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Inequality problems of quasi-hemivariational type involving set-valued operators and a nonlinear term

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Abstract The aim of this paper is to establish the existence of at least one solution for a general inequality of quasi-hemivariational type, whose solution is sought in a subset K of a real Banach space E. First, we prove the existence of solutions in the case of compact convex subsets and the case of bounded closed and convex subsets. Finally, the case when K is the whole space is analyzed and necessary and sufficient conditions for the existence of solutions are stated. Our proofs rely essentially on the Schauder's fixed point theorem and a version of the KKM principle due to Ky Fan (Math Ann 266:519–537, 1984).

Keywords Quasi-hemivariational inequality · Set-valued operator · Lower semicontinuous set-valued operator · Clarke's generalized gradient · Generalized monotonicity · KKM mapping

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1 Introduction and preliminaries

The study of inequality problems captured special attention in the last decades, one of the most recent and general type of inequalities being the *hemivariational inequalities*. The notion of hemivariational inequality was introduced by P.D. Panagiotopoulos at the beginning of the 1980s (see e.g. [27,28]) as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to *variational inequalities* which were studied earlier by many authors (see e.g. Fichera [13] or Hartman and Stampacchia [18]).

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Having a life of almost thirty years now, the theory of hemivariational inequalities has produced an abundance of important results both in pure and applied mathematics as well as in other domains such as mechanics and engineering sciences (see e.g. the monographs [14,16,17,24–26,29,32,33]) as it allowed mathematical formulations for new classes of interesting problems (see e.g. [1,6–8,11,12,19–22]).

The aim of this paper is to establish the existence of at least one solution for a general class of inequalities of quasi-hemivariational type. For the proof of the main results we shall use Schauder's fixed point theorem and a version of the well known KKM Principle due to Ky Fan [10].

For the convenience of the reader we present next some notations and preliminary results from functional analysis that will be used throughout the paper. For a given Banach space $(X, \|\cdot\|_X)$ we denote by X^* its dual space and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X.

We recall that a functional $\phi : X \to \mathbb{R}$ is called *locally Lipschitz* if for every $u \in X$ there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u \|w - v\|_X, \quad \text{for all} \quad v, w \in U.$$

Definition 1.1 Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized derivative of ϕ at $u \in X$ in the direction $v \in X$, denoted $\phi^0(u; v)$, is defined by

$$\phi^0(u; v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda v) - \phi(w)}{\lambda}.$$

Lemma 1.1 *Let $\phi : X \to \mathbb{R}$ be locally Lipschitz of rank L_u near the point $u \in X$. Then*

(a) *The function $v \mapsto \phi^0(u; v)$ is finite, positively homogeneous, subadditive and satisfies*

$$|\phi^0(u; v)| \leq L_u \|v\|_X;$$

(b) *$\phi^0(u; v)$ is upper semicontinuous as a function of (u, v).*

The proof can be found in Clarke [5], Proposition 2.1.1.

Definition 1.2 The generalized gradient of a locally Lipschitz functional $\phi : X \to \mathbb{R}$ at a point $u \in X$, denoted $\partial\phi(u)$, is the subset of X^* defined by

$$\partial\phi(u) = \{\zeta \in X^* : \phi^0(u; v) \geq \langle\zeta, v\rangle_X, \quad \text{for all} \quad v \in X\}.$$

We point out the fact that for each $u \in X$ we have $\partial\phi(u) \neq \emptyset$. In order to see that it suffices to apply the Hahn-Banach theorem (see e.g. Brezis [3], p. 1).

The next lemma points out important properties of generalized gradients.

Lemma 1.2 *Let $\phi : X \to \mathbb{R}$ be locally Lipschitz of rank L_u near the point $u \in X$. Then*

(a) *$\partial\phi(u)$ is a convex, weak* compact subset of X^* and*

$$\|\zeta\|_{X^*} \leq L_u, \quad \textit{for all } \zeta \in \partial\phi(u);$$

(b) *For each $v \in X$, one has*

$$\phi^0(u; v) = \sup\{\langle\zeta, v\rangle_X : \zeta \in \partial\phi(u)\}.$$

The proof can be found in Clarke [5], Proposition 2.1.2.

Definition 1.3 A set-valued operator $T : X \to 2^Y$ (X, Y Hausdorff topological spaces) is said to be lower semicontinuous (Vietoris lower semicontinuous) at $u_0 \in X$ (l.s.c. at u_0 for short), if for all $V \subset Y$ open such that $T(u_0) \cap V \neq \emptyset$, we can find U a neighborhood of u_0 such that $T(u) \cap V \neq \emptyset$ for all $u \in U$.

If this is true at every $u_0 \in X$, we say that T is lower semicontinuous (l.s.c. for short).

It is clear from Definition 1.3, that when A is single-valued, the notion of lower semicontinuity coincides with the usual notion of continuity of a map between two Hausdorff topological spaces.

The following proposition gives an useful characterization of lower semicontinuity in terms of generalized sequences (see e.g. Papageorgiou and Kyritsi-Yiallourou [31], p. 457).

Proposition 1.1 *Given a set-valued operator $T : X \to 2^Y$, the following statements are equivalent:*

(a) *T is l.s.c.;*
(b) *If $u \in X$, $\{u_\lambda\}_{\lambda \in J} \subset X$ is a net in X such that $u_\lambda \to u$ and $u^* \in T(u)$, then for each $\lambda \in J$ we can find $u^*_\lambda \in T(u_\lambda)$ such that $u^*_\lambda \to u^*$ in Y.*

We close this section with two theorems that will play a key role in the proof of our main results. The first is the Schauder fixed point iheorem (for the proof see Berger [2], p. 90) while the second represents a version of the KKM Principle due to Ky Fan [10].

Theorem 1.1 *Let X be a Banach space and let K be a nonempty, bounded, closed and convex subset of X. Let $S : K \to K$ be a completely continuous operator. Then S has at least one fixed point in the set K.*

Theorem 1.2 *Let K be a nonempty subset of a Hausdorff topological vector space X and let $\Theta : K \to 2^X$ be a set-valued mapping satisfying the following properties:*

- *Θ is a KKM mapping;*
- *$\Theta(u)$ is closed in X for every $u \in K$;*
- *there exists $u_0 \in K$ such that $\Theta(u_0)$ is compact in X.*

Then $\bigcap_{u \in K} \Theta(u) \neq \emptyset$.

We recall that a set-valued mapping $\Theta : K \to 2^X$ is said to be a *KKM mapping* if for any $\{u_1, \ldots, u_n\} \subset K$, $\mathrm{co}\{u_1, \ldots, u_n\} \subset \bigcup_{j=1}^n \Theta(u_j)$, where $\mathrm{co}\{u_1, \ldots, u_n\}$ denotes the convex hull of $\{u_1, \ldots, u_n\}$.

2 Formulation of the problem

Let $(E, \|\cdot\|_E)$ be a real Banach space which is continuously embedded in $L^p(\Omega; \mathbb{R}^n)$, for some $1 < p < +\infty$ and $n \geq 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m \geq 1$. Let i be the canonical injection of E into $L^p(\Omega; \mathbb{R}^n)$ and denote by $i^* : L^q(\Omega; \mathbb{R}^n) \to E^*$ the adjoint operator of i ($1/p + 1/q = 1$).

Throughout this paper $A : E \to 2^{E^*}$ is a nonlinear set-valued mapping, $F : E \to E^*$ is a nonlinear operator and $J : L^p(\Omega; \mathbb{R}^n) \to \mathbb{R}$ is a locally Lipschitz functional. We also assume that $h : E \to \mathbb{R}$ is a given nonnegative functional.

The aim of this paper is to study the existence of solutions for the following *quasi-hemivariational inequality*:

(P) Find $u \in E$ and $u^* \in A(u)$ such that

$$\langle u^*, v\rangle_E + h(u)J^0(iu; iv) \geq \langle Fu, v\rangle_E, \quad \text{for all} \quad v \in E.$$

The above problem is called a quasi-hemivariational inequality because, in general, we cannot determine a function G such that $\partial G(u) = h(u)\partial J(u)$.

As we will see next problem **(P)** can be rewritten equivalently as an *inclusion* in the following way:

$(\mathcal{P})$ Find $u \in E$ such that

$$Fu \in A(u) + h(u)i^*\partial J(iu).$$

An element $u \in E$ is called a solution of $(\mathcal{P})$ if there exist $u^* \in A(u)$ and $\zeta \in \partial J(iu)$ such that

$$\langle u^*, v\rangle_E + h(u)\langle i^*\zeta, v\rangle_E = \langle Fu, v\rangle_E, \quad \text{for all} \quad v \in E. \tag{2.1}$$

Proposition 2.1 *An element $u \in E$ is a solution of problem $(\mathcal{P})$ if and only if it solves problem* **(P)**.

Proof

(P) $\Rightarrow(\mathcal{P})$. Let $u \in E$ be a solution of **(P)**. Lemma 1.2 implies that there exists $\zeta_u \in \partial J(iu)$ such that for all $w \in L^p(\Omega; \mathbb{R}^n)$ we have

$$J^0(iu; w) = \langle \zeta_u, w\rangle_{L^q \times L^p} = \sup\{\langle \zeta, w\rangle_{L^q \times L^p} : \zeta \in \partial J(iu)\}.$$

Taking $w = iv$ and using the fact that u is a solution of **(P)** we obtain that

$$\langle u^*, v\rangle_E + h(u)\langle i^*\zeta_u, v\rangle_E \geq \langle Fu, v\rangle_E, \quad \text{for all} \quad v \in E.$$

Taking $-v$ instead of v in the above relation we deduce that (2.1) holds therefore u is a solution of problem $(\mathcal{P})$.

$(\mathcal{P})$ $\Rightarrow$**(P)**. Let $u \in E$ be a solution of $\mathcal{P}$. Then, there exist $u^* \in A(u)$ and $\zeta \in \partial J(iu)$ such that (2.1) takes place. As $\zeta \in \partial J(iu)$ we obtain that

$$\langle \zeta, w\rangle_{L^q \times L^p} \leq J^0(iu; w), \quad \text{for all} \quad w \in L^p(\Omega; \mathbb{R}^n).$$

For a fixed $v \in E$ we define $w = iv$ and taking into account that h is nonnegative we get

$$h(u)\langle i^*\zeta, v\rangle_E = h(u)\langle \zeta, iv\rangle_{L^q \times L^p} \leq h(u)J^0(iu; iv) \tag{2.2}$$

Combining (2.1) and (2.2) we obtain that u solves inequality problem **(P)**.

qed

Sometimes, due to some technical reasons, it is useful to study hemivariational inequalities of the type **(P)** whose solution is sought in a nonempty, closed and convex subset K of E. This leads us to the study of the following inequality problem:

($\mathbf{P_K}$) Find $u \in K$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u\rangle_E + h(u)J^0(iu; iv - iu) \geq \langle Fu, v - u\rangle_E, \quad \text{for all} \quad v \in K.$$

We point out the fact that, unlike problem **(P)**, the above problem cannot be rewritten as an inclusion and this is one of the reasons for which we prefer the hemivariational approach. However, the formulation in terms of hemivariational inequalities has a great advantage: that the hemivariational inequalities express a physical principle, the principle of *virtual work or power*.

We shall study three cases regarding the set K:

1. K is a nonempty, compact and convex subset of the space E;
2. K is a nonempty, bounded, closed and convex subset of the space E;
3. K is an unbounded, closed and convex subset of E (for simplicity we shall consider that K is the the whole space E; in this case problems $(\mathbf{P_K})$ and **(P)** are one and the same).

The novelty of our inequality problem consists in the following things:

- the operator A is multi-valued;
- in order to prove the existence of at least one solution in the case of bounded, closed and convex sets we ask A not to be monotone (as in most papers dealing with hemivariational inequalities), but to be relaxed α monotone which is rather a weak condition compared to monotonicity;
- the presence of the nonlinear term in the right-hand side of the inequality which depends on the unknown variable u;
- it is a general inequality since it contains several particular cases which lead to various known inequalities arising in many fields such as mechanics, engineering sciences, numerical analysis.

In order to highlight the generality of our inequality problem we present below several particular cases.

Case 1. The operator A is multi-valued.

(1.a) $F \equiv 0$ and $h \equiv 0$. In this case problem $(\mathbf{P_K})$ becomes Find $u \in K$ and $u^* \in A(u)$ such that

$$\langle u^*, v - u\rangle_E \geq 0, \quad \text{for all } v \in K,$$

which is called the *generalized variational inequality* (see e.g. Minty [23] or Browder [4]);

(1.b) $Au = \partial_C \phi(u)$ and $h \equiv 0$, where $\phi : E \to (-\infty, +\infty]$ is a proper, convex functional and $\partial_C : E \to 2^{E^*}$ is the convex subdifferential of ϕ, i.e.

$$\partial_C \phi(u) = \{\eta \in E^* : \phi(v) - \phi(u) \geq \langle \eta, v - u\rangle_E, \text{ for all } v \in E\}.$$

In this case problem $(\mathbf{P_K})$ becomes Find $u \in K$ such that

$$\langle -Fu, v - u\rangle_E + \phi(v) - \phi(u) \geq 0, \quad \text{for all } v \in K,$$

which is called the *mixed variational inequality* (see e.g. Glowinski, Lions and Trèmoliéres [15]);

(1.c) $Au = \partial W(u) + \partial_C \phi(u)$, where $W : E \to \mathbb{R}$ is a locally Lipschitz functional and $\phi : E \to (-\infty, +\infty]$ is a proper, convex functional. In this case problem $(\mathbf{P_K})$ becomes

$$W^0(u; v - u) + \phi(v) - \phi(u) + h(u)J^0(iu; iv - iu) \\ \geq \langle Fu, v - u\rangle_E, \quad \text{for all } v \in K,$$

which is called *general quasi-hemivariational inequality* (see Costea [9]);

Case 2. The operator A is single-valued.

(2.a) $h \equiv 0$ and $F \equiv 0$. In this case problem ($\mathbf{P_K}$) becomes Find $u \in K$ such that

$$\langle Au, v-u\rangle_E \geq 0, \quad \text{for all } v \in K,$$

which is called the *standard variational inequality* (see e.g. Hartman and Stampacchia [18]);

(2.b) $h \equiv 1$ and $F \equiv 0$. In this case problem ($\mathbf{P_K}$) becomes Find $u \in K$ such that

$$\langle Au, v-u\rangle_E + J^0(iu; iv - iu) \geq 0, \quad \text{for all } v \in K,$$

which is called the *Hartman–Stampacchia hemivariational inequality* (see Panagiotopoulos, Fundo and Rădulescu [30]);

(2.c) $F \equiv 0$. In this case problem ($\mathbf{P_K}$) becomes Find $u \in K$ such that

$$\langle Au, v-u\rangle_E + h(u)J^0(iu; iv - iu) \geq 0, \quad \text{for all } v \in K,$$

which is called the *standard quasi-hemivariational inequality* (see e.g. Naniewicz and Panagiotopoulos [26]);

In conclusion, we do not deal with a classical hemivariational inequality and consequently several difficulties occur in determining the existence of solutions since the classical methods fail to be applied directly.

3 Main results

The first main result of this paper is given by the following theorem.

Theorem 3.1 *Let K be a nonempty compact convex subset of the real Banach space E. Assume that:*

- *$A : E \to 2^{E^*}$ is l.s.c. with respect to the weak* topology of E^*;*
- *$h : E \to \mathbb{R}$ is a continuous nonnegative functional;*
- *$F : E \to E^*$ is an operator such that $\limsup_{n\to\infty}\langle Fu_n, v-u_n\rangle_E \geq \langle Fu, v-u\rangle_E$, whenever $u_n \to u$.*

Then the inequality problem ($\mathbf{P_K}$) has at least one solution.

Proof Arguing by contradiction, let us assume that problem ($\mathbf{P_K}$) has no solution. Then, for each $u \in K$, there exists $v \in K$ such that

$$\sup_{u^* \in A(u)} \langle u^*, v-u\rangle_E + h(u)J^0(iu; iv-iu) < \langle Fu, v-u\rangle_E. \tag{3.1}$$

We introduce the set-valued mapping $\Lambda : K \to 2^K$ defined by

$$\Lambda(v) = \left\{ u \in K : \inf_{u^* \in A(u)} \langle u^*, v-u\rangle_E + h(u)J^0(iu; iv-iu) \geq \langle Fu, v-u\rangle_E \right\}.$$

Claim 1. The set $\Lambda(v)$ is nonempty and closed for each $v \in K$.
The fact that $\Lambda(v)$ is nonempty is obvious as $v \in \Lambda(v)$ for each $v \in K$.
In order to prove the above claim let us fix $v \in K$ and consider a sequence

$\{u_n\}_{n\geq 1} \subset \Lambda(v)$ which converges to some $u \in K$. We shall prove that $u \in \Lambda(v)$. As $u_n \in \Lambda(v)$, for each $n \geq 1$ we get that

$$\begin{aligned}\langle u_n^*, v - u_n\rangle_E + h(u_n)J^0(iu_n; iv - iu_n)\\ \geq \langle Fu_n, v - u_n\rangle_E, \quad \text{for all } u_n^* \in A(u_n).\end{aligned} \tag{3.2}$$

Let $u^* \in A(u)$ be fixed and let $\bar{u}_n^* \in A(u_n)$ such that $\bar{u}_n^* \rightharpoonup u^*$ in E^* (the existence of such a sequence is ensured by Proposition 1.1 and the fact that A is l.s.c. with respect to the weak* topology of E^*). On the other hand, using the continuous embedding of E into $L^p(\Omega; \mathbb{R}^n)$ we obtain that $iu_n \to iu$ in $L^p(\Omega; \mathbb{R}^n)$. Passing to lim sup as $n \to \infty$ in (3.2) we obtain the following estimates:

$$\begin{aligned}\langle Fu, v - u\rangle_E &\leq \limsup_{n\to\infty}\langle Fu_n, v - u_n\rangle_E\\ &\leq \limsup_{n\to\infty}\left[\langle \bar{u}_n^*, v - u_n\rangle_E + h(u_n)J^0(iu_n; iv - iu_n)\right]\\ &\leq \limsup_{n\to\infty}\langle \bar{u}_n^*, v - u_n\rangle_E\\ &\quad + \limsup_{n\to\infty}\,[h(u_n) - h(u) + h(u)]\,J^0(iu_n; iv - iu_n)\\ &\leq \langle u^*, v - u\rangle_E + \limsup_{n\to\infty}\,[h(u_n) - h(u)]\,J^0(iu_n; iv - iu_n)\\ &\quad + \limsup_{n\to\infty} h(u)J^0(iu_n; iv - iu_n)\\ &\leq \langle u^*, v - u\rangle_E + h(u)J^0(iu; iv - iu).\end{aligned}$$

This shows that $u \in \Lambda(v)$ hence $\Lambda(v)$ is a closed set and the proof of the claim is now complete.

According to (3.1) for each $u \in K$ there exists $v \in K$ such that $u \in [\Lambda(v)]^c = E - \Lambda(v)$. This means that the family $\{[\Lambda(v)]^c\}_{v\in K}$ is an open covering of the compact set K. Therefore there exists a finite subset $\{v_1, \ldots, v_N\}$ of K such that $\{[\Lambda(v_j)]^c\}_{1\leq j\leq N}$ is a finite subcover of K. For each $j \in \{1, \ldots, N\}$ let $\delta_j(u)$ be the distance between u and the set $\Lambda(v_j)$ and define $\beta_j : K \to \mathbb{R}$ as follows:

$$\beta_j(u) = \frac{\delta_j(u)}{\sum_{k=1}^N \delta_k(u)}.$$

Clearly, for each $j \in \{1, \ldots, N\}$, β_j is a Lipschitz continuous function that vanishes on $\Lambda(v_j)$ and $0 \leq \beta_j(u) \leq 1$, for all $u \in K$. Moreover, $\sum_{j=1}^N \beta_j(u) = 1$. Let us consider next the operator $S : K \to K$ defined by

$$S(u) = \sum_{j=1}^N \beta_j(u)v_j.$$

We shall prove that S is a completely continuous operator. We have

$$\begin{aligned}\|Su_1 - Su_2\|_E &= \left\| \sum_{j=1}^{N} (\beta(u_1) - \beta(u_2)) v_j \right\|_E \\ &\le \sum_{j=1}^{N} \|v_j\|_E \ \|\beta(u_1) - \beta(u_2)\|_E \\ &\le \sum_{j=1}^{N} \|v_j\|_E \ L_j \ \|u_1 - u_2\|_E \\ &\le L \ \|u_1 - u_2\|_E,\end{aligned}$$

which shows that S is Lipschitz continuous hence continuous.

Let M be a bounded subset of K. As $\overline{S(M)}$ is a closed subset of the compact set K we conclude that $S(M)$ is relatively compact, hence S maps bounded sets into relatively compact sets which shows that S is a compact map. Thus, by Schauder's fixed point theorem, there exists $u_0 \in K$ such that $S(u_0) = u_0$.

Let us define next the functional $g : K \to \mathbb{R}$

$$g(u) = \inf_{u^* \in A(u)} \langle u^*, S(u) - u\rangle_E + h(u) J^0(iu, iS(u) - iu) - \langle Fu, S(u) - u\rangle_E.$$

Taking into account Lemma 1.1 and the way the operator S was constructed, for each $u \in K$, we have:

$$\begin{aligned}g(u) = &\inf_{u^* \in A(u)} \left\langle u^*, \sum_{j=1}^{N} \beta_j(u)(v_j - u)\right\rangle_E + h(u) J^0 \left(iu, \sum_{j=1}^{N} \beta_j(u)(iv_j - iu)\right) \\ &- \left\langle Fu, \sum_{j=1}^{N} \beta_j(u)(v_j - u)\right\rangle_E \\ &\le \sum_{j=1}^{N} \beta_j(u) \left[\inf_{u^* \in A(u)} \langle u^*, v_j - u\rangle_E + h(u) J^0(iu, iv_j - iu) - \langle Fu, v_j - u\rangle_E \right].\end{aligned}$$

Let $u \in K$ be arbitrary fixed. For each index $j \in \{1, \ldots, N\}$ we distinguish the following possibilities:

- $u \in \left[\Lambda(v_j)\right]^c$. In this case we have

$$\beta_j(u) > 0$$

and

$$\inf_{u^* \in A(u)} \langle u^*, v_j - u\rangle_E + h(u) J^0(iu, iv_j - iu) - \langle Fu, v_j - u\rangle_E < 0.$$

- $u \in \Lambda(v_j)$. In this case we have

$$\beta_j(u) = 0$$

and

$$\inf_{u^* \in A(u)} \langle u^*, v_j - u\rangle_E + h(u) J^0(iu, iv_j - iu) - \langle Fu, v_j - u\rangle_E \ge 0.$$

Taking into account that $K \subseteq \cup_{j=1}^{N} \left[\Lambda(v_j)\right]^c$ we deduce that there exists at least one index $j_0 \in \{1, \ldots, N\}$ such that $u \in \left[\Lambda(v_{j_0})\right]^c$. This shows that $g(u) < 0$ for all $u \in K$.

On the other hand, $g(u_0) = 0$ and thus we have obtained a contradiction that completes the proof. □

We point out the fact that in the above case when K is a compact convex subset of E we do not impose any monotonicity conditions on A, nor we assume E to be a reflexive space. However, in applications, most problems lead to an inequality whose solution is sought in a closed and convex subset of the space E. Weakening the hypotheses on K by assuming that K is only bounded, closed and convex, we need to impose certain monotonicity properties on A and assume in addition that E is reflexive. We shall use a kind of generalized monotonicity, so called *relaxed* α *monotonicity*. We recall the following definition.

Definition 3.1 A set-valued mapping $T : E \to 2^{E^*}$ is said to be relaxed α monotone if there exists a functional $\alpha : E \to \mathbb{R}$ such that for all $u, v \in E$ we have

$$\langle v^* - u^*, v - u\rangle_E \geq \alpha(v - u), \quad \text{for all } v^* \in T(v) \text{ and all } u^* \in T(u). \tag{3.3}$$

SPECIAL CASES.

- If $\alpha(u) = m\|u\|_E^2$, with $m > 0$ constant, then (3.3) becomes

$$\langle v^* - u^*, v - u\rangle_E \geq m\|v - u\|_E^2, \quad \text{for all } v^* \in T(v) \text{ and all } u^* \in T(u),$$

  and T is said to be *strongly monotone;*
- If $\alpha(u) \equiv m$, with $m > 0$ constant, then (3.3) becomes

$$\langle v^* - u^*, v - u\rangle_E \geq m > 0, \quad \text{for all } u \neq v,\ v^* \in T(v),\ u^* \in T(u),$$

  and T is said to be *strictly monotone;*
- If $\alpha(u) \equiv 0$, then (3.3) becomes

$$\langle v^* - u^*, v - u\rangle_E \geq 0, \quad \text{for all } v^* \in T(v) \text{ and all } u^* \in T(u),$$

  and T is said to be *monotone;*
- If $\alpha(u) = -m\|u\|_E^2$, with $m > 0$ constant, then (3.3) becomes

$$\langle v^* - u^*, v - u\rangle_E \geq -m\|v - u\|_E^2, \quad \text{for all } v^* \in T(v) \text{ and all } u^* \in T(u),$$

  and T is said to be *relaxed monotone.*

From the above definitions, we have the following implications (and the inverse of every implication is not true):

strongly monotone $\Rightarrow$ strictly monotone $\Rightarrow$ monotone $\Rightarrow$ relaxed monotone $\Rightarrow$ relaxed α monotone

We are now able to formulate another main result concerning the existence of solutions on bounded, closed and convex subsets.

Theorem 3.2 *Let K be a nonempty, bounded, closed and convex subset of the real reflexive Banach space E which is compactly embedded in $L^p(\Omega; \mathbb{R}^n)$. Assume that:*

- *$A : E \to 2^{E^*}$ is l.s.c. with respect to the weak topology of E^* and relaxed α monotone;*
- *$\alpha : E \to \mathbb{R}$ is a functional such that $\limsup_{n\to\infty} \alpha(u_n) \geq \alpha(u)$ whenever $u_n \rightharpoonup u$ and $\lim_{t\downarrow 0} \frac{\alpha(tu)}{t} = 0$;*
- *$h : E \to \mathbb{R}$ is a nonnegative sequentially weakly continuous functional;*
- *$F : E \to E^*$ is an operator such that the application $u \mapsto \langle Fu, v-u\rangle_E$ is weakly lower semicontinuous.*

Then the inequality problem $(\mathbf{P_K})$ *has at least one solution in K.*

Proof Let us define the set-valued mapping $\Theta : K \to 2^K$

$$\Theta(v) = \left\{ u\in K : \inf_{v^*\in A(v)} \langle v^*, v-u\rangle_E + h(u)J^0(iu; iv-iu) - \langle Fu, v-u\rangle_E \geq \alpha(v-u) \right\}.$$

Claim 2. The set $\Theta(v)$ is weakly closed for each $v \in K$.
In order to prove the above claim let us fix $v \in K$ and consider a sequence $\{u_n\}_{n\geq 1} \subset \Theta(v)$ such that $u_n \rightharpoonup u$ in E. We must prove that $u \in \Theta(v)$. First we observe that the compactness of the embedding operator i implies that the sequence $\{iu_n\}_{n\geq 1}$ converges strongly to iu in $L^p(\Omega, \mathbb{R}^n)$.
For each $v^* \in A(v)$ we have

$$\begin{aligned}\alpha(v-u) &\leq \limsup_{n\to\infty} \alpha(v-u_n)\\ &\leq \limsup_{n\to\infty} \Big[\langle v^*, v-u_n\rangle_E + h(u_n)J^0(iu_n; iv-iu_n) - \langle Fu_n, v-u_n\rangle_E\Big]\\ &\leq \langle v^*, v-u\rangle_E + h(u)J^0(iu, iv-iu) - \langle Fu, v-u\rangle_E,\end{aligned}$$

which shows that $u \in \Theta(v)$ and thus the proof of the claim is complete.

Claim 3. Θ is a KKM mapping.
Arguing by contradiction let us assume that Θ is not a KKM mapping. According to the definition of a KKM mapping there exists a finite subset $\{v_1, \ldots, v_N\} \subset K$ and $u_0 = \sum_{j=1}^N \lambda_j v_j$, with $\lambda_j \in [0, 1]$ and $\sum_{j=1}^N \lambda_j = 1$ such that $u_0 \notin \bigcup_{j=1}^N \Theta(v_j)$. This is equivalent to

$$\begin{aligned}&\inf_{v_j^*\in A(v_j)} \langle v_j^*, v_j - u_0\rangle_E + h(u_0)J^0(iu_0; iv_j - iu_0)\\ &\quad - \langle Fu_0, v_j - u_0\rangle_E < \alpha(v_j - u_0), \qquad (3.4)\end{aligned}$$

for all $j \in \{1, \ldots, N\}$.
On the other hand, A is a relaxed α monotone operator and thus, for each $j \in \{1, \ldots, N\}$ we have

$$\begin{aligned}&\langle u_0^* - v_j^*, v_j - u_0\rangle_E \leq -\alpha(v_j - u_0),\\ &\qquad \text{for all } u_0^* \in A(u_0) \text{ and all } v_j^* \in A(v_j).\end{aligned} \qquad (3.5)$$

Combining (3.4) and (3.5) we are led to

$$\begin{aligned}&\langle u_0^*, v_j - u_0\rangle_E + h(u_0)J^0(iu_0; iv_j - iu_0)\\ &\quad - \langle Fu_0, v_j - u_0\rangle_E < 0, \quad \text{for all } u_0^* \in A(u_0). \qquad (3.6)\end{aligned}$$

Using (3.6) and the fact that $J^0(iu_0;\cdot)$ is subadditive (see Lemma 1.1), for a fixed $u_0^* \in A(u_0)$ we have

$$\begin{aligned}0 &= \langle u_0^*, u_0 - u_0\rangle_E + h(u_0)J^0(iu_0; iu_0 - iu_0) - \langle Fu_0, u_0 - u_0\rangle_E\\ &= \left\langle u_0^*, \sum_{j=1}^N \lambda_j(v_j - u_0)\right\rangle_E + h(u_0)J^0\left(iu_0; \sum_{j=1}^N \lambda_j(iv_j - iu_0)\right)\\ &\quad - \left\langle Fu_0, \sum_{j=1}^N \lambda_j(v_j - u_0)\right\rangle_E \le \sum_{j=1}^N \lambda_j\left[\langle u_0^*, v_j - u_0\rangle_E\right.\\ &\quad \left.+ h(u_0)J^0(iu_0; iv_j - iu_0) - \langle Fu_0, v_j - u_0\rangle_E\right] < 0,\end{aligned}$$

which obviously is a contradiction and thus the proof of the claim is complete.

We already know from Claim 2 that $\Theta(v)$ is a weakly closed subset of K, for each $v \in K$. On the other hand, K is a weakly compact set as it is a bounded, closed and convex subset of the real reflexive Banach space E. Therefore $\Theta(v)$ it is weakly compact for each $v \in K$. Thus we can apply the KKM Principle to conclude that $\bigcap_{v\in K}\Theta(v) \neq \emptyset$.

Let $u_0 \in \bigcap_{v\in K}\Theta(v)$. This implies that for each $w \in K$ we have

$$\inf_{w^*\in A(w)} \langle w^*, w - u_0\rangle_E + h(u_0)J^0(iu_0; iw - iu_0) - \langle Fu_0, w - u_0\rangle_E \ge \alpha(w - u_0).$$

Let $v \in K$ be fixed and define $w_\lambda = u_0 + \lambda(v - u_0)$, $\lambda \in (0, 1)$. Using the fact that $w_\lambda \in K$ and taking into account the above relation and Lemma 1.1 we deduce that

$$\begin{aligned}\langle w_\lambda^*, v - u_0\rangle_E &+ h(u_0)J^0(iu_0, iv - iu_0) - \langle Fu_0, v - u_0\rangle_E\\ &\ge \frac{\alpha(\lambda(v - u_0))}{\lambda}, \text{ for all } w_\lambda^* \in A(w_\lambda).\end{aligned}$$

Letting $\lambda \to 0$ and using the l.s.c. of A we obtain that u_0 solves problem $(\mathbf{P_K})$. □

As we have seen above the boundedness of the set K played a key role in proving that problem $(\mathbf{P_K})$ admits at least one solution. In the case when K is the whole space E, assuming that the same hypotheses as in Theorem 3.2 hold, we shall need an extra condition to overcome the lack of boundedness. For each real number $R > 0$ taking $K = \bar{B}(0; R) = \{u \in E : \|u\|_E \le R\}$ we know from Theorem 3.2 that problem

$(\mathbf{P_R})$ Find $u_R \in \bar{B}(0; R)$ and $u_R^* \in A(u_R)$ such that

$$\langle u_R^*, v - u_R\rangle_E + h(u_R)J^0(iu_R; iv - iu_R) \ge \langle Fu_R, v - u_R\rangle_E, \quad \text{for all} \quad v \in \bar{B}(0; R),$$

admits at least one solution.

Theorem 3.3 *Assume that the same hypotheses as in Theorem 3.2 hold in the case $K = E$. Then problem $(\mathbf{P})$ admits at least one solution if and only if the following condition holds true:*

- *There exists $R > 0$ such that at least one solution u_R of problem $(\mathbf{P_R})$ satisfies $u_R \in$ $\operatorname{int} \bar{B}(0; R)$.*

Proof The necessity is obvious.

In order to prove the sufficiency let us fix $v \in E$. We shall prove that u_R is a solution of $(\mathbf{P})$. First we define

$$\lambda = \begin{cases} 1 & \text{if } u_R = v\\ \frac{R - \|u_R\|_E}{\|v - u_R\|_E} & \text{otherwise .}\end{cases}$$

Since $u_R \in \operatorname{int} B(0; R)$ we conclude that $\lambda > 0$ and that $w_\lambda = u_R + \lambda(v - u_R) \in \bar{B}(0; R)$. Using that u_R solves problem $(\mathbf{P_R})$ we find

$$\begin{aligned}\langle Fu_R, \lambda(v - u_R)\rangle_E &= \langle Fu_R, w_\lambda - u_R\rangle_E \\ &\le \langle u_R^*, w_\lambda - u_R)\rangle_E + h(u_R)J^0(iu_R; iw_\lambda - iu_R) \\ &= \langle u_R^*, \lambda(v - u_R)\rangle_E + h(u_R)J^0(iu_R; \lambda(iv - iu_R)) \\ &= \lambda\left[\langle u_R^*, v - u_R\rangle_E + h(u_R)J^0(iu_R; iv - iu_R)\right].\end{aligned}$$

Dividing by $\lambda > 0$ we conclude that u_R solves problem $(\mathbf{P})$. □

Corollary 3.1 *Let us assume that the same hypotheses as in Theorem 3.2 hold in the case $K = E$. Then a sufficient condition for problem* $(\mathbf{P})$ *to admit a solution is:*

- *There exists $R_0 > 0$ such that for each $u \in E \setminus \bar{B}(0; R_0)$ there exists $v \in \operatorname{int} \bar{B}(0; R_0)$ with the property that*

$$\sup_{u^* \in A(u)} \langle u^*, v - u\rangle_E + h(u)J^0(iu; iv - iu) < \langle Fu, v - u\rangle_E.$$

Proof Let us fix $R > R_0$. According to Theorem 3.2 there exists $u_R \in \bar{B}(0, R)$ and $\bar{u}_R^* \in A(u_R)$ such that

$$\langle \bar{u}_R^*, v - u_R\rangle_E + h(u_R)J^0(iu_R; iv - iu_R) \ge \langle Fu_R, v - u_R\rangle_E, \quad \text{for all } v \in \bar{B}(0; R). \tag{3.7}$$

Case 1. $u_R \in \operatorname{int} \bar{B}(0; R)$.
In this case we have nothing to prove, Theorem 3.3 showing that u_R is a solution of problem $(\mathbf{P})$.

Case 2. $u_R \in \partial \bar{B}(0; R)$.
In this case $\|u_R\|_E = R > R_0$ and thus $u_R \in E \setminus \bar{B}(0; R_0)$. According to our hypothesis there exists $\bar{v} \in \operatorname{int} \bar{B}(0; R_0)$ such that

$$\sup_{u_R^* \in A(u_R)} \langle u_R^*, \bar{v} - u_R\rangle_E + h(u_R)J^0(iu_R; i\bar{v} - iu_R) < \langle Fu_R, \bar{v} - u_R\rangle_E. \tag{3.8}$$

Let us fix $v \in E$. Defining

$$\lambda = \begin{cases} 1 & \text{if } v = \bar{v} \\ \frac{R - R_0}{\|v - \bar{v}\|_E} & \text{otherwise,}\end{cases}$$

we observe that $w_\lambda = \bar{v} + \lambda(v - \bar{v}) \in \bar{B}(0; R)$. On the other hand we observe that

$$\begin{aligned} w_\lambda - u_R &= \bar{v} - u_R + \lambda(v - \bar{v}) + \lambda u_R - \lambda u_R \\ &= \lambda(v - u_R) + (1 - \lambda)(\bar{v} - u_R).\end{aligned}$$

Taking w_λ instead of v in (3.7) and using (3.8) we are led to the following estimates

$$\begin{aligned}&\langle Fu_R, \lambda(v - u_R) + (1 - \lambda)(\bar{v} - u_R)\rangle_E = \langle Fu_R, w_\lambda - u_R\rangle_E \\ &\le \langle \bar{u}_R^*, w_\lambda - u_R\rangle_E + h(u_R)J^0(iu_R; iw_\lambda - iu_R) \\ &\le \lambda\left[\langle \bar{u}_R^*, v - u_R\rangle_E + h(u_R)J^0(iu_R; iv - iu_R)\right] \\ &\quad +(1 - \lambda)\left[\langle \bar{u}_R^*, \bar{v} - u_R\rangle_E + h(u_R)J^0(iu_R; i\bar{v} - iu_R)\right] \\ &\le \lambda\left[\langle \bar{u}_R^*, v - u_R\rangle_E + h(u_R)J^0(iu_R; iv - iu_R)\right] \\ &\quad +(1 - \lambda)\langle Fu_R, \bar{v} - u_R\rangle_E.\end{aligned}$$

This shows that

$$\langle \bar{u}^*_R, v - u_R \rangle_E + h(u_R) J^0(iu_R; iv - iu_R) \geq \langle Fu_R; v - u_R \rangle_E, \quad \text{for all } v \in E,$$

which means that u_R solves problem **(P)** and thus the proof is complete. □

Corollary 3.2 *Let us assume that the same hypotheses as in Theorem 3.2 hold in the case $K = E$. Assume in addition that:*

- *A is coercive, i.e. there exists a function $c : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\lim\limits_{r\to\infty} c(r) = +\infty$ such that*

$$\inf_{u^* \in A(u)} \langle u^*, u \rangle_E \geq c(\|u\|_E)\|u\|_E;$$

- *there exists a constant $k > 0$ such that $h(v)J^0(iv; -iv) \leq k\|v\|_E$ for all $v \in E$;*
- *there exists a constant $m > 0$ such that $\|Fu\|_{E^*} \leq m$ for all $u \in E$;*

Then the inequality problem **(P)** *has at least one solution.*

Proof For each $R > 0$ Theorem 3.2 guarantees that there exist $u_R \in E$ and $u^*_R \in A(u_R)$ such that

$$\langle u^*_R, v - u_R \rangle_E + h(u_R) J^0(iu; iv - iu_R) \geq \langle Fu_R, v - u_R \rangle_E, \quad \text{for all } v \in \bar{B}(0; R). \tag{3.9}$$

We shall prove that there exists $R_0 > 0$ such that $u_{R_0} \in \operatorname{int} \bar{B}(0; R_0)$. According to Theorem 3.3, this is equivalent to the fact that u_R is a solution of problem **(P)**.

Arguing by contradiction let us assume that $u_R \in \partial \bar{B}(0; R)$ for all $R > 0$. Taking $v = 0$ in (3.9) we have

$$\begin{aligned} c(R)R &= c(\|u_R\|)\|u_R\|_E \\ &\leq \langle u^*_R, u_R \rangle_E \\ &\leq \langle Fu_R, u_R \rangle_E + h(u_R) J^0(iu_R; -iu_R) \\ &\leq \|Fu_R\|_{E^*}\|u_R\|_E + k\,\|u_R\|_E \\ &= (m + k)R. \end{aligned}$$

Dividing by $R > 0$ we obtain that $c : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded from above which contradicts the fact that $\lim_{R\to\infty} c(R) = +\infty$. □

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