



Ground State Solutions for Robin Concave–Convex Problems with Unbalanced Growth

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Abstract. We consider a nonlinear Robin problem driven by a differential operator with unbalanced growth and a reaction which exhibits the competing effects of a parametric concave (sublinear) term and of a convex (superlinear) term. Using the Nehari method, we show that for all small values of the parameter, the problem has two bounded, ground state (least energy) solutions. In the process of the proof, we establish some auxiliary results which are of independent interest.

Mathematics Subject Classification. 35J75, 35J97, 46E30, 58E07.

Keywords. Unbalanced growth, Concave–convex nonlinearities, Generalized Orlicz spaces, Nehari manifold, Ground state solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper, we study the following nonlinear Robin problem

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) + \xi_1(z)u(z)^{p-1} + \xi_2(z)u(z)^{q-1} = \lambda u(z)^{\tau-1} + \eta(z)u(z)^{r-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n_\theta} + \beta(z)u^{p-1} = 0 & \text{on } \partial\Omega, \lambda > 0, u > 0, 1 < \tau < q < p < r. \end{cases}$$

We denote by $C^{0,1}(\overline{\Omega})$ the space of Lipschitz continuous functions defined on $\overline{\Omega}$. Given $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$ with $a(z) \geq 0$ for all $z \in \overline{\Omega}$ and $1 < s < \infty$, by Δ_s^a we denote the weighted s -Laplace differential operator with weight $a(\cdot)$, defined by

$$\Delta_s^a u = \operatorname{div}(a(z)|Du|^{s-2}Du).$$

If $a \equiv 1$, then $\Delta_s^a = \Delta_s$.

We consider two distinct cases depending on the behavior of $a(\cdot)$. If $\min_{\overline{\Omega}} a = c_0 > 0$, then Δ_s^a is a nonautonomous version of the classical s -Laplace differential operator and it can be treated using the same tools as for Δ_s . In problem $(\mathcal{P}_\lambda)$, the

differential operator $u \rightarrow -\Delta_p^a u - \Delta_q u$ ($1 < q < p$) has a density function $\theta(z, t)$ defined by

$$\theta(z, t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

With $a(\cdot)$ bounded away from zero (recall that $\min_{\bar{\Omega}} a = c_0 > 0$), we see that $\theta(z, \cdot)$ exhibits balanced growth, namely we have

$$c_0 t^p \leq \theta(z, t) \leq c_1(1 + t^p) \quad \text{for all } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_1 > 0.$$

So, in this case the growth of $\theta(z, \cdot)$ is trapped between two same powers of $t \geq 0$. This property leads to the global (that is, up to the boundary) regularity theory of Lieberman [13]. This theory provides many powerful tools for the analysis of $(\mathcal{P}_\lambda)$. We mention some characteristic of these tools.

- (i) The equivalence of Hölder and Sobolev local minimizers of the energy functional. This result was first proved for semilinear equations driven by the Dirichlet Laplacian, by Brezis and Nirenberg [2]. It was extended to p -Laplacian Dirichlet equations by García Azorero et al. [5] and to nonlinear nonhomogeneous Robin problems by Papageorgiou and Rădulescu [19].
- (ii) Nonlinear extensions of the Hopf maximum principle. We refer to the book of Pucci and Serrin [27] and for singular anisotropic (p, q) -equations by Papageorgiou et al. [24].
- (iii) Strong comparison results. We refer to the paper of Papageorgiou et al. [22] for Dirichlet equations driven by a nonlinear, nonhomogeneous differential operator.
- (iv) The analysis of problem $(\mathcal{P}_\lambda)$ can be conducted using the classical Lebesgue and Sobolev spaces.

The situation changes if the weight $a(\cdot)$ is not bounded away from zero, that is, if $\min_{\bar{\Omega}} a = 0$. In this case, the density function $\theta(z, \cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \theta(z, t) \leq c_1(1 + t^p) \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

Now the growth of $\theta(z, \cdot)$ is trapped between two different powers of $t \geq 0$. This means that the global regularity theory of Lieberman [13] is not valid and consequently the tools coming with it and mentioned above can not be used. In addition, in this case the classical Lebesgue and Sobolev spaces do not provide an adequate functional framework for the analysis of $(\mathcal{P}_\lambda)$ and we have to pass to the broader class of generalized Orlicz spaces (see Harjulehto and Hästö [7]).

In the reaction (right-hand side) of $(\mathcal{P}_\lambda)$, we have the combined effects of a parametric strictly $(p-1)$ -sublinear term (“concave term”) $u \rightarrow \lambda u^{\tau-1}$ ($\lambda > 0$ is the parameter and recall that $1 < \tau < q$) and of a $(p-1)$ -superlinear term (“convex term”) $u \rightarrow \eta(z)u^{r-1}$ (recall that $p < r$). So, we have a “concave–convex” reaction. The study of such problems, started with the paper of Ambrosetti et al. [1] on semilinear problems driven by the Dirichlet Laplacian. Their work was extended to equations driven by the Dirichlet p -Laplacian by García Azorero et al. [5]. In the context of unbalanced growth equations, concave–convex problems were studied only with Dirichlet boundary conditions. We refer to the works of Gasiński and Papageorgiou [6], Joe et al. [8], Kim et al. [9], Leonardi and Papageorgiou [11, 12],

and Papageorgiou et al. [20]. For related contributions, we refer to [18, 23, 26, 28]. There are no works on Robin equations and none of the aforementioned works, produces ground state (least energy) solutions.

In the boundary condition, we have

$$\frac{\partial u}{\partial n_\theta} = (a(z)|Du|^{p-2} + |Du|^{q-2}) \frac{\partial u}{\partial n},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

We mention that problems with unbalanced growth, were first considered by Marcellini [14, 15] and Zhikov [29–31], in the context of problems of the calculus of variations (including the Lavrentiev phenomenon) and of problems of nonlinear elasticity theory. More recently, there have been efforts to develop a regularity theory for such problems. There have been several local regularity results, but a global regularity theory remains elusive. The survey papers of Marcellini [16] and of Mingione and Rădulescu [17] provide an informative overview of the existing results.

We employ the Nehari method and we show that for all small values of the parameter $\lambda > 0$, problem $(\mathcal{P}_\lambda)$ has at least two ground state (least energy) solutions, one with positive energy and the second with negative energy.

2. Mathematical Background and Hypotheses

Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null set.

Throughout this paper, we will use the following density functions

$$\theta(z, t) = a(z)t^p + t^q, \quad \theta_0(z, t) = a(z)t^p, \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

Let $\mu = \theta$ or $\mu = \theta_0$. Then the generalized Lebesgue–Orlicz space $L^\mu(\Omega)$ is defined by

$$L^\mu(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\mu(u) = \int_\Omega \mu(z, |u|) dz < \infty \right\}.$$

The function $\rho_\mu(\cdot)$ is known as the modular function corresponding to the density μ . The space $L^\mu(\Omega)$ is equipped with the so-called “Luxemburg norm” $\|\cdot\|_\mu$ defined by

$$\|u\|_\mu = \inf \left\{ \lambda > 0 : \rho_\mu\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Then $L^\mu(\Omega)$ becomes a Banach space which is separable and reflexive. In fact $L^\mu(\Omega)$ is uniformly convex, since $\mu(z, \cdot)$ is a uniformly convex function. Recall that a uniformly convex Banach space is reflexive (Milman–Pettis theorem).

Using $L^\mu(\Omega)$, we can define the corresponding generalized Sobolev–Orlicz space $W^{1,\mu}(\Omega)$ by

$$W^{1,\mu}(\Omega) = \{u \in L^\mu(\Omega) : |Du| \in L^\mu(\Omega)\}.$$

Here Du denotes the weak gradient of u . The norm $\|\cdot\|_{1,\mu}$ of $W^{1,\mu}(\Omega)$ is defined by

$$\|u\|_{1,\mu} = \|u\|_\mu + \|Du\|_\mu \quad \text{for all } u \in W^{1,\mu}(\Omega).$$

Note that $\|Du\|_\mu = \| |Du| \|_\mu$. Also, we set

$$W_0^{1,\mu}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mu}},$$

with $C_c^\infty(\Omega)$ being the space of C^∞ -functions on Ω with compact support.

The Poincaré inequality is valid on $W_0^{1,\mu}(\Omega)$. In what follows by $\|\cdot\|$ we denote the norm of $W^{1,\theta}(\Omega)$, that is

$$\|u\| = \|u\|_\theta + \|Du\|_\theta \quad \text{for all } u \in W^{1,\theta}(\Omega). \quad (1)$$

A function $a \in L^0(\Omega)$ is a “weight”, if $a \in L_{\text{loc}}^1(\Omega)$, $a(z) > 0$ for a.a. $z \in \Omega$. For $1 < p < \infty$, by $\mathcal{A}_p$ we denote the class of p -Muckenhoupt weights (see Harjulehto and Hästö [7, p. 106]). In the sequel by $\hookrightarrow$ we denote continuous dense embedding and by $\xhookrightarrow{c}$ a compact, dense embedding. If $a \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p$, then we have

$$W^{1,\theta}(\Omega) \hookrightarrow W^{1,\theta_0}(\Omega) \xhookrightarrow{c} L^{\theta_0}(\Omega). \quad (2)$$

For the last compact embedding, we refer to Chua et al. [3].

Note that in this case the norm $\|\cdot\|_{\theta_0}$ coincides with the weighted norm $|\cdot|_a$ defined by $|h|_a = \left(\int_\Omega a(z)|h|^p dz \right)^{\frac{1}{p}}$ for all $h \in L^{\theta_0}(\Omega)$.

In order to guarantee embeddings of $W^{1,\theta_0}(\Omega)$ into classical Lebesgue spaces, we will need to consider a subset of $\mathcal{A}_p$. So, let $s \in \left[\frac{1}{p-1}, +\infty \right) \cap \left(\frac{N}{p}, +\infty \right)$ and set

$$\mathcal{A}_p^s = \{a \in \mathcal{A}_p : a^{-s} \in L^1(\Omega)\}.$$

The weight $a(z) = |z|^l$ with $-\frac{N}{s} < l < \frac{N}{s}$ belongs in $\mathcal{A}_p^s$ provided $1 < p < N$. We set $p_s = \frac{sp}{s+1}$ and $p_s^* = \frac{Np_s}{N-p_s}$ for $1 < p < N$. We have (see [4]):

Proposition 1. *If $a \in \mathcal{A}_p^s$ with $1 < p < N$ and $s \in \left[\frac{1}{p-1}, +\infty \right) \cap \left(\frac{N}{p}, +\infty \right)$, then $W^{1,\theta_0}(\Omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leq r \leq p_s^*$ and the last embedding is compact if $1 \leq r < p_s^*$.*

Also we have

Proposition 2. *If $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $a(z) \geq 0$ for all $z \in \overline{\Omega}$, then*

- (a) $L^\theta(\Omega) \hookrightarrow L^r(\Omega)$, $W^{1,\theta}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ for all $1 \leq r \leq q$;
- (b) $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ if $1 \leq r \leq q^*$ and the embedding is compact if $1 \leq r < q^*$;
- (c) $L^p(\Omega) \hookrightarrow L^\theta(\Omega)$.

There is a close relation between the norm $\|\cdot\|$ (see (1)) and the function $u \rightarrow \hat{\rho}_\theta(u) = \rho_\theta(u) + \rho_\theta(Du)$.

Proposition 3. *If $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $a(z) \geq 0$ for all $z \in \overline{\Omega}$, then*

- (a) $\|u\| = t \iff \hat{\rho}_\theta\left(\frac{u}{t}\right) = 1$;
- (b) $\|u\| < 1$ (resp. $= 1, > 1$) $\iff \hat{\rho}_\theta(u) < 1$ (resp. $= 1, > 1$);
- (c) $\|u\| < 1 \implies \|u\|^p \leq \hat{\rho}_\theta(u) \leq \|u\|^q$;
- (d) $\|u\| > 1 \implies \|u\|^q \leq \hat{\rho}_\theta(u) \leq \|u\|^p$;
- (e) $\|u\| \rightarrow 0$ (resp. $\rightarrow +\infty$) $\iff \hat{\rho}_\theta(u) \rightarrow 0$ (resp. $\rightarrow +\infty$).

Let $V : W^{1,\theta}(\Omega) \rightarrow (W^{1,\theta}(\Omega))^*$ be the nonlinear operator defined by

$$\langle V(u), h \rangle = \int_{\Omega} [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,\theta}(\Omega).$$

This operator has the following properties.

Proposition 4. *If $a \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $a(z) \geq 0$ for all $z \in \overline{\Omega}$, then $V(\cdot)$ is bounded (maps bounded sets to bounded ones), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W^{1,\theta}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle V(u_n), u_n - u \rangle \leq 0,$$

$$\text{then } u_n \rightarrow u \text{ in } W^{1,\theta}(\Omega).”$$

Given $u \in L^0(\Omega)$, we set

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W^{1,\theta}(\Omega)$, then $u^{\pm} \in W^{1,\theta}(\Omega)$. Also we write $0 \prec u$ if and only if for all $K \subseteq \Omega$ compact, $0 < c_K \leq u(z)$ for a.a. $z \in K$.

In the sequel by $\sigma(\cdot)$ we denote the $(N-1)$ -dimensional Hausdorff measure (surface measure) on $\partial\Omega$.

Our hypotheses on the data of $(\mathcal{P}_{\lambda})$ are the following:

H : $a \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p^s$ with $\frac{N}{p-1} < s$, $\eta \in L^{\infty}(\Omega)$, $\eta(z) > 0$ for a.a. $z \in \Omega$, $\xi_1 \in L^{\infty}(\Omega)$ with $\xi_1(z) \geq 0$ for a.a. $z \in \Omega$, $\beta \in L^{\infty}(\partial\Omega)$ with $\beta(z) \geq 0$ for σ -a.a. $z \in \partial\Omega$, $\xi_1 \not\equiv 0$ or $\beta \not\equiv 0$, $\xi_2(z) \in L^{\infty}(\Omega) \setminus \{0\}$, $\xi_2(z) \geq 0$ for a.a. $z \in \Omega$, $1 < \tau < q < p < p_s^* \leq N$ (recall that $p_s = \frac{sp}{s+1}$ and $p_s^* = \frac{Np_s}{N-p_s}$), $\frac{p}{q} < 1 + \frac{1}{N}$.

The last inequality above is common to problems with unbalanced growth. It leads to local regularity results and it also implies that $p < q^*$.

Remark 5. These hypotheses include also the Neumann problem (when $\beta \equiv 0$, then $\xi_1 \not\equiv 0$).

Let $\gamma : W^{1,\theta}(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional defined by

$$\gamma(u) = \rho_{\theta_0}(Du) + \int_{\Omega} \xi_1(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Also for $\lambda > 0$, let $\varphi_{\lambda} : W^{1,\theta}(\Omega) \rightarrow \mathbb{R}$ be the energy functional for problem $(\mathcal{P}_{\lambda})$ defined by

$$\varphi_{\lambda}(u) = \frac{1}{p}\gamma(u) + \frac{1}{q} \left(\|Du\|_q^q + \int_{\Omega} \xi_2(z)|u|^q dz \right) - \frac{\lambda}{\tau} \|u\|_{\tau}^{\tau} - \frac{1}{r} \int_{\Omega} \eta(z)|u|^r dz.$$

Evidently $\varphi_{\lambda} \in C^1(W^{1,\theta}(\Omega))$. To simplify our notation, we set

$$\rho_{\xi_1}(u) = \int_{\Omega} \xi_1(z)|u|^p dz, \quad \rho_{\xi_2,q}(u) = \int_{\Omega} \xi_2(z)|u|^q dz, \quad \rho_{\beta}^{\partial\Omega}(u) = \int_{\partial\Omega} \beta(z)|u|^p d\sigma, \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

The Nehari manifold for the energy functional $\varphi_{\lambda}(\cdot)$ is defined by

$$\mathcal{N}_{\lambda} = \{u \in W^{1,\theta}(\Omega) : \langle \varphi'_{\lambda}(u), u \rangle = 0, u \neq 0\}.$$

Note that, if $u \in W^{1,\theta}(\Omega)$ is a nontrivial solution of $(\mathcal{P}_{\lambda})$, then $u \in \mathcal{N}_{\lambda}$. The Nehari manifold $\mathcal{N}_{\lambda}$ is much smaller than the ambient space $W^{1,\theta}(\Omega)$ and so $\varphi_{\lambda}|_{\mathcal{N}_{\lambda}}$ can have properties which fail globally.

The Nehari manifold is closely related to the properties of the so-called fibering functions. More precisely, given $u \in W^{1,\theta}(\Omega) \setminus \{0\}$, the fibering function $\mu_\lambda^u : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($\mathbb{R}_+ = [0, +\infty)$, $\lambda > 0$) corresponding to u is defined by

$$\mu_\lambda^u(t) = \varphi_\lambda(tu) \quad \text{for all } t \geq 0.$$

Evidently $\mu_\lambda^u \in C^2(\mathbb{R}_+^0)$ ($\mathbb{R}_+^0 = (0, +\infty)$) and we have

$$(\mu_\lambda^u)'(t) = t^{p-1}\gamma(u) + t^{q-1}(\|Du\|_q^q + \rho_{\xi_2,q}(u)) - \lambda t^{\tau-1}\|u\|_\tau^\tau - t^{r-1} \int_\Omega \eta(z)|u|^r dz$$

and

$$\begin{aligned} (\mu_\lambda^u)''(t) &= (p-1)t^{p-2}\gamma(u) + (q-1)t^{q-2}(\|Du\|_q^q + \rho_{\xi_2,q}(u)) - \lambda(\tau-1)t^{\tau-2}\|u\|_\tau^\tau \\ &\quad - (r-1)t^{r-2} \int_\Omega \eta(z)|u|^r dz. \end{aligned}$$

We see that

$$tu \in \mathcal{N}_\lambda \iff (\mu_\lambda^u)'(t) = 0. \quad (3)$$

So, from (3) we infer that $\mathcal{N}_\lambda$ consists at those nontrivial elements of $W^{1,\theta}(\Omega)$, for which the corresponding function $\mu_\lambda^u(\cdot)$ has a critical point at $t = 1$. This leads to the following partition of $\mathcal{N}_\lambda$:

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : (\mu_\lambda^u)''(1) > 0\} && \text{(local minimizers),} \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : (\mu_\lambda^u)''(1) < 0\} && \text{(local maximizers),} \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : (\mu_\lambda^u)''(1) = 0\} && \text{(points of inflection).} \end{aligned}$$

In the next section, we prove some auxiliary results, which will be crucial in the implementation of the Nehari method (see Sect. 4).

3. Auxiliary Results

Consider the function $|\cdot| : W^{1,\theta_0}(\Omega) \rightarrow \mathbb{R}_+$ defined by

$$|u| = \|Du\|_{\theta_0} + \rho_{\xi_1}(u)^{\frac{1}{p}} + \rho_{\beta}^{\partial\Omega}(u)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,\theta_0}(\Omega).$$

By $\|\cdot\|_{1,\theta_0}$ we denote the norm of $W^{1,\theta_0}(\Omega)$. Recall that

$$\|u\|_{1,\theta_0} = \|u\|_{\theta_0} + \|Du\|_{\theta_0} \quad \text{for all } u \in W^{1,\theta_0}(\Omega).$$

Proposition 6. $|\cdot|$ and $\|\cdot\|_{1,\theta_0}$ are equivalent norms on $W^{1,\theta_0}(\Omega)$.

Proof. First, we verify that $|\cdot|$ is indeed a norm on $W^{1,\theta_0}(\Omega)$.

If $|u| = 0$, then $\|Du\|_{\theta_0} = 0$. So, from the definition of the Luxemburg norm, we have

$$\begin{aligned} \int_\Omega a(z)|Du|^p dz &= 0 \\ \implies u &= c \in \mathbb{R}. \end{aligned}$$

Then $\rho_{\xi_1}(u) + \rho_{\beta}^{\partial\Omega}(u) = 0$. So, $0 = |c|^p \left[\int_{\Omega} \xi_1(z) dz + \int_{\partial\Omega} \beta(z) d\sigma \right]$. But on account of hypotheses H , we have

$$0 < \int_{\Omega} \xi_1(z) dz + \int_{\partial\Omega} \beta(z) d\sigma.$$

Therefore $c = 0$ and so $u = 0$.

Next we check the triangle inequality. Evidently $\|\cdot\|_{\theta_0}$ satisfies the triangle inequality. We check that $\rho_{\xi_1}(\cdot)^{\frac{1}{p}}$ satisfies the triangle inequality and the proof for $\rho_{\beta}^{\partial\Omega}(\cdot)^{\frac{1}{p}}$ is similar. We have

$$\begin{aligned} \rho_{\xi_1}(u+v) &= \int_{\Omega} \xi_1(z) |u+v|^p dz \\ &= \int_{\Omega} \xi_1(z)^{\frac{1}{p'}} |u+v|^{p-1} \xi_1(z)^{\frac{1}{p}} |u+v| dz \quad \left(\frac{1}{p'} + \frac{1}{p} = 1 \right) \\ &\leq \int_{\Omega} \xi_1(z)^{\frac{1}{p'}} |u+v|^{p-1} \xi_1(z)^{\frac{1}{p}} |u| dz + \int_{\Omega} \xi_1(z)^{\frac{1}{p'}} |u+v|^{p-1} \xi_1(z)^{\frac{1}{p}} |v| dz. \end{aligned} \quad (4)$$

Note that

$$\xi_1(\cdot)^{\frac{1}{p'}} |u+v|^{p-1} \in L^{p'}(\Omega), \quad \xi_1(\cdot)^{\frac{1}{p}} |u|, \quad \xi_1(\cdot)^{\frac{1}{p}} |v| \in L^p(\Omega).$$

So, applying Hölder's inequality, we have

$$\int_{\Omega} \xi_1(z)^{\frac{1}{p'}} |u+v|^{p-1} \xi_1(z)^{\frac{1}{p}} |u| dz \leq \rho_{\xi_1}(u+v)^{\frac{1}{p'}} \rho_{\xi_1}(u)^{\frac{1}{p}} \quad (5)$$

and

$$\int_{\Omega} \xi_1(z)^{\frac{1}{p'}} |u+v|^{p-1} \xi_1(z)^{\frac{1}{p}} |v| dz \leq \rho_{\xi_1}(u+v)^{\frac{1}{p'}} \rho_{\xi_1}(v)^{\frac{1}{p}}. \quad (6)$$

We return to (4) and use (5), (6). Then

$$\begin{aligned} \rho_{\xi_1}(u+v) &\leq \rho_{\xi_1}(u+v)^{\frac{1}{p'}} \left[\rho_{\xi_1}(u)^{\frac{1}{p}} + \rho_{\xi_1}(v)^{\frac{1}{p}} \right], \\ \implies \rho_{\xi_1}(u+v)^{\frac{1}{p}} &\leq \rho_{\xi_1}(u)^{\frac{1}{p}} + \rho_{\xi_1}(v)^{\frac{1}{p}} \quad \left(\text{recall that } \frac{1}{p'} + \frac{1}{p} = 1 \right). \end{aligned}$$

This proves that $\rho_{\xi_1}(\cdot)^{\frac{1}{p}}$ satisfies the triangle inequality. Similarly for $\rho_{\beta}^{\partial\Omega}(\cdot)^{\frac{1}{p}}$. We conclude that $|\cdot|$ is a norm on $W^{1,\theta_0}(\Omega)$.

We show that $\|\cdot\|_{1,\theta_0}$ and $|\cdot|$ are equivalent norms on $W^{1,\theta_0}(\Omega)$. First we show that there exists $c_2 > 0$ such that

$$|u| \leq c_2 \|u\|_{1,\theta_0} \quad \text{for all } u \in W^{1,\theta_0}(\Omega). \quad (7)$$

If (7) is not true, then we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta_0}(\Omega)$ such that

$$n \|u_n\|_{1,\theta_0} < |u_n| \quad \text{for all } n \in \mathbb{N}. \quad (8)$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,\theta_0}}$, $n \in \mathbb{N}$. Then $\|y_n\|_{1,\theta_0} = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,\theta_0}(\Omega), \quad y_n \rightarrow y \text{ in } L^p(\Omega), \quad L^p(\partial\Omega) \quad (9)$$

(note that since $\frac{N}{p-1} < s$, we have $p < \frac{(N-1)p_s}{N-p_s}$ and so the trace map is compact, see Proposition 1). From (8) we have

$$n < |y_n| \quad \text{for all } n \in \mathbb{N}. \quad (10)$$

On account of (9), we have

$$\rho_{\xi_1}(y_n) \rightarrow \rho_{\xi_1}(y), \quad \rho_{\beta}^{\partial\Omega}(y_n) \rightarrow \rho_{\beta}^{\partial\Omega}(y). \quad (11)$$

Then from (10) and (11), we infer that

$$\|Dy_n\|_{\theta_0} \rightarrow +\infty.$$

But $\|Dy_n\|_{\theta_0} \leq \|y_n\|_{1,\theta_0} = 1$ for all $n \in \mathbb{N}$, a contradiction. So, (7) is true.

Next we show that there exists $c_3 > 0$ such that

$$\|u\|_{\theta_0} \leq c_3|u| \quad \text{for all } u \in W^{1,\theta_0}(\Omega). \quad (12)$$

Again we argue by contradiction. So, suppose that (12) is not true. Then we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta_0}(\Omega)$ such that

$$n|u_n| < \|u_n\|_{\theta_0} \quad \text{for all } n \in \mathbb{N}. \quad (13)$$

We set $v_n = \frac{u_n}{\|u_n\|_{\theta_0}}$, $n \in \mathbb{N}$. Then $\|v_n\|_{\theta_0} = 1$ for all $n \in \mathbb{N}$. From (13), we have

$$\begin{aligned} |v_n| &< \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \\ \implies \|Dv_n\|_{\theta_0} &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (14)$$

Therefore $\{v_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta_0}(\Omega)$ is bounded and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W^{1,\theta_0}(\Omega), \quad v_n \rightarrow v \text{ in } L^{\theta_0}(\Omega), \quad L^p(\Omega), \quad L^p(\partial\Omega) \quad (\text{see (2)}). \quad (15)$$

From (14) and (15), we obtain

$$\begin{aligned} \|Dv\|_{\theta_0} + \rho_{\xi_1}(v)^{\frac{1}{p}} + \rho_{\beta}^{\partial\Omega}(v)^{\frac{1}{p}} &\leq 0, \\ \implies \|Dv\|_{\theta_0} &= 0 \quad \text{and so } v \equiv c \in \mathbb{R}. \end{aligned}$$

We have

$$\begin{aligned} \rho_{\xi_1}(v)^{\frac{1}{p}} + \rho_{\beta}^{\partial\Omega}(v)^{\frac{1}{p}} &= 0, \\ \implies |c| \left[\left(\int_{\Omega} \xi_1(z) dz \right)^{\frac{1}{p}} + \left(\int_{\partial\Omega} \beta(z) d\sigma \right)^{\frac{1}{p}} \right] &= 0, \\ \implies c &= 0 \quad (\text{see hypotheses } H). \end{aligned}$$

So, we have

$$v_n \rightarrow 0 \text{ in } L^{\theta_0}(\Omega) \quad (\text{see (15)}),$$

contradicting the fact that $\|v_n\|_{\theta_0} = 1$ for all $n \in \mathbb{N}$. Therefore (12) is true. From (12), we obtain

$$\|u\|_{1,\theta_0} \leq c_4|u| \quad \text{for all } u \in W^{1,\theta_0}(\Omega), \quad \text{some } c_4 > 0. \quad (16)$$

From (7) and (16) we conclude that $\|\cdot\|_{1,\theta_0}$ and $|\cdot|$ are equivalent norms on $W^{1,\theta_0}(\Omega)$. $\square$

On $W^{1,q}(\Omega)$ we have the usual norm $\|\cdot\|_{1,q}$ defined by

$$\|u\|_{1,q} = \|u\|_q + \|Du\|_q \quad \text{for all } u \in W^{1,q}(\Omega).$$

Also, we introduce the function $|\cdot|_0 : W^{1,q}(\Omega) \rightarrow \mathbb{R}_+$ defined by

$$|u|_0 = \left[\|Du\|_q^q + \int_{\Omega} \xi_2(z)|u|^q dz \right]^{\frac{1}{q}} \quad \text{for all } u \in W^{1,q}(\Omega).$$

Reasoning as in the proof of Proposition 6, we obtain the following result.

Proposition 7. $|\cdot|_0$ and $\|\cdot\|_{1,q}$ are equivalent norms on $W^{1,q}(\Omega)$.

Also, we have:

Proposition 8. If hypotheses H hold, then

$$c_5\|u\|_{1,\theta_0}^p \leq \gamma(u) \leq c_6\|u\|_{1,\theta_0}^p \quad \text{for all } u \in W^{1,\theta_0}(\Omega), \quad \text{some } 0 < c_5 \leq c_6.$$

Proof. We have

$$\begin{aligned} \gamma(u)^{\frac{1}{p}} &= [\rho_{\theta_0}(Du) + \rho_{\xi_1}(u) + \rho_{\beta}^{\partial\Omega}(u)]^{\frac{1}{p}} \\ &= [\|Du\|_{\theta_0}^p + \rho_{\xi_1}(u) + \rho_{\beta}^{\partial\Omega}(u)]^{\frac{1}{p}} \\ &\leq \|Du\|_{\theta_0} + \rho_{\xi_1}(u)^{\frac{1}{p}} + \rho_{\beta}^{\partial\Omega}(u)^{\frac{1}{p}} = |u|, \\ &\implies \gamma(u) \leq |u|^p \quad \text{for all } u \in W^{1,\theta_0}(\Omega). \end{aligned} \quad (17)$$

On the other hand, note that

$$\begin{aligned} |u| &\leq 3\gamma(u)^{\frac{1}{p}} \quad \text{for all } u \in W^{1,\theta_0}(\Omega), \\ &\implies \frac{1}{3^p}|u|^p \leq \gamma(u) \quad \text{for all } u \in W^{1,\theta_0}(\Omega). \end{aligned} \quad (18)$$

Then from (17) and (18), it follows that

$$\frac{1}{3^p}|u|^p \leq \gamma(u) \leq |u|^p \quad \text{for all } u \in W^{1,\theta_0}(\Omega). \quad (19)$$

From Proposition 6, we know that $|\cdot|$ and $\|\cdot\|_{1,\theta_0}$ are equivalent norms on $W^{1,\theta_0}(\Omega)$. Therefore, from (19), we see that we can find $0 < c_5 \leq c_6$ such that

$$c_5\|u\|_{1,\theta_0}^p \leq \gamma(u) \leq c_6\|u\|_{1,\theta_0}^p \quad \text{for all } u \in W^{1,\theta_0}(\Omega).$$

The proof is now complete. $\square$

Proposition 8 above implies that

$$\gamma(u) + \|u\|_{1,q}^q \geq c_5 \|u\|_{1,\theta_0}^p + \|u\|_{1,q}^q \geq c_7 \left(\|u\|_{1,\theta_0}^p + \|u\|_{1,q}^q \right) \quad (20)$$

for all $u \in W^{1,\theta}(\Omega)$, with $c_7 = \min\{1, c_5\}$.

From (20), we see that we need to estimate $\|u\|_{1,\theta_0}^p + \|u\|_{1,q}^q$ in terms of the norm $\|\cdot\|$ of $W^{1,\theta}(\Omega)$. From (1), we know that

$$\|u\| = \|u\|_\theta + \|Du\|_\theta \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Proposition 9. *If hypotheses H hold, then there exists $c_8 > 0$ such that*

$$c_8 \min\{\|u\|^p, \|u\|^q\} \leq \|u\|_{1,\theta_0}^p + \|u\|_{1,q}^q \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Proof. We proceed indirectly. So, suppose that the assertion of the proposition is not true. Then we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ such that

$$\|u_n\|_{1,\theta_0}^p + \|u_n\|_{1,q}^q < \frac{1}{n} \min\{\|u_n\|^p, \|u_n\|^q\} \quad \text{for all } n \in \mathbb{N}. \quad (21)$$

If, for at least a subsequence, we have $\|u_n\| \leq 1$ for all $n \in \mathbb{N}$, then from (21), we have

$$\|u_n\|_{1,\theta_0}^p + \|u_n\|_{1,q}^q < \frac{1}{n} \|u_n\|^p \quad \text{for all } n \in \mathbb{N}. \quad (22)$$

We set $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so from (22), we obtain

$$\begin{aligned} \|y_n\|_{1,\theta_0}^p + \frac{1}{\|u_n\|^{p-q}} \|y_n\|_{1,q}^q &< \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,\theta_0}(\Omega). \end{aligned} \quad (23)$$

Moreover, since $\|u_n\| \leq 1$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|y_n\|_{1,q} &< \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,q}(\Omega). \end{aligned} \quad (24)$$

Then (23) and (24) imply that

$$\begin{aligned} \rho_\theta(Dy_n) + \rho_\theta(y_n) &\rightarrow 0, \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,\theta}(\Omega) \quad (\text{see Proposition 3}), \end{aligned}$$

a contradiction, since $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

Next suppose that, at least for a subsequence, we have $\|u_n\| > 1$ for all $n \in \mathbb{N}$. From (21), we obtain

$$\|u_n\|_{1,\theta_0}^p + \|u_n\|_{1,q}^q < \frac{1}{n} \|u_n\|^q \quad \text{for all } n \in \mathbb{N}. \quad (25)$$

As before, we set $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so from (25) it follows that

$$\begin{aligned} \|u_n\|^{p-q} \|y_n\|_{1,\theta_0}^p + \|y_n\|_{1,q}^q &< \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,q}(\Omega). \end{aligned} \quad (26)$$

Moreover, since $\|u_n\| > 1$ for all $n \in \mathbb{N}$, we also have

$$\begin{aligned} \|y_n\|_{1,\theta_0}^p &< \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,\theta_0}(\Omega). \end{aligned} \quad (27)$$

Then (26) and (27) imply that

$$\begin{aligned} \rho_\theta(Dy_n) + \rho_\theta(y_n) &\rightarrow 0 \quad (\text{see Proposition 2}), \\ \implies y_n &\rightarrow 0 \quad \text{in } W^{1,\theta}(\Omega) \quad (\text{see Proposition 3}), \end{aligned}$$

again a contradiction, since $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

We conclude that there exists $c_8 > 0$ such that

$$c_8 \min \{\|u\|^p, \|u\|^q\} \leq \|u\|_{1,\theta_0}^p + \|u\|_{1,q}^q \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

This completes the proof. $\square$

To have the boundedness of solutions of $(\mathcal{P}_\lambda)$, we will use the following lemma. It is an outgrowth of some estimates of Ladyzhenskaya and Ural'tseva [10, pp. 66, 73–74].

Given $u \in L^0(\Omega)$ and $k > 0$, we define

$$E_k = \{z \in \Omega : |u| > k\}.$$

Also by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Lemma 10. *If $\ell, d > 1, u \in L^\ell(\Omega)$ and*

$$\begin{aligned} s \left[\int_{E_k} (|u| - k)^\ell dz \right]^{\frac{1}{d}} &\leq c \left[\int_{E_k} (|u| - k)^\ell dz + k^\ell |E_k|_N \right], \\ \text{for } k &\geq k^* > 0, \text{ with } c > 0, \quad k^* > 0, \end{aligned}$$

then $u \in L^\infty(\Omega)$.

Proof. Let $i > 2k^*$ and consider the sequence $\{k_n\}_{n \in \mathbb{N}_0}$ defined by

$$k_n = i \left(1 - \frac{1}{2^{n+1}} \right) \quad \text{for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Evidently $\{k_n\}_{n \in \mathbb{N}_0} \subseteq \mathring{\mathbb{R}}_+ = (0, +\infty)$ is a strictly increasing sequence and $k_n < i$ for all $n \in \mathbb{N}_0$. Then

$$\left(\int_{E_{k_{n+1}}} (|u| - k_{n+1})^\ell dz \right)^{\frac{1}{d}} \leq c \left[\int_{E_{k_n}} (|u| - k_n)^\ell dz + k_{n+1}^\ell |E_{k_n}|_N \right]. \quad (28)$$

We have

$$\begin{aligned} (k_{n+1} - k_n)^\ell |E_{k_{n+1}}|_N &\leq \int_{E_{k_n}} (|u| - k_n)^\ell dz, \\ \implies k_{n+1}^\ell |E_{k_{n+1}}|_N &\leq \left(\frac{k_{n+1}}{k_{n+1} - k_n} \right)^\ell \int_{E_{k_n}} (|u| - k_n)^\ell dz \\ &\leq (2^{n+2} - 1)^\ell \int_{E_{k_n}} (|u| - k_n)^\ell dz. \end{aligned} \quad (29)$$

We use (29) in (28) and obtain

$$\begin{aligned} \int_{E_{k_{n+1}}} (|u| - k_{n+1})^\ell dz &\leq \left[c \left(1 + (2^{n+2} - 1)^\ell \right) \int_{E_{k_n}} (|u| - k_n)^\ell dz \right]^d \\ &\leq \hat{c} \chi^n \left[\int_{E_{k_n}} (|u| - k_n)^\ell dz \right]^{1+\delta} \end{aligned}$$

with $\chi = 2^{\ell d} > 1$, $\delta = d - 1$ and $\hat{c} = \hat{c}(c, \ell, d) > 0$. We introduce the sequence

$$\left\{ v_n = \int_{E_{k_n}} (|u| - k_n)^\ell dz \right\}_{n \in \mathbb{N}_0}.$$

Lemma 4.7 of Ladyzhenskaya and Ural'tseva [10, p. 66] says that

$$\text{“ if } v_0 \leq \hat{c}^{-\frac{1}{\delta}} \chi^{-\frac{1}{\delta^2}}, \text{ then } v_n \rightarrow 0 \text{ as } n \rightarrow \infty. \text{”} \quad (30)$$

Note that

$$\begin{aligned} v_0 &= \int_{E_{k_0}} (|u| - k_0)^\ell dz = \int_{E_{k_0}} \left(|u| - \frac{i}{2} \right)^\ell dz, \\ &\implies v_0 \rightarrow 0 \text{ as } i \rightarrow +\infty. \end{aligned}$$

Therefore for $i > 2k^*$ large, we have

$$\begin{aligned} v_0 &\leq \hat{c}^{-\frac{1}{\delta}} \chi^{-\frac{1}{\delta^2}}, \\ &\implies v_n \implies 0 \text{ as } n \implies \infty \quad (\text{see (30)}). \end{aligned} \quad (31)$$

Observe that

$$\begin{aligned} \int_{\Omega} (|u| - \ell)^+ dz &\leq \int_{\Omega} (|u| - k_n)^+ dz = \int_{E_{k_n}} (|u| - k_n)^+ dz \leq c_9 \int_{E_{k_n}} ((|u| - k_n)^+)^{\ell} dz \\ &\text{for all } n \in \mathbb{N}, \text{ some } c_9 > 0 \text{ (since } L^{\ell}(\Omega) \hookrightarrow L^1(\Omega)), \\ &\implies \int_{\Omega} (|u| - \ell)^+ dz = 0, \\ &\implies |u(z)| \leq \ell \text{ for a.a. } z \in \Omega \text{ and so } u \in L^{\infty}(\Omega). \end{aligned}$$

The proof is now complete. $\square$

4. Ground State Solutions

As we already said in Sect. 2, $\varphi_{\lambda}|_{\mathcal{N}_{\lambda}}$ may have properties which fail globally. In our case the presence of the $(p - 1)$ -superlinear term $\frac{1}{r} \int_{\Omega} \eta(z) |u|^r dz$ (recall that $p < r$), implies that the energy functional $\varphi_{\lambda}(\cdot)$ is unbounded below and so there is no global minimizer for $\varphi_{\lambda}(\cdot)$. However, in the next proposition, we show that $\varphi_{\lambda}|_{\mathcal{N}_{\lambda}}$ is coercive and so we can expect that $\varphi_{\lambda}|_{\mathcal{N}_{\lambda}}$ has a minimizer. This observation highlights the importance of the Nehari manifold.

Proposition 11. *If hypotheses H hold, $\lambda > 0$, and $\mathcal{N}_{\lambda} \neq \emptyset$, then $\varphi_{\lambda}|_{\mathcal{N}_{\lambda}}$ is coercive.*

Proof. Let $u \in \mathcal{N}_\lambda$ with $\|u\| \geq 1$. From the definition of $\mathcal{N}_\lambda$, we have

$$\gamma(u) + \|Du\|_q^q + \rho_{\xi_2, q}(u) = \lambda \|u\|_\tau^\tau + \int_\Omega \eta(z) |u|^r dz. \quad (32)$$

Using (32), we obtain

$$\begin{aligned} \varphi_\lambda(u) &= \left(\frac{1}{p} - \frac{1}{r}\right) \gamma(u) + \left(\frac{1}{q} - \frac{1}{r}\right) (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - \frac{\lambda}{\tau} \left(\frac{1}{\tau} - \frac{1}{r}\right) \|u\|_\tau^\tau \\ &\geq \left(\frac{1}{p} - \frac{1}{r}\right) (\gamma(u) + c_{10} \|u\|_{1, q}^q) - \frac{\lambda}{\tau} \left(\frac{1}{\tau} - \frac{1}{r}\right) \|u\|_\tau^\tau \\ &\quad \text{for some } c_{10} > 0 \quad (\text{see Proposition 7 and recall that } q < p) \\ &\geq c_{11} \left(\|u\|_{1, \theta}^p + \|u\|_{1, q}^q\right) - \lambda c_{12} \|u\|^\tau \\ &\quad \text{for some } c_{11} > 0, c_{12} > 0 \quad (\text{see Propositions 2 and 8}) \\ &\geq c_{13} \|u\|^q - \lambda c_{12} \|u\|^\tau \\ &\quad \text{for some } c_{13} > 0 \quad (\text{see Proposition 9 and recall that } \|u\| \geq 1). \end{aligned}$$

Since $\tau < q$, we conclude that $\varphi_\lambda|_{\mathcal{N}_\lambda}$ is coercive. $\square$

For every $\lambda > 0$, we have that

$$\varphi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.} \quad (33)$$

Also, since $\mathcal{N}_\lambda^+ \subseteq \mathcal{N}_\lambda$, then using Proposition 11, we see that if $\lambda > 0$ and $\mathcal{N}_\lambda^+ \neq \emptyset$, then

$$\varphi_\lambda|_{\mathcal{N}_\lambda^+} \text{ is coercive.} \quad (34)$$

Then (33) and (34) suggest that we consider the following minimization problem

$$m_\lambda^+ = \inf_{\mathcal{N}_\lambda^+} \varphi_\lambda.$$

Proposition 12. *If hypotheses H hold, $\lambda > 0$, and $\mathcal{N}_\lambda^+ \neq \emptyset$, then $-\infty < m_\lambda^+ < 0$.*

Proof. On account of (34), we have that $\varphi_\lambda|_{\mathcal{N}_\lambda^+}$ is bounded below. Therefore

$$-\infty < m_\lambda^+.$$

If $u \in \mathcal{N}_\lambda^+$, then from the definition of $\mathcal{N}_\lambda^+$, we have

$$(p-1)\gamma(u) + (q-1)(\|Du\|_q^q + \rho_{\xi_2, q}(u)) - (r-1) \int_\Omega \eta(z) |u|^r dz > \lambda(\tau-1) \|u\|_\tau^\tau. \quad (35)$$

In addition, since $u \in \mathcal{N}_\lambda$ (recall that $\mathcal{N}_\lambda^+ \subseteq \mathcal{N}_\lambda$), we have

$$\gamma(u) + \|Du\|_q^q + \rho_{\xi_2, q}(u) - \int_\Omega \eta(z) |u|^r dz = \lambda \|u\|_\tau^\tau. \quad (36)$$

From (35) and (36), it follows that

$$\begin{aligned} p\gamma(u) + q(\|Du\|_q^q + \rho_{\xi_2, q}(u)) - r \int_{\Omega} \eta(z)|u|^r dz &> \lambda\tau\|u\|_{\tau}^{\tau}, \\ \implies (p - \tau)\gamma(u) + (q - \tau)(\|Du\|_q^q + \rho_{\xi_2, q}(u)) &> (r - \tau) \int_{\Omega} \eta(z)|u|^r dz \quad (\text{see (36)}), \\ \implies \int_{\Omega} \eta(z)|u|^r dz &< \frac{p - \tau}{r - \tau}\gamma(u) + \frac{q - \tau}{r - \tau}(\|Du\|_q^q + \rho_{\xi_2, q}(u)). \end{aligned} \quad (37)$$

Then we have

$$\begin{aligned} \varphi_{\lambda}(u) &= \left(\frac{1}{p} - \frac{1}{\tau}\right)\gamma(u) + \left(\frac{1}{q} - \frac{1}{\tau}\right)(\|Du\|_q^q + \rho_{\xi_2, q}(u)) - \left(\frac{1}{r} - \frac{1}{\tau}\right) \\ &\quad \times \int_{\Omega} \eta(z)|u|^r dz \quad (\text{see (36)}) \\ &< \left(\frac{r - \tau}{r\tau} \frac{p - \tau}{r - \tau} - \frac{p - \tau}{p\tau}\right)\gamma(u) + \left(\frac{r - \tau}{r\tau} \frac{q - \tau}{r - \tau} - \frac{q - \tau}{q\tau}\right) \\ &\quad (\|Du\|_q^q + \rho_{\xi_2, q}(u)) \quad (\text{see (37)}) \\ &= \frac{p - \tau}{\tau} \left(\frac{1}{r} - \frac{1}{p}\right)\gamma(u) + \frac{q - \tau}{\tau} \left(\frac{1}{r} - \frac{1}{q}\right)(\|Du\|_q^q + \rho_{\xi_2, q}(u)) \\ &< 0 \quad (\text{since } 1 < \tau < q < p < r), \\ \implies m_{\lambda}^+ &< 0. \end{aligned}$$

The proof is now complete. $\square$

Proposition 13. *If hypotheses H hold, then there exists $\lambda^* > 0$ such that $\mathcal{N}_{\lambda}^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$.*

Proof. We argue by contradiction. So, suppose that we can find $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathring{\mathbb{R}}_+$ such that

$$\lambda_n \rightarrow 0^+ \quad \text{and} \quad u_n \in \mathcal{N}_{\lambda_n}^0 \quad \text{for all } n \in \mathbb{N}.$$

From the definition of $\mathcal{N}_{\lambda_n}^0$, we have

$$\begin{aligned} (p - 1)\gamma(u_n) + (q - 1)(\|Du_n\|_q^q + \rho_{\xi_2, q}(u_n)) \\ = \lambda_n(\tau - 1)\|u_n\|_{\tau}^{\tau} + (r - 1) \int_{\Omega} \eta(z)|u_n|^r dz \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (38)$$

We know that $u_n \in \mathcal{N}_{\lambda_n}$ (since $\mathcal{N}_{\lambda_n}^0 \subseteq \mathcal{N}_{\lambda_n}$ for all $n \in \mathbb{N}$). Hence

$$\begin{aligned} (r - 1)\gamma(u_n) + (r - 1)(\|Du_n\|_q^q + \rho_{\xi_2, q}(u_n)) \\ = \lambda_n(r - 1)\|u_n\|_{\tau}^{\tau} + (r - 1) \int_{\Omega} \eta(z)|u_n|^r dz \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (39)$$

Subtracting (38) from (39), we obtain

$$\begin{aligned}
 (r-p)\gamma(u_n) + (r-q)(\|Du_n\|_q^q + \rho_{\xi_2,q}(u_n)) &= \lambda_n(r-\tau)\|u_n\|_\tau^\tau \quad \text{for all } n \in \mathbb{N}, \\
 \implies (r-p)(\gamma(u_n) + c_{14}\|u_n\|_{1,q}^q) &\leq \lambda_n c_{15}\|u_n\|^\tau \\
 &\quad \text{for some } c_{14} > 0, \ c_{15} > 0 \quad (\text{see Propositions 1 and 7}), \\
 \implies (r-p)c_{16}(\|u_n\|_{1,\theta_0}^p + \|u_n\|_{1,q}^q) &\leq \lambda_n c_{15}\|u_n\|^\tau \\
 &\quad \text{for some } c_{16} > 0, \text{ all } n \in \mathbb{N} \quad (\text{see Proposition 8}), \\
 \implies c_{17} \min\{\|u_n\|^p, \|u_n\|^q\} &\leq \lambda_n \frac{c_{15}}{r-p}\|u_n\|^\tau \\
 &\quad \text{for some } c_{17} > 0, \text{ all } n \in \mathbb{N} \quad (\text{see Proposition 9}).
 \end{aligned}$$

Since $1 < \tau < q < p$ and $\lambda_n \rightarrow 0^+$, it follows that

$$u_n \rightarrow 0 \quad \text{in } W^{1,\theta}(\Omega) \quad \text{as } n \rightarrow \infty. \quad (40)$$

Recall that $u_n \in \mathcal{N}_{\lambda_n}$, $n \in \mathbb{N}$. Therefore

$$\begin{aligned}
 (\tau-1)\gamma(u_n) + (\tau-1)(\|Du_n\|_q^q + \rho_{\xi_2,q}(u_n)) \\
 = \lambda_n(\tau-1)\|u_n\|_\tau^\tau + (\tau-1) \int_\Omega \eta(z)|u_n|^r dz \quad \text{for all } n \in \mathbb{N}.
 \end{aligned} \quad (41)$$

We subtract (41) from (38). Then

$$\begin{aligned}
 (p-\tau)\gamma(u_n) + (q-\tau)(\|Du_n\|_q^q + \rho_{\xi_2,q}(u_n)) &= (r-\tau) \int_\Omega \eta(z)|u_n|^r dz \\
 &\quad \text{for all } n \in \mathbb{N}, \\
 \implies (q-\tau)(\gamma(u_n) + c_{18}\|u_n\|_{1,q}^q) &\leq (r-\tau)c_{19}\|u_n\|^r \\
 &\quad \text{for some } c_{18} > 0, \ c_{19} > 0, \text{ all } n \in \mathbb{N} \quad (\text{see Propositions 1 and 7}), \\
 \implies \min\{\|u_n\|^p, \|u_n\|^q\} &\leq c_{20}\|u_n\|^r \\
 &\quad \text{for some } c_{20} > 0, \text{ all } n \in \mathbb{N} \quad (\text{see Proposition 9}), \\
 \implies 0 < c_{21} \leq \|u_n\| &\text{ for some } c_{21} > 0, \text{ all } n \in \mathbb{N} \quad (\text{recall that } q < p < r). \quad (42)
 \end{aligned}$$

Comparing (42) and (40), we have a contradiction. Therefore there exists $\lambda^* > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$. $\square$

So far, we do not know if the sets $\mathcal{N}_\lambda^\pm$ are nonempty. In the next proposition we show that for all $\lambda > 0$ small, $\mathcal{N}_\lambda^\pm \neq \emptyset$ (and so $\mathcal{N}_\lambda \neq \emptyset$ too).

Proposition 14. *If hypotheses H hold, then there exists $\hat{\lambda}^* \in (0, \lambda^*]$ such that $\mathcal{N}_\lambda^\pm = \emptyset$ for all $0 < \lambda < \hat{\lambda}^*$.*

Proof. Let $u \in W^{1,\theta}(\Omega) \setminus \{0\}$ and consider the functional $\ell_u : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$\ell_u(t) = t^{p-\tau}\gamma(u) - t^{r-\tau} \int_\Omega \eta(z)|u|^r dz \quad \text{for all } t \geq 0.$$

Since $\tau < p < r$, we see that

$$\ell_u(t) > 0 \quad \text{for } t \in (0, 1) \text{ small}, \quad \ell_u(t) < 0 \quad \text{for } t > 1 \text{ large}.$$

Moreover, $\ell_u(0) = 0$. Therefore we can find $t_0 = t_0(u, \lambda) > 0$ such that

$$\begin{aligned}\ell_u(t_0) &= \max_{t \geq 0} \ell_u(t), \\ \implies \ell'_u(t_0) &= 0, \\ \implies (p - \tau)t_0^{p-\tau-1} \gamma(u) &= (r - \tau)t_0^{r-\tau-1} \int_{\Omega} \eta(z) |u|^r dz, \\ \implies t_0 &= \left[\frac{(p - \tau) \gamma(u)}{(r - \tau) \int_{\Omega} \eta(z) |u|^r dz} \right]^{\frac{1}{r-p}}.\end{aligned}$$

We have

$$\begin{aligned}\ell_u(t_0) &= \left[\frac{(p - \tau) \gamma(u)}{(r - \tau) \int_{\Omega} \eta(z) |u|^r dz} \right]^{\frac{p-\tau}{r-p}} \gamma(u) - \left[\frac{(p - \tau) \gamma(u)}{(r - \tau) \int_{\Omega} \eta(z) |u|^r dz} \right]^{\frac{r-\tau}{r-p}} \int_{\Omega} \eta(z) |u|^r dz \\ &= \left(\frac{p - \tau}{r - \tau} \right)^{\frac{p-\tau}{r-p}} \left(\frac{1}{\int_{\Omega} \eta(z) |u|^r dz} \right)^{\frac{p-\tau}{r-p}} \gamma(u)^{\frac{r-\tau}{r-p}} \\ &\quad - \left(\frac{p - \tau}{r - \tau} \right)^{\frac{r-\tau}{r-p}} \left(\frac{1}{\int_{\Omega} \eta(z) |u|^r dz} \right)^{\frac{p-\tau}{r-p}} \gamma(u)^{\frac{r-\tau}{r-p}} \\ &= \left(\frac{p - \tau}{r - \tau} \right)^{\frac{p-\tau}{r-p}} \left(1 - \frac{p - \tau}{r - \tau} \right) \frac{\gamma(u)^{\frac{r-\tau}{r-p}}}{(\int_{\Omega} \eta(z) |u|^r dz)^{\frac{p-\tau}{r-p}}} \\ &\geq \left(\frac{p - \tau}{r - \tau} \right)^{\frac{p-\tau}{r-p}} \frac{r - p}{r - \tau} c_{22} \|u\|_{1, \theta_0}^{\tau} \quad \text{for some } c_{22} > 0 \\ &\quad \text{(see Propositions 1 and 8).}\end{aligned}\tag{43}$$

Then

$$\begin{aligned}\ell_u(t_0) - \lambda \|u\|_{\tau}^{\tau} &\geq \ell_u(t_0) - \lambda c_{23} \|u\|_{1, \theta_0}^{\tau} \quad \text{for some } c_{23} > 0 \\ &\quad \text{(see Proposition 1 and note } \tau < p_s^*) \\ &\geq \left[\left(\frac{p - \tau}{r - \tau} \right)^{\frac{p-\tau}{r-p}} \frac{r - p}{r - \tau} c_{22} - \lambda c_{23} \right] \|u\|_{1, \theta_0}^{\tau} \quad \text{(see (43)).}\end{aligned}$$

So, we can find $\hat{\lambda}^* \in (0, \lambda^*]$ such that

$$\ell_u(t_0) - \lambda \|u\|_{\tau}^{\tau} > 0 \quad \text{for all } 0 < \lambda < \hat{\lambda}^*.\tag{44}$$

Then define

$$\hat{\ell}_u(t) = \ell_u(t) + t^{q-\tau} (\|Du\|_q^q + \rho_{\xi_2, q}(u)).$$

Evidently $\hat{\ell}_u(t) \geq \ell_u(t)$ for all $t \geq 0$. Moreover, as we did for $\ell_u(\cdot)$, we can find $\hat{t}_0 = \hat{t}_0(u, \lambda) > 0$ such that

$$\hat{\ell}_u(\hat{t}_0) = \max_{t \geq 0} \hat{\ell}_u(t) \geq \hat{\ell}_u(t_0) \geq \ell_u(t_0).\tag{45}$$

Therefore, if we set

$$\hat{k}_u^{\lambda}(t) = \hat{\ell}_u(t) - \lambda \|u\|_{\tau}^{\tau} \quad \text{for all } t \geq 0,$$

then we have

$$\hat{k}_u^\lambda(\hat{t}_0) = \hat{\ell}_u(\hat{t}_0) - \lambda \|u\|_\tau^\tau \geq \ell_u(t_0) - \lambda \|u\|_\tau^\tau > 0 \quad \text{for all } 0 < \lambda < \hat{\lambda}^* \quad (\text{see (45), (44)}). \quad (46)$$

Since $\hat{\ell}_u(0) = 0$, we see that

$$\hat{k}_u^\lambda(t) < 0 \quad \text{for all } t \in (0, 1) \text{ small}. \quad (47)$$

We know that $\hat{\ell}_u(t) < 0$ for $t > 1$ large (since $1 < \tau < q < p < r$). This fact combined with (46) and (47), implies that there exist $t_1 < \hat{t}_0 < t_2$ (both depending on $\lambda \in (0, \hat{\lambda}^*)$) such that

$$\hat{k}_u^\lambda(t_1) = \hat{k}_u^\lambda(t_2) = 0 < \hat{k}_u^\lambda(\hat{t}_0) \quad (48)$$

and

$$\left(\hat{k}_u^\lambda\right)'(t_2) < 0 < \left(\hat{k}_u^\lambda\right)'(t_1). \quad (49)$$

From (48), we have

$$t_1^{p-\tau} \gamma(u) + t_1^{q-\tau} (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - t_1^{r-\tau} \int_\Omega \eta(z) |u|^r dz = \lambda \|u\|_\tau^\tau. \quad (50)$$

On the other hand from (49) we have

$$\begin{aligned} 0 &< (p-\tau)t_1^{p-\tau-1} \gamma(u) + (q-\tau)t_1^{q-\tau-1} (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - (r-\tau)t_1^{r-\tau-1} \int_\Omega \eta(z) |u|^r dz \\ &= t_1^{-1} \left[(p-\tau)t_1^{p-\tau} \gamma(u) + (q-\tau)t_1^{q-\tau} (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - (r-\tau)t_1^{r-\tau} \int_\Omega \eta(z) |u|^r dz \right] \\ &= t_1^{-1} \left[(p-1)t_1^{p-\tau} \gamma(u) + (q-1)t_1^{q-\tau} (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - (r-1)t_1^{r-\tau} \int_\Omega \eta(z) |u|^r dz \right. \\ &\quad \left. - \lambda(\tau-1) \|u\|_\tau^\tau \right] \quad (\text{see (50)}) \\ &= t_1^{1-\tau} \left[(p-1)t_1^{p-2} \gamma(u) + (q-1)t_1^{q-2} (\|Du\|_q^q + \rho_{\xi_2, q}(u)) - (r-1)t_1^{r-2} \int_\Omega \eta(z) |u|^r dz \right. \\ &\quad \left. - \lambda(\tau-1)t_1^{\tau-2} \|u\|_\tau^\tau \right] \\ &= t_1^{1-\tau} (\mu_\lambda^u)''(t_1), \\ &\implies 0 < (\mu_\lambda^u)''(t_1), \\ &\implies t_1 u \in \mathcal{N}_\lambda^+ \neq \emptyset \quad \text{for all } 0 < \lambda < \hat{\lambda}^*. \end{aligned}$$

Similarly we show that $\mathcal{N}_\lambda^- \neq \emptyset$ for all $0 < \lambda < \hat{\lambda}^*$. □

Recall that

$$m_\lambda^+ = \inf_{\mathcal{N}_\lambda^+} \varphi_\lambda.$$

We will show that m_λ^+ is reached on $\mathcal{N}_\lambda^+$.

Proposition 15. *If hypotheses H hold and $0 < \lambda < \hat{\lambda}^*$, then there exists $u_0^\lambda \in \mathcal{N}_\lambda^+$ such that $m_\lambda^+ = \varphi_\lambda(u_0^\lambda)$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{N}_\lambda^+$ be such that

$$\varphi_\lambda(u_n) \downarrow m_\lambda^+.$$

From (34) we see that $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ is bounded. Therefore, we may assume that

$$u_n \xrightarrow{w} u_0^\lambda \text{ in } W^{1,\theta}(\Omega), \quad u_n \rightarrow u_0^\lambda \text{ in } L^{\theta_0}(\Omega), \quad L^r(\Omega), \quad L^p(\partial\Omega) \quad (51)$$

(see (2) and as before note that $s > \frac{N}{p-1}$ implies $p < \frac{(N-1)p_s}{N-p_s}$ and so $W^{1,p_s}(\Omega) \xhookrightarrow{c} L^p(\partial\Omega)$ via the trace map).

Claim: $u_n \rightarrow u_0^\lambda$ in $W^{1,\theta}(\Omega)$ as $n \rightarrow \infty$.

We argue indirectly. So, suppose that the Claim is not true. Then at least one of the following inequalities holds

$$\rho_{\theta_0}(Du_0^\lambda) < \liminf_{n \rightarrow \infty} \rho_{\theta_0}(Du_n), \quad |u_0^\lambda|_0 < \liminf_{n \rightarrow \infty} |u_n|_0. \quad (52)$$

On account of (51), we have

$$\begin{cases} \|u_n\|_\tau \rightarrow \|u_0^\lambda\|_\tau \quad (\text{recall that } \tau < r), \\ \int_\Omega \eta(z)|u_n|^r dz \rightarrow \int_\Omega \eta(z)|u_0^\lambda|^r dz, \quad \int_{\partial\Omega} \beta(z)|u_n|^p d\sigma \rightarrow \int_{\partial\Omega} \beta(z)|u_0^\lambda|^p d\sigma. \end{cases} \quad (53)$$

From (52), (53), and Proposition 12, we have

$$\begin{aligned} \varphi_\lambda(0) = 0 > m_\lambda^+ &= \lim_{n \rightarrow \infty} \varphi_\lambda(u_n) \\ &> \frac{1}{p} \gamma(u_0^\lambda) + \frac{1}{q} (\|Du_0^\lambda\|_q^q + \rho_{\xi_2,q}(u_0^\lambda)) - \frac{\lambda}{\tau} \|u_0^\lambda\|_\tau^\tau - \frac{1}{r} \int_\Omega \eta(z)|u_0^\lambda|^r dz \\ &= \varphi_\lambda(u_0^\lambda), \\ &\implies u_0^\lambda \neq 0 \text{ and } \varphi_\lambda(u_0^\lambda) < m_\lambda^+. \end{aligned} \quad (54)$$

From the proof of Proposition 14, we know that there exists $t_1^\lambda > 0$ such that $t_1^\lambda u_0^\lambda \in \mathcal{N}_\lambda^+$. Then

$$\left(\mu_\lambda^{u_0^\lambda} \right)'(t_1^\lambda) = 0. \quad (55)$$

By passing to a subsequence if necessary, we can say that

$$\liminf_{n \rightarrow \infty} (\mu_\lambda^{u_n})'(t_1^\lambda) = \lim_{n \rightarrow \infty} (\mu_\lambda^{u_n})'(t_1^\lambda). \quad (56)$$

Then (56) and (52) imply that

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\mu_\lambda^{u_n})'(t_1^\lambda) &> \left(\mu_\lambda^{u_0^\lambda} \right)'(t_1^\lambda) = 0 \quad (\text{see (55)}), \\ &\implies (\mu_\lambda^{u_n})'(t_1^\lambda) > 0 \text{ for all } n \geq n_0. \end{aligned} \quad (57)$$

Since $u_n \in \mathcal{N}_\lambda^+$ for all $n \in \mathbb{N}$, we have

$$(\mu_\lambda^{u_n})'(t) \leq (\mu_\lambda^{u_n})'(1) = 0 \text{ for all } t \in (0, 1), \text{ all } n \in \mathbb{N}. \quad (58)$$

From (57) and (58), it follows that

$$1 < t_1^\lambda. \quad (59)$$

We know that $\mu_\lambda^{u_0^\lambda}(\cdot)$ is nonincreasing on $(0, t_1^\lambda]$. Hence

$$\begin{aligned} m_\lambda^+ &\leq \varphi_\lambda(t_1^\lambda u_0^\lambda) \leq \varphi_\lambda(u_0^\lambda) \quad (\text{see (59)}) \\ &< m_\lambda^+ \quad (\text{see (54)}), \end{aligned}$$

a contradiction. This proves the Claim.

The Nehari manifold is closed in $W^{1,\theta}(\Omega) \setminus \{0\}$. Therefore from the Claim it follows that

$$u_0^\lambda \in \mathcal{N}_\lambda.$$

Recall that $u_n \in \mathcal{N}_\lambda^+$ for all $n \in \mathbb{N}$. So, we have

$$\begin{aligned} (\mu_\lambda^{u_n})''(1) &> 0 \quad \text{for all } n \in \mathbb{N}, \\ \implies (\mu_\lambda^{u_0^\lambda})''(1) &\geq 0 \quad (\text{see the Claim}). \end{aligned}$$

From Proposition 13 we know that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \hat{\lambda}^*)$. Therefore

$$\begin{aligned} (\mu_\lambda^{u_0^\lambda})''(1) &> 0, \\ \implies u_0^\lambda &\in \mathcal{N}_\lambda^+ \quad \text{and} \quad \varphi_\lambda(u_0^\lambda) = m_\lambda^+ \quad (\text{see (54)}). \end{aligned}$$

The proof is now complete. $\square$

Let

$$K_{\varphi_\lambda} = \{u \in W^{1,\theta}(\Omega) : \varphi'_\lambda(u) = 0\} \quad (\text{critical set of } \varphi_\lambda).$$

In the next proposition, we show that $u_0^\lambda \in \mathcal{N}_\lambda^+$ is a critical point of φ_λ .

Proposition 16. *If hypotheses H hold and $0 < \lambda < \hat{\lambda}^*$, then $u_0^\lambda \in K_{\varphi_\lambda}$.*

Proof. Note that $\mathcal{N}_\lambda^+$ is an open subset of $\mathcal{N}_\lambda$. Also u_0^λ is a minimizer of $\varphi_\lambda|_{\mathcal{N}_\lambda^+}$. Therefore u_0^λ is a local minimizer of $\varphi_\lambda|_{\mathcal{N}_\lambda}$. We can apply the Lagrange multiplier rule (see Papageorgiou et al. [21, p. 422]) and obtain $b \in \mathbb{R}$ such that

$$\varphi'_\lambda(u_0^\lambda) = b\psi'_\lambda(u_0^\lambda) \quad \text{in } W^{1,\theta}(\Omega), \quad (60)$$

where $\psi_\lambda(u) = \langle \varphi'_\lambda(u), u \rangle$ for all $u \in W^{1,\theta}(\Omega) \setminus \{0\}$ (the constraint function defining $\mathcal{N}_\lambda$). From (60), we have

$$\begin{aligned} 0 &= \langle \varphi'_\lambda(u_0^\lambda), u_0^\lambda \rangle = b \langle \psi'_\lambda(u_0^\lambda), u_0^\lambda \rangle \quad (\text{recall that } u_0^\lambda \in \mathcal{N}_\lambda^+ \subseteq \mathcal{N}_\lambda), \\ \implies b &\left[p\gamma(u_0^\lambda) + q(\|Du_0^\lambda\|_q^q + \rho_{\xi_2,q}(u_0^\lambda)) - \lambda\tau\|u_0^\lambda\|_\tau^\tau - r \int_\Omega \eta(z)|u_0^\lambda|^r dz \right] = 0, \\ \implies b &\left[(p-1)\gamma(u_0^\lambda) + (q-1)(\|Du_0^\lambda\|_q^q + \rho_{\xi_2,q}(u_0^\lambda)) - \lambda(\tau-1)\|u_0^\lambda\|_\tau^\tau \right. \\ &\quad \left. - (r-1) \int_\Omega \eta(z)|u_0^\lambda|^r dz \right] = 0, \quad (\text{recall that } u_0^\lambda \in \mathcal{N}_\lambda), \\ \implies b &(\mu_\lambda^{u_0^\lambda})''(1) = 0. \end{aligned} \quad (61)$$

Since $u_0^\lambda \in \mathcal{N}_\lambda^+$, we have $\left(\mu_\lambda^{u_0^\lambda}\right)''(1) > 0$. Therefore from (54) it follows that

$$\begin{aligned} b &= 0, \\ \implies \varphi'_\lambda(u_0^\lambda) &= 0 \quad \text{in } W^{1,\theta}(\Omega) \quad (\text{see (60)}), \\ \implies u_0^\lambda &\in K_{\varphi_\lambda}. \end{aligned}$$

The proof is complete. $\square$

In the next proposition, we prove a regularity property for the elements of K_{φ_λ} . In particular, using Lemma 10, we show that the elements of K_{φ_λ} are bounded functions.

Proposition 17. *If hypotheses H hold, $\lambda > 0$, $u \in K_{\varphi_\lambda}$ and $u \neq 0$, then $u \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$, $0 \prec u$ or $0 \prec -u$.*

Proof. Note that we can always replace u by $|u|$ and so we may assume that $u \neq 0, 0 \leq u$. We have

$$\begin{aligned} -\Delta_p^a u - \Delta_q u + \xi_1(z)u^{p-1} + \xi_2(z)u^{q-1} &\geq 0 \quad \text{in } \Omega, \\ \implies 0 &\prec u \quad (\text{see Papageorgiou et al. [25]}). \end{aligned}$$

So, we have $0 < u(z)$ for a.a. $z \in \Omega$. Also

$$\begin{aligned} \langle V(u), h \rangle + \int_\Omega (\xi_1(z)u^{p-1} + \xi_2(z)u^{q-1}) h dz + \int_{\partial\Omega} \beta(z)u^{p-1} h d\sigma \\ = \int_\Omega (\lambda u^{\tau-1} + \eta(z)u^{r-1}) h dz \quad \text{for all } h \in W^{1,\theta}(\Omega). \end{aligned} \quad (62)$$

Let $k > 1$ be large so that

$$\|(u - k)^+\| \leq 1. \quad (63)$$

In (62), we choose the test function $h = (u - k)^+ \in W^{1,\theta}(\Omega)$. We obtain

$$\begin{aligned} \gamma((u - k)^+) + (\|D(u - k)^+\|_q^q + \rho_{\xi_2,q}((u - k)^+)) \\ = \int_\Omega \lambda u^{\tau-1}(u - k)^+ dz + \int_\Omega \eta(z)u^{r-1}(u - k)^+ dz. \end{aligned} \quad (64)$$

Set $E_k = \{z \in \Omega : u(z) > k\}$ ($k > 1$). Then on E_k we have

$$u^\tau \leq u^r = (u - k + k)^r \leq 2^{r-1}[(u - k)^r + k^r]. \quad (65)$$

We know that $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ (see Proposition 1 and recall that $r < p_s^*$). So, from (64) we obtain

$$\begin{aligned} \left[\int_{E_k} (u - k)^r dz \right]^{\frac{p}{r}} &\leq c_{24} \|(u - k)^+\|^p \quad \text{for some } c_{24} > 0 \\ &\leq c_{25} \left[\int_{E_k} (u - k)^r dz + k^r |E_k|_N \right] \\ &\quad \text{for some } c_{25} = c_{25}(\lambda) > 0 \quad (\text{see (65)}), \\ &\implies u \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad (\text{see Lemma 10}). \end{aligned}$$

This completes the proof. $\square$

Now we are ready to produce the first bounded positive solution for $(\mathcal{P}_\lambda)$ (if $\lambda \in (0, \hat{\lambda}^*)$).

Proposition 18. *If hypotheses H hold and $\lambda \in (0, \hat{\lambda}^*)$, then problem $(\mathcal{P}_\lambda)$ has a positive ground state solution*

$$u_0^\lambda \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega), \quad 0 \prec u_0^\lambda, \quad \varphi_\lambda(u_0^\lambda) = m_\lambda^+ < 0.$$

Proof. From Proposition 15, we know that there exists $u_0^\lambda \in \mathcal{N}_\lambda^+ \subseteq \mathcal{N}_\lambda$ such that

$$\varphi_\lambda(u_0^\lambda) = m_\lambda^+ < 0.$$

Also from Proposition 16, we know that $u_0^\lambda \in K_{\varphi_\lambda}$. We can assume that $u_0^\lambda \geq 0$. We have

$$\langle \varphi'_\lambda(u_0^\lambda), h \rangle = 0 \quad \text{for all } h \in W^{1,\theta}(\Omega), \quad u_0^\lambda \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega), \quad 0 \prec u_0^\lambda. \quad (66)$$

From (66), using the nonlinear Green's identity (see Papageorgiou et al. [21, p. 34]), we conclude that u_0^λ is a solution of $(\mathcal{P}_\lambda)$. $\square$

Next, working with $\mathcal{N}_\lambda^-$, we will produce a second bounded positive solution of $(\mathcal{P}_\lambda)$ when $\lambda > 0$ is small.

Proposition 19. *If hypotheses H hold, then there exists $\lambda_0^* \in (0, \hat{\lambda}^*]$ such that*

$$\varphi_\lambda|_{\mathcal{N}_\lambda^-} > 0 \quad \text{for all } 0 < \lambda < \lambda_0^*.$$

Proof. Let $v \in \mathcal{N}_\lambda^-$. From the definition of $\mathcal{N}_\lambda^-$, we have

$$\begin{aligned} (\mu_\lambda^v)''(1) &< 0, \\ \implies (p-1)\gamma(v) + (q-1)(\|Dv\|_q^q + \rho_{\xi_2,q}(v)) \\ &< \lambda(\tau-1)\|v\|_\tau^\tau + (r-1) \int_\Omega \eta(z)|v|^r dz. \end{aligned} \quad (67)$$

In addition, since $\mathcal{N}_\lambda^- \subseteq \mathcal{N}_\lambda$, we have

$$\gamma(v) + (\|Dv\|_q^q + \rho_{\xi_2,q}(v)) - \int_\Omega \eta(z)|v|^r dz = \lambda\|v\|_\tau^\tau. \quad (68)$$

Using (68) in (67), we obtain

$$\begin{aligned} (p-\tau)\gamma(v) + (q-\tau)(\|Dv\|_q^q + \rho_{\xi_2,q}(v)) &< (r-\tau) \int_\Omega \eta(z)|v|^r dz, \\ \implies (p-\tau)[\gamma(v) + (\|Dv\|_q^q + \rho_{\xi_2,q}(v))] &< (r-\tau) \int_\Omega \eta(z)|v|^r dz \quad (\text{since } q < p), \\ \implies \|v\|_{1,\theta_0}^p + \|v\|_{1,q}^q &\leq c_{26}\|v\|_r^r \quad \text{and} \quad \|v\|_{1,\theta_0}^p + \|v\|_{1,q}^q \leq c_{26}\|v\|^r \\ &\quad \text{for some } c_{26} > 0 \quad (\text{see Propositions 1, 6, and 7}), \\ \implies \min\{\|v\|^p, \|v\|^q\} &\leq c_{27}\|v\|_r^r \quad \text{and} \quad \min\{\|v\|^p, \|v\|^q\} \leq c_{27}\|v\|^r \\ &\quad \text{for some } c_{27} > 0 \quad (\text{see Proposition 3}). \end{aligned}$$

Since $q < p < r$, we infer that

$$0 < c_{28} \leq \|v\|, \|v\|_r, \text{ for all } v \in \mathcal{N}_\lambda^-, \text{ some } c_{28} > 0. \quad (69)$$

Now suppose that the assertion of the proposition is not true. Then for every $\lambda \in (0, \hat{\lambda}^*)$, we can find $v_\lambda \in \mathcal{N}_\lambda^-$ such that

$$\begin{aligned} \varphi_\lambda(v_\lambda) &\leq 0, \\ \implies \frac{1}{p}\gamma(v_\lambda) + \frac{1}{q}(\|Dv_\lambda\|_q^q + \rho_{\xi_2, q}(v_\lambda)) &\leq \frac{\lambda}{\tau}\|v_\lambda\|_\tau^\tau + \frac{1}{r} \int_\Omega \eta(z)|v_\lambda|^r dz, \\ \implies \left(\frac{1}{p} - \frac{1}{r}\right)\gamma(v_\lambda) + \left(\frac{1}{q} - \frac{1}{r}\right)(\|Dv_\lambda\|_q^q + \rho_{\xi_2, q}(v_\lambda)) &\leq \lambda \left(\frac{1}{\tau} - \frac{1}{r}\right)\|v_\lambda\|_\tau^\tau \\ &\quad (\text{since } v_\lambda \in \mathcal{N}_\lambda, \text{ see (68)}), \\ \implies \min\{\|v_\lambda\|^p, \|v_\lambda\|^q\} &\leq \lambda c_{29}\|v_\lambda\|^\tau \text{ for some } c_{29} > 0 \\ &\quad (\text{see Propositions 1, 3, 6, and 7}). \end{aligned} \quad (70)$$

Then from (70) and since $W^{1, \theta}(\Omega) \hookrightarrow L^r(\Omega)$ (see Proposition 1), $\tau < q < p$, we have

$$\|v_\lambda\|_r, \|v_\lambda\| \leq \hat{c}(\lambda) \text{ with } \hat{c}(\lambda) \rightarrow 0^+ \text{ as } \lambda \rightarrow 0^+. \quad (71)$$

Comparing (71) and (69), we have a contradiction. Therefore we can find $\lambda_0^* \in (0, \hat{\lambda}^*]$ such that

$$0 < \varphi_\lambda|_{\mathcal{N}_\lambda^-} \text{ for all } 0 < \lambda < \lambda_0^*.$$

The proof is now complete. $\square$

We consider the following constrained minimization problem

$$m_\lambda^- = \inf_{\mathcal{N}_\lambda^-} \varphi_\lambda.$$

Proposition 19 above implies that $m_\lambda^- \geq 0$.

Proposition 20. *If hypotheses H hold and $\lambda \in (0, \lambda_0^*)$, then there exists $v_0^\lambda \in \mathcal{N}_\lambda^-$ such that $m_\lambda^- = \varphi_\lambda(v_0^\lambda) > 0$.*

Proof. Let $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{N}_\lambda^-$ be such that

$$\varphi_\lambda(v_n) \downarrow m_\lambda^- \geq 0.$$

From Proposition 11, it follows that $\{v_n\}_{n \in \mathbb{N}} \subseteq W^{1, \theta}(\Omega)$ is bounded. So, using Proposition 1 and since $p < \frac{(N-1)p_s}{N-p_s}$, we obtain

$$v_n \xrightarrow{w} v_0^\lambda \text{ in } W^{1, \theta}(\Omega) \text{ and } v_n \rightarrow v_0^\lambda \text{ in } L^{\theta_0}(\Omega), L^r(\Omega), L^p(\partial\Omega). \quad (72)$$

From (69) and (72), we see that

$$\begin{aligned} 0 < c_{28} &\leq \|v_0^\lambda\|_r, \\ \implies v_0^\lambda &\neq 0. \end{aligned}$$

Claim: $v_n \rightarrow v_0^\lambda$ in $W^{1,\theta}(\Omega)$ as $n \rightarrow \infty$.

We argue as in the “Claim” in the proof of Proposition 15. So, suppose that Claim is not true. Then at least one of the following strict inequalities is true

$$\rho_{\theta_0}(Dv_0^\lambda) < \liminf_{n \rightarrow \infty} \rho_{\theta_0}(Dv_n), \quad |v_0^\lambda|_0 < \liminf_{n \rightarrow \infty} |v_n|_0 \quad (\text{see (72)}). \quad (73)$$

From (72), we have

$$\begin{cases} \rho_{\xi_2,q}(v_n) \rightarrow \rho_{\xi_2,q}(v_0^\lambda), & \|v_n\|_\tau \rightarrow \|v_0^\lambda\|_\tau, \\ \int_\Omega \eta(z)|v_n|^r dz \rightarrow \int_\Omega \eta(z)|v_0^\lambda|^r dz. \end{cases} \quad (74)$$

We know that $v_0^\lambda \neq 0$. So, we can find $\hat{t}_2 > 0$ such that

$$\hat{t}_2 v_0^\lambda \in \mathcal{N}_\lambda^- \quad (\text{see the proof of Proposition 14}). \quad (75)$$

From (73), (74), and (75), it follows that

$$\begin{aligned} \varphi_\lambda(\hat{t}_2 v_0^\lambda) &< \liminf_{n \rightarrow \infty} \left[\frac{\hat{t}_2^p}{p} \rho_{\theta_0}(Dv_n) + \frac{\hat{t}_2^q}{q} (\|Dv_n\|_q^q + \rho_{\xi_2,q}(v_n)) - \frac{\lambda \hat{t}_2^\tau}{\tau} \|v_n\|_\tau^\tau \right. \\ &\quad \left. - \frac{\hat{t}_2^r}{r} \int_\Omega \eta(z)|v_n|^r dz \right]. \end{aligned} \quad (76)$$

Since $v_n \in \mathcal{N}_\lambda^- \subseteq \mathcal{N}_\lambda$, we have

$$\begin{aligned} \varphi_\lambda(\hat{t}_2 v_n) &\leq \varphi_\lambda(v_n) \quad \text{for all } n \in \mathbb{N}, \\ \implies m_\lambda^- &\leq \varphi_\lambda(\hat{t}_2 v_0^\lambda) < \liminf_{n \rightarrow \infty} \varphi_\lambda(v_n) = m_\lambda^- \quad (\text{see (76)}), \end{aligned}$$

a contradiction. This proves the Claim.

Using the Claim, we see that

$$v_0^\lambda \in \mathcal{N}_\lambda \quad \text{and} \quad \varphi_\lambda(v_0^\lambda) = m_\lambda^-. \quad (77)$$

We know that

$$\begin{aligned} (\mu_\lambda^{v_n})''(1) &< 0 \quad \text{for all } n \in \mathbb{N} \quad (\text{recall that } v_n \in \mathcal{N}_\lambda^-), \\ \implies (\mu_\lambda^{v_0^\lambda})''(1) &\leq 0 \quad (\text{see the Claim}). \end{aligned} \quad (78)$$

From Proposition 13 we know that $\mathcal{N}_\lambda^0 = \emptyset$ for all $0 < \lambda < \lambda_0^*$. Therefore from (78), we obtain

$$\begin{aligned} (\mu_\lambda^{v_0^\lambda})''(1) &< 0 \quad \text{for all } 0 < \lambda < \lambda_0^*, \\ \implies v_0^\lambda &\in \mathcal{N}_\lambda^- \quad \text{for all } 0 < \lambda < \lambda_0^*. \end{aligned}$$

Suppose that $m_{\lambda}^- = 0$. Then

$$\begin{aligned} & \left(\frac{1}{p} - \frac{1}{r} \right) \gamma(v_0^\lambda) + \left(\frac{1}{q} - \frac{1}{r} \right) (\|Dv_0^\lambda\|_q^q + \rho_{\xi_2, q}(v_0^\lambda)) = \lambda \left(\frac{1}{\tau} - \frac{1}{r} \right) \|v_0^\lambda\|_\tau^\tau \\ & \quad (\text{since } v_0^\lambda \in \mathcal{N}_\lambda^- \subseteq \mathcal{N}_\lambda), \\ & \implies \min\{\|v_0^\lambda\|^p, \|v_0^\lambda\|^q\} \leq \lambda c_{30} \|v_0^\lambda\|^\tau \text{ for some } c_{30} > 0 \\ & \quad (\text{see Propositions 1 and 9}), \\ & \implies v_0^\lambda \rightarrow 0 \text{ in } W^{1, \theta}(\Omega) \text{ as } \lambda \rightarrow 0^+, \end{aligned}$$

which contradicts (60). Therefore $m_{\lambda}^- > 0$ for all $0 < \lambda < \lambda_0^*$. $\square$

Reasoning as in the proof of Proposition 16, via the Lagrange multiplier rule, we obtain the following result.

Proposition 21. *If hypotheses H hold and $\lambda \in (0, \lambda_0^*)$, then $v_0^\lambda \in K_{\varphi_\lambda}$.*

This leads to a second bounded positive solution for problem $(\mathcal{P}_\lambda)$, this time with positive energy.

Proposition 22. *If hypotheses H hold and $\lambda \in (0, \lambda_0^*)$, then problem $(\mathcal{P}_\lambda)$ has another positive ground state solution*

$$v_0^\lambda \in W^{1, \theta}(\Omega) \cap L^\infty(\Omega), \quad 0 \prec v_0^\lambda, \quad \varphi_\lambda(v_0^\lambda) = m_{\lambda}^- > 0.$$

Propositions 18 and 22 lead to the following multiplicity theorem for ground state (least energy) solutions for problem $(\mathcal{P}_\lambda)$, when the parameter $\lambda > 0$ is small.

Theorem 23. *If hypotheses H hold, then for all $\lambda > 0$ small problem $(\mathcal{P}_\lambda)$ has at least two ground state solutions*

$$u_0^\lambda, v_0^\lambda \in W^{1, \theta}(\Omega) \cap L^\infty(\Omega), \quad 0 \prec u_0^\lambda, \quad 0 \prec v_0^\lambda, \quad \varphi_\lambda(u_0^\lambda) < 0 < \varphi_\lambda(v_0^\lambda).$$

Acknowledgements

Peng Jin was supported by the China Scholarship Council (No. 202406370154) and by the Hunan Province Graduate Research Innovation Project (No. CX20240163). N.S. Papageorgiou was supported by the Innovation Research Team of the Natural Science Foundation of Guangxi (Grant No. 2025GXNSFGA06900) and the Bagui Scholar Project of Guangxi. The research of V.D. Rădulescu was supported by the AGH University of Kraków under Grant No. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education.

Author contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Data Availability Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Ethical approval Not applicable.

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Received: February 5, 2026.

Accepted: May 8, 2026.