}$$

(see Granas and Dugundji [10, p. 407]). From (36), (37), (38) we see that in order to prove the proposition, we need to show that

$$H_k(C_0^1(\overline{\Omega}), \varphi_c^a) = 0 \text{ for all } k \in \mathbb{N}_0.$$

To this end, let $C \subseteq \varphi_c^a$ be a compact set.

Claim 3.6. *For $a < 0$ with $|a| > 0$ big, the set C is contractible in φ_c^a.*

In what follows by $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(C_0^1(\overline{\Omega})^*, C_0^1(\overline{\Omega}))$. Also, let $i : C_0^1(\overline{\Omega}) \to W_0^{1,p}(\Omega)$ be the continuous embedding map. We have

$$\begin{aligned} & \varphi_c = \varphi \circ i \\ \Rightarrow\ & \varphi_c'(u) = i^* \varphi'(u) \text{ for all } u \in C_0^1(\overline{\Omega}). \end{aligned} \tag{39}$$

Let $u \in \varphi_c^a$. Then for $t > 0$ we have

$$\begin{aligned} & \frac{d}{dt}\varphi_c(tu) \\ =\ & \langle \varphi_c'(tu), u \rangle_0 \quad \text{(by the chain rule)} \\ =\ & \langle \varphi'(tu), u \rangle \quad \text{(see (39))} \\ =\ & \frac{1}{t} \langle \varphi'(tu), tu \rangle \\ =\ & \frac{1}{t} \left[t^p ||Du||_p^p + t^2 ||Du||_2^2 - \int_\Omega f(z, tu^+)(tu^+) dz - \int_\Omega f(z, -tu^-)(-tu^-) dz \right] \\ \leqslant\ & \frac{1}{t} \left[t^p ||Du||_p^p + t^2 ||Du||_2^2 - \int_\Omega pF(z, tu^+) dz - \int_\Omega pF(z, -tu^-) dz + c_3 \right] \\ & \qquad \text{with } c_3 = ||\gamma_0||_1 + c_0 |\Omega|_N > 0 \text{ (see hypotheses } H(\text{iii}),(\text{iv})) \\ \leqslant\ & \frac{1}{t} \left[t^p ||Du||_p^p + \frac{p}{2} t^2 ||Du||_2^2 - \int_\Omega pF(z, tu) dz + c_3 \right] \quad (\text{since } p \geqslant 2) \\ =\ & \frac{1}{2} \left[p\varphi_c(tu) + c_3 \right], \\ & \Rightarrow \frac{d}{dt}\varphi_c(tu)\bigg|_{t=1} \leqslant p\varphi_c(u) + c_3 \leqslant pa + c_3 \quad (\text{recall } u \in \varphi_c^a). \end{aligned}$$

Therefore

$$a < -\frac{c_3}{p} \Rightarrow \frac{d}{dt}\varphi_c(tu)\bigg|_{t=1} < 0.$$

So, if $\varphi_c(u) \in (a-1, a]$, then we can find a unique $k(u) > 0$ such that $\varphi_c(k(u)u) = a - 1$. If $u \in \varphi_c^{a-1}$, then we set $k(u) = 1$. The implicit function theorem implies that $k \in C(\varphi_c^a, (0,1])$. We consider the deformation $h_1 : [0,1] \times C \to \varphi_c^a$ defined by

$$h_1(t,u) = ((1-t) + tk(u))u\,.$$

Let $C_1 = h_1(1, C) \subseteq \varphi_c^{a-1}$. The set $C_1 \subseteq C_0^1(\overline{\Omega})$ is compact. So, we can find $M_9 > 0$ such that

$$\left|\frac{\partial u}{\partial n}(z)\right| \leqslant M_9 \;\text{ for all } z \in \overline{\Omega},\; \text{all } u \in C_1. \tag{40}$$

Given $\epsilon > 0$, we can find $\tilde{h}_\epsilon \in \operatorname{int} C_+$ such that

$$\frac{\partial \tilde{h}\epsilon}{\partial n}(z) < -M_9 \;\text{ for all } z \in \partial\Omega \quad\text{and}\quad (u + \tilde{h}_\epsilon)^+ \neq 0.$$

To see this, set $\hat{d}(z) = d(z, \partial\Omega)$ and define

$$\hat{h}_\epsilon(z) = \begin{cases} \hat{M}\hat{d}(z) & \text{if } \hat{d}(z) \leqslant \epsilon \\ \hat{M}\epsilon & \text{if } \epsilon < \hat{d}(z) \end{cases} \quad \text{with } \hat{M} > 0.$$

Approximate $\hat{h}_\epsilon$ by a $C_0^1(\overline{\Omega})$-function $\tilde{h}_\epsilon$ and choose $\hat{M} > 0$ big enough so that $\tilde{h}_\epsilon \in \operatorname{int} C_+$ has the desired properties.

We have $C_1 \subseteq \varphi_c^{a-1}$. Hence, if we choose $\epsilon > 0$ small, then the deformation $h_2 : [0,1] \times C_1 \to \varphi_c^a$ defined by

$$h_2(t,u) = u + t\tilde{h}_\epsilon \;\text{ for all } (t,u) \in [0,1] \times C_1,$$

is well-defined.

Let $C_2 = h_2(1, C_1)$ and pick $u \in C_2$. Then $u^+ \neq 0$ and we have

$$\varphi_c(u) = \varphi_c(u^+) + \varphi_c(-u^-) \leqslant a.$$

From the previous considerations we know that $t \longmapsto \varphi_c(tu)$ is decreasing on $[1, \infty)$. Because $C_2 \subseteq C_0^1(\overline{\Omega})$ is compact, we can find $t_* \geqslant 1$ such that

$$\varphi_c(tu^+) \leqslant a \;\text{ for all } t \geqslant t_*,\; \text{all } u \in C_2. \tag{41}$$

We introduce the deformation $h_3 : [0,1] \times C_2 \to \varphi_c^a$ defined by

$$h_3(t,u) = (1 - t + tt_*)u \;\text{ for all } (t,u) \in [0,1] \times C_2.$$

Evidently this is a well-defined deformation and if $C_3 = h_3(1, C_2)$, then

$$\varphi_c(u^+) \leqslant a \;\text{ for all } u \in C_3 \text{ (see (41))}. \tag{42}$$

The set $C_3 = h_3(1, C_2) \subseteq C_0^1(\overline{\Omega})$ is compact. So, we can find $M_{10} > 0$ such that

$$\varphi_c(s(-u^-)) \leqslant M_{10} \text{ for all } u \in C_3, \text{ all } s \in [0,1]. \tag{43}$$

From (42) and since $t \longmapsto \varphi_c(tu^+)$ is decreasing on $[1, \infty)$, we can find $\hat{t}_* \geqslant 1$ big such that

$$\varphi_c(\hat{t}_* u^+) \leqslant a - M_{10} \text{ for all } u \in C_3.$$

We consider the deformation $h_4 : [0,1] \times C_3 \to \varphi_c^a$ defined by

$$h_4(t,u) = (1 - t + t\hat{t}_*)u^+ + u^-.$$

This deformation too is well-defined. We set $C_4 = h_3(1, C_3)$ and have

$$\begin{gathered} C_4 \subseteq C_0^1(\overline{\Omega}) \text{ is compact} \\ C_4 \subseteq \varphi_c^a \cap \{u \in C_0^1(\overline{\Omega}) : \varphi_c(u^+) \leqslant a - M_{10}\} \text{ (see (43)).} \end{gathered} \tag{44}$$

Using $C_4 \subseteq C_0^1(\overline{\Omega})$, we will deform to a compact subset of positive functions in φ_c^a. To this end, let $h_5 : [0,1] \times C_4 \to \varphi_c^a$ be the deformation defined by

$$h_5(t,u) = u^+ + (1-t)(-u^-) \text{ for all } (t,u) \in [0,1] \times C_4.$$

We have

$$\begin{aligned} \varphi_c(h_5(t,u)) &= \varphi_c(u^+ + (1-t)(-u^-)) \\ &= \varphi_c(u^+) + \varphi_c((1-t)(-u^-)) \\ &\leqslant a - M_{10} + M_{10} = a \text{ (see (3) and (43)),} \\ \Rightarrow \quad & h_5 \text{ is well defined.} \end{aligned}$$

So, if $C_5 = h(1, C_4)$, then we have

$$\begin{aligned} & C_5 \subseteq \varphi_c^a \quad \text{and} \quad C_5 \subseteq C_+, \\ \Rightarrow \quad & C_5 \subseteq \varphi_c^a \cap C_+ = C_+^a. \end{aligned} \tag{45}$$

Let $\partial B_+^c = \{u \in C_0^1(\overline{\Omega}) : ||u||_{C_0^1(\overline{\Omega})} = 1\} \cap C_+$. From the first part of the proof we have

$$C_+^a = \{tu : u \in \partial B_+^c, t \geqslant \hat{k}(u)\}$$

with $\hat{k}(u) > 0$ being the unique real such that $\varphi_c(\hat{k}(u)u) = a$. Using the radial retraction, we see that C_+^a and ∂B_+^c are homotopy equivalent. We consider the deformation $h_+ : [0,1] \times \partial B_+^c \to \partial B_+^c$ defined by

$$h_+(t,u) = \frac{(1-t)u + t\hat{u}_1(p)}{||(1-t)u + t\hat{u}_1(p)||_{C_0^1(\overline{\Omega})}} \text{ for all } (t,u) \in [0,1] \times \partial B_+^c.$$

Note that

$$\begin{aligned} & h_+(1,u) = \frac{\hat{u}_1(p)}{||\hat{u}_1(p)||_{C_0^1(\overline{\Omega})}} \in \partial B_+^c, \\ \Rightarrow \quad & \partial B_+^c \text{ is contractible,} \\ \Rightarrow \quad & C_+^a \text{ is contractible.} \end{aligned}$$

Then from (45) we infer that C_5 is contractible. Since C was deformed to C_5 by successive deformations, we conclude that C is contractible in φ_c^a for $a < 0$ with $|a| > 0$ big. This proves the Claim.

Let $* \in \dot{\varphi}_c^a$. For $a < \inf \varphi(K_\varphi)$, we have

$$H_k(\varphi_c^a, *) = H_k(\dot{\varphi}_c^a, *) \ \text{ for all } k \in \mathbb{N}_0 \tag{46}$$

(see Granas and Dugundji [10, p. 407]).

The Banach space $C_0^1(\overline{\Omega})$ is separable. So, we can find a sequence $\{V_n\}_{n\geqslant 1}$ of increasing finite dimensional subspaces of $C_0^1(\overline{\Omega})$ such that

$$C_0^1(\overline{\Omega}) = \overline{\bigcup_{n\geqslant 1} V_n} \, .$$

From the Claim we have

$$H_k(\dot{\varphi}_c^a, *) = H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a \cap \bar{B}_n^{V_n}) \ \text{ for all } k \in \mathbb{N}_0, \tag{47}$$

where $\bar{B}_n^{V_n} = \{u \in V_n : ||u||_{C_0^1(\overline{\Omega})} \leqslant n\}$, $* \in \bar{B}_n^{V_n}$. From Palais [19] (Corollary p. 5) (see also Granas and Dugundji [10, Theorem D.6, p. 615]), we have

$$0 = H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a) = \varinjlim_{n} H_k(\dot{\varphi}_c^a, \dot{\varphi}_c^a \cap \bar{B}_n^{V_n})$$

where $\varinjlim_{n}$ denotes the inductive limit. So, from (47), we infer that

$$H_k(\varphi_c^a, *) = 0 \ \text{ for all } k \in \mathbb{N}_0. \tag{48}$$

Consider the following triple of sets:

$$\{*\} \subseteq \varphi_c^a \subseteq C_0^1(\overline{\Omega}).$$

For this triple, we introduce corresponding long exact sequence of singular homology groups

$$\cdots \to H_k(\varphi_c^a, *) \xrightarrow{i_*} H_k(C_0^1(\overline{\Omega}), \varphi_c^a) \xrightarrow{\partial_*} H_{k-1}(\varphi_c^a, *) \to \cdots \tag{49}$$

Here i_* is the homomorphism induced by the inclusion $i : (\varphi_c^a, *) \to (C_0^1(\overline{\Omega}), \varphi_c^a)$ and ∂_* is the boundary homomorphism. From (48) and the exactness of (49), we see that

$$\begin{aligned} & H_k(C_0^1(\overline{\Omega}), \varphi_c^a) = 0 \ \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\varphi_c, \infty) = 0 \ \text{ for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\varphi, \infty) = 0 \ \text{ for all } k \in \mathbb{N}_0. \end{aligned}$$

$\square$

Proposition 3.7. *If hypotheses H hold, then $u = 0$ is a local minimizer of the functional φ.*

Proof. Hypotheses H(i),(iv) imply that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(f'_x(z,0)+\epsilon)x^2 + \frac{c_\epsilon}{r}|x|^r \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{50}$$

Then for all $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}\varphi(u) &\geqslant \frac{1}{p}||Du||_p^p + \frac{1}{2}\left[||Du||_2^2 - \int_\Omega f'_x(z,0)u^2dz\right] - \frac{\epsilon}{2}\frac{1}{\hat{\lambda}_1(2)}||Du||_2^2 - c_4||u||^r \\ &\qquad \text{for some } c_4 = c_4(\epsilon) > 0 \text{ (see (50) and (3))} \\ &\geqslant \frac{1}{p}||Du||_p^p + \frac{1}{2}\left[\hat{c} - \frac{\epsilon}{\hat{\lambda}_1(2)}\right]||Du||_2^2 - c_4||u||^r \text{ (see Lemma 2.2).}\end{aligned}$$

Choosing $\epsilon \in (0, \hat{\lambda}_1(2)c_6)$, we have

$$\varphi(u) \geqslant \frac{1}{p}||u||^p + c_5||u||^2 - c_4||u||^r \text{ for some } c_5 > 0, \text{ all } u \in W_0^{1,p}(\Omega)$$

Because $2 \leqslant p < r$, for $\rho \in (0,1)$ small we have

$$\begin{aligned}&\varphi(u) > 0 = \varphi(0) \text{ for all } u \in W_0^{1,p}(\Omega) \text{ with } 0 < ||u|| \leqslant \rho, \\ \Rightarrow\ &u = 0 \text{ is a (strict) local minimizer of } \varphi.\end{aligned}$$

□

Now we are ready to produce two nontrivial constant sign solutions.

Proposition 3.8. *If hypotheses H hold, then problem* (1) *has at least two constant sign solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

Proof. Let $\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_+(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega F(z,u^+)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 3.9. *The functional φ_+ satisfies the C-condition.*

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_+(u_n)| \leqslant M_{11} \text{ for some } M_{11} > 0, \text{ all } n \geqslant 1 \tag{51}$$

$$(1+||u_n||)\varphi'(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{52}$$

From (52) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f(z, u_n^+)hdz\right| \leqslant \frac{\epsilon_n||h||}{1+||u_n||} \tag{53}$$

for all $h \in W_0^{1,p}(\Omega)$ with $\epsilon_n \to 0^+$.

In (53) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & ||Du_n^-||_p^p + ||Du_n^-||_2^2 \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \quad & u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{54}$$

From (51) and (54) it follows that

$$\varphi_+(u_n^+) \leqslant M_{12} \text{ for some } M_{12} > 0, \text{ for all } n \in \mathbb{N}. \tag{55}$$

In (53) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-||Du_n^+||_p^p - ||Du_n^+||_2^2 + \int_\Omega f(z, u_n^+)u_n^+dz \leqslant \epsilon_n \text{ for all } n \in \mathbb{N}. \tag{56}$$

From (55) and since $2 \leqslant p$, we have

$$||Du_n^+||_p^p + ||Du_n^+||_2^2 - \int_\Omega pF(z, u_n^+)dz \leqslant pM_{12} \text{ for all } n \in \mathbb{N}. \tag{57}$$

Adding (56) and (57), we obtain

$$\int_\Omega \xi(z, u_n^+)dz \leqslant M_{13} \text{ for some } M_{13} > 0, \text{ all } n \in \mathbb{N}. \tag{58}$$

Using (58) and reasoning as in the Claim in the proof of Proposition 3.3, we obtain that

$$\begin{aligned} & \{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow \quad & \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (54)).} \end{aligned}$$

From this, as in the proof of Proposition 3.3, via the $(S)_+$-property of the map A_p (see Proposition 2.4), we conclude that φ_+ satisfies the C-condition. This proves Claim 3.9.

It is straightforward to check that

$$u \in K_{\varphi_+} \Rightarrow u \geqslant 0.$$

So, we may assume that K_{φ_+} is finite or otherwise we already have a sequence of distinct positive solutions for problem (1).

A careful reading of the proof of Proposition 3.7, reveals that $u = 0$ is also a local minimizer for φ_+. So, we can find $\rho \in (0, 1)$ small such that

$$\varphi_+(0) = 0 < \inf[\varphi_+(u) : ||u|| = \rho] = m_\rho^+ \tag{59}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29).

Finally note that hypothesis H(ii) implies that

$$\varphi_+(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to +\infty. \tag{60}$$

The Claim and (59) and (60), permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\varphi_+} \quad \text{and} \quad m_\rho^+ \leqslant \varphi_+(u_0). \tag{61}$$

From (59) and (61), we see that $u_0 \neq 0$, $u_0 \geqslant 0$. Also, we have

$$\begin{aligned} & A_p(u_0) + A(u_0) = N_f(u_0) \ \text{ in } W^{-1,p'}(\Omega), \\ \Rightarrow \ & -\Delta_p(u_0)(z) - \Delta u_0(z) = f(z, u_0(z)) \ \text{ for almost all } z \in \Omega,\ u_0|_{\partial\Omega} = 0. \end{aligned} \tag{62}$$

From Ladyzhenskaya and Uraltseva [11, Theorem 7.1, p. 286], we have that $u_0 \in L^\infty(\Omega)$. So, we can apply Theorem 1 of Lieberman [12] and infer that $u_0 \in C_+ \backslash \{0\}$.

Let $a(y) = |y|^{p-2} + y$ for all $y \in \mathbb{R}^N$. Evidently $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$\begin{aligned} & \nabla a(y) = |y|^{p-2}\left[I + (p-2)\frac{y \otimes y}{|y|^2}\right] + I \ \text{ for all } y \in \mathbb{R}^N, \\ \Rightarrow \quad & (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant |\xi|^2 \ \text{ for all } y, \xi \in \mathbb{R}^N. \end{aligned}$$

Note that

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \ \text{ for all } u \in W_0^{1,p}(\Omega).$$

So, we can use the tangency principle of Pucci and Serrin [25, Theorem 2.5.2, p. 35] and have

$$u_0(z) > 0 \ \text{ for all } z \in \Omega.$$

For $\rho = ||u_0||_\infty$, let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H(iv). From (62) we have

$$\begin{aligned} & -\Delta_p u_0(z) - \Delta u_0(z) + \hat{\xi}_\rho u_0(z)^{p-1} \geqslant 0 \ \text{ for almost all } z \in \Omega, \\ \Rightarrow \quad & \Delta_p u_0(z) + \Delta u_0(z) \leqslant \hat{\xi}_\rho u_0(z)^{p-1} \ \text{ for almost all } z \in \Omega. \end{aligned}$$

Then the boundary point theorem of Pucci and Serrin [25, Theorem 5.5.1, p. 120] implies that $u_0 \in \operatorname{int} C_+$.

Next we produce a negative solution. For this purpose let

$$f_-(z, x) = f(z, -x^-), \qquad F_-(z, x) = \int_0^x f_-(z, s)ds$$

and let $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_-(u) = \frac{1}{p}||Du||_p^p + \frac{1}{2}||Du||_2^2 - \int_\Omega F_-(z,u)dz \ \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 3.10. *The functional φ_- satisfies the C-condition.*

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_-(u_n)| \leqslant M_{14} \ \text{ for some } M_{14} > 0, \text{ all } n \in \mathbb{N}, \tag{63}$$

$$(1 + ||u_n||)\varphi'_-(u_n) \to 0 \ \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{64}$$

From (64) we have

$$\left|\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega f_-(z,u_n)hdz\right| \leqslant \frac{\epsilon_n||h||}{1 + ||u_n||} \tag{65}$$
$$\text{for all } h \in W^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+.$$

In (65) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$||Du_n^+||_p^p + ||Du_n^+||_2^2 \leqslant \epsilon_n \ \text{ for all } n \in \mathbb{N},$$
$$\Rightarrow \quad u_n^+ \to 0 \ \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{66}$$

Then using (66), inequality (65) becomes

$$\left|\left\langle A_p(-u_n^-), h\right\rangle + \left\langle A(-u_n^-), h\right\rangle - \int_\Omega f(z,-u_n^-)hdz\right| \leqslant \epsilon'_n||h||$$
$$\text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon'_n \to 0^+.$$

Reasoning as in the last part of the proof of Proposition 3.3 (see the part of the proof after (29)), we obtain

$$\{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \ \text{ is bounded},$$
$$\Rightarrow \quad \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \ \text{ is bounded (see (66))},$$
$$\Rightarrow \quad \varphi_- \text{ satisfies the } C\text{-condition (as before using Proposition 2.4).}$$

This proves Claim 3.10.

As we did for φ_+, a critical inspection of the proof of Proposition 3.7, reveals that $u = 0$ is a local minimizer of φ_-. Also, it is easy to see that $K_{\varphi_-} \subseteq -C_+$ and so we may assume that K_{φ_-} is finite or otherwise we already have a whole sequence of distinct negative solutions of (1). These facts imply that we can find $\rho \in (0,1)$ small such that

$$\varphi_-(0) = 0 < \inf[\varphi_-(u) : ||u|| = \rho] = m_\rho^- \tag{67}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29).

Note that hypothesis H(iv) implies that

$$\varphi_-(t\hat{u}_1(p)) \to -\infty \quad \text{as } t \to -\infty. \tag{68}$$

Then Claim 3.10 and (67), (68) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $v_0 \in W_0^{1,p}(\Omega)$ such that

$$v_0 \in K_{\varphi_-} \quad \text{and} \quad m_\rho^- \leqslant \varphi_-(v_0). \tag{69}$$

From (67) and (69) we see that

$$v_0 \in (-C_+)\backslash\{0\} \text{ (see Lieberman [12]).}$$

In fact as we did for u_0, using the tangency principle and the boundary point theorem of Pucci and Serrin [25, pp. 35 and 120], we have

$$v_0 \in -\operatorname{int} C_+. \qquad \square$$

Next we compute the critical groups of φ at these solutions.

Proposition 3.11. *If hypotheses H hold and K_φ is finite, then $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof. Let $h_+(t,u) = (1-t)\varphi_+(u) + t\varphi(u)$ for all $(t,u) \in [0,1] \times W_0^{1,p}(\Omega)$. Suppose that we can find $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, u_n \to u_0 \;\text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad (h_+)'_u(t_n, u_n) = 0 \;\text{ for all } n \in \mathbb{N}. \tag{70}$$

From (70) we have

$$\begin{aligned} & A_p(u_n) + A(u_n) = N_f(u_n^+) + t_n N_f(-u_n^-), \\ \Rightarrow \quad & -\Delta_p u_n(z) - \Delta u_n(z) = f(z, u_n^+(z)) + t_n f(z, -u_n^-(z)) \\ & \qquad \text{for almost all } z \in \Omega, u_n|_{\partial\Omega} = 0. \end{aligned} \tag{71}$$

From Theorem 7.1, p. 286 of Ladyzhenskaya and Uraltseva [11], we can find $M_{15} > 0$ such that

$$||u_n||_\infty \leqslant M_{15} \;\text{ for all } n \in \mathbb{N}.$$

Invoking Theorem 1 of Lieberman [12], we can find $\beta \in (0,1)$ and $M_{16} > 0$ such that

$$u_n \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad ||u_n||_{C_0^{1,\beta}(\overline{\Omega})} \leqslant M_{16} \;\text{ for all } n \in \mathbb{N}. \tag{72}$$

Since $C_0^{1,\beta}(\overline{\Omega})$ is embedded compactly into $C_0^1(\overline{\Omega})$, from (70) and (72) we infer that

$$u_n \to u_n \;\text{ in } C_0^1(\overline{\Omega})$$

Recall that $u_0 \in \operatorname{int} C_+$ (see Proposition 3.8). So, we have

$$\begin{aligned} & u_n \in \operatorname{int} C_+ \text{ for all } n \geqslant n_0, \\ \Rightarrow \quad & \{u_n\}_{n \geqslant n_0} \subseteq K_\varphi \text{ (see (71))}, \end{aligned}$$

which contradicts our hypothesis that K_φ is finite. So, (65) cannot hold. Since for every $t \in [0,1]$ and every bounded set $D \subseteq W_0^{1,p}(\Omega)$, $h_+(t,\cdot)$ satisfies the C-condition on D (see Proposition 2.4), using Theorem 5.2 of Corvellec and Hantoute [8] (the homotopy invariance of the critical groups), we have

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \text{ for all } k \in \mathbb{N}_0. \tag{73}$$

From the proof of Proposition 3.8, we know that u_0 is a critical point of φ_+ of mountain pass-type. Then from Proposition 6.10, p. 176 of Motreanu, Motreanu and Papageorgiou [17], we have

$$\begin{aligned} & C_1(\varphi_+, u_0) \neq 0, \\ \Rightarrow \quad & C_1(\varphi, u_0) \neq 0 \text{ (see (73))}. \end{aligned}$$

But $\varphi \in C^2(W_0^{1,p}(\Omega))$. So, from Papageorgiou and Smyrlis [23] (see also Papageorgiou and Rădulescu [21]), we have

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0.$$

Similarly for $v_0 \in -\operatorname{int} C_+$, using this time the functional φ_-. □

Now we are ready for the multiplicity theorem concerning problem (1).

Theorem 3.12. *If hypotheses H hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C_0^1(\overline{\Omega}).$$

Proof. From Proposition 3.8, we already have two constant sign solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

Suppose $K_\varphi = \{0, u_0, v_0\}$. From Proposition 3.11, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{74}$$

From Proposition 3.7 we know that $u = 0$ is a local minimizer of φ. Hence

$$C_k(\varphi, u) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{75}$$

Moreover, from Proposition 3.5 we have

$$C_k(\varphi, \infty) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{76}$$

From (74), (75), (76) and the Morse relation with $t = -1$ (see (4)), we have

$$\begin{aligned} & (-1)^0 + 2(-1)^1 = 0, \\ \Rightarrow \quad & (-1)^1 = 0 \quad \text{a contradiction.} \end{aligned}$$

So, there exists $y_0 \in K_\varphi$, $y_0 \notin \{0, u_0, v_0\}$. Then y_0 is a third nontrivial solution of problem (1) and the nonlinear regularity theory (see Lieberman [12]), implies that $y_0 \in C_0^1(\overline{\Omega})$. $\square$

Remark 3.13. When $p = 2$, Theorem 3.12 is related to the multiplicity theorems of Recova and Rumbos [26], [27] who produce three nontrivial solutions under more restrictive regularity conditions on the reaction $f(z, x)$ and using the Ambrosetti-Rabinowitz condition to express the superlinearity condition in the positive direction. A precise improvement of the works of Recova and Rumbos [26], [27], in fact to Robin problems with an indefinite potential, can be found in the paper of Papageorgiou and Rădulescu [22].

References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu: Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, Mem. Amer. Math. Soc. 915 (2008) 1–70.

[2] A. Ambrosetti, P. Rabinowitz: Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] V. Benci, P. D'Avenia, D. Fortunato, L. Pisani: Solutions in several space dimensions: Dirrick's problem and infinitely many solutions, Arch. Ration. Mech. Anal. 154 (2000) 297–324.

[4] L. Cherfils, Y. Ilyasov: On the stationary solutions of generalized reaction diffusion equations with $p\&q$ Laplacian, Commun. Pure Appl. Anal. 4 (2005) 9–22.

[5] S. Cingolani, M. Degiovanni: Nontrivial solutions for p-Laplace equations with right-hand side having p-linear growth, Commun. Partial Differ. Equations 30 (2005) 1191–1203.

[6] S. Cingolani, M. Degiovanni, G. Vannella: On the critical polynomial of functionals related to p-area $(1 < p < \infty)$ and p-Laplace $(1 < p \leqslant 2)$ type operators, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. 26 (2015) 49–56.

[7] S. Cingolani, G. Vannella: Critical groups computations on a class of Sobolev Banach spaces via Morse index, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 20 (2003) 271–292.

[8] J. N. Corvellec, A. Hantoute: Homotopical stability of isolated critical points of continuous functionals, Set Valued Anal. 10 (2002) 143–164.

[9] L. Gasinski, N. S. Papageorgiou: Nonlinear Analysis, Chapman & Hall/CRC, Boca Raton (2006).

[10] A. Granas, J. Dugundji: Fixed Point Theory, Springer, New York (2003).

[11] O. Ladyzhenskaya, N. Uraltseva: Linear and Quasilinear Elliptic Equations, Academic Press, New York (1968).

[12] G. Lieberman: Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.

[13] G. Li, C. Yang: The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of a p-Laplacian type without the Ambrosetti-Rabinowitz condition, Nonlinear Anal. 72 (2010) 4602–4613.

[14] O. H. Miyagaki, M. A. S. Souto: Superlinear problems without Ambrosetti and Rabinowitz growth condition, J. Differ. Equations 245 (2008) 3628–3638.

[15] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou: Multiple solutions for Dirichlet problems which are superlinear at $+\infty$ and sublinear at $-\infty$, Commun. Appl. Anal. 13 (2009) 341–358.

[16] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou: On p-Laplacian equations with concave terms and asymmetric perturbations, Proc. R. Soc. Edinb., Sect. A, Math. 141 (2011) 171–192.

[17] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou: Topological and Variational Methods with Applications to Boundary Value Problems, Springer, New York (2014).

[18] D. Mugnai, N. S. Papageorgiou: Wang's multiplicity result for superlinear (p,q)-equations without the Ambrosetti-Rabinowitz condition, Trans. Amer. Math. Soc. 366 (2014) 4919–4937.

[19] R. Palais: Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 1–16.

[20] F. O. de Paiva, A. Presoto: Semilinear elliptic problems with asymmetric nonlinearities, J. Math. Anal. Appl. 409 (2014) 254–262.

[21] N. S. Papageorgiou, V. D. Rădulescu: Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance, Appl. Math. Optim 69 (2014) 393–430.

[22] N. S. Papageorgiou, V. D. Rădulescu: Robin problems with indefinite and unbounded potential, resonant at $-\infty$, superlinear at $+\infty$, Tohoku Math. J. 69 (2017) 261–286.

[23] N. S. Papageorgiou, G. Smyrlis: On nonlinear nonhomogeneous resonant Dirichlet equations, Pac. J. Math. 264 (2013) 421–453.

[24] N. S. Papageorgiou, P. Winkert: Resonant $(p,2)$-equations with concave terms, Appl. Anal. 94 (2015) 342–360.

[25] P. Pucci, J. Serrin: The Maximum Principle, Birkhäuser, Basel (2007).

[26] L. Recova, A. Rumbos: Multiple solutions to asymmetric semilinear elliptic problems via Morse theory, Electron. J. Differ. Equ. 2014(207) (2014) 1–29.

[27] L. Recova, A. Rumbos: An asymmetric superlinear elliptic problem at resonance, Nonlinear Anal. 112 (2015) 181–198.